Алгебра и анализ
Том. 17 (2005), вып. 5

St. Petersburg Math. J.
Vol. 17 (2006), No. 5, Pages 773–796
S 1061-0022(06)00929-0
Article electronically published on July 20, 2006

ON THE EXISTENCE OF EXTREMAL FUNCTIONS IN SOBOLEV EMBEDDING THEOREMS WITH CRITICAL EXPONENTS

A. V. DEMYANOV AND A. I. NAZAROV

In memory of Ol'ga Aleksandrovna Ladyzhenskaya

Abstract. Sufficient conditions for the existence of extremal functions in Sobolev-type inequalities on manifolds with or without boundary are established. Some of these conditions are shown to be sharp. Similar results for embeddings in some weighted L_q-spaces are obtained.

§1. Introduction

Let $n \geq 2$, and let Ω be a smooth compact n-dimensional Riemannian manifold (with or without boundary) with metric g. In the sequel we shall omit the words "smooth", "compact" and "Riemannian". For $1 < p < n$, we denote by $p^* = \frac{np}{n-p}$ the Sobolev conjugate of p, i.e., the critical exponent for the embedding $W_p^1(\Omega) \hookrightarrow L_q(\Omega)$.

Since the embedding operator $W_p^1(\Omega) \hookrightarrow L_{p^*}(\Omega)$ is noncompact, the problem as to whether this operator attains its norm (i.e., the existence problem for an extremal function in the embedding theorem) is nontrivial. This problem was treated in many papers (see, e.g., [Br, LPT] and references therein).

In this paper we establish sufficient conditions for the existence of an extremal function in four embedding theorems:

$$\text{(I)} \qquad \lambda_1(n,p,\Omega) = \inf_{v \in \overset{\circ}{W}{}_p^1(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^*,\Omega}} > 0;$$

$$\text{(II)} \qquad \lambda_2(n,p,\Omega) = \inf_{v \in W_p^1(\Omega)\setminus\{0\}} \frac{\|v\|_{W_p^1(\Omega)}}{\|v\|_{p^*,\Omega}} > 0$$

(the norm in the numerator is defined as $\|v\|^p_{W_p^1(\Omega)} = \|\nabla v\|^p_{p,\Omega} + \|v\|^p_{p,\Omega}$);

$$\text{(III)} \qquad \lambda_3(n,p,\Omega) = \inf_{v \in W_p^1(\Omega)\setminus\{c\}} \frac{\|\nabla v\|_{p,\Omega}}{\|v - \overline{v}\|_{p^*,\Omega}} > 0$$

(here we use the notation $\overline{v} = |\Omega|^{-1} \int_\Omega v \, dV_g$);

$$\text{(IV)} \qquad \lambda_4(n,p,\Omega) = \inf_{v \in W_p^1(\Omega)\setminus\{c\}} \sup_{a \in \mathbb{R}} \frac{\|\nabla v\|_{p,\Omega}}{\|v - a\|_{p^*,\Omega}} > 0.$$

2000 *Mathematics Subject Classification.* Primary 49J10, 35J20, 35J60.

Key words and phrases. Minimizers, critical exponent, Sobolev inequality, Sobolev–Poincaré inequality, Hardy–Sobolev inequality, p-Laplacian.

Partially supported by the RF Ministry of Education (project no. 4733), and by RFBR (grant no. 05–01–01063).

It is well known that if Ω is a bounded domain in $\mathbb{R}^n$, then the infimum in (I) is *not attained*, does not depend on Ω, and is equal to $\frac{1}{K(n,p)}$, where

$$K(n,p) = \sup_{v\in\mathcal{C}_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\|v\|_{p^*,\mathbb{R}^n}}{\|\nabla v\|_{p,\mathbb{R}^n}} = \omega_{n-1}^{-\frac{1}{n}}\cdot k(n,p), \tag{1}$$

and

$$k(n,p) = n^{-\frac{1}{p}}\Big(\frac{p-1}{n-p}\Big)^{\frac{1}{p'}}\Big(\mathcal{B}\Big(\frac{n}{p},\frac{n}{p'}+1\Big)\Big)^{-\frac{1}{n}}$$

is the sharp constant in the Bliss inequality [Bl]. The second identity in (1) was obtained in [Aub, Tal]. Note that the supremum in (1) is only attained on radially symmetric functions

$$w_\varepsilon(r) \equiv (\varepsilon + r^{p'})^{1-\frac{n}{p}}, \quad \varepsilon\in\mathbb{R}_+, \tag{2}$$

with *noncompact* support, and on their translates and dilations.

On the other hand (see [LPT]), the infimum in (I) can be attained, under certain additional assumptions, if we deal with functions in $W_p^1(\Omega)$ that do not belong to $\overset{\circ}{W}{}_p^1(\Omega)$ but vanish on some part of $\partial\Omega$.

In some particular cases, results on the attainability or nonattainability of the infimum in (II) were obtained for domains Ω in $\mathbb{R}^n$ [AM, W], and for manifolds Ω without boundary [Dr]. Note that the existence problem for extremal functions in (II) is closely related to the optimal constants in Sobolev inequalities [Aub, HV, Dr].

Finally, inequalities (III) and (IV) on manifolds without boundary were studied in [Zhu]. However, the proofs of Theorems 1.1 and 1.3 in [Zhu] have some gaps. Moreover, in §6 we shall disprove Theorem 1.3 in [Zhu].

The following proposition is of crucial importance for the discussions to follow.

Proposition 1.1. *Let Ω be a manifold with boundary* (*problem* (I)) *or without boundary* (*problems* (II)–(IV)). *Suppose that the infimum of the corresponding problem satisfies the inequality*

$$\lambda_m(n,p,\Omega) < \frac{1}{K(n,p)}.$$

Then the infimum is attained.

Proof. Consider a minimizing sequence $\{v_k\}$ normalized in $L_{p^*}(\Omega)$. Without loss of generality, we may assume that $v_k \rightharpoonup v$ in $W_p^1(\Omega)$. By the Lions theorem (see [Ls, Part 1] and also [LPT, Lemma 2.2]), we have

$$|v_k|^{p^*} \rightharpoonup |v|^{p^*} + \sum_j c_j\delta(x-x_j), \quad |\nabla v_k|^p \rightharpoonup |\nabla v|^p + \sum_j C_j\delta(x-x_j)$$

(with convergence in the space of measures on $\overline{\Omega}$), where $\{x_j\}$ is an at most countable set of distinct points in $\overline{\Omega}$, and c_j, C_j are positive constants.

Since $\{v_k\}$ is a minimizing sequence, a word-for-word repetition of the proof of Theorem 2.2 in [LPT] yields the following alternative: either v is a minimizer of the corresponding problem and the set $\{x_j\}$ is empty, or $v=0$ and the set $\{x_j\}$ contains a unique point x_0. In the second case we have $c_0=1$ and $C_0=\lambda_m^p(n,p,\Omega)$; moreover, $x_0\notin\partial\Omega$ in the case of problem (I).

If the second case occurs, then, arguing as in Corollary 2.1 in [LPT], we multiply v_k by a cut-off function with a sufficiently small support and show easily that $\lambda_m(n,p,\Omega)\geq \frac{1}{K(n,p)}$. This contradiction proves the claim. $\square$

Thus, to prove that the infimum in (I)–(IV) is attained, it suffices to present a function for which the corresponding quotient is less than $\frac{1}{K(n,p)}$. Following [LPT] and assuming certain conditions on the exponent p and on the manifold Ω, we construct such a function in the form of a function having small support and simulating the behavior of $w_\varepsilon(r)$. It is interesting to note that, in some cases, this very specific approach leads to sharp results.

This paper is organized as follows. §2 contains auxiliary integral estimates. Some of them were obtained in [Dr, Zhu]; we present these estimates for the reader's convenience. Problem (I) for manifolds with boundary is considered in §3. Problems (II), (III), and (IV) for manifolds without boundary are treated in §4, 5, and 6, respectively. §7 is devoted to problems (II)–(IV) for a domain in $\mathbb{R}^n$ and for a manifold with boundary. In §8 we investigate analogs of problems (I)–(IV) that correspond to critical embeddings into Lebesgue spaces with singular weights. Finally, in §9 we generalize the results of §§7–8 to various definitions of the norms occurring in the numerators in (I)–(IV).

We fix some notation: $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere in $\mathbb{R}^n$; $p' = \frac{p}{p-1}$ is the Hölder conjugate exponent to p; $\mathcal{B}$ is the Euler beta function. We write $o_\rho(1)$ for a quantity that tends to zero as $\rho \to 0$.

We use the letter C to denote various positive constants. To indicate that C depends on some parameters, we write $C(\cdots)$.

§2. Auxiliary calculations

Let $x_0 \in \Omega$. We denote $r = \operatorname{dist}_g(x, x_0)$. For sufficiently small $\varepsilon > 0$ and $\rho > 0$, we introduce the function

$$u(x) = u(r) = \varphi(r) w_\varepsilon(r), \tag{3}$$

where w_ε is the function defined in (2), and $\varphi(r)$ is a smooth cut-off function such that

$$\varphi(r) \begin{cases} = 1 & \text{if } r \le \frac{\rho}{2}, \\ \le 1 & \text{if } \frac{\rho}{2} \le r \le \rho, \\ = 0 & \text{if } r \ge \rho. \end{cases} \tag{4}$$

We note (see, e.g., [Dr]) that, when integrating a function depending only on r, for the volume form in a neighborhood of x_0 we have

$$dV_g = \omega_{n-1}\Big(r^{n-1} - \frac{R_g(x_0)}{6n} r^{n+1} + o(r^{n+1})\Big)\, dr,$$

where $R_g(x_0)$ stands for the scalar curvature at x_0. Thus, all estimations reduce to manipulations with one-dimensional integrals.

2.1. Estimates of $\|\nabla u\|_{p,\Omega}$ for $1 < p \le \frac{n+2}{3}$. We apply the estimate

$$|\nabla(fg)|^p \le |f\nabla g|^p + C\left(|\nabla f\, g|^p + |\nabla f\, g| \cdot |f \nabla g|^{p-1}\right) \tag{5}$$

to ∇u. Since $\nabla\varphi(r)$ is nonzero only for $\rho/2 < r < \rho$, we have

$$\begin{aligned} &\int_\Omega \left(|\nabla\varphi(r) w_\varepsilon(r)|^p + |\nabla\varphi(r) w_\varepsilon(r)| \cdot |\varphi(r)\nabla w_\varepsilon(r)|^{p-1}\right) dV_g \\ &\qquad \le C \int_{\rho/2}^{\rho} \Big[\rho^{-p}(\varepsilon + r^{p'})^{p-n} + \rho^{-1} r (\varepsilon + r^{p'})^{1-n}\Big] r^{n-1}\, dr \le C\rho^{\frac{p-n}{p-1}}. \end{aligned} \tag{6}$$

Furthermore,

$$|\varphi(r)\nabla w_\varepsilon(r)|^p \le |\nabla w_\varepsilon(r)|^p = \Big(\frac{n-p}{p-1}\Big)^p r^{p'} (\varepsilon + r^{p'})^{-n}.$$

Inequalities (5) and (6) imply

$$\int_\Omega |\nabla u|^p\,dV_g \le I_1 - I_2(R_g(x_0)+o_\rho(1)) + C\rho^{\frac{p-n}{p-1}}, \tag{7}$$

where

$$I_1 = \omega_{n-1}\int_0^\rho \Big(\frac{n-p}{p-1}\Big)^p r^{p'}(\varepsilon + r^{p'})^{-n} r^{n-1}\,dr,$$

$$I_2 = \frac{\omega_{n-1}}{6n}\int_0^{\frac{\rho}{2}} \Big(\frac{n-p}{p-1}\Big)^p r^{p'}(\varepsilon + r^{p'})^{-n} r^{n+1}\,dr.$$

It is easily seen that

$$I_1 \le E_1\varepsilon^{1-\frac{n}{p}}, \tag{8}$$

where

$$\begin{aligned} E_1 &= \omega_{n-1}\Big(\frac{n-p}{p-1}\Big)^p \int_0^\infty (1+t^{p'})^{-n} t^{p'+n-1}\,dt \\ &= \frac{\omega_{n-1}}{p'}\Big(\frac{n-p}{p-1}\Big)^p \cdot \mathcal{B}\Big(\frac{n}{p'}+1, \frac{n}{p}-1\Big). \end{aligned} \tag{9}$$

Next,

$$I_2 = \frac{\omega_{n-1}}{6n}\Big(\frac{n-p}{p-1}\Big)^p \varepsilon^{1-\frac{n}{p}+\frac{2}{p'}} \int_0^{\frac{\rho}{2}\varepsilon^{-\frac{1}{p'}}} (1+t^{p'})^{-n} t^{p'+n+1}\,dt.$$

If $p < \frac{n+2}{3}$, we have

$$I_2 \ge E_2\varepsilon^{1-\frac{n}{p}+\frac{2}{p'}} - C\rho^{\frac{p-n}{p-1}+2}. \tag{10}$$

