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A MINIMAL AREA PROBLEM FOR NONVANISHING FUNCTIONS

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Abstract. The minimal area covered by the image of the unit disk is found for nonvanishing univalent functions normalized by the conditions $f(0) = 1$, $f'(0) = \alpha$. Two different approaches are discussed, each of which contributes to the complete solution of the problem. The first approach reduces the problem, via symmetrization, to the class of typically real functions, where the well-known integral representation can be employed to obtain the solution upon *a priori* knowledge of the extremal function. The second approach, requiring smoothness assumptions, leads, via some variational formulas, to a boundary value problem for analytic functions, which admits an explicit solution.

§1. Introduction

The class of nonvanishing analytic functions, with the exponential e^z as a typical example, is one of the standard classes studied in complex analysis. The nonlinearity of this class causes difficulties when working with extremal problems. To give an example, we mention that Krzyz's coefficient conjecture $|a_n| \le 2/e$ for zero-free analytic functions bounded by 1 in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ has remained open for more than 30 years. The present work was motivated by a recent paper [1], where the authors studied a general extremal problem involving minimal area related to nonvanishing functions.

For $\alpha > 0$, let N_α denote the set of analytic functions

$$f(z) = 1 + a_1(f)z + a_2(f)z^2 + \cdots$$

normalized by the condition $a_1(f) = \alpha$ that are univalent and *zero-free* in $\mathbb{D}$. For univalent functions, the Dirichlet integral

$$D(f) = \int_{\mathbb{D}} |f'|^2 \, d\sigma = \pi \sum_{n=1}^{\infty} n \, |a_n(f)|^2 \tag{1.1}$$

measures the area of the image $f(\mathbb{D})$. From (1.1), it is immediate that

$$D(f) \ge \pi\alpha^2,$$

with the sign of equality only for the linear polynomial $p_\alpha(z) = 1+\alpha z$, which is in N_α for $0 < \alpha \le 1$. Thus if $0 < \alpha \le 1$, the minimal area problem for N_α is trivial. In addition, the Koebe 1/4-theorem implies that the class N_α is empty for $\alpha > 4$ and is trivial for $\alpha = 4$. Thus, we are left with the so-called *nontrivial range* $1 < \alpha < 4$.

For this range, the minimal area problem for N_α is solved by the following

Theorem 1. *For $1 < \alpha < 4$, let $f \in N_\alpha$. Then*

$$D(f) \ge \pi\alpha a^2(a + \sqrt{a^2-1})^2(\alpha a^2 - 2\sqrt{a^2-1}(a + \sqrt{a^2-1})), \tag{1.2}$$

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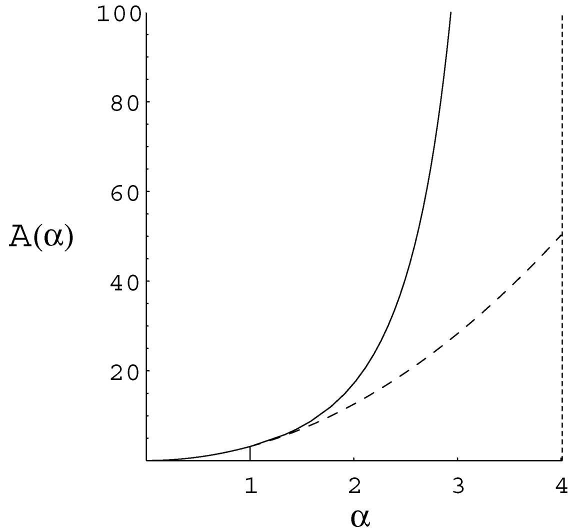

FIGURE 1. The graph of $A(\alpha)$.

where $a = a(\alpha)$ is the solution to the equation

$$\alpha^{-1} = a^2 \left(1 - \sqrt{a^2-1}(a+\sqrt{a^2-1})^3 \log \frac{(a+\sqrt{a^2-1})^4}{16a^2(a^2-1)}\right), \tag{1.3}$$

which is unique in the interval $1 < a < \infty$.

Equality occurs in (1.2) *if and only if $f = f_\alpha$, where f_α is a univalent function defined by*

$$f_\alpha(z) = \int_{-1}^{z} g_\alpha(\zeta) \frac{dz}{z} \tag{1.4}$$

with

$$\zeta = \frac{ia}{2} \frac{1-z}{\sqrt{z}} \tag{1.5}$$

and

$$g_\alpha(\zeta) = -\beta \frac{\sqrt{\zeta^2 - a^2}}{(\zeta+\sqrt{\zeta^2-1})^2 (a\sqrt{\zeta^2-1} + \zeta\sqrt{a^2-1})} \tag{1.6}$$

with the principal branches of the radicals and

$$\beta = \alpha a^2 (a + \sqrt{a^2-1}). \tag{1.7}$$

For $0 < \alpha < 4$, let

$$A(\alpha) = \min_{f \in N_\alpha} D(f)$$

denote the minimal area covered by the images of functions of class N_α. Thus $A(\alpha) = \pi\alpha^2$ if $0 < \alpha \le 1$ and $A(\alpha)$ is given by the right-hand side of (1.2) if $1 < \alpha < 4$. The graph of $A(\alpha)$ displayed in Figure 1 suggests that $A(\alpha)$ is strictly increasing and convex on $0 < \alpha < 4$. These properties can be derived from the explicit form of $A(\alpha)$ given by formulas (1.2) and (1.3). Alternatively, in Section 3 we derive some of these properties using geometric and variational arguments. An advantage of the latter method is that it does not require explicit formulas and can be used when such formulas are not known or when they are complicated.

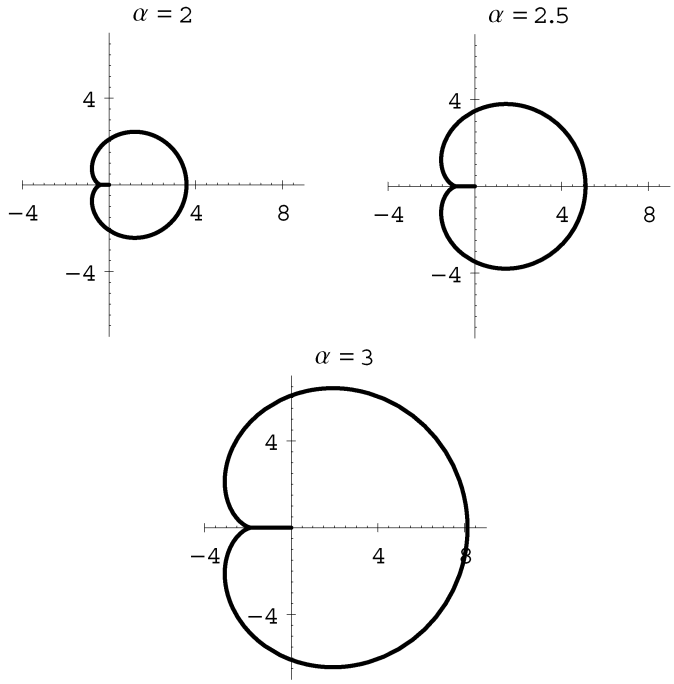

FIGURE 2. Extremal domains for some typical values of α.