Here

$$\begin{aligned} E_2 &= \frac{\omega_{n-1}}{6n}\Big(\frac{n-p}{p-1}\Big)^p \int_0^\infty (1+t^{p'})^{-n} t^{p'+n+1}\,dt \\ &= \frac{\omega_{n-1}}{6np'}\Big(\frac{n-p}{p-1}\Big)^p \cdot \mathcal{B}\Big(\frac{n+2}{p'}+1, \frac{n+2}{p}-3\Big). \end{aligned} \tag{11}$$

But if $p = \frac{n+2}{3}$, we obtain

$$I_2 \ge C\big(1 - |\ln\rho| + \ln\varepsilon^{-\frac{1}{p'}}\big). \tag{12}$$

2.2. Estimates of $\|u\|_{p^*,\Omega}$ for $1 < p < \frac{n+2}{2}$. We have

$$\int_\Omega u^{p^*}\,dV_g \ge \int_{r<\rho/2} w_\varepsilon^{p^*}(r)\,dV_g = J_1 - J_2(R_g(x_0)+o_\rho(1)), \tag{13}$$

where

$$J_1 = \omega_{n-1}\int_0^{\frac{\rho}{2}} (\varepsilon + r^{p'})^{-n} r^{n-1}\,dr,$$

$$J_2 = \frac{\omega_{n-1}}{6n}\int_0^{\frac{\rho}{2}} (\varepsilon + r^{p'})^{-n} r^{n+1}\,dr.$$

Setting

$$D_1 = \omega_{n-1}\int_0^\infty (1+t^{p'})^{-n} t^{n-1}\,dt = \frac{\omega_{n-1}}{p'}\cdot\mathcal{B}\Big(\frac{n}{p'}, \frac{n}{p}\Big), \tag{14}$$

$$D_2 = \frac{\omega_{n-1}}{6n}\int_0^\infty (1+t^{p'})^{-n} t^{n+1}\,dt = \frac{\omega_{n-1}}{6np'}\cdot\mathcal{B}\Big(\frac{n+2}{p'}, \frac{n+2}{p}-2\Big), \tag{15}$$

we arrive at the inequalities

$$J_1 \ge D_1\varepsilon^{-\frac{n}{p}} - C\rho^{-\frac{n}{p-1}}, \quad J_2 \le \varepsilon^{-\frac{n}{p}+\frac{2}{p'}} D_2. \tag{16}$$

2.3. Estimates for $\|u\|_{p,\Omega}$. The relation $dV_g \le Cr^{n-1}\,dr$ yields

$$\begin{aligned}\int_\Omega u^p\,dV_g &\le C\int_0^\rho w_\varepsilon^p(r)r^{n-1}\,dr = C\varepsilon^{\frac{p^2-n}{p}}\int_0^{\rho\varepsilon^{-\frac{1}{p'}}}(1+t^{p/(p-1)})^{p-n}t^{n-1}\,dt\\ &\le \begin{cases} C\varepsilon^{\frac{p^2-n}{p}} & \text{if } 1<p<\sqrt{n},\\ C\big(1+|\ln\rho|+\ln\varepsilon^{-\frac{1}{p'}}\big) & \text{if } p=\sqrt{n},\\ C & \text{if } p>\sqrt{n}.\end{cases}\end{aligned} \tag{17}$$

2.4. Estimates for $\int_\Omega u\,dV_g$. We have

$$\begin{aligned}\int_\Omega u\,dV_g &\le C\int_0^\rho w_\varepsilon(r)r^{n-1}\,dr = C\varepsilon^{\frac{np+p-2n}{p}}\int_0^{\rho\varepsilon^{-\frac{1}{p'}}}(1+t^{p'})^{1-\frac{n}{p}}t^{n-1}\,dt\\ &\le \begin{cases} C\varepsilon^{\frac{np+p-2n}{p}} & \text{if } 1<p<\frac{2n}{n+1},\\ C\big(1+|\ln\rho|+\ln\varepsilon^{-\frac{1}{p'}}\big) & \text{if } p=\frac{2n}{n+1},\\ C & \text{if } p>\frac{2n}{n+1}.\end{cases}\end{aligned} \tag{18}$$

Suppose $p>\frac{2n}{n+1}$. Then the relation $dV_g \ge Cr^{n-1}\,dr$ implies the estimate

$$\int_\Omega u\,dV_g \ge \int_0^{\rho/2} w_\varepsilon(r)\,dr = C\varepsilon^{\frac{np+p-2n}{p}}\int_0^{\frac{\rho}{2}\varepsilon^{-\frac{1}{p'}}}(1+t^{p'})^{1-\frac{n}{p}}t^{n-1}\,dt \ge C\rho^{\frac{np+p-2n}{p-1}} \tag{19}$$

for all $\rho\le 1$ and all sufficiently small ε.

2.5. Estimates for $\int_\Omega u^{p^*-1}\,dV_g$. We have

$$\begin{aligned}\int_\Omega u^{p^*-1}\,dV_g &\le C\int_0^\rho w_\varepsilon^{p^*-1}(r)r^{n-1}\,dr\\ &= C\varepsilon^{-1}\int_0^{\rho\varepsilon^{-\frac{1}{p'}}}(1+t^{p'})^{-\frac{np-n+p}{p}}t^{n-1}\,dt \le C\varepsilon^{-1}.\end{aligned} \tag{20}$$

On the other hand, given ρ, for ε sufficiently small we obtain

$$\begin{aligned}\int_\Omega u^{p^*-1}\,dV_g &\ge C\int_0^{\rho/2} w_\varepsilon^{p^*-1}(r)r^{n-1}\,dr\\ &= C\varepsilon^{-1}\int_0^{\frac{\rho}{2}\varepsilon^{-\frac{1}{p'}}}(1+t^{p'})^{-\frac{np-n+p}{p}}t^{n-1}\,dt \ge C\varepsilon^{-1}.\end{aligned} \tag{21}$$

2.6. The case of a domain in $\mathbb{R}^n$. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$. We take $x_0\in\partial\Omega$ and consider the function u defined in (3) with $r=|x-x_0|$.

When integrating a function depending only on r, for the volume form in a neighborhood of x_0 we have

$$dx = \frac{1}{2}\left(\omega_{n-1}r^{n-1} - H(x_0)\omega_{n-2}r^n + o(r^n)\right)\,dr,$$

where $H(x_0)$ stands for the mean curvature of $\partial\Omega$ at x_0 with respect to the inward normal.

Arguing as above, we see that

$$\int_\Omega |\nabla u|^p\,dx \le \begin{cases} \frac12 E_1\varepsilon^{1-\frac{n}{p}} - \widetilde{E}_2(H(x_0)+o_\rho(1))\varepsilon^{1-\frac{n}{p}+\frac{1}{p'}} + C\rho^{\frac{p-n}{p-1}} & \text{if } p<\frac{n+1}{2},\\ \frac12 E_1\varepsilon^{1-\frac{n}{p}} - C(H(x_0)+o_\rho(1))\ln\varepsilon^{-\frac{1}{p'}} + C\rho^{\frac{p-n}{p-1}} & \text{if } p=\frac{n+1}{2},\end{cases} \tag{22}$$

$$\int_\Omega |u|^{p^*}\,dx \ge \frac12 D_1\varepsilon^{-\frac{n}{p}} - \widetilde{D}_2(H(x_0)+o_\rho(1))\varepsilon^{-\frac{n}{p}+\frac{1}{p'}} - C\rho^{-\frac{n}{p-1}}, \tag{23}$$

where E_1, D_1 are given by (9), (14), and

$$\begin{aligned}\widetilde{E}_2 &= \frac{\omega_{n-2}}{2}\Big(\frac{n-p}{p-1}\Big)^p\int_0^\infty (1+t^{p'})^{-n}t^{p'+n}\,dt\\ &= \frac{\omega_{n-2}}{2p'}\Big(\frac{n-p}{p-1}\Big)^p\cdot\mathcal{B}\Big(\frac{n+1}{p'}+1,\frac{n+1}{p}-2\Big),\\ \widetilde{D}_2 &= \frac{\omega_{n-2}}{2}\int_0^\infty (1+t^{p'})^{-n}t^{n}\,dt\\ &= \frac{\omega_{n-2}}{2p'}\cdot\mathcal{B}\Big(\frac{n+1}{p'},\frac{n+1}{p}-1\Big).\end{aligned}$$

Moreover, as in the case of manifolds, estimates (17)–(21) are valid.

§3. The Sobolev inequality on a manifold with boundary, $n \ge 2$

Theorem 3.1. *Let $n \ge 2$, and let Ω be an n-dimensional manifold with boundary. Suppose that the scalar curvature is positive at some point of Ω. Then, for some $\beta > 0$ and all $1 < p < \frac{n+2}{3} + \beta$, the infimum in* (I) *is attained.*

Proof. Let $x_0 \in \Omega$ be a point with positive scalar curvature, and let u be the function defined in (3). Using (7), (8), (10), and (12), we get

$$\int_\Omega |\nabla u|^p\,dV_g \le \begin{cases} E_1\varepsilon^{1-\frac{n}{p}} - E_2\varepsilon^{1-\frac{n}{p}+\frac{2}{p'}}(R_g(x_0)+o_\rho(1)) + C\rho^{\frac{p-n}{p-1}} & \text{if } p<\frac{n+2}{3};\\ E_1\varepsilon^{1-\frac{n}{p}} - C(R_g(x_0)+o_\rho(1))\ln\varepsilon^{-\frac{1}{p'}} + C\rho^{\frac{p-n}{p-1}} & \text{if } p=\frac{n+2}{3},\end{cases} \tag{24}$$

where E_1 and E_2 are the quantities defined in (9) and (11).

Estimates (13) and (16) yield

$$\int_\Omega u^{p^*}\,dV_g \ge D_1\varepsilon^{-\frac{n}{p}} - D_2(R_g(x_0)+o_\rho(1))\varepsilon^{-\frac{n}{p}+\frac{2}{p'}} - C\rho^{-\frac{n}{p-1}} \tag{25}$$

for $1 < p < \frac{n+2}{2}$, where D_1 and D_2 are the quantities defined in (14) and (15).

For $p < \frac{n+2}{3}$, inequalities (24) and (25) result in

$$\frac{\|\nabla u\|_{p,\Omega}^p}{\|u\|_{p^*,\Omega}^p} \le \frac{E_1}{D_1^{p/p^*}}\Big[1+\Big(\frac{p}{p^*}\frac{D_2}{D_1}-\frac{E_2}{E_1}\Big)(R_g(x_0)+o_\rho(1))\varepsilon^{\frac{2}{p'}} + o(\varepsilon^{\frac{2}{p'}})\Big].$$

Direct computations show that if $p < \frac{n+2}{3}$, then

$$\frac{p}{p^*}\frac{D_2}{D_1} - \frac{E_2}{E_1} = -\frac{p}{2n^2}\cdot\frac{\mathcal{B}(\frac{n+2}{p'},\frac{n+2}{p}-3)}{\mathcal{B}(\frac{n}{p'},\frac{n}{p}-1)} < 0. \tag{26}$$

Consequently, for sufficiently small ε and ρ we have

$$\frac{\|\nabla u\|_{p,\Omega}}{\|u\|_{p^*,\Omega}} < \frac{E_1^{1/p}}{D_1^{1/p^*}} = \frac{1}{K(n,p)}. \tag{27}$$

For $p = \frac{n+2}{3}$, we also obtain (27) for ε and ρ sufficiently small. By continuity, for some $\beta > 0$ and $p < \frac{n+2}{3} + \beta$ we have

$$\lambda_1(n, p, \Omega) < \frac{1}{K(n,p)}.$$

Applying Proposition 1.1, we complete the proof. □

Now we show that the statement of Theorem 3.1 is sharp.

Theorem 3.2. *For any $\beta > 0$, there exists $\theta_* > 0$ such that if $p \geq \frac{n+2}{3} + \beta$, then the infimum in* (I) *on the spherical "hat"*

$$\Omega = \{(\theta, \phi_1, \dots, \phi_{n-1}) \in \mathbb{S}^n : 0 < \theta < \theta_*\} \tag{28}$$

is not attained.

Proof. Let $u \in \overset{\circ}{W}{}^1_p(\Omega)$ be a minimizer. Spherical symmetrization arguments allow us to assume that u depends only on θ and is monotone decreasing. Then $u(\theta_*) = 0$ and

$$\frac{\|\nabla u\|_{p,\Omega}}{\|u\|_{p^*,\Omega}} = \omega_{n-1}^{\frac{1}{n}} \cdot \frac{\left(\int_0^{\theta_*} |u'(\theta)|^p \sin^{n-1}(\theta)\, d\theta\right)^{\frac{1}{p}}}{\left(\int_0^{\theta_*} |u(\theta)|^{p^*} \sin^{n-1}(\theta)\, d\theta\right)^{\frac{1}{p^*}}}.$$

Set $\mu = \frac{n-1}{p-1} > 1$. Then the change of variables

$$x(\theta) = \int_\theta^{\theta_*} \sin^{-\mu}(t)\, dt$$

yields

$$\frac{\|\nabla u\|_{p,\Omega}}{\|u\|_{p^*,\Omega}} = \omega_{n-1}^{\frac{1}{n}} \cdot \frac{\left(\int_0^{\infty} |u'(x)|^p\, dx\right)^{\frac{1}{p}}}{\left(\int_0^{\infty} |u(x)|^{p^*} f(x)\, dx\right)^{\frac{1}{p^*}}}, \tag{29}$$

where $f(x) = \sin^{\mu p}(\theta)$.

Similarly, for the function $v(x) = u(\varkappa x)$ we have

$$\begin{aligned} \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^*,\Omega}} &= \omega_{n-1}^{\frac{1}{n}} \cdot \frac{\left(\int_0^{\infty} |v'(x)|^p\, dx\right)^{\frac{1}{p}}}{\left(\int_0^{\infty} |v(x)|^{p^*} f(x)\, dx\right)^{\frac{1}{p^*}}} \\ &= \omega_{n-1}^{\frac{1}{n}} \cdot \frac{\left(\int_0^{\infty} |u'(x)|^p\, dx\right)^{\frac{1}{p}}}{\left(\int_0^{\infty} |u(x)|^{p^*} f(\frac{x}{\varkappa})\, dx\right)^{\frac{1}{p^*}}} \cdot \varkappa^{1-\frac{1}{n}}. \end{aligned} \tag{30}$$

We claim that the following inequality is true for $\varkappa < 1$:

$$f\left(\frac{x}{\varkappa}\right) > f(x)\varkappa^{\frac{(n-1)p}{n-p}}, \quad x \in \mathbb{R}_+. \tag{31}$$

Then, (29) and (30) show that $\frac{\|\nabla u\|_{p,\Omega}}{\|u\|_{p^*,\Omega}} > \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^*,\Omega}}$, which contradicts the minimality of u, and the theorem follows.

Let θ_1 be defined by the relation $\int_{\theta_1}^{\theta_*} \sin^{-\mu}(t)\, dt = \frac{x}{\varkappa}$. Then $f(\frac{x}{\varkappa}) = \sin^{\mu p}(\theta_1)$, and (31) can be rewritten as

$$\frac{\sin^{\mu-1}(\theta_1)}{\sin^{\mu-1}(\theta)} > \varkappa = \int_\theta^{\theta_*} \sin^{-\mu}(t)\, dt \cdot \left(\int_{\theta_1}^{\theta_*} \sin^{-\mu}(t)\, dt\right)^{-1}.$$

Since $\varkappa < 1$, we have $\theta_1(x) = \theta(\frac{x}{\varkappa}) < \theta(x)$. Hence, the above inequality is equivalent to the fact that the function

$$h(\theta) = \sin^{\mu-1}(\theta) \cdot \int_\theta^{\theta_*} \sin^{-\mu}(t)\, dt$$

is strictly monotone decreasing. Obviously,

$$h'(\theta) < 0 \iff (\mu-1)\sin^{\mu-1}(\theta)\cos(\theta) \int_\theta^{\theta_*} \sin^{-\mu}(t)\, dt < 1. \tag{32}$$

Observe that

$$(\mu-1)\int_\theta^{\theta_*} \sin^{-\mu}(t)\, dt = \frac{\cos(\theta)}{\sin^{\mu-1}(\theta)} - \frac{\cos(\theta_*)}{\sin^{\mu-1}(\theta_*)} + (\mu-2)\int_\theta^{\theta_*} \sin^{2-\mu}(t)\, dt,$$

and (32) is equivalent to the inequality

$$1 - \cos^2(\theta) > \sin^{\mu-1}(\theta)\cos(\theta) \cdot \left[(\mu-2)\int_\theta^{\theta_*} \sin^{2-\mu}(t)\, dt - \frac{\cos(\theta_*)}{\sin^{\mu-1}(\theta_*)}\right].$$

For $p > \frac{n+2}{3}$ we have $\mu < 3$. Therefore, $\sin^{2-\mu}(t)$ is integrable in a neighborhood of zero, and the expression in the square brackets is negative for $0 < \theta < \theta_*$ if θ_* is sufficiently small. This implies (32). □

Remark 1. If $\theta_* = \pi/2$ (the case of a hemisphere), inequality (32) is valid for $\mu \le 2$, i.e., for $p \ge \frac{n+1}{2}$.

§4. The limit embedding theorem on manifolds without boundary, $n \ge 5$

Theorem 4.1. *Let $n \ge 5$, and let Ω be an n-dimensional manifold without boundary. Suppose that the scalar curvature is positive at some point of Ω. Then, for some $\beta > 0$ and all $2 < p < \frac{n+2}{3} + \beta$, the infimum in* (II) *is attained.*

Proof. Let $x_0 \in \Omega$ be a point with positive scalar curvature, and let u be the function defined in (3). Estimates (24) and (17) imply

$$\|u\|^p_{W^1_p(\Omega)} \le \begin{cases} E_1\varepsilon^{1-\frac{n}{p}} - E_2(R_g(x_0)+o_\rho(1))\varepsilon^{1-\frac{n}{p}+\frac{2}{p'}} + C\varepsilon^{\frac{p^2-n}{p}} + C(\rho) & \text{if } 1<p<\sqrt{n}, \\ E_1\varepsilon^{1-\frac{n}{p}} - E_2(R_g(x_0)+o_\rho(1))\varepsilon^{1-\frac{n}{p}+\frac{2}{p'}} + C\ln\varepsilon^{-\frac{1}{p'}} + C(\rho) & \text{if } p=\sqrt{n}, \\ E_1\varepsilon^{1-\frac{n}{p}} - E_2(R_g(x_0)+o_\rho(1))\varepsilon^{1-\frac{n}{p}+\frac{2}{p'}} + C(\rho) & \text{if } \sqrt{n}<p<\frac{n+2}{3}, \\ E_1\varepsilon^{1-\frac{n}{p}} - C(R_g(x_0)+o_\rho(1))\ln\varepsilon^{-\frac{1}{p'}} + C\rho^{\frac{p-n}{p-1}} & \text{if } p=\frac{n+2}{3}\ , \end{cases}$$

where E_1 and E_2 are as in (9) and (11).

Since $1 - \frac{n}{p} + \frac{2}{p'} < \frac{p^2-n}{p}$ for $p > 2$, taking (25) into account, for $2 < p < \frac{n+2}{3}$ we get

$$\frac{\|u\|^p_{W^1_p(\Omega)}}{\|u\|^p_{p^*,\Omega}} \le \frac{E_1}{D_1^{p/p^*}}\Big[1 + \Big(\frac{p}{p^*}\frac{D_2}{D_1} - \frac{E_2}{E_1}\Big)(R_g(x_0)+o_\rho(1))\varepsilon^{\frac{2}{p'}} + o(\varepsilon^{\frac{2}{p'}})\Big].$$

Recalling (26), we obtain

$$\frac{\|u\|_{W^1_p(\Omega)}}{\|u\|_{p^*,\Omega}} < \frac{E_1^{1/p}}{D_1^{1/p^*}} = \frac{1}{K(n,p)}$$

for ε and ρ sufficiently small. Similarly, we have

$$\lambda_2(n,p,\Omega) < \frac{1}{K(n,p)}$$

for $p = \frac{n+2}{3}$. By continuity, this inequality remains valid for $p < \frac{n+2}{3} + \beta$ with some $\beta > 0$.

Applying Proposition 1.1, we complete the proof. □

Remark 2. For $2 < p < \sqrt{n}$, a statement equivalent to Theorem 4.1 was proved in [Dr]. Note that for $n \geq 5$ we have $\sqrt{n} < \frac{n+2}{3}$.

Corollary 4.1. *Under the assumptions of Theorem* 4.1, *the Neumann problem for the equation*

$$-\Delta_p u = |u|^{p^*-2}u - |u|^{p-2}u \tag{33}$$

(*here* $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ *is the p-Laplacian*) *has a nonconstant positive solution in* Ω.

Proof. In [NShch, Proposition 1.3] it was shown that for $p > 2$ the minimizer of (II) cannot be a constant. Therefore, under the assumptions of Theorem 4.1, problem (II) has a nonconstant minimizer u. After multiplying u by a suitable constant, the Euler equation for (II) reduces to the form (33). The Neumann condition is the natural boundary condition for (II). Finally, the positivity of u can be proved by the standard argument involving the Harnack inequality (see [Tr]). □

We introduce the manifold $\Omega(\varkappa)$ as a "dilation" of Ω with the metric $g(\varkappa) = \varkappa^2 g$. Since the quotient in (II) is not homogeneous with respect to dilations, the attainability of the infimum, generally speaking, depends on $\varkappa$.

Let $n \geq 2$, and let Ω be an arbitrary n-dimensional manifold without boundary or with a strictly Lipschitz boundary. Then for any $1 < p < n$ there exists $\varkappa_* > 0$ such that the infimum in (II) is attained on $\Omega(\varkappa)$ for $\varkappa < \varkappa_*$ (see [NShch, Theorem 1.1]). On the other hand, in [Aub, Theorem 8] it was shown that for the standard sphere $\Omega = \mathbb{S}^n$ and for $1 < p < 2$, the so-called *optimal Sobolev inequality*

$$\|v\|^p_{p^*,\Omega} \leq K^p(n,p) \cdot \|\nabla v\|^p_{p,\Omega} + C(p) \cdot \|v\|^p_{p,\Omega}, \quad v \in W^1_p(\Omega), \tag{34}$$

is true with some $C(p) > 0$. For $n \geq 3$ and $p = 2$ this is true for an arbitrary manifold Ω without boundary [HV]. In a similar way, in [Dr] it was shown that the optimal Sobolev inequality is valid for any $1 < p < n$ on the torus $\Omega = \mathbb{T}^n$ and on the hyperbolic manifold $\mathbb{H}^n$ without boundary.

It is easily seen that (34) implies that the infimum in (II) on $\Omega(\varkappa)$ is not attained for sufficiently large $\varkappa$. We can make two conclusions: first, the positivity of the scalar curvature at some point of Ω is necessary for the infimum to be attained; second, the lower bound of the interval for p in Theorem 4.1 cannot be reduced in general.

Conjecture. *Let* $n \geq 3$, *let* Ω *be an* n*-dimensional manifold without boundary, and let* $\frac{n+2}{3} < p < n$. *Then there exists* $\varkappa^* > 0$ *such that the infimum in (II) is not attained on* $\Omega(\varkappa)$ *with* $\varkappa > \varkappa^*$.

§5. The Sobolev–Poincaré inequality on manifolds without boundary, $n \geq 3$

Theorem 5.1. *Let* $n \geq 3$, *and let* Ω *be an* n*-dimensional manifold without boundary. Suppose that the scalar curvature is positive at some point of* Ω. *Then, for some* $\beta > 0$ *and all* $1 < p < \frac{n+2}{3} + \beta$, *the infimum in* (III) *is attained.*

Proof. To construct a function in Ω with zero mean and with the quotient (III) less than $\frac{1}{K(n,p)}$, we modify the function (3). For large p, we subtract a suitable function with small support, while for small p we subtract a constant.

1. First, let $p > \frac{2n}{n+1}$. Let $x_1 \in \Omega$ be a point with positive scalar curvature, and let x_2 be another point of the manifold. Consider the function u_1 defined as in (3), with r being the distance from x to x_1, and with parameters ε_1 and ρ_1. In the same way, we define a function u_2, with r being the distance from x to x_2 and with parameters ε_2 and ρ_2. Choosing ρ_1 and ρ_2 such that the supports of u_1 and u_2 are disjoint, we introduce the following function with zero mean:

$$u = u_1 - u_2 \cdot \frac{\int_\Omega u_1 \, dV_g}{\int_\Omega u_2 \, dV_g}. \tag{35}$$

Estimates (18) and (19) imply that

$$\int_\Omega u_1 \, dV_g \le C(\rho_2) \int_\Omega u_2 \, dV_g. \tag{36}$$

Hence, by (24),

$$\int_\Omega |\nabla u|^p \, dV_g \le \begin{cases} E_1 \varepsilon_1^{1-\frac{n}{p}} - E_2(R_g(x_1) + o_{\rho_1}(1)) \varepsilon_1^{1-\frac{n}{p}+\frac{2}{p'}} + C(\rho_1, \rho_2, \varepsilon_2) & \text{if } p < \frac{n+2}{3}, \\ E_1 \varepsilon_1^{1-\frac{n}{p}} - C(R_g(x_1) + o_{\rho_1}(1)) \ln \varepsilon_1^{-\frac{1}{p'}} + C(\rho_1, \rho_2, \varepsilon_2), & \text{if } p = \frac{n+2}{3}. \end{cases} \tag{37}$$

Combining (37), (25) and (26), we obtain (27) provided ε_1 and ρ_1 are sufficiently small. Thus, we have

$$\lambda_3(n, p, \Omega) < \frac{1}{K(n, p)} \tag{38}$$

for $p \le \frac{n+2}{3}$. By continuity, this inequality remains valid for $p < \frac{n+2}{3} + \beta$ with some $\beta > 0$.