Figure 2 demonstrates the extremal shapes $D_\alpha = f_\alpha(\mathbb{D})$ for some typical values of α. The boundary ∂D_α consists of the so-called *free boundary* L_{fr}, the closure of which is a closed Jordan curve, and the *nonfree boundary* L_{nf}, which is a straight line segment on the negative real axis. The precise definitions will be postponed until §3. Here we want to mention one important property of the free boundary L_{fr}: if f_α is an extremal function in N_α and $f_\alpha(e^{i\theta}) \in L_{\mathrm{fr}}$, then $|f'_\alpha(e^{i\theta})| = \beta$, where the constant $\beta = \beta(\alpha)$ is defined by (1.7).

Our main goal in this paper is to demonstrate and compare two different approaches developed in [3] and [5], respectively, each of which upon using the other gives a complete solution to the minimal area problem on N_α.

First in §2, we consider the minimal area problem on the class T_α of typically real nonvanishing functions (which are *not necessarily univalent*). In this new setting we can use the linear structure of T_α. The latter immediately implies uniqueness of the extremal function and leads to a simple sufficient condition for the extremality of the corresponding linearized functional. Theorem 2 in §2 gives a complete solution to the minimal area problem on T_α. The final step of its proof is to verify that the conjectured extremal function, again given by formulas (1.4)–(1.7), satisfies the mentioned sufficient condition. The method explained above was developed in [3]. It can be applied in extremal problems on T_α if we *a priori* know the explicit form of the extremal function. What is missing here is that the method gives no clues as to how to construct the extremal function.

At this step another approach developed in [5] turns out to be useful. This method, employed in §§3 and 4, is based on a local variation. Assuming a sufficient smoothness of the free boundary of an extremal domain, we can apply a variant of Julia's variation, which leads to certain boundary conditions for the extremal analytic function. For the problem studied in this paper and in many other problems this knowledge of the boundary is sufficient to recover the extremal function itself. In particular, we use this approach in §4 to find the extremal function of Theorem 1.

A disadvantage of this variational method (and of any other method based on the Julia variation) is that it requires some *a priori* smoothness of the boundary of the extremal configuration. In this paper, we achieve the desired smoothness by exploiting the geometrical content of the main parameter $\alpha = a_1(f)$. Namely, since α is equal to the conformal radius of $f(\mathbb{D})$ at $w = 1$, we can apply suitable symmetrizations to obtain the desired boundary characteristics while keeping control of $a_1(f)$. This approach is not as successful by itself when working with other constraints, for instance, in problems with a fixed nth coefficient with $n \geq 2$.

Nevertheless, by combining both approaches we can overcome the disadvantages of each of them as follows:

• Assuming regularity of the boundary, we apply the variational method to find an explicit form of the extremal function. At this step we obtain a "conditional" solution of the extremal problem in question. Upon justifying the required boundary smoothness, the conditional proof becomes a true proof.

• Next, to finish the proof, we verify that the function recovered at the first step satisfies the sufficient condition of extremality, which also leads then to a complete solution of the problem.

§2. Typically real nonvanishing functions

The standard class T of typically real functions consists of functions f analytic in $\mathbb{D}$, normalized by the conditions $f(0) = 0$, $f'(0) = 1$, and satisfying an additional constraint

$$\operatorname{Im} z \cdot \operatorname{Im} f(z) \geq 0 \quad \text{for all } z \in \mathbb{D}; \tag{2.1}$$

see [7].

In this section we consider typically real functions with a different normalization. For $\alpha > 0$, let T_α be the class of functions f analytic and nonvanishing in $\mathbb{D}$, normalized by the conditions $f(0) = 1$, $f'(0) = \alpha$, and also satisfying (2.1). Since $f \in T_\alpha$ is nonvanishing, it easily follows that $(-\infty, 0] \subset \mathbb{C} \setminus f(\mathbb{D})$. Therefore, $0 < \alpha \leq 4$ by the subordination principle; cf. [7].

If $f, f_0 \in T_\alpha$, then $f_\varepsilon = (1-\varepsilon)f_0 + \varepsilon f$ is in T_α for all $0 \leq \varepsilon \leq 1$. Thus, T_α is a convex compact subset of the set of analytic functions. The latter easily implies that for every α there is exactly one function f_α minimizing the Dirichlet integral over T_α. Indeed, let f_1 and f_2 be two minimizers. Then

$$\begin{aligned} D((f_1+f_2)/2) &= \frac{1}{4}\int_{\mathbb{D}} |f_1' + f_2'|^2 \, d\sigma \\ &\leq \frac{1}{2}\left(\int_{\mathbb{D}} |f_1'|^2 \, d\sigma + \int_{\mathbb{D}} |f_2'|^2 \, d\sigma \right) = (1/2)(D(f_1) + D(f_2)), \end{aligned}$$

with the sign of equality if and only if $f_1'(z) \equiv f_2'(z)$, which implies the uniqueness of the minimizer. Now an elementary variational argument used in [3] implies the following lemma.

Lemma 1 (cf. [3, Lemma 1])**.** *The function $f_\alpha \in T_\alpha$ such that f_α' is continuous on $\overline{\mathbb{D}}$ minimizes the Dirichlet integral $D(f)$ on the class T_α if and only if f_α minimizes on T_α the linear functional*

$$L(f) := \operatorname{Re} \int_{\mathbb{D}} \overline{f_\alpha'}(z) f'(z) \, d\sigma. \tag{2.2}$$

As is well known, the class T of typically real functions has an integral representation (see [7, Theorem 2.20]). Adapting this representation to our situation, we see that $f \in T_\alpha$ if and only if there is a probability measure μ_f on $\mathbb{T} = \{z : |z| = 1\}$ such that $d\mu_f(-t) = d\mu_f(t)$ and the following two conditions are satisfied:

$$f(z) = 1 + \frac{\alpha z}{1 - z^2} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu_f(t) = 1 + 2\alpha \int_0^{\pi} \frac{z}{1 - 2z\cos t + z^2}\, d\mu_f(t) \tag{2.3}$$

and

$$\int_0^{\pi} \sec^2(t/2)\, d\mu_f(t) \le 2\alpha^{-1}. \tag{2.4}$$

Inequality (2.4), which is equivalent to the inequality $f(-1) \ge 0$, guarantees that f is nonvanishing.

Since we can differentiate (2.3) with respect to z, we obtain

$$f'(z) = 2\alpha \int_0^{\pi} \frac{1 - z^2}{(1 - 2z\cos t + z^2)^2}\, d\mu_f(t). \tag{2.5}$$

The following lemma is a modification of Lemma 8 in [3].