2. Now we consider the case where $1 < p \le \frac{2n}{n+1}$. Let $x_0 \in \Omega$ be a point with positive scalar curvature. We define $\widehat{u}$ as in (3) and put $\rho = \varepsilon^\gamma$ with some $\gamma \in (0, \gamma^*)$, where

$$\gamma^* = \frac{1}{p'} \Big(1 - \frac{2(p-1)}{n-p} \Big). \tag{39}$$

Relation (25) yields

$$\int_\Omega \widehat{u}^{p^*} \, dV_g \ge D_1 \varepsilon^{-\frac{n}{p}} - D_2(R_g(x_0) + o_\varepsilon(1)) \varepsilon^{-\frac{n}{p}+\frac{2}{p'}} - C \varepsilon^{-\gamma \frac{n}{p-1}}.$$

Immediate computations show that

$$-\gamma^* \frac{n}{p-1} > -\frac{n}{p} + \frac{2}{p'}.$$

Consequently, for $\gamma \in (0, \gamma^*)$ we have

$$\int_\Omega \widehat{u}^{p^*} \, dV_g \ge D_1 \varepsilon^{-\frac{n}{p}} - D_2(R_g(x_0) + o_\varepsilon(1)) \varepsilon^{-\frac{n}{p}+\frac{2}{p'}}. \tag{40}$$

Furthermore, (24) implies the estimate

$$\int_\Omega |\nabla \widehat{u}|^p \, dV_g \le E_1 \varepsilon^{1-\frac{n}{p}} - E_2(R_g(x_0) + o_\varepsilon(1)) \varepsilon^{1-\frac{n}{p}+\frac{2}{p'}} + C \varepsilon^{-\gamma \frac{n-p}{p-1}}.$$

By direct calculations,

$$-\gamma^* \frac{n-p}{p-1} = 1 - \frac{n}{p} + \frac{2}{p'}.$$

Thus, if $\gamma \in (0, \gamma^*)$, then

$$\int_\Omega |\nabla \widehat{u}|^p \, dV_g \le E_1 \varepsilon^{1-\frac{n}{p}} - E_2(R_g(x_0) + o_\varepsilon(1)) \varepsilon^{1-\frac{n}{p}+\frac{2}{p'}}. \tag{41}$$

Now, we introduce the following function with zero mean:

$$u = \widehat{u} - \frac{1}{|\Omega|}\int_\Omega \widehat{u}\, dV_g. \tag{42}$$

By (40),

$$\|\widehat{u}\|_{p^*,\Omega} \geq D_1^{1/p^*}\varepsilon^{-\frac{n}{pp^*}}\Big(1 - \frac{D_2}{p^*D_1}(R_g(x_0) + o_\varepsilon(1))\varepsilon^{\frac{2}{p'}}\Big).$$

Therefore, using (18) and the Minkowski inequality, we obtain

(43)

$$\begin{aligned}
\|u\|_{p^*,\Omega} &\geq \left(\int_{\operatorname{supp}\widehat{u}} |u|^{p^*}\, dV_g\right)^{\frac{1}{p^*}} \geq \|\widehat{u}\|_{p^*,\Omega} - \frac{|\operatorname{supp}\widehat{u}|^{\frac{1}{p^*}}}{|\Omega|}\int_\Omega \widehat{u}\, dV_g \\
&\geq D_1^{1/p^*}\varepsilon^{-\frac{n}{pp^*}} \\
&\quad\times \begin{cases} 1 - \frac{D_2}{p^*D_1}\left(R_g(x_0) + o_\varepsilon(1)\right)\varepsilon^{\frac{2}{p'}} - C\cdot\varepsilon^{\frac{n}{pp^*}+\gamma\frac{n}{p^*}+\frac{np+p-2n}{p}} & \text{if } 1 < p < \frac{2n}{n+1}, \\ 1 - \frac{D_2}{p^*D_1}\left(R_g(x_0) + o_\varepsilon(1)\right)\varepsilon^{\frac{2}{p'}} - C\cdot\varepsilon^{\frac{n}{pp^*}+\gamma\frac{n}{p^*}}(1 + \ln\varepsilon^{\gamma-\frac{1}{p'}}) & \text{if } p = \frac{2n}{n+1}. \end{cases}
\end{aligned}$$

Immediate computations show that, for $n \geq 3$ and $1 < p \leq \frac{2n}{n+1}$, the inequality

$$\frac{n}{pp^*} + \gamma\frac{n}{p^*} + \frac{np+p-2n}{p} > \frac{2}{p'}$$

is true for $\gamma = \gamma^*$, and therefore also for some $\gamma \in (0, \gamma^*)$. Fixing such a γ, from (43) we deduce that

$$\|u\|_{p^*,\Omega}^p \geq D_1^{p/p^*}\varepsilon^{-\frac{n}{p^*}}\Big(1 - \frac{p}{p^*}\frac{D_2}{D_1}(R_g(x_0) + o_\varepsilon(1))\varepsilon^{\frac{2}{p'}}\Big). \tag{44}$$

Relations (44), (41), and (26) imply (27) provided ε is sufficiently small. Thus, inequality (38) is valid in this case. Applying Proposition 1.1, we complete the proof. □

Remark 3. For $n \geq 4$ and $\frac{n}{n-1} < p < \frac{1}{4}(1 + \sqrt{1+8n})$, Theorem 5.1 was proved in [Zhu, Theorem 1.2]. Observe that for $n \geq 2$ we have $\frac{1}{4}(1+\sqrt{1+8n}) < \frac{n+2}{3}$. Moreover, Theorem 1.1 in [Zhu] claims that the infimum in (III) for $n \geq 2$ and $1 < p < \frac{1}{4}(1 + \sqrt{1+8n})$ is attained in the case where $\Omega = \mathbb{S}^n$. However, as was already mentioned in §1, the proof of that theorem has a gap.

§6. Inequality for the best approximation by a constant on manifolds without boundary, $n \geq 3$

Theorem 6.1. *Let $n \geq 3$, and let Ω be an n-dimensional manifold without boundary. Suppose that the scalar curvature is positive at some point of Ω. Then, for*

$$1 < p < \max\{p_1, p_2\} \tag{45}$$

with

$$\begin{aligned}
p_1 &= 2n + 1 - \sqrt{3n^2 + 2n + 1}, \\
p_2 &= \frac{1}{2(5n+4)}\cdot(n^2 + 6n + 2 + \sqrt{n^4 + 12n^3 - 8n + 4}),
\end{aligned}$$

the infimum in (IV) *is attained.*

Remark 4. The exponents p_1 and p_2 are monotone functions of n and have linear growth for large n. Moreover, for $3 \leq n \leq 5$ these exponents satisfy $1 < p_1 < p_2$, while for $n \geq 7$ they satisfy $2 < p_2 < p_1$; for $n = 6$ we have $p_1 = p_2 = 2$.

Proof. First of all, we note that the maximum in (IV) with respect to a is attained whenever

$$\int_\Omega |u|^{p^*-2} u \, dV_g = 0, \tag{46}$$

where $u = v - a$.

We modify the function (3) so as to ensure (46). As in the preceding section, we make this modification in two ways; the first way works for $1 < p < p_1$, and the second works for $1 < p < p_2$. The comparison of p_1 and p_2 can be made by elementary calculations.

The first way. Let $x_1 \in \Omega$ be a point with positive scalar curvature, and let x_2 be another point of the manifold. We define functions u_1 and u_2 as in the proof of Theorem 5.1. Let ρ_1 and ρ_2 be such that the supports of u_1 and u_2 are disjoint, and let $\varepsilon_2 = \varepsilon_1^\gamma$ with some $\gamma \in (0,1)$.

Now we construct a function satisfying (46):

$$u = u_1 - u_2 \cdot \left(\frac{\int_\Omega u_1^{p^*-1} \, dV_g}{\int_\Omega u_2^{p^*-1} \, dV_g} \right)^{\frac{1}{p^*-1}}. \tag{47}$$

Inequalities (20) and (21) imply that

$$\int_\Omega u_1^{p^*-1} \, dV_g \le C \varepsilon_1^{\gamma-1} \int_\Omega u_2^{p^*-1} \, dV_g. \tag{48}$$

For $p < \frac{n+2}{3}$, we use (24) to obtain

$$\begin{aligned} \int_\Omega |\nabla u|^p \, dV_g \le{}& E_1 \varepsilon_1^{1-\frac{n}{p}} - E_2 (R_g(x_1) + o_{\rho_1}(1)) \varepsilon_1^{1-\frac{n}{p}+\frac{2}{p'}} \\ &+ C \varepsilon_1^{(\gamma-1)\frac{p}{p^*-1} + \gamma(1-\frac{n}{p})} (1 + o_{\varepsilon_1}(1)). \end{aligned} \tag{49}$$

Immediate computations show that if $n \ge 3$ and $1 < p < p_1$, then the inequality

$$(\gamma - 1) \frac{p}{p^* - 1} + \gamma \Big(1 - \frac{n}{p} \Big) > 1 - \frac{n}{p} + \frac{2}{p'}$$

is fulfilled for $\gamma = 0$ and, therefore, for some $\gamma > 0$. Fixing this γ and combining (49), (25), and (26), we obtain (27), provided ε_1 and ρ_1 are sufficiently small.

The second way. Let $x_0 \in \Omega$ be a point with positive scalar curvature. We define $\widehat{u}$ as in (3) and put $\rho = \varepsilon^\gamma$ with some $\gamma \in (0, \gamma^*)$; here γ^* is defined by (39).

Now we introduce the function $u = \widehat{u} - C_\varepsilon$ and choose the constant C_ε so as to ensure (46). Estimate (20) yields $C_\varepsilon \le C \varepsilon^{-\frac{1}{p^*-1}}$. Arguing as in the proof of (43), from (40) we deduce that

$$\|u\|_{p^*,\Omega} \ge D_1^{1/p^*} \varepsilon^{-\frac{n}{pp^*}} \Big(1 - \frac{D_2}{p^* D_1} (R_g(x_0) + o_\varepsilon(1)) \varepsilon^{\frac{2}{p'}} - C \cdot \varepsilon^{\frac{n}{pp^*} + \gamma \frac{n}{p^*} - \frac{1}{p^*-1}} \Big). \tag{50}$$

For $n \ge 3$ and $1 < p < p_2$, direct computations show that the inequality

$$\frac{n}{pp^*} + \gamma \frac{n}{p^*} - \frac{1}{p^* - 1} > \frac{2}{p'}$$

is true for $\gamma = \gamma^*$ and, therefore, for some $\gamma \in (0, \gamma^*)$. Fixing such a γ, we see that inequalities (49), (25), and (26) imply (27), provided ε is sufficiently small.

Applying Proposition 1.1, we complete the proof. □

Condition (45) in Theorem 6.1 cannot be viewed as quite satisfactory and is likely to admit relaxation. However, we claim that some upper bound for p is necessary.

Theorem 6.2. *Let $n \ge 2$. Then for $p \ge \frac{n+1}{2}$ the infimum in* (IV) *on the sphere $\mathbb{S}^n$ is not attained.*

Proof. Let $u \in W_p^1(\mathbb{S}^n)$ be a minimizer satisfying (46) and normalized in $L_{p^*}(\mathbb{S}^n)$. By spherical symmetrization arguments, we can assume that u depends only on θ and is monotone decreasing. By (46), there exists θ_* such that $u(\theta_*) = 0$. Replacing (if necessary) $u(\theta)$ by $-u(\pi - \theta)$, we assume that $\theta_* \le \pi/2$. Standard arguments of the calculus of variations show that u is a weak solution of the Euler equation

$$-\Delta_p u = \lambda_4^p(n, p, \mathbb{S}^n) \cdot |u|^{p^*-2} u.$$

Multiplying by u and integrating over the spherical "hat" defined in (28), we obtain

$$\begin{aligned}\lambda_1^p(n, p, \Omega) \le \frac{\|\nabla u\|_{p,\Omega}^p}{\|u\|_{p^*,\Omega}^p} &= \lambda_4^p(n, p, \mathbb{S}^n) \cdot \|u\|_{p^*,\Omega}^{p^*-p} \\ &< \lambda_4^p(n, p, \mathbb{S}^n) \le \frac{1}{K^p(n, p)}.\end{aligned}$$

By Proposition 1.1, the infimum in (I) on Ω is attained. This contradicts Theorem 3.2 (we recall Remark 1). □

Remark 5. In particular, this theorem refutes Theorem 1.3 in [Zhu].

Using Theorem 6.2, we can study the symmetry breaking of the extremal in the embedding theorem on the sphere:

$$\widetilde{\lambda}_4(n, p, q) = \inf_{v \in W_p^1(\mathbb{S}^n) \setminus \{c\}} \sup_{a \in \mathbb{R}} \frac{\|\nabla v\|_{p,\mathbb{S}^n}}{\|v - a\|_{q,\mathbb{S}^n}} > 0 \tag{IVa}$$

for the subcritical q. As in (46), the maximum in (IVa) with respect to a is attained whenever

$$\int_{\mathbb{S}^n} |u|^{q-2} u \, dV_g = 0, \tag{51}$$

where $u = v - a$.