Lemma 2. *If $f'_\alpha(z)$ is continuous on $\overline{\mathbb{D}}$, then the linear functional* (2.2) *can be represented as*

$$L(f) = \int_0^{\pi} K_\alpha(t)\, d\mu_f(t) \tag{2.6}$$

with the kernel

$$K_\alpha(t) = \frac{2\pi\alpha}{\sin t} \operatorname{Im}\left(e^{it} f'_\alpha(e^{it})\right). \tag{2.7}$$

Proof. Using (2.5), we can represent (2.2) as

$$\begin{aligned} L(f) &= 2\alpha \lim_{r\to 1^-} \operatorname{Re} \int_{\mathbb{D}_r} \bar{f}'_\alpha(z) \left(\int_0^{\pi} \frac{1 - z^2}{(1 - 2z\cos t + z^2)^2}\, d\mu_f(t) \right) d\sigma \\ &= 2\alpha \lim_{r\to 1^-} \operatorname{Re} \int_0^{\pi} \left(\int_{\mathbb{D}_r} \bar{f}'_\alpha(z) \frac{1 - z^2}{(1 - 2z\cos t + z^2)^2}\, d\sigma \right) d\mu_f(t). \end{aligned} \tag{2.8}$$

Applying Green's formula to the second area integral in (2.8), we obtain

$$L(f) = \alpha \lim_{r\to 1^-} \int_0^{\pi} \left(\operatorname{Im} \int_{|z|=r} \bar{f}_\alpha(z) \frac{1 - z^2}{(1 - 2z\cos t + z^2)^2}\, dz \right) d\mu_f(t). \tag{2.9}$$

Let $J_1(r)$ denote the inner integral in (2.9). Then

$$\begin{aligned} \operatorname{Im} J_1(r) &= \operatorname{Im} \int_{|z|=r} \bar{f}_\alpha(z) \frac{1 - z^2}{(1 - 2z\cos t + z^2)^2}\, dz \\ &= -\operatorname{Im} \int_{|z|=r} f_\alpha(z) \frac{1 - \bar{z}^2}{(1 - 2\bar{z}\cos t + \bar{z}^2)^2}\, d\bar{z} \\ &= r^2 \operatorname{Im} \int_{|z|=r} f_\alpha(z) \frac{z^2 - r^4}{(z^2 - 2r^2 z\cos t + r^4)^2}\, dz. \end{aligned} \tag{2.10}$$

Let $H(z)$ denote the integrand of the last integral in (2.10). Then H has second order poles at the points $z = r^2 e^{it}$ and $z = r^2 e^{-it}$. Computing the residues of H at the poles, we obtain

$$\operatorname{Res}[H, r^2 e^{it}] = -\frac{ie^{it}}{2\sin t} f'_\alpha(r^2 e^{it}), \quad \operatorname{Res}[H, r^2 e^{-it}] = \frac{ie^{-it}}{2\sin t} f'_\alpha(r^2 e^{-it}).$$

Now applying the residue theorem, we find

$$\operatorname{Im} J_1(r) = \frac{2\pi r^2}{\sin t} \operatorname{Im}\left(e^{it} f'_\alpha(r^2 e^{it})\right). \tag{2.11}$$

Substituting (2.11) into (2.9) and taking the limit inside the integral, we obtain (2.6) and (2.7). □

The following theorem shows that for all $0 < \alpha < 4$ the minimal area problems for T_α and for N_α share the same extremal function (cf. [3, Theorem 3]).

Theorem 2. *For $1 < \alpha < 4$, let $f \in T_\alpha$. Then inequality* (1.2) *of Theorem* 1 *holds true with the same cases of equality.*

Proof. Let f_α be the extremal function in T_α, and let $D_\alpha = f_\alpha(\mathbb{D})$. First let us show that $(0,1] \subset D_\alpha$ for every $1 < \alpha < 4$. If not, then, for all $\tau < 1$ sufficiently close to 1, the function

$$\tilde{f}(z) = 1 - \tau^{-1} + \tau^{-1} f_\alpha(\tau z) \tag{2.12}$$

is in T_α. We also have

$$\begin{aligned} D(\tilde{f}) &= \pi\left(\alpha^2 + \sum_{k=2}^{\infty} k|a_k(\tilde{f})|^2\right) = \pi\left(\alpha^2 + \sum_{k=2}^{\infty} k\tau^{2k-1}|a_k(f_\alpha)|^2\right) \\ &\le \pi\left(\alpha^2 + \sum_{k=2}^{\infty} k|a_k(f_\alpha)|^2\right) = D(f_\alpha) \end{aligned} \tag{2.13}$$

with the sign of equality if and only if $f_\alpha(z) = 1 + \alpha z$. Since for $\alpha > 1$ the polynomial $p_\alpha(z) = 1 + \alpha z$ is not in T_α, we conclude that $(0,1] \subset D_\alpha$. Since f_α is typically real and zero-free, it follows that $f_\alpha(-1) = 0$.

The latter observation when combined with Lemmas 1 and 2 implies that f_α minimizes the Dirichlet integral on T_α if and only if f_α minimizes on T_α the linear functional (2.6) under the following linear constraints:

$$2\int_0^\pi d\mu_f(t) = 1 \quad \text{and} \quad \int_0^\pi \sec^2(t/2)\, d\mu_f(t) = 2\alpha^{-1}.$$

The above consideration allows us to use well-known results about extremal problems for linear functionals with linear restrictions, which say that f_α is extremal if and only if the kernel (2.7) satisfies the conditions

$$K_\alpha(t) = \lambda_0 + \lambda_1 \sec^2(t/2) \quad \text{for all } t \in \operatorname{Support}(\mu_{f_\alpha}) \tag{2.14}$$

and

$$K_\alpha(t) \ge \lambda_0 + \lambda_1 \sec^2(t/2) \quad \text{for all } t \notin \operatorname{Support}(\mu_{f_\alpha}), \tag{2.15}$$

where λ_0 and λ_1 are real constants and μ_{f_α} is the probability measure representing f_α by (2.3); cf. Lemmas 3 and 4 in [3].

It follows from the first integral representation in (2.3) that the function

$$h_\alpha(z) = \frac{1 - z^2}{\alpha z}\,(f_\alpha(z) - 1) \tag{2.16}$$

has positive real part on $\mathbb{D}$. Therefore, the corresponding probability measure μ_{f_α} is given by the following formula; see [7]:

$$d\mu_{f\alpha}(t) = \frac{1}{2\alpha\pi} \operatorname{Re}\left(\frac{1-z^2}{z} f_\alpha(z)\right) dt, \quad z = e^{it}. \tag{2.17}$$

For $1 < \alpha < 4$, let $\theta_\alpha = 2\arcsin(1/a)$, where $a = a(\alpha)$ is defined by equation (1.3), and let $l_\alpha = \{e^{i\theta} : \theta_\alpha < \theta < 2\pi - \theta_\alpha\}$. Using formulas (1.4)–(1.7), one can show that f_α takes negative real values on $l_\alpha \setminus \{0\}$; see §4 for the details of such a computation. The latter property together with (2.17) implies that

$$d\mu_\alpha(t) = 0 \quad \text{for all } \theta_\alpha < t < 2\pi - \theta_\alpha. \tag{2.18}$$

Since h_α defined by (2.16) has positive real part on $\mathbb{D}$, we have

$$d\mu_\alpha(t) \geq 0 \quad \text{for all } 0 \leq t \leq \pi. \tag{2.19}$$

The latter inequality follows also from formulas (1.4)–(1.7) and (2.17) after direct computation.

Now to finish the proof we must show that the kernel $K_\alpha(t)$ constructed for f_α by formula (2.7) satisfies conditions (2.14) and (2.15).

Differentiating (1.4) and taking into account (1.6), we obtain

$$zf'_\alpha(z) = -\beta \frac{\sqrt{\zeta^2 - a^2}}{(\zeta + \sqrt{\zeta^2 - 1})^2 (a\sqrt{\zeta^2 - 1} + \zeta\sqrt{a^2 - 1})}, \tag{2.20}$$

where ζ and β are given by (1.5) and (1.7).