Theorem 6.3. 1. *Suppose $n \ge 2$, $1 < p < \infty$, and $q = p$. Then the extremal function of* (IVa) *satisfying* (51) *is symmetric with respect to θ, i.e., $u(\theta) = -u(\pi - \theta)$ for $0 < \theta < \pi/2$.*

2. *Suppose $n \ge 2$ and $\frac{n+1}{2} \le p < n$. Then there exists $\beta > 0$ such that for $q \in (p^* - \beta, p^*)$ the extremal function of* (IVa) *satisfying* (51) *is not symmetric with respect to θ.*

Proof. 1. Let $u \in W_p^1(\mathbb{S}^n)$ be a minimizer of (IVa) with $q = p$, and let u satisfy (51). As in Theorem 6.2, u depends only on θ and is monotone decreasing. Moreover, there exists $\theta_* \le \pi/2$ such that $u(\theta_*) = 0$. If $\theta_* < \pi/2$, we introduce the function

$$\widetilde{u}(\theta) = \begin{cases} u(\theta) & \text{if } \theta < \theta_*, \\ -u(\pi - \theta) & \text{if } \theta > \pi - \theta_*, \\ 0 & \text{if } \theta_* \le \theta \le \pi - \theta_*. \end{cases}$$

Then, obviously, $\widetilde{u}$ satisfies (51). The Euler equation for u has the form

$$-\Delta_p u = \widetilde{\lambda}_4^p(n, p, p) \cdot |u|^{p-2} u.$$

Multiplication by u and integration over the spherical "hat" defined in (28) yields

$$\frac{\|\nabla \widetilde{u}\|_{p,\mathbb{S}^n}^p}{\|\widetilde{u}\|_{p,\mathbb{S}^n}^p} = \frac{\|\nabla u\|_{p,\Omega}^p}{\|u\|_{p,\Omega}^p} = \widetilde{\lambda}_4^p(n, p, p).$$

Thus, $\widetilde{u}$ is also a minimizer of (IVa); therefore, it satisfies the same Euler equation. Since $\widetilde{u}$ is a nonpositive sub-p-harmonic function on the complement of the "hat" (28),

the Harnack inequality (see [Tr]) shows that $\hat{u}$ cannot vanish at interior points. This contradiction proves the first statement.

2. Let $u_q \in W_p^1(\mathbb{S}^n)$ be minimizers of (IVa) satisfying (51). Suppose that there exists a sequence of q's tending to p^* and such that $u_q(\pi-\theta) = -u_q(\theta)$. Then the corresponding functions u_q also satisfy (46) and therefore give rise to a minimizing sequence for (IV). We may assume that this sequence is normalized in $L_{p^*}(\mathbb{S}^n)$ and weakly converges in $W_p^1(\mathbb{S}^n)$ to some u. Theorem 6.2 implies that the function u is not a minimizer of (IV). Hence, arguing as in the proof of Theorem 2.2 in [LPT], we conclude that $|u_q|^{p^*} \rightharpoonup \delta(x-x_0)$ in the sense of measures on $\mathbb{S}^n$. But this is impossible, contradicting the symmetry of u_q. Thus, the second statement is also proved. □

Remark 6. If $n=1$, then for all $1<p,q<\infty$ the extremal function of (IVa) satisfying (51) is symmetric with respect to θ.

§7. The case of a domain in $\mathbb{R}^n$, $n \geq 2$

The proofs in the case of a domain are similar to those in the case of a manifold. Instead of Proposition 1.1 we use the following statement.

Proposition 7.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega$ of class $\mathcal{C}^2$. Suppose that the infimum in* (II), (III), *or* (IV) *satisfies*

$$\lambda_m(n,p,\Omega) < \frac{2^{-\frac{1}{n}}}{K(n,p)}. \tag{52}$$

Then the infimum is attained.

Proof. Word-for-word repetition of the proof of [LPT, Corollary 2.1]. □

Under certain assumptions on the exponent p, we obtain (52) for $m = 2,3,4$ by constructing a function with small support that simulates the behavior of $w_\varepsilon(r)$ in a half-space.

7.1. The critical embedding theorem.

Theorem 7.1. *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$. Then, for some $\beta > 0$ and all $1 < p < \frac{n+1}{2} + \beta$, the infimum in* (II) *is attained.*

Proof. Consider the smallest ball containing Ω. Let $x_0 \in \partial\Omega$ be a point of contact of Ω with this ball. Then all the principal curvatures of $\partial\Omega$ at x_0 are positive, whence $H(x_0) > 0$.

For the function u defined in (3), from (22) and (17) we deduce the estimate

$$\|u\|^p_{W_p^1(\Omega)} \leq \begin{cases} \frac{1}{2}E_1\varepsilon^{1-\frac{n}{p}} - \widetilde{E}_2(H(x_0)+o_\rho(1))\varepsilon^{1-\frac{n}{p}+\frac{1}{p'}} + C\varepsilon^{\frac{p^2-n}{p}} + C(\rho) & \text{if } 1<p<\sqrt{n}, \\ \frac{1}{2}E_1\varepsilon^{1-\frac{n}{p}} - \widetilde{E}_2(H(x_0)+o_\rho(1))\varepsilon^{1-\frac{n}{p}+\frac{1}{p'}} + C\ln\varepsilon^{-\frac{1}{p'}} + C(\rho) & \text{if } p=\sqrt{n}, \\ \frac{1}{2}E_1\varepsilon^{1-\frac{n}{p}} - \widetilde{E}_2(H(x_0)+o_\rho(1))\varepsilon^{1-\frac{n}{p}+\frac{1}{p'}} + C(\rho) & \text{if } \sqrt{n}<p<\frac{n+1}{2}, \\ \frac{1}{2}E_1\varepsilon^{1-\frac{n}{p}} - C(H(x_0)+o_\rho(1))\ln\varepsilon^{-\frac{1}{p'}} + C\rho^{\frac{p-n}{p-1}} & \text{if } p=\frac{n+1}{2}. \end{cases}$$

Since $1-\frac{n}{p}+\frac{1}{p'} < \frac{p^2-n}{p}$ for $p>1$, we can use (23) to get

$$\frac{\|u\|^p_{W_p^1(\Omega)}}{\|u\|^p_{p^*,\Omega}} \leq 2^{\frac{p}{p^*}-1}\frac{E_1}{D_1^{p/p^*}}\Big[1+2\Big(\frac{p}{p^*}\frac{\widetilde{D}_2}{D_1} - \frac{\widetilde{E}_2}{E_1}\Big)(H(x_0)+o_\rho(1))\varepsilon^{\frac{1}{p'}} + o(\varepsilon^{\frac{1}{p'}})\Big]$$

for $1 < p < \frac{n+1}{2}$.

Immediate computations show that

$$\frac{p}{p^*}\frac{\widetilde{D}_2}{D_1}-\frac{\widetilde{E}_2}{E_1}=-\frac{p}{n}\frac{\omega_{n-2}}{\omega_{n-1}}\cdot\frac{\mathcal{B}(\frac{n+1}{p'},\frac{n+1}{p}-2)}{\mathcal{B}(\frac{n}{p'},\frac{n}{p}-1)}<0 \tag{53}$$

for $p<\frac{n+1}{2}$.

Consequently, if ε and ρ are sufficiently small, then

$$\frac{\|u\|_{W_p^1(\Omega)}}{\|u\|_{p^*,\Omega}}<2^{-\frac{1}{n}}\frac{E_1^{1/p}}{D_1^{1/p^*}}=\frac{2^{-\frac{1}{n}}}{K(n,p)}.$$

Similarly, the inequality

$$\lambda_2(n,p,\Omega)<\frac{2^{-\frac{1}{n}}}{K(n,p)}$$

is valid for $p=\frac{n+1}{2}$. By continuity, it is also valid for $p<\frac{n+1}{2}+\beta$ with some $\beta>0$.

Applying Proposition 7.1, we complete the proof. □

Remark 7. For $n\geq 3$ and $p=2$ this result was proved in [AM, W].

Now consider the image $\Omega(\varkappa)$ of Ω under the dilation with coefficient $\varkappa$. For an arbitrary bounded domain $\Omega\subset\mathbb{R}^n$ with a strictly Lipschitz boundary, and for any $1<p<n$, the infimum in (II) is attained on $\Omega(\varkappa)$ if $\varkappa$ is sufficiently small, as in the case of a manifold. For large $\varkappa$, in general, the smoothness of $\partial\Omega$ is essential.

Theorem 7.2. *Let $n\geq 2$, let Ω be a polyhedron in $\mathbb{R}^n$, and let $1<p<n$. Then there exists $\varkappa^*>0$ such that for $\varkappa>\varkappa^*$ the infimum in* (II) *is not attained on $\Omega(\varkappa)$.*

Proof. An equivalent statement for $\Omega=\mathbb{T}^n$ was proved in [Dr, Theorem 2]. We follow the idea of [Dr] but simplify the arguments considerably.

Put $\lambda_{2,\varkappa}=\lambda_2(n,p,\Omega(\varkappa))$. Let x_0 be a polyhedron vertex with the smallest solid angle Ψ. In [NShch, Proposition 1.2] it was shown that the function $w_\varepsilon(r)$ defined in (2) with $r=|x-x_0|$ satisfies

$$\lim_{\varepsilon\to 0}\frac{\|w_\varepsilon\|_{W_p^1(\Omega)}}{\|w_\varepsilon\|_{p^*,\Omega}}=\frac{\phi^{\frac{1}{n}}}{K(n,p)},$$

where $\phi=\frac{\Psi}{\omega_{n-1}}$. Therefore,

$$\lambda_{2,\varkappa}\leq\frac{\phi^{\frac{1}{n}}}{K(n,p)}. \tag{54}$$

Suppose that there exists an unbounded sequence of $\varkappa$'s such that the infimum in (II) on $\Omega(\varkappa)$ is attained. Then, by a standard argument, the corresponding minimizers $u_\varkappa$ are positive on $\Omega(\varkappa)$. Consider the functions $v_\varkappa(x)=C(\varkappa)u_\varkappa(\frac{x}{\varkappa})$ on Ω, with $C(\varkappa)$ given by the normalization condition $\|v_\varkappa\|_{p^*,\Omega}=1$. Then $v_\varkappa$ is a solution of the Neumann problem for the Euler equation

$$-\Delta_p v_\varkappa+\varkappa^p\cdot v_\varkappa^{p-1}=\lambda_{2,\varkappa}^p\cdot v_\varkappa^{p^*-1}. \tag{55}$$

There is no loss of generality in assuming that $v_\varkappa\rightharpoonup v$ in $W_p^1(\Omega)$. Multiplying (55) by $v_\varkappa$ and integrating over Ω, we obtain

$$\|\nabla v_\varkappa\|_{p,\Omega}^p+\varkappa^p\cdot\|v_\varkappa\|_{p,\Omega}^p=\lambda_{2,\varkappa}^p, \tag{56}$$

and (54) implies $v=0$. Arguing as in the proof of Theorem 2.2 in [LPT], we deduce that $|v_\varkappa|^{p^*}\rightharpoonup\delta(x-x_0)$ in the space of measures on $\overline{\Omega}$; therefore,

$$\|v_\varkappa\|_{p^*,\Omega\setminus B_{\frac{\rho}{2}}(x_0)}\to 0 \tag{57}$$

for any given $\rho>0$.

Put $\eta(x) = \eta(r) = 1 - \varphi(r)$, where φ is the function defined in (4), and $r = |x - x_0|$. For the function $w = \eta v_\varkappa^{\frac{s+p-1}{p}}$, $s \in [1, \frac{n}{n-p})$, we write the limit embedding theorem:

$$\|w\|^p_{p^*,\Omega} \le C \cdot \Big(\|\nabla w\|^p_{p,\Omega} + \|w\|^p_{p,\Omega} \Big).$$

Using the inequality $(a+b)^p \le 2^p(a^p + b^p)$ and estimating ∇w, we get

$$\|w\|^p_{p^*,\Omega} \le C(s) \int_\Omega \eta^p v_\varkappa^{s-1} |\nabla v_\varkappa|^p \, dx + C \int_\Omega v_\varkappa^{s+p-1} \, dx. \tag{58}$$

To bound the first term on the right in (58), we multiply (55) by $\eta^p v_\varkappa^s$ and integrate over Ω. Omitting a positive term, we obtain

$$\begin{aligned} s \int_\Omega \eta^p v_\varkappa^{s-1} |\nabla v_\varkappa|^p \, dx + p \int_\Omega v_\varkappa^s \eta^{p-1} |\nabla v_\varkappa|^{p-2} \langle \nabla \eta, \nabla v_\varkappa \rangle \, dx \\ = - \int_\Omega \eta^p v_\varkappa^s \cdot \Delta_p v_\varkappa \, dx \le \lambda^p_{2,\varkappa} \int_\Omega \eta^p v_\varkappa^{s+p^*-1} \, dx, \end{aligned} \tag{59}$$

whence, by (54),

$$\|w\|^p_{p^*,\Omega} \le C(s) \int_\Omega \eta^p v_\varkappa^{s+p^*-1} \, dx + C(s) \|v_\varkappa\|^s_{sp,\Omega} \|\nabla v_\varkappa\|^{p-1}_{p,\Omega} + C \int_\Omega v_\varkappa^{s+p-1} \, dx. \tag{60}$$

The Hölder inequality yields

$$\begin{aligned} \int_\Omega \eta^p v_\varkappa^{s+p^*-1} \, dx &= \int_{\Omega \setminus B_{\frac{\rho}{2}}(x_0)} \eta^p v_\varkappa^{s+p^*-1} \, dx \\ &\le \|w\|^p_{p^*,\Omega} \cdot \|v_\varkappa\|^{p^*-p}_{p^*,\Omega \setminus B_{\frac{\rho}{2}}(x_0)}. \end{aligned}$$

Therefore, by (60),

$$\Big[1 - C(s) \|v_\varkappa\|^{p^*-p}_{p^*,\Omega \setminus B_{\frac{\rho}{2}}(x_0)} \Big] \cdot \|w\|^p_{p^*,\Omega} \le C(s) \|v_\varkappa\|^s_{sp,\Omega} \|\nabla v_\varkappa\|^{p-1}_{p,\Omega} + C \int_\Omega v_\varkappa^{s+p-1} \, dx. \tag{61}$$

Relation (57) implies that, for any fixed s, the expression in square brackets exceeds $1/2$ if $\varkappa$ is sufficiently large. Moreover, since $s < \frac{n}{n-p}$, we have

$$s + p - 1 < p^* \quad \text{and} \quad sp < p^*.$$

Since $v_\varkappa \rightharpoonup 0$ in $W^1_p(\Omega)$, it follows that $\|v_\varkappa\|_{s+p\,1,\Omega} \to 0$ and $\|v_\varkappa\|_{sp,\Omega} \to 0$. By (61), we have $\|w\|_{p^*,\Omega} \to 0$, whence $\|v_\varkappa\|_{p^*+\varepsilon,\Omega \setminus B_\rho(x_0)} \to 0$ as $\varkappa \to \infty$, for some $\varepsilon > 0$.

Now, in the same way as in [LaU, Chapter IV, Theorem 7.1], we check that

$$\sup_{\Omega \setminus B_\rho(x_0)} v_\varkappa \to 0, \quad \varkappa \to \infty, \tag{62}$$

for any $\rho > 0$.

Set $\varphi_2(r) = \varphi(\frac{r}{2})$. Since in a neighborhood of x_0 the boundary $\partial\Omega$ is a conical surface, Theorem 2.1 in [LPT] shows that

$$\|\varphi_2 v_\varkappa\|_{p^*,\Omega} \le \frac{K(n,p)}{\phi^{1/n}} \cdot \|\nabla(\varphi_2 v_\varkappa)\|_{p,\Omega}$$

if ρ is sufficiently small. Then inequality (5) and the normalization of $v_\varkappa$ yield

$$\begin{aligned} \|v_\varkappa\|^{p^*}_{p^*,\Omega \cap B_\rho(x_0)} &\le \|v_\varkappa\|^p_{p^*,\Omega \cap B_\rho(x_0)} \\ &\le \frac{K(n,p)^p}{\phi^{p/n}} \cdot \|\nabla v_\varkappa\|^p_{p,\Omega} + C \Big[\|v_\varkappa\|^p_{p,\Omega} + \|v_\varkappa\|_{p,\Omega} \, \|\nabla v_\varkappa\|^{p-1}_{p,\Omega \setminus B_\rho(x_0)} \Big]. \end{aligned}$$

Substituting $\|\nabla v_\varkappa\|_{p,\Omega}^p$ from (56) and recalling (54), we obtain

$$\varkappa^p \frac{K(n,p)^p}{\phi^{p/n}} \cdot \|v_\varkappa\|_{p,\Omega}^p \\ \le 1 - \|v_\varkappa\|_{p^*,\Omega\cap B_\rho(x_0)}^{p^*} + C\left[\|v_\varkappa\|_{p,\Omega}^p + \|v_\varkappa\|_{p,\Omega}\|\nabla v_\varkappa\|_{p,\Omega\setminus B_\rho(x_0)}^{p-1}\right],$$

whence

$$\varkappa^p \frac{K(n,p)^p}{\phi^{p/n}} \le \frac{\|v_\varkappa\|_{p^*,\Omega\setminus B_\rho(x_0)}^{p^*}}{\|v_\varkappa\|_{p,\Omega}^p} + C\Big(\frac{\|\nabla v_\varkappa\|_{p,\Omega\setminus B_\rho(x_0)}}{\|v_\varkappa\|_{p,\Omega}}\Big)^{p-1} + C. \tag{63}$$

By (62), we have

$$\frac{\|v_\varkappa\|_{p^*,\Omega\setminus B_\rho(x_0)}^{p^*}}{\|v_\varkappa\|_{p,\Omega}^p} \le \sup_{\Omega\setminus B_\rho(x_0)} v_\varkappa^{p^*-p} \to 0, \quad \varkappa\to\infty. \tag{64}$$

To bound the second term on the right in (63), we put $s=1$ in (59). Recalling (54), we see that

$$\|\eta\nabla v_\varkappa\|_{p,\Omega}^p \le C\left(\|v_\varkappa\|_{p,\Omega}\,\|\eta\nabla v_\varkappa\|_{p,\Omega}^{p-1} + \|v_\varkappa\|_{p^*,\Omega\setminus B_\rho(x_0)}^{p^*}\right),$$

whence, by (64),

$$\Big(\frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\Big)^p \le C\Big(\frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}}\Big)^{p-1} + o(1), \quad \varkappa\to\infty.$$

Therefore,

$$\frac{\|\nabla v_\varkappa\|_{p,\Omega\setminus B_\rho(x_0)}}{\|v_\varkappa\|_{p,\Omega}} \le \frac{\|\eta\nabla v_\varkappa\|_{p,\Omega}}{\|v_\varkappa\|_{p,\Omega}} \le C.$$

Thus, inequality (63) leads to a contradiction if $\varkappa$ is sufficiently large. This completes the proof. □

Remark 8. In the case where $n\ge 3$, $p=2$, and Ω is a rectangular parallelepiped, this result was proved in [Lin, Theorem 1.2], by using subtle properties of the solution of the corresponding boundary value problem.

Remark 9. Under the assumptions of Theorem 7.1, let $p>2$ (this is possible only for $n\ge 3$). Then the arguments in Corollary 4.1 show that the Neumann problem for equation (33) has a nonconstant positive solution in Ω.

In [NShch] it was proved that, for $1<p\le 2$ and small $\varkappa$, the minimizer of (II) on $\Omega(\varkappa)$ is a constant function, and that for large $\varkappa$ it cannot be constant. Therefore, for $1<p\le\min\{2,\frac{n+1}{2}\}$ the Neumann problem for equation (33) has a nonconstant positive solution in $\Omega(\varkappa)$ provided that $\varkappa$ is sufficiently large.

7.2. The Sobolev–Poincaré inequality.

Theorem 7.3. *Let $n\ge 2$, and let $\Omega\subset\mathbb{R}^n$ be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$. Then, for some $\beta>0$ and all $1<p<\frac{n+1}{2}+\beta$, the infimum in* (III) *is attained.*

Proof. 1. Suppose $p>\frac{2n}{n+1}$. Let $x_1\in\partial\Omega$ be a point with positive mean curvature, and let x_2 be another point of the boundary. We define functions u_1 and u_2 as in the proof of Theorem 5.1. Choosing ρ_1 and ρ_2 such that the supports of u_1 and u_2 are disjoint, we use formula (35) to introduce a function u with zero mean.

Inequalities (22) and (36) lead to the following estimate:

$$\int_\Omega |\nabla u|^p\,dx \le \begin{cases} \frac12 E_1\varepsilon_1^{1-\frac{n}{p}} - \widetilde{E}_2(H(x_1)+o_{\rho_1}(1))\varepsilon_1^{1-\frac{n}{p}+\frac{1}{p'}} + C(\rho_1,\rho_2,\varepsilon_2) & \text{if } p<\frac{n+1}{2},\\ \frac12 E_1\varepsilon_1^{1-\frac{n}{p}} - C(H(x_1)+o_{\rho_1}(1))\ln\varepsilon_1^{-\frac{1}{p'}} + C(\rho_1,\rho_2,\varepsilon_2) & \text{if } p=\frac{n+1}{2}. \end{cases} \tag{65}$$

Combining (65), (23), and (53) and assuming that ε_1 and ρ_1 are sufficiently small, we conclude that the inequality

$$\lambda_3(n,p,\Omega) < \frac{2^{-\frac{1}{n}}}{K(n,p)} \tag{66}$$

is true for $p \le \frac{n+1}{2}$. By continuity, it is also valid for $p < \frac{n+1}{2} + \beta$ with some $\beta > 0$.

2. Now, suppose that $1 < p \le \frac{2n}{n+1}$. Let $x_0 \in \partial\Omega$ be a point with positive mean curvature. We define $\widehat{u}$ as in (3), and put $\rho = \varepsilon^\gamma$ with some $\gamma \in (0, \gamma^{**})$, where

$$\gamma^{**} = \frac{1}{p'}\Big(1 - \frac{(p-1)}{n-p}\Big). \tag{67}$$

Relation (23) implies

$$\int_\Omega \widehat{u}^{p^*}\,dx \ge \frac{1}{2}D_1\varepsilon^{-\frac{n}{p}} - \widetilde{D}_2(H(x_0) + o_\varepsilon(1))\varepsilon^{-\frac{n}{p}+\frac{1}{p'}} - C\varepsilon^{-\gamma\frac{n}{p-1}}.$$

Immediate calculations show that

$$-\gamma^{**}\frac{n}{p-1} > -\frac{n}{p} + \frac{1}{p'}.$$

Hence, for $\gamma \in (0, \gamma^{**})$,

$$\int_\Omega \widehat{u}^{p^*}\,dx \ge \frac{1}{2}D_1\varepsilon^{-\frac{n}{p}} - \widetilde{D}_2(H(x_0) + o_\varepsilon(1))\varepsilon^{-\frac{n}{p}+\frac{1}{p'}}. \tag{68}$$

Furthermore, (22) implies the estimate

$$\int_\Omega |\nabla\widehat{u}|^p\,dx \le \frac{1}{2}E_1\varepsilon^{1-\frac{n}{p}} - \widetilde{E}_2(H(x_0) + o_\varepsilon(1))\varepsilon^{1-\frac{n}{p}+\frac{1}{p'}} + C\varepsilon^{-\gamma\frac{n-p}{p-1}}.$$

By direct calculations, we have

$$-\gamma^{**}\frac{n-p}{p-1} = 1 - \frac{n}{p} + \frac{1}{p'}.$$

Consequently, for $\gamma \in (0, \gamma^{**})$ we obtain

$$\int_\Omega |\nabla\widehat{u}|^p\,dx \le \frac{1}{2}E_1\varepsilon^{1-\frac{n}{p}} - \widetilde{E}_2(H(x_0) + o_\varepsilon(1))\varepsilon^{1-\frac{n}{p}+\frac{1}{p'}}. \tag{69}$$

Now, let u be the function with zero mean defined by formula (42). Arguing as in the proof of (43), we see that

(70)

$$\|u\|_{p^*} \ge \Big(\frac{D_1}{2}\Big)^{\frac{1}{p^*}}\varepsilon^{-\frac{n}{pp^*}} \times \begin{cases} 1 - \frac{2\widetilde{D}_2}{p^*D_1}(H(x_0) + o_\varepsilon(1))\varepsilon^{\frac{1}{p'}} - C\cdot\varepsilon^{\frac{n}{pp^*}+\gamma\frac{n}{p^*}+\frac{np+p-2n}{p}} & \text{if } 1 < p < \frac{2n}{n+1}, \\ 1 - \frac{2\widetilde{D}_2}{p^*D_1}(H(x_0) + o_\varepsilon(1))\varepsilon^{\frac{1}{p'}} - C\cdot\varepsilon^{\frac{n}{pp^*}+\gamma\frac{n}{p^*}}(1 + \ln\varepsilon^{\gamma-\frac{1}{p'}}) & \text{if } p = \frac{2n}{n+1}. \end{cases}$$

Direct computations show that the inequality

$$\frac{n}{pp^*} + \gamma\frac{n}{p^*} + \frac{np+p-2n}{p} > \frac{1}{p'}$$

is true for $p > 1$ and $\gamma > 0$. Hence, by (70),

$$\|u\|_{p^*}^p \ge \Big(\frac{D_1}{2}\Big)^{\frac{p}{p^*}}\varepsilon^{-\frac{n}{p^*}}\Big(1 - \frac{2p}{p^*}\frac{\widetilde{D}_2}{D_1}(H(x_0) + o_\varepsilon(1))\varepsilon^{\frac{1}{p'}}\Big). \tag{71}$$

Using relations (71), (69), and (53) and taking ε sufficiently small, we see that in this case inequality (66) is also true. Applying Proposition 7.1, we complete the proof. $\square$

7.3. An inequality for the best approximation by a constant.

Theorem 7.4. *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$. Then the infimum in* (IV) *is attained if*

$$1 < p < \max\{\widetilde{p}_1, \widetilde{p}_2\},$$

where

$$\widetilde{p}_1 = \frac{1}{2} \cdot (3n + 1 - \sqrt{5n^2 + 2n + 1}),$$
$$\widetilde{p}_2 = \frac{1}{2(3n+2)} \cdot (n^2 + 3n + 1 + \sqrt{n^4 + 6n^3 - n^2 - 2n + 1}).$$

Remark 10. The exponents $\widetilde{p}_1$ and $\widetilde{p}_2$ are monotone functions of n with linear growth for large n. Moreover, for $2 \leq n \leq 8$ these exponents satisfy $1 \leq \widetilde{p}_1 < \widetilde{p}_2$, while for $n \geq 9$ they satisfy $1 < \widetilde{p}_2 < \widetilde{p}_1$.