From (2.7) and (2.20) we find

$$K_\alpha(t) = -\frac{2\pi\alpha\beta}{\sin t} \operatorname{Im} \frac{\sqrt{\zeta^2 - a^2}}{(\zeta + \sqrt{\zeta^2 - 1})^2 (a\sqrt{\zeta^2 - 1} + \zeta\sqrt{a^2 - 1})}. \tag{2.21}$$

Now we consider two cases. First, let $0 < t < \theta_\alpha$. Then $\zeta(e^{it}) = a\sin(t/2)$ by (1.5), and therefore

$$\sqrt{\zeta^2 - a^2} = ia\cos(t/2) \quad \text{and} \quad \sqrt{\zeta^2 - 1} = i\sqrt{1 - a^2\sin^2(t/2)}.$$

Substituting this into (2.21), after some algebra we obtain

$$K_\alpha(t) = \lambda_0 + \lambda_1 \sec^2(t/2) \tag{2.22}$$

with

$$\lambda_0 = 2\pi\beta^2 \quad \text{and} \quad \lambda_1 = \pi\alpha\beta(2a + \sqrt{a^2 - 1} - 2a^2\sqrt{a^2 - 1} - 2a^3). \tag{2.23}$$

Now we consider the case where $\theta_\alpha < t < \pi$. Computing $K_\alpha(t)$ as above, we obtain

$$K_\alpha(t) = \lambda_0 + \lambda_1 \sec^2(t/2) + \Phi(t),$$

where λ_0 and λ_1 are given by (2.23) and

$$\Phi(t) = \frac{2\pi\alpha\beta}{\sin t\cos(t/2)} \sqrt{a^2\sin^2(t/2) - 1} \left(2a(a + \sqrt{a^2 - 1})\sin^2(t/2) - 1\right).$$

We recall that $a^{-1} \leq \sin(t/2) \leq 1$ in the case under consideration, which easily implies that

$$2a(a + \sqrt{a^2 - 1})\sin^2(t/2) - 1 \geq 0.$$

Thus, $\Phi(t) \geq 0$. Therefore,

$$K_\alpha(t) \geq \lambda_0 + \lambda_1 \sec^2(t/2) \tag{2.24}$$

for every t, $\theta_\alpha \leq t \leq \pi$.

Equations (2.18), (2.19), (2.22), and (2.24) show that the kernel $K_\alpha(t)$ satisfies conditions (2.14) and (2.15). This implies that the function f_α defined by (1.4)–(1.7) is extremal for the minimal area problem for typically real nonvanishing functions.

To complete the proof of Theorem 2, we need to compute the minimal area $A(\alpha) = \text{area}(f_\alpha(\mathbb{D}))$ and show that equation (1.3) has a unique solution in the interval $1 < a < \infty$. These computations will be postponed until §4. $\square$

§3. Extremal domains

In this section, we prove symmetry properties of our domains extremal for the minimal area problem on N_α and then derive some implications.

Let $C_r(z_0) = \{z : |z - z_0| = r\}$ with $0 \le r \le \infty$. Let $\gamma_\theta(z_0) = \{z = z_0 + te^{i\theta}, 0 \le t < \infty\}$. By circular symmetrization of a domain $D \subset \overline{\mathbb{C}}$ with respect to $\gamma_\theta(z_0)$ we mean the domain D^* such that $C_r(z_0) \subset D^*$ whenever $C_r(z_0) \subset D$ and if $C_r(z_0) \not\subset D$ for $0 < r < \infty$, then $D^* \cap C_r(z_0)$ is a proper single arc of $C_r(z_0)$ centered at $z_0 + re^{i\theta}$ and such that $\operatorname{meas}(D^* \cap C_r(z_0)) = \operatorname{meas}(D \cap C_r(z_0))$.

A domain D is said to be *starlike* with respect to $z_0 \in D$ if D contains the segment $[z_0, z]$ for any point z in D. A domain D^* starlike with respect to z_0 is called the radial symmetrization of D with respect to z_0 if for some $\varepsilon > 0$ such that the disk $\mathbb{D}_\varepsilon(z_0) := \{z : |z - z_0| < \varepsilon\}$ is in D and for all $\theta \in \mathbb{R}$ we have $\lambda((D^* \setminus \mathbb{D}_\varepsilon(z_0)) \cap \gamma_\theta(z_0)) = \lambda((D \setminus \mathbb{D}_\varepsilon(z_0)) \cap \gamma_\theta(z_0))$, where for any $E \subset \gamma_\theta(z_0)$, $\lambda(E)$ denotes the logarithmic measure of E:

$$\lambda(E) = \int_E |z - z_0|^{-1}\, |dz|.$$

Now we define the *polarization* of a domain $D \subset \overline{\mathbb{C}}$ with respect to the directed line $l_\theta(z_0) := \{z = z_0 + te^{i\theta} : -\infty < t < \infty\}$ (see [6]). Let H^+ and H^- be the left and right half-planes with respect to $l_\theta(z_0)$, and let D^* denote the reflection of D in $l_\theta(z_0)$. Then the polarization of D with respect to $l_\theta(z_0)$ is defined by

$$D_p = ((D \cup D^*) \cap \overline{H}^+) \cup ((D \cap D^*) \cap H^-).$$

Note that D_p is open but might be disconnected and contain multiply connected components even if D is a simply connected domain.

It is necessary to emphasize that circular symmetrization and polarization preserve the area, while radial symmetrization diminishes it. All of these transformations increase the inner radius of a domain evaluated at appropriate points; see [6, 8]. We recall that the inner radius $R(D, z_0)$ of a domain $D \subset \mathbb{C}$ having Green's function $g(z, z_0)$ with singularity at $z_0 \in D$ is defined by

$$\log R(D, z_0) = \lim_{z \to z_0} (g(z, z_0) + \log |z - z_0|)$$

(see [6]). For simply connected domains the inner radius coincides with the conformal radius.

Lemma 3. *For every $1 < \alpha < 4$, there is at least one function $f_\alpha \in N_\alpha$ minimizing $D(f)$ over N_α.*

If $f \in N_\alpha$ is extremal, then the image $f(\mathbb{D})$ is a bounded domain starlike with respect to $w = 1$ and possessing circular symmetry with respect to the rays $l_\tau = \{z = x + iy : y = 0, x \ge \tau\}$ for all $0 \le \tau \le 1$.

In addition, the minimal area $A(\alpha) := D(f_\alpha)$ strictly increases in the interval $1 < \alpha < 4$.

Proof. Since this lemma is standard (cf. [2], [5], [4]), we only outline its proof.

For a given $1 < \alpha < 4$, the class N_α is compact with respect to the uniform convergence on compact subsets of $\mathbb{D}$. Since the Dirichlet integral $D(f)$ is lower semicontinuous, the latter implies the existence of an extremal function f_α, at least one for each α.

Let $\alpha_1 < \alpha_2$ and suppose that f_{α_2} is extremal for N_{α_2}. Then $\hat{f}(z) = f_{\alpha_2}((\alpha_1/\alpha_2)z)$ is in N_{α_1} and therefore $A(\alpha_1) \le D(\hat{f}) < D(f_{\alpha_2})$, which implies the strict monotonicity of $A(\alpha)$.