Proof. As in the proof of Theorem 6.1, we modify the function (3) to satisfy (46) in two ways; the first way works for $1 < p < \widetilde{p}_1$, while the second works for $1 < p < \widetilde{p}_2$.

The first way. Let $x_1 \in \partial\Omega$ be a point with positive mean curvature, and let x_2 be another point of the boundary. We define functions u_1 and u_2 as in the proof of Theorem 5.1. We choose ρ_1 and ρ_2 such that the supports of u_1 and u_2 are disjoint, and put $\varepsilon_2 = \varepsilon_1^\gamma$ with some $\gamma \in (0,1)$.

Let u be the function defined by formula (47); then u satisfies (46). Relations (22) and (48) imply the estimate

$$\begin{aligned}\int_\Omega |\nabla u|^p\,dx \leq \frac{1}{2}E_1\varepsilon_1^{1-\frac{n}{p}} - \widetilde{E}_2(H(x_1) + o_{\rho_1}(1))\varepsilon_1^{1-\frac{n}{p}+\frac{1}{p'}} \\ + C\varepsilon_1^{(\gamma-1)\frac{p}{p^*-1}+\gamma(1-\frac{n}{p})}(1 + o_{\varepsilon_1}(1))\end{aligned} \tag{72}$$

for $p < \frac{n+1}{2}$.

Immediate calculations show that if $n \geq 2$ and $1 < p < \widetilde{p}_1$, then the inequality

$$(\gamma - 1)\frac{p}{p^* - 1} + \gamma\Big(1 - \frac{n}{p}\Big) > 1 - \frac{n}{p} + \frac{1}{p'}$$

is true for $\gamma = 0$, and therefore, for some $\gamma > 0$. Choosing such a γ and combining (72), (23), and (53), we see that

$$\frac{\|\nabla u\|_p}{\|u\|_{p^*}} < \frac{2^{-\frac{1}{n}}}{K(n,p)} \tag{73}$$

if ε_1 and ρ_1 are sufficiently small.

The second way. Let $x_0 \in \partial\Omega$ be a point with positive mean curvature. We take $\widehat{u}$ as in (3) and put $\rho = \varepsilon^\gamma$ with some $\gamma \in (0, \gamma^{**})$, where γ^{**} is defined in (67).

Now we construct the function $u = \widehat{u} - C_\varepsilon$, choosing a constant C_ε so as to ensure (46). By (20), we have $C_\varepsilon \leq C\varepsilon^{-\frac{1}{p^*-1}}$. Using (68) and arguing as in the proof of (43), we get

$$\|u\|_{p^*} \geq \Big(\frac{D_1}{2}\Big)^{\frac{1}{p^*}} \varepsilon^{-\frac{n}{pp^*}} \Big(1 - \frac{2\widetilde{D}_2}{p^* D_1}(H(x_0) + o_\varepsilon(1))\varepsilon^{\frac{1}{p'}} - C \cdot \varepsilon^{\frac{n}{pp^*}+\gamma\frac{n}{p^*}-\frac{1}{p^*-1}}\Big). \tag{74}$$

Direct computations show that for $n \geq 2$ and $1 < p < \widetilde{p}_2$ the inequality

$$\frac{n}{pp^*} + \gamma\frac{n}{p^*} - \frac{1}{p^* - 1} > \frac{1}{p'}$$

is true for $\gamma = \gamma^{**}$, and therefore, for some $\gamma \in (0, \gamma^{**})$. Choosing such a γ and combining (74), (69), and (53), we obtain (73) provided that ε is sufficiently small.

To complete the proof, it remains to apply Proposition 7.1. □

Remark 11. Theorems 7.1, 7.3, and 7.4 are also valid for an arbitrary manifold Ω with smooth boundary if $\partial\Omega$ contains a point with positive mean curvature (with respect to the inward normal). The proofs need no modifications.

§8. Problems with singular weights

Let $x_0 \in \overline{\Omega}$. Consider the weighted Lebesgue space $L_{q,\sigma}(\Omega)$ with the norm

$$\|v\|_{q,\sigma,\Omega} = \|r^{\sigma-1}v\|_{q,\Omega}, \tag{75}$$

where $r = \operatorname{dist}_g(x, x_0)$.

Dilation invariance implies that, for a fixed $0 \le \sigma \le 1$, the critical exponent of the embedding $W^1_p(\Omega) \hookrightarrow L_{q,\sigma}(\Omega)$ is equal to $p^*_\sigma = \frac{np}{n-\sigma p}$. For the corresponding embedding theorem in $\mathbb{R}^n$,

$$\sup_{v\in\mathcal{C}^\infty_0(\mathbb{R}^n)\setminus\{0\}} \frac{\|v\|_{p^*_\sigma,\sigma,\mathbb{R}^n}}{\|\nabla v\|_{p,\mathbb{R}^n}} < \infty, \tag{76}$$

the case of $\sigma = 1$ leads to the Sobolev inequality, while the case of $\sigma = 0$ provides the Hardy inequality. Therefore, (76) is often referred to as the Hardy–Sobolev inequality.

Lieb [Lb] proved that, for any $0 < \sigma < 1$, for $p = 2$ and $n \ge 3$ the supremum in (76) is only attained on the radial functions

$$w_{\varepsilon,\sigma}(r) \equiv \Big(\varepsilon + r^{p'\frac{\sigma p^*_\sigma}{p^*}}\Big)^{-\frac{n}{\sigma p^*_\sigma}}, \quad \varepsilon \in \mathbb{R}_+ \tag{77}$$

with noncompact support. In [GhY] this result was generalized to arbitrary $1 < p < n$. The value of the supremum in (76) is equal to

$$K(n,p,\sigma) = \omega_{n-1}^{-\frac{\sigma}{n}}\Big(\frac{\sigma}{n}\Big)^{\frac{1}{p}}\Big(\frac{p-1}{n-p}\Big)^{\frac{1}{p'}}\Big(\frac{p^*}{\sigma p^*_\sigma}\Big)^{\frac{1}{p^*_\sigma}}\Big(\mathcal{B}\Big(\frac{n}{\sigma p}, \frac{n}{\sigma p'} + 1\Big)\Big)^{-\frac{\sigma}{n}}.$$

Problems (I)–(IV) can be generalized naturally to the case of the embedding theorems $W^1_p(\Omega) \hookrightarrow L_{p^*_\sigma,\sigma}(\Omega)$. Namely, replace the norm $\|\cdot\|_{p^*,\Omega}$ in the denominators of the ratios (I)–(IV) by the norm $\|\cdot\|_{p^*_\sigma,\sigma,\Omega}$ and denote the resulting problems by (I_σ)–(IV_σ). Let $\lambda_1(n,p,\sigma,\Omega),\dots,\ \lambda_4(n,p,\sigma,\Omega)$ denote the infimum values in these problems.

Theorem 8.1. *Let $n \ge 2$. Suppose that Ω is an n-dimensional manifold with boundary. Let $x_0 \notin \partial\Omega$ be a point with positive scalar curvature. Then, for any $0 < \sigma < 1$ there exists $\beta > 0$ such that for $1 < p < \frac{n+2}{3} + \beta$ the infimum in (I_σ) is attained.*

Let $n \ge 5$. Suppose that Ω is an n-dimensional manifold without boundary. Let x_0 be a point with positive scalar curvature. Then, for any $0 < \sigma < 1$ there exists $\beta > 0$ such that for $2 < p < \frac{n+2}{3} + \beta$ the infimum in (II_σ) is attained.

Let $n \ge 3$. Suppose that Ω is an n-dimensional manifold without boundary. Let x_0 be a point with positive scalar curvature. Then, for any $0 < \sigma < 1$ there exists $\beta > 0$ such that for $1 < p < \frac{n+2}{3} + \beta$ the infimum in (III_σ) is attained.

Proof. The statements of the theorem can be proved in the same way as Theorems 3.1, 4.1, and 5.1, respectively. An appropriate analog of Proposition 1.1 should be used, and the function $w_\varepsilon(r)$ defined in (3) should be replaced by the function $w_{\varepsilon,\sigma}(r)$. □

Theorem 8.2. *For any $\beta > 0$, there exists $\theta_* > 0$ such that if Ω is the spherical "hat" (28) and x_0 is the pole $\theta = 0$, then for $p \ge \frac{n+2}{3} + \beta$ and for any $0 < \sigma < 1$ the infimum in (I_σ) is not attained.*

Proof. As in Theorem 3.2, the claim reduces to the fact that the function

$$h_\sigma(\theta) = \frac{\sin^{\alpha+\mu-1}(\theta)}{\theta^\alpha}\cdot\int_\theta^{\theta_*}\sin^{-\mu}(t)\,dt = \Big(\frac{\sin(\theta)}{\theta}\Big)^\alpha\cdot h(\theta), \quad \alpha = \frac{n(1-\sigma)}{n-\sigma}, \tag{78}$$

is monotone decreasing. Since the first factor in (78) is monotone decreasing on $[0,\pi]$, inequality (32) implies the monotonicity of h_σ. □

Theorem 8.3. *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with strictly Lipschitz boundary. Suppose $\partial\Omega$ is of class $\mathcal{C}^2$ in a neighborhood of a point $x_0 \in \partial\Omega$ and $H(x_0) > 0$. Then for any $0 < \sigma < 1$ there exists $\beta > 0$ such that for $1 < p < \frac{n+1}{2} + \beta$ the infimum in (II_σ) is attained.*

Under the same assumptions, for any $0 < \sigma < 1$ there exists $\beta > 0$ such that for $1 < p < \frac{n+1}{2} + \beta$ the infimum in (III_σ) is attained.

Proof. The statements of the theorem can be proved by the same arguments as in the proof of Theorems 7.1 and 7.3, respectively. An appropriate analog of Proposition 7.1 should be used, and the function $w_\varepsilon(r)$ defined in (3) should be replaced by the function $w_{\varepsilon,\sigma}(r)$. □

Remark 12. Theorem 8.3 is also true if Ω is an arbitrary manifold with a strictly Lipschitz boundary.

Remark 13. Analogs of Theorems 6.1 and 7.4 for problem (IV_σ) can be proved by the same method. However, for $\sigma \neq 1$, the restrictions imposed on p turn out to be even "wilder".

Remark 14. In [GhK], besides the problem of minimizing a functional under the zero Dirichlet boundary conditions, problem (II_σ) was also considered in the case where Ω is a domain $\mathbb{R}^n$, $n \geq 3$, $x_0 \in \partial\Omega$, and $p = 2$. Essentially, the technique of [GhK] is the same as that used in the present paper. The last part of [GhK] contains Theorem 5.4 (without proof); that theorem deals with the case of an arbitrary $1 < p < n$. We conjecture that this statement is false without the additional requirement $p < \frac{n+1}{2} + \beta$; under this assumption, it coincides with the first part of our Theorem 8.3.

§9. Further generalizations

When $\Omega \subset \mathbb{R}^n$, we can consider the ratios (I)–(IV) with a more general definition of $\|\nabla u\|_{p,\Omega}$. Namely, consider an arbitrary norm $\mathcal{M}(x)$ in $\mathbb{R}^n$. Assume, for simplicity, that the function $\mathcal{M}$ is strictly convex in the nonradial directions, and that $\mathcal{M} \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\})$. We denote by

$$\mathcal{M}_o(x) = \sup_{\mathcal{M}(\xi)\leq 1} \langle x \cdot \xi \rangle = \sup_{\xi \neq 0} \frac{\langle x \cdot \xi \rangle}{\mathcal{M}(\xi)}$$

the dual norm. In particular, if

$$\mathcal{M}(x) = |x|_q \equiv \left(\sum_{k=1}^n |x_k|^q \right)^{\frac{1}{q}}, \quad 1 < q < \infty,$$

then $\mathcal{M}_o(x) = |x|_{q'}$.

Let $\omega_{n-1,\mathcal{M}}$ stand for the area of the unit sphere $\{x \in \mathbb{R}^n : \mathcal{M}(x) = 1\}$. In particular, for $\mathcal{M}(x) = |x|_q$ we have $\omega_{n-1,\mathcal{M}} = \omega_{n-1,q} = \frac{q[2\Gamma(\frac{1}{q}+1)]^n}{\Gamma(\frac{n}{q})}$.

Introducing an equivalent seminorm $\|\nabla v\|_{p,\mathcal{M},\Omega}$ on $W_p^1(\Omega)$ by

$$\|\nabla v\|_{p,\mathcal{M},\Omega}^p = \int_\Omega \mathcal{M}^p(\nabla v)\, dx,$$

we replace $\|\nabla v\|_{p,\Omega}$ in the numerators in (I)–(IV) by $\|\nabla v\|_{p,\mathcal{M},\Omega}$. Furthermore, for $0 < \sigma \leq 1$, we replace the norm $\|\cdot\|_{p^*,\Omega}$ in the denominators by the weighted norm $\|\cdot\|_{p^*_\sigma,\sigma,\Omega}$, defined in (75) with $r = \mathcal{M}_o(x - x_0)$; the resulting problems will be denoted

by $(\mathrm{I}_{\sigma,\mathcal{M}})$–$(\mathrm{IV}_{\sigma,\mathcal{M}})$. Let $\lambda_1(n,p,\sigma,\mathcal{M},\Omega),\dots,\lambda_4(n,p,\sigma,\mathcal{M},\Omega)$ denote the infimum values for these problems.