To establish the circular symmetry, we assume that $D_\alpha = f_\alpha(\mathbb{D})$ is not circularly symmetric with respect to a ray l_τ for some $0 \le \tau \le 1$. Then let D_α^* be the circular

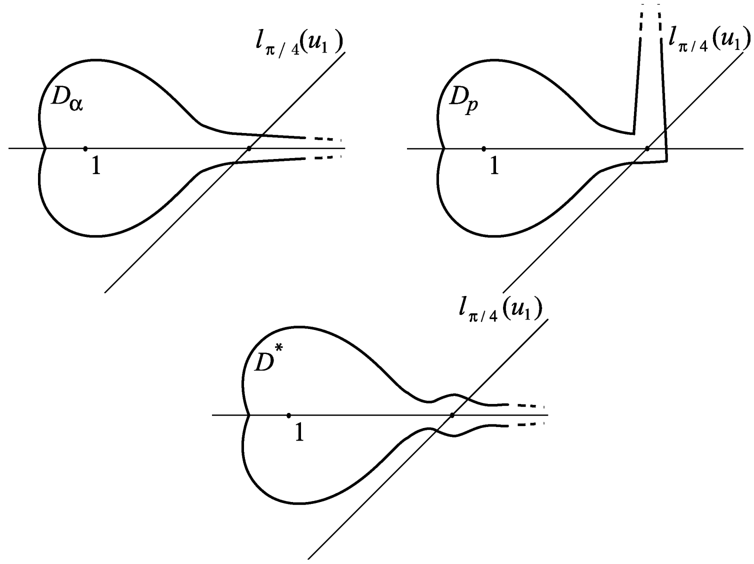

FIGURE 3. Boundedness of D_α via polarization.

symmetrization of D_α with respect to l_τ, and let $\tilde{f}$ map $\mathbb{D}$ conformally onto D_α^* in such a way that $\tilde{f}(0) = 1$, $\tilde{f}'(0) = \beta > 0$. Then $\operatorname{area}(D_\alpha^*) = \operatorname{area}(D_\alpha)$, and $\beta > \alpha$ by the principle of symmetrization (see [6]). Since $0 \notin D_\alpha^*$, we have $\tilde{f} \in N_\beta$, contradicting the monotonicity property of $A(\alpha)$.

Now, assuming circular symmetry, the same symmetrization argument works for radial symmetrization with respect to $w = 1$ (see [6]). The latter yields the starlike property of D_α.

Finally, to show that D_α is bounded we may use the polarization as in the proof of Lemma 2.2 in [5]. Namely, assuming that D_α is not bounded, we must have $l_1 \subset D_\alpha$ since D_α is circularly symmetric with respect to l_1. Since $\operatorname{area}(D_\alpha) < \infty$, for a given $\varepsilon > 0$, there is $u_0 > 1$ such that $|\operatorname{Im} w| < \varepsilon$ for all $w = u + iv \in \partial D_\alpha$ such that $u \geq u_0$. For $u_1 > u_0$, let H^- be the right half-plane with respect to the line $l_{\pi/4}(u_1)$. If u_1 is large enough, the set $D_\alpha \cap H^-$ lies in the horizontal strip between $l_0(-i\varepsilon)$ and $l_0(i\varepsilon)$, and therefore the polarization D_p of D_α with respect to $l_{\pi/4}(u_1)$ lies in the half-plane $\{w : \operatorname{Re} w < u_1 + \varepsilon\}$. Let D^* be the circular symmetrization of D_p with respect to l_1. Note that D^* is simply connected and $0 \notin D^*$ if u_0 is sufficiently large. Let $F(z) = 1 + \beta z + a_2 z^2 + \ldots$, where $\beta > 0$, map $\mathbb{D}$ conformally onto D^*. Note that $\operatorname{area} D^* = \operatorname{area} D_\alpha$ and $\beta > \alpha$ by the principles of symmetrization and polarization (see [6]). Since $0 \notin D^*$, we have $F \in N_\beta$, contradicting the monotonicity property of $A(\alpha)$. Figure 3 shows how polarization and symmetrization affect the domain D_α. $\square$

Combining the results of Lemma 3 and Theorem 2, we can prove inequality (1.2) of Theorem 1 as follows. For $1 < \alpha < 4$, let $f_\alpha \in N_\alpha$ be an extremal function, which exists by Lemma 3. The same lemma shows that f_α maps $\mathbb{D}$ onto a domain D_α symmetric with respect to the real axis. The latter implies that f_α is typically real, and therefore f_α is in the class T_α.

Now the conclusion of Theorem 1 follows from Theorem 2.

The symmetries established by Lemma 1 show that for every extremal domain D_α the boundary ∂D_α may contain a “nonfree” part L_{nf}, which, if it exists, must be a closed segment (possibly degenerate) lying on the ray $\{w : \operatorname{Re} w \leq 1\}$. The rest of the boundary, $L_{\mathrm{fr}} = \partial D_\alpha \backslash L_{\mathrm{nf}}$ is usually called a "free" boundary. Accordingly, the preimages $l_{\mathrm{nf}} = \{e^{i\theta} : f_\alpha(e^{i\theta}) \in L_{\mathrm{nf}}\}$ and $l_{\mathrm{fr}} = \{e^{i\theta} : f_\alpha(e^{i\theta}) \in L_{\mathrm{fr}}\}$ will be called a *nonfree arc* and a *free arc*, respectively. The following lemma describes some useful properties of these arcs.

Lemma 4. (1) *For* $1 < \alpha < 4$ *and every extremal domain* $D_\alpha = f_\alpha(\mathbb{D})$*, the free boundary* L_{fr} *is a Jordan rectifiable arc symmetric with respect to the real axis, which begins and ends at some point* $c_\alpha \le 0$*.*

In addition, L_{fr} *satisfies the following Lavrent′ev condition:*

$$\mathrm{length}(J(w_1, w_2)) \le C|w_1 - w_2| \quad \text{for } w_1, w_2 \in L_{\mathrm{fr}},$$

where C *is a constant independent of* w_1, w_2*, and* $J(w_1, w_2)$ *denotes the arc of* L_{fr} *between* w_1 *and* w_2*.*

(2) *Suppose that* $c_\alpha < 0$*. Then* $l_{\mathrm{nf}} = \{e^{i\theta} : \theta_\alpha \le \theta \le 2\pi - \theta_\alpha\}$ *for some* $0 < \theta_\alpha < \pi$*, and* $|f'_\alpha(e^{i\theta})|$ *strictly decreases from some* $\beta > 0$ *to* 0 *as* θ *runs from* θ_α *to* π*.*

Proof. Part (1), except for the inequality $c_\alpha \le 0$, easily follows from the symmetry properties of D_α; see [2, Lemma 8] or [4, Lemma 2.2].

Let us show that $c_\alpha \le 0$. If not, then the elementary variation $\tilde{f}$ defined by (2.12) is in N_α. By inequality (2.13), we have $D(\tilde{f}) \le D(f_\alpha)$ with the sign of equality if and only if $f_\alpha(z) = 1 + \alpha z$. Since for $\alpha > 1$, the polynomial $p_\alpha(z) = 1 + \alpha z$ is not in N_α, we conclude that $(0, 1] \subset D_\alpha$ and therefore $c_\alpha \le 0$.

To prove part (2), we note that D_α, being circularly symmetric with respect to l_0, possesses *the polarization property* with respect to the vertical lines $v_u := \{w = u + it : -\infty < t < \infty\}$ for all $c_\alpha \le u \le 0$. The latter means that $(D^{u-})^* \subset D^{u+}$ for all $c_\alpha \le u \le 0$, where $D^{u-} = D_\alpha \cap \{w : \mathrm{Re}\, w < u\}$, $D^{u+} = D_\alpha \cap \{w : \mathrm{Re}\, w > u\}$, and $(D^{u-})^*$ denotes the reflection of D^{u-} with respect to v_u. Now the strict monotonicity of $|f'(e^{i\theta})|$ for $\theta_\alpha < \theta < \pi$ follows from Lemma 2.4 in [5]. In addition, since the domain D_α has an inner angle 2π at $w = c_\alpha$, it follows that $f'(-1) = 0$. □

§4. Boundary variation and extremal functions

In this section, we show how the function f_α, extremal for the minimal area problem on N_α, can be recovered from its boundary values, which in turn reflect some variational properties of the extremal domains.

Lemma 5. *For* $1 < \alpha < 4$*, let* f *be an extremal function for the minimal area problem for* N_α*, and let* $l_{\mathrm{fr}} = \{e^{i\theta} : |\theta| < \theta_0(\alpha)\}$*, with* $0 < \theta_0(\alpha) < \pi$*, be the corresponding free arc. Then* $f'(z)$ *is continuous on* $\overline{\mathbb{D}}$ *and* $|f'(z)| = \beta$ *for some* $\beta > \alpha$ *for all* $z \in l_{\mathrm{fr}}$*.*

Proof. First we show that $|f'(z)|$ is constant a.e. on l_{fr}. Since L_{fr} is Jordan rectifiable by Lemma 4, it follows that the nonzero finite limit

$$f'(\zeta) = \lim_{z \to \zeta, z \in \overline{\mathbb{U}}} \frac{f(z) - f(\zeta)}{z - \zeta} \ne 0, \infty \tag{4.1}$$

exists a.e. on l_{fr} (see [10, Theorem 6.8, Exercise 6.4.5]). Assume that

$$0 < \beta_1 = |f'(e^{i\theta_1})| < |f'(e^{i\theta_2})| = \beta_2 < \infty \tag{4.2}$$

for $e^{i\theta_1}, e^{i\theta_2} \in l_{\mathrm{fr}}$. Note that (4.1) and (4.2) allow us to apply the two-point variational formulas of [5, Lemma 10]. Namely, for fixed positive k_1, k_2 such that $0 < k_1 < 1 < k_2$ and $k_1\beta_1^{-1} > k_2\beta_2^{-1}$ and fixed $\varphi > 0$ small enough, we consider the two-point variation $\tilde{D}$ of D centered at $w_1 = f(e^{i\theta_1})$ and $w_2 = f(e^{i\theta_2})$ with inclinations φ and radii $\varepsilon_1 = k_1\varepsilon$,

$\varepsilon_2 = k_2\varepsilon$, respectively (see [5, Section 3]). Computing the change in the area by [5, formula (3.32)], we find

$$\text{Area}\,\tilde{D} - \text{Area}\,D = \frac{2\pi\varphi - \sin 2\pi\varphi}{2\sin^2 \pi\varphi}\varepsilon^2(k_1^2 - k_2^2) + o(\varepsilon^2) < 0 \tag{4.3}$$

for all $\varepsilon > 0$ small enough. Similarly, applying [5, formula (3.31)], we get

$$\log \frac{R(\tilde{D}, 0)}{R(D, 0)} = \left[\frac{\varphi(2+\varphi)}{6(1+\varphi)^2}\frac{k_1^2}{\beta_1^2} - \frac{\varphi(2-\varphi)}{6(1-\varphi)^2}\frac{k_2^2}{\beta_2^2}\right]\varepsilon^2 + o(\varepsilon^2) > 0 \tag{4.4}$$

for all $\varepsilon > 0$ small enough and φ chosen so that the expression in the brackets is positive.

Inequalities (4.3) and (4.4) lead to a contradiction to the extremality of f for $A(\alpha)$, via a standard subordination argument. Thus, $|f'(e^{i\theta})| = \beta$ a.e. on $l_{\rm fr}$ with some $\beta > 0$.

To prove that $|f'(e^{i\theta})| < \beta$ for all $e^{i\theta} \in l_{\rm nf}$, we assume that $\beta = |f'(e^{i\theta_1})| < |f'(e^{i\theta_2})| = \beta_2$ with $e^{i\theta_1} \in l_{\rm fr}$ and some $e^{i\theta_2} \in l_{\rm nf}$. Then applying the two-point variation as above, we get inequalities (4.3) and (4.4), contradicting the extremality of f, again via a subordination argument. Hence, $|f'(e^{i\theta})| \le \beta$ for all $e^{i\theta} \in l_{\rm nf}$, which, when combined with the strict monotonicity property of $|f'|$ established in Lemma 4, leads to the strict inequality $|f'(e^{i\theta})| < \beta$ for $e^{i\theta} \in l_{\rm nf}$.

To prove that $|f'| = \beta$ everywhere on $l_{\rm fr}$, we consider the function $g = f^{1/2}$ with the principal branch of the radical. Lemma 4 implies that $D_g = g(\mathbb{D})$ has a Jordan rectifiable boundary. Moreover, since $L_{\rm fr}$ satisfies the Lavrent′ev condition, it follows that D_g is a Lavrent′ev domain and hence a Smirnov domain (see [10, Subsections 7.3, 7.4]). Thus, $\log|g'|$ can be represented by the Poisson integral

$$\log|f'(z)/(2f^{1/2}(z))| = \log|g'(z)| = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\log|g'(e^{it})|\,dt \tag{4.5}$$

with boundary values defined a.e. on $\mathbb{T}$ (see [10, p. 155]). Formula (4.5) easily implies that $|g'(e^{i\theta})| = \beta/(2|f(e^{i\theta})|^{1/2})$, and therefore $|f'(e^{i\theta})| = \beta$ for all $e^{i\theta} \in l_{\rm fr}$. In addition, (4.5) implies that $\log f'$ is bounded on $\overline{\mathbb{D}}$ outside any neighborhood of the point $z = -1$.

To show that f' is continuous at $e^{\pm i\theta_0}$, we note that, by the reflection principle, f can be continued analytically through $l_{\rm nf}^+ = \{z \in l_{\rm nf} : \operatorname{Im} z > 0\}$ and f' can be continued analytically through $l_{\rm fr}$. This implies that f can be considered as a function analytic in a slit disk $\{z : |z - e^{i\theta_0}| < \varepsilon\} \setminus [e^{i\theta_0}, (1+\varepsilon)e^{i\theta_0}]$ with $\varepsilon > 0$ small enough.

Using the Julia–Wolff lemma (see [10, Proposition 4.13]), the boundedness of $\log f'$, and the well-known properties of the angular derivatives (see [10, Propositions 4.7, 4.9]), one can prove that f' has a finite limit $f'(e^{i\theta_0})$, $|f'(e^{i\theta_0})| = \beta$, along any path in $\overline{\mathbb{D}}$ ending at $e^{i\theta_0}$. The details of this proof are similar to the arguments in [5, Lemma 13].

Since $|f'|$ takes its maximal values on $\mathbb{T}$, it follows that $|f'(z)| < \beta$ for all $z \in \mathbb{D}$. In particular, $\alpha = |f'(0)| < \beta$. The proof is complete. □

Summing up the results of this section we can prove Lemma 6 below, which allows us to find a closed form for the extremal functions.

For real τ and s such that $\tau < s$, let $\Pi(\tau, s)$ denote a simply connected domain obtained from the half-strip

$$H(s) = \{w = u + iv : u < s, |v| < 3\pi/2\}$$

by deleting two rays $l_+(\tau) = \{w = u + i\pi : u \le \tau\}$ and $l_-(\tau) = \{w = u - i\pi : u \le \tau\}$.