Proposition 9.1. *Let $n \geq 2$. Suppose Ω is a bounded domain in $\mathbb{R}^n$, $1 < p < n$, and $0 < \sigma \leq 1$. For $\sigma < 1$ we assume in addition that $x_0 \in \Omega$. Then the infimum in $(\mathrm{I}_{\sigma,\mathcal{M}})$ is not attained, does not depend on Ω, and is equal to $\frac{1}{K(n,p,\sigma,\mathcal{M})}$, where*

$$K(n,p,\sigma,\mathcal{M}) = \sup_{v\in\mathcal{C}_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\|v\|_{p^*_\sigma,\sigma,\mathbb{R}^n}}{\|\nabla v\|_{p,\mathcal{M},\mathbb{R}^n}} = \Big(\frac{\omega_{n-1}}{\omega_{n-1,\mathcal{M}}}\Big)^{\frac{\sigma}{n}} \cdot K(n,p,\sigma). \tag{79}$$

Proof. Without loss of generality, we may assume that $x_0 = 0$. Obviously, the dilation invariance of $(\mathrm{I}_{\sigma,\mathcal{M}})$ implies that the infimum does not depend on Ω. For $\sigma = 1$, the supremum in (79) was calculated in [AFTL, Corollary 3.2] by using convex symmetrization. We note that the same statement was obtained in [CNV] by using the mass transportation approach (the generalized Monge–Kantorovich problem). In [AFTL] and [CNV] it was also shown that the supremum in (79) is only attained on the functions $w_\varepsilon(r)$ defined by (2), with $r = \mathcal{M}_o(x)$. For a bounded domain, this means that the infimum in $(\mathrm{I}_{1,\mathcal{M}})$ is not attained. For $\sigma < 1$, arguing in the same way, we arrive at the function $w_{\varepsilon,\sigma}(r)$ defined in (77). □

Corollary 9.1. *Suppose that a bounded domain Ω is situated in the half-space $\{x \in \mathbb{R}^n : \langle x\cdot\xi\rangle > 0\}$ and $\partial\Omega$ contains an open part Γ situated in the plane $\langle x\cdot\xi\rangle = 0$ and containing a point x_0. Let $\mathcal{V}$ be the set of functions $v \in W^1_p(\Omega)$ vanishing on $\{x \in \partial\Omega : \langle x\cdot\xi\rangle > 0\}$. Then the infimum of the ratio $(\mathrm{I}_{\sigma,\mathcal{M}})$ over $\mathcal{V}$ is not attained and is equal to $\frac{2^{-\frac{\sigma}{n}}}{K(n,p,\sigma,\mathcal{M})}$.*

Proof. It suffices to use the even reflection with respect to the plane $\langle x\cdot\xi\rangle = 0$ and to apply Proposition 9.1. □

Proposition 9.2. *Let $n \geq 2$, and let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega$ of class $\mathcal{C}^2$. For $\sigma < 1$, we assume in addition that $x_0 \in \partial\Omega$. Suppose that for the infimum in $(\mathrm{II}_{\sigma,\mathcal{M}})$, $(\mathrm{III}_{\sigma,\mathcal{M}})$, or $(\mathrm{IV}_{\sigma,\mathcal{M}})$ we have*

$$\lambda_m(n,p,\sigma,\mathcal{M},\Omega) < \frac{2^{-\frac{\sigma}{n}}}{K(n,p,\sigma,\mathcal{M})}.$$

Then the infimum is attained.

Proof. This can be proved by arguments similar to those in [LPT, Corollary 2.1] and involving Corollary 9.1. □

Theorem 9.1. *For problems $(\mathrm{II}_{1,\mathcal{M}})$, $(\mathrm{III}_{1,\mathcal{M}})$, and $(\mathrm{IV}_{1,\mathcal{M}})$, the respective statements of Theorems 7.1, 7.3, and 7.4 are valid.*

Proof. We define a point x_0 in the same way as in the proof of Theorem 7.1 and assume that x_0 coincides with the origin. When integrating a function that depends only on $r = \mathcal{M}_o(x)$, for the volume form in a neighborhood of x_0 we have

$$dx = \frac{1}{2}\left(\omega_{n-1,\mathcal{M}} r^{n-1} - \mathbf{Sp}(\mathfrak{K}(x_0)\mathbb{M}) r^n + o(r^n)\right)\, dr,$$

where $\mathfrak{K}(x_0)$ stands for the curvature matrix of $\partial\Omega$ at x_0, and $\mathbb{M}$ denotes the second moment matrix of the intersection of the unit sphere $\{x \in \mathbb{R}^n : \mathcal{M}_o(x) = 1\}$ and the tangent plane to $\partial\Omega$ at x_0. Since both matrices are positive definite, the coefficient of r^n is negative.

Now the proofs of Theorems 7.1, 7.3, and 7.4 can be repeated without changes, with the use of Proposition 9.2 in place of Proposition 7.1. □

The next statement can be proved in a similar way.

Theorem 9.2. *Let $x_0 \in \partial\Omega$. Suppose that $\mathbf{Sp}(\mathfrak{K}(x_0)\mathbb{M}) > 0$. Then the statement of Theorem 8.3 is true for problems $(\mathrm{II}_{\sigma,\mathcal{M}})$ and $(\mathrm{III}_{\sigma,\mathcal{M}})$ with $0 < \sigma < 1$.*

Theorems 9.1 and 9.2 imply the solvability of the Euler equations for the corresponding functionals with the natural boundary conditions. Using a suitable normalization, these equations and boundary conditions can be written as follows:

$$\begin{cases} -\Delta_{p,\mathcal{M}} u = r^{(\sigma-1)p^*_\sigma}|u|^{p^*_\sigma-2}u - |u|^{p-2}u & \text{for } (\mathrm{II}_{\sigma,\mathcal{M}}), \\ -\Delta_{p,\mathcal{M}} u = r^{(\sigma-1)p^*_\sigma}|u|^{p^*_\sigma-2}u - C & \text{for } (\mathrm{III}_{\sigma,\mathcal{M}}), \\ -\Delta_{p,\mathcal{M}} u = r^{(\sigma-1)p^*_\sigma}|u|^{p^*_\sigma-2}u & \text{for } (\mathrm{IV}_{\sigma,\mathcal{M}})\ ; \end{cases} \tag{80}$$

$$\langle \mathbf{n}\cdot\nabla\mathcal{M}(\xi)\rangle\Big|_{\substack{\xi=\nabla u\\ x\in\partial\Omega}} = 0.$$

Here

$$\Delta_{p,\mathcal{M}} u = \operatorname{div}\left(\mathcal{M}^{p-1}(\xi)\nabla\mathcal{M}(\xi)\Big|_{\xi=\nabla u}\right)$$

is the generalized p-Laplacian generated by $\mathcal{M}$. In particular, for $\mathcal{M}(x) = |x|_q$ we have

$$\Delta_{p,\mathcal{M}} u = \Delta_{p,q} u \equiv \sum_{j=1}^{n} \left(|\nabla u|_q^{p-q}|u_{x_j}|^{q-2}u_{x_j}\right)_{x_j}.$$

Observe that $\Delta_{p,2}$ is the usual p-Laplacian, and $\Delta_{p,p}$ is the so-called pseudo-p-Laplacian (see, e.g., [BK] and [IK]). The properties of solutions of the Dirichlet problem for equations with the general operator $\Delta_{p,\mathcal{M}}$ and a subcritical right-hand side were studied in [BFK].

Remark 15. The statements of Remark 9 remain true for the first boundary value problem in (80) with $\sigma = 1$.

Theorem 9.3. *Let $n \geq 2$, let Ω be the flat torus $\mathbb{T}^n$ or a polyhedron in $\mathbb{R}^n$, and let $1 < p < n$. Then there exists $\varkappa^* > 0$ such that for $\varkappa > \varkappa^*$ the infimum in $(\mathrm{II}_{1,\mathcal{M}})$ on $\Omega(\varkappa)$ is not attained.*

Proof. In the case of a polyhedron, the proof of Theorem 7.2 works without any modifications. The proof for the torus is similar. □

We would like to thank Professors A. Alvino and B. Kawohl, who kindly provided us with the texts of their papers. Also, we are grateful to the participants of the V. I. Smirnov seminar (St. Petersburg Branch of Steklov Mathematical Institute) for critical remarks that helped us to improve the paper.

References

[AM] Adimurthi and G. Mancini, *The Neumann problem for elliptic equations with critical nonlinearity*, Nonlinear Analysis, Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa, 1991, pp. 9–25. MR1205370 (94d:35043)

[AFTL] A. Alvino, V. Ferone, G. Trombetti, and P.-L. Lions, *Convex symmetrization and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 275–293. MR1441395 (98d:35068)

[Aub] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geom. **11** (1976), 573–598. MR0448404 (56:6711)

[BK] M. Belloni and B. Kawohl, *The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$*, ESAIM Contrôle Optim. Calc. Var. **10** (2004), 28–52. MR2084254 (2005g:35227)

[BFK] M. Belloni, V. Ferone, and B. Kawohl, *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators*, Z. Angew. Math. Phys. **54** (2003), 771–783. MR2019179 (2005e:35177)

[Bl] G. A. Bliss, *An integral inequality*, J. London Math. Soc. **5** (1930), 40–46.

[Br] H. Brezis, *Some variational problems with lack of compactness*, Nonlinear Functional Analysis and its Applications, Part 1 (Berkeley, CA, 1983), Proc. Sympos. Pure Math., vol. 45, Part 1, Amer. Math. Soc., Providence, RI, 1986, pp. 165–201. MR0843559 (87h:35095)

[CNV] D. Cordero-Erausquin, B. Nazaret, and C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo–Nirenberg inequalities*, Adv. Math. **182** (2004), 307–332. MR2032031 (2005b:26023)

[Dr] O. Druet, *Optimal Sobolev inequalities of arbitrary order on compact Riemannian manifolds*, J. Funct. Anal. **159** (1998), 217–242. MR1654123 (99m:53076)

[GhK] N. Ghoussoub and X. S. Kang, *Hardy–Sobolev critical elliptic equations with boundary singularities*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), 767–793. MR2097030 (2005i:35086)

[GhY] N. Ghoussoub and C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. **352** (2000), 5703–5743. MR1695021 (2001b:35109)

[HV] E. Hebey and M. Vaugon, *Meilleures constantes dans le théorème d'inclusion de Sobolev*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 57–93. MR1373472 (96m:46054)

[IK] T. Ishibashi and S. Koike, *On fully nonlinear PDEs derived from variational problems of L^p norms*, SIAM J. Math. Anal. **33** (2001), 545–569. MR1871409 (2002j:49037)

[LaU] O. A. Ladyzhenskaya and N. N. Ural′tseva, *Linear and quasilinear equations of elliptic type*, 2nd ed., "Nauka", Moscow, 1973; English transl. of 1st ed., Acad. Press, New York–London, 1968. MR0509265 (58:23009); MR0244627 (39:5941)

[Lb] E. H. Lieb, *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities*, Ann. of Math. (2) **118** (1983), no. 2, 349–374. MR0717827 (86i:42010)

[Lin] C. S. Lin, *Locating the peaks of solutions via the maximum principle.* I. *The Neumann problem*, Comm. Pure Appl. Math. **54** (2001), 1065–1095. MR1835382 (2002d:35052)

[Ls] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case*. 1, 2, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 109–145, 223–283. MR0778970 (87e:49035a); MR0778974 (87e:49035b)

[LPT] P.-L. Lions, F. Pacella, and M. Tricarico, *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. **37** (1988), no. 2, 301–324. MR0963504 (89i:46036)

[NShch] A. I. Nazarov and A. P. Shcheglova, *On some properties of extremals in a variational problem generated by the Sobolev embedding theorem*, Nonlinear Problems and Function Theory, Probl. Mat. Anal., vyp. 27, Novosibirsk, 2004, pp. 109–136; English transl., J. Math. Sci. **120** (2004), no. 2, 1125–1144.

[Tal] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372. MR0463908 (57:3846)

[Tr] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747. MR0226198 (37:1788)

[W] X. J. Wang, *Neumann problems of semilinear elliptic equations involving critical Sobolev exponents*, J. Differential Equations **93** (1991), no. 2, 283–310. MR1125221 (92j:35072)

[Zhu] M. Zhu, *On the extremal functions of Sobolev–Poincaré inequality*, Pacific J. Math. **214** (2004), no. 1, 185–199. MR2039132 (2004m:58037)

St. Petersburg State University, Russia
E-mail address: alex@ad9503.spb.edu

St. Petersburg State University, Russia
E-mail address: an@AN4751.spb.edu

Received 30/NOV/2004

Translated by A. I. NAZAROV