Lemma 6. *Let f be an extremal function for the minimal area problem on N_α, $1 < \alpha < 4$. Then $\varphi(z) = \log(zf'(z))$ maps the slit disk $\mathbb{D}' = \mathbb{D} \setminus (-1, 0]$ conformally and univalently onto the domain $\Pi(\tau, s)$ with some $\tau < s$ and $s = \log\beta$, where $\beta = |f'(1)|$.*

Proof. Let $l_{\rm fr} = \{e^{i\theta} : |\theta| < \theta_0\}$, $0 < \theta_0 < \pi$, and $l_{\rm nf} = \mathbb{T} \setminus \overline{l_{\rm fr}}$ be the free arc and the nonfree arc corresponding to the extremal function f. To show that φ maps $\mathbb{D}^+ = \{z \in \mathbb{D} : \operatorname{Im} z > 0\}$ univalently onto $\Pi^+(\tau, s) = \{\zeta \in \Pi(\tau, s) : \operatorname{Im} \zeta > 0\}$, we will consider the boundary values of φ on $\partial\mathbb{D}^+$.

First we notice that φ maps the arc $l^+_{\rm fr} = \{z \in l_{\rm fr} : \operatorname{Im} z \geq 0\}$ one-to-one onto the vertical segment $I_s = [s, s + (3/2)\pi i]$. Indeed, since $|f'| < \beta$ in $\mathbb{D}$ and $|f'| = \beta$ on $l_{\rm fr}$, it follows that $f''(e^{i\theta}) \neq 0$ on $l_{\rm fr}$. Thus, f' is locally univalent on $l_{\rm fr}$, and therefore $\arg f'(e^{i\theta})$ is monotone on $l_{\rm fr}$. Since $\varphi(1) = \log\beta > 0$ and $\varphi(e^{i\theta_0}) = \log(e^{i\theta_0} f'(e^{i\theta_0})) = \log\beta + (3/2)\pi i$ it follows that φ maps $l^+_{\rm fr}$ one-to-one onto I_s.

Next, it follows from the monotonicity property of Lemma 4(2) that φ maps the arc $l^+_{\rm nf} = \{z \in l_{\rm nf} : \operatorname{Im} z > 0\}$ one-to-one onto the ray $\{w = u + (3/2)\pi i : u \leq \log\beta\}$.

Considering the values of φ on the negative radius $I = [-1, 0]$, we first notice that $\operatorname{Im}(\varphi(z)) = \pi$ for all $z \in I$ and that $\varphi(z)$ approaches $-\infty + \pi i$ as z approaches -1 or 0 along I. Then, by Lemma 4(2) and Lemma 5, $\operatorname{Re}\varphi(e^{i\theta})$ decreases on $0 < \theta < \pi$. Thus $\frac{\partial}{\partial\theta}\operatorname{Re}\varphi(z) \leq 0$ on $\mathbb{T}^+$. Because φ is symmetric with respect to the real axis, it follows that

$$\frac{\partial}{\partial\theta}\operatorname{Re}\varphi(re^{i\theta})\bigg|_{\theta=0} = \frac{\partial}{\partial\theta}\operatorname{Re}\varphi(re^{i\theta})\bigg|_{\theta=\pi} = 0.$$

Since $\frac{\partial}{\partial\theta}\operatorname{Re}\varphi(re^{i\theta})$ is harmonic on $\mathbb{D}^+$, the maximum principle implies that $\frac{\partial}{\partial\theta}\operatorname{Re}\varphi(re^{i\theta}) \leq 0$ whenever $re^{i\theta} \in \mathbb{D}^+$.

Since φ is symmetric with respect to the real axis, the latter inequality shows that the function $-\log|zf'(z)|$ takes its maximal value on the circle $\{z : |z| = r\}$ at the point $z = -r$. Thus,

$$M(r) := \max_{0\leq\theta<2\pi}(-\log|rf'(re^{i\theta})|) = -\log(r|f'(-r)|).$$

Since $-\log|zf'(z)|$ is harmonic on the punctured disk $0 < |z| < 1$, the maximum $M(r)$ is a strictly convex function of $\log r$ on $0 < r < 1$ (see [9, Theorem 2.13]). Since $M(r) \to +\infty$ as $r \to 0$ or $r \to 1$, the latter implies that there is r_0, $0 < r_0 < 1$, such that $\operatorname{Re}\varphi(t)$ strictly increases from $-\infty$ to some value $\tau < \log\beta$ as t runs from -1 to $-r_0$ and $\operatorname{Re}\varphi(t)$ strictly decreases from τ to $-\infty$ as t runs from $-r_0$ to 0.

Finally, we consider the values of φ on the positive radius $[0, 1]$. Our consideration above shows that $|f'(re^{i\theta})|$ takes its maximal values on the circle $\{z : |z| = r\}$ at the point $z = r$. Then [9, Theorem 2.13] implies that $\log|rf'(r)|$ strictly increases from $-\infty$ to $\log\beta$ as r runs from 0 to 1.

In conclusion, since φ is analytic on $\mathbb{D}^+$ and maps its boundary $\partial\mathbb{D}^+$ one-to-one onto the boundary of $H^+(\tau, s)$, the principle of boundary correspondence implies that φ maps $\mathbb{D}^+$ conformally and univalently onto $H^+(\tau, s)$. Finally, applying the Schwarz reflection principle we finish the proof of the lemma. □

Now we are ready to find the extremal function explicitly. Let $w = \varphi_{\tau,s}(z)$ be a univalent function that maps the slit disk $\mathbb{D}'$ conformally onto $\Pi(\tau, s)$ and satisfies

$$\begin{aligned} \varphi_{\tau,s}(0) &= -\infty + i0, \\ \varphi_{\tau,s}(-1 + i0) &= -\infty + (3/2)\pi i, \\ \varphi_{\tau,s}(-1 - i0) &= -\infty - (3/2)\pi i \end{aligned}$$

in the sense of the boundary correspondence. The configuration $\Pi(\tau, s)$ depends on two parameters τ and s. Accordingly, we introduce two related parameters θ_0 and r_0, which characterize the preimage of this configuration in the z-plane. Namely, we put $e^{i\theta_0} = \varphi^{-1}_{\tau,s}(s + 3\pi i/2)$ and $r_0 = -\varphi^{-1}_{\tau,s}(\tau + \pi i)$. Then

$$0 < \theta_0 < \pi, \quad 0 < r_0 < 1.$$

By symmetry, we also have $e^{-i\theta_0} = \varphi_{\tau,s}^{-1}(s - 3\pi i/2)$ and $r_0 = -\varphi_{\tau,s}^{-1}(\tau - \pi i)$.

We represent $\varphi_{\tau,s}$ as a composition of two functions: $\varphi_{\tau,s} = \varphi_2 \circ \varphi_1$. Here

$$\varphi_1(z) = \frac{i}{2\sin(\theta_0/2)} \frac{1-z}{\sqrt{z}} \tag{4.6}$$

with the principal branch of the radical. The function φ_1 maps $\mathbb{D}'$ conformally onto the upper half-plane $\mathbb{H}_+ = \{w : \operatorname{Im} w > 0\}$ so that

$$\varphi_1(0) = \infty, \quad \varphi_1(1) = 0, \quad \varphi_1(e^{\pm i\theta_0}) = \pm 1.$$

Then the function φ_2 is given by the Schwarz–Christoffel integral:

$$\varphi_2(\zeta) = Ci \int_0^\zeta \frac{t^2 - b^2}{(t^2 - a^2)\sqrt{1-t^2}}\, dt + s \tag{4.7}$$

with the principal branch of the radical and with parameters a, b, and C defined as follows:

$$a = \csc(\theta_0/2), \quad b = (a/2)(r_0^{1/2} + r_0^{-1/2}),$$

and

$$C = \frac{3\pi}{2} \left(\int_0^1 \frac{t^2 - b^2}{(t^2 - a^2)\sqrt{1-t^2}}\, dt \right)^{-1}. \tag{4.8}$$

Integrating (4.7) and (4.8), we find

$$\begin{aligned}\varphi_2(\zeta) = Ci\Big(& \arcsin\zeta \\ & + \frac{b^2 - a^2}{2ai\sqrt{a^2-1}} \Big(\log \frac{(a+\zeta)(a\zeta - 1 - i\sqrt{a^2-1}\sqrt{1-\zeta^2})}{(a-\zeta)(a\zeta + 1 + i\sqrt{a^2-1}\sqrt{1-\zeta^2})} - \pi i \Big) \Big) + s\end{aligned} \tag{4.9}$$

and

$$C = 3\Big(1 + \frac{b^2 - a^2}{a\sqrt{a^2-1}}\Big)^{-1} = \frac{3a\sqrt{a^2-1}}{b^2 - a^2 + a\sqrt{a^2-1}}. \tag{4.10}$$

For $\varepsilon > 0$ sufficiently small, let γ_ε denote the semicircle $\{\zeta : |\zeta - 1| = \varepsilon, \operatorname{Im}\zeta > 0\}$ oriented in the clockwise direction. Using (4.7), we can find the change $\Delta(\gamma_\varepsilon)$ in $p = \varphi_2(\zeta)$ when ζ runs along γ_ε:

$$\Delta(\gamma_\varepsilon) = \lim_{\varepsilon \to 0^+} \left(Ci \int_{\gamma_\varepsilon} \frac{t^2 - b^2}{(t^2 - a^2)\sqrt{1-t^2}}\, dt \right) = -\frac{3\pi i}{2} \frac{b^2 - a^2}{b^2 - a^2 + a\sqrt{a^2-1}}. \tag{4.11}$$

Since φ_2 maps the upper half-plane onto $\Pi(\tau, s)$, it follows that

$$\Delta(\gamma_\varepsilon) = -\frac{\pi i}{2}. \tag{4.12}$$

Equating (4.11) and (4.12), we express b as a function of a:

$$b = \sqrt{a^2 + (a/2)\sqrt{a^2-1}}. \tag{4.13}$$

Substituting this into (4.10), we obtain

$$C = 2. \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.9) and simplifying, we can represent φ_2 as

$$\varphi_2(\zeta) = -\log \frac{(\zeta + \sqrt{\zeta^2-1})^2 (a\sqrt{\zeta^2-1} + \zeta\sqrt{a^2-1})}{\sqrt{\zeta^2 - a^2}} + \pi i + s. \tag{4.15}$$

Now we can write the closed form of the extremal function f_α. By Lemma 6, we have $\log(zf'_\alpha(z)) = \varphi_{\tau,s}(z)$ for some τ and $s = \log\beta$. Using equation (4.15) we obtain

$$f'_\alpha(z) = -\beta z^{-1}\frac{\sqrt{\zeta^2-a^2}}{(\zeta+\sqrt{\zeta^2-1})^2(a\sqrt{\zeta^2-1}+\zeta\sqrt{a^2-1})}, \tag{4.16}$$

where $\zeta = \varphi_1(z)$ is defined by (4.6). The parameter β can be found from the equation

$$\lim_{z\to 0}(\varphi_2(\zeta) - \log z) = \log\alpha. \tag{4.17}$$

Solving (4.6) for z, we obtain

$$z = -a^{-2}(\zeta - \sqrt{\zeta^2-a^2})^2 \tag{4.18}$$

with the principal branch of the radical. Using (4.15) and (4.18) and taking the limit in (4.17), we obtain equation (1.7):

$$\beta = \alpha a^2(a+\sqrt{a^2-1}). \tag{4.19}$$

Integrating (4.16), we find

$$\begin{aligned} f_\alpha(z) &= -\beta\int_{-1}^{z} z^{-1}\frac{\sqrt{\zeta^2-a^2}}{(\zeta+\sqrt{\zeta^2-1})^2(a\sqrt{\zeta^2-1}+\zeta\sqrt{a^2-1})}\,dz \\ &= \frac{2\alpha a^2}{(a-\sqrt{a^2-1})^2} \\ &\quad\times\Bigg(\frac{a}{a+\sqrt{a^2-1}} - \frac{\zeta}{\zeta+\sqrt{\zeta^2-1}} \\ &\qquad + \frac{\sqrt{a^2-1}}{a-\sqrt{a^2-1}}\log\frac{2a\sqrt{a^2-1}(\zeta+\sqrt{\zeta^2-1})}{(a+\sqrt{a^2-1})(a\sqrt{\zeta^2-1}+\zeta\sqrt{a^2-1})}\Bigg) \end{aligned} \tag{4.20}$$

with β given by (4.19) and $\zeta = \zeta(z)$ defined by (4.18). Thus, (4.20) gives an integrated form of the extremal function (1.4).

To express a as a function of α, we shall use the normalization $f_\alpha(0) = 1$. From (4.20) we obtain

$$f_\alpha(0) = \beta\Big((a-\sqrt{a^2-1}) + (a+\sqrt{a^2-1})(1-a^2-a\sqrt{a^2-1})\log\frac{(a+\sqrt{a^2-1})^4}{16a^2(a^2-1)}\Big).$$

Since $f_\alpha(0) = 1$, this leads to equation (1.3).

To show that the right-hand side of (1.3) is monotone in $1 < a < \infty$, we change variables via

$$a = \frac{x}{\sqrt{x^2-1}},$$

which expresses a as a monotone function of x, $1 < x < \infty$. Then the right-hand side of (1.3) becomes

$$T(x) = \frac{x^2}{x^2-1}\left(1 - \frac{x+1}{(x-1)^2}\log\frac{(x+1)^4}{16x^2}\right).$$

Differentiating, we find

$$T'(x) = \frac{x(2+x)}{(x-1)^4(x+1)^2}T_1(x),$$

where

$$T_1(x) = (1+x)^2\log\frac{(x+1)^4}{16x^2} - 2(x-1)^2.$$

To show that $T_1(x) > 0$ for $x > 1$, we put

$$T_2(x) = (1+x)^{-2}T_1(x).$$

Then $T_2(1) = 0$ and

$$T_2'(x) = \frac{2}{x}\left(\frac{x-1}{x+1}\right)^2 > 0$$

for all $x > 1$. Thus, $T_1(x) > 0$ for $x > 1$. Therefore, $T(x)$ strictly increases in $1 < x < \infty$. Since

$$\lim_{a\to\infty} a^2\left(1 - \sqrt{a^2-1}(a+\sqrt{a^2-1})^3 \log\frac{(a+\sqrt{a^2-1})^4}{16a^2(a^2-1)}\right) = \frac{1}{4}$$

and

$$\lim_{a\to 1^-} a^2\left(1 - \sqrt{a^2-1}(a+\sqrt{a^2-1})^3 \log\frac{(a+\sqrt{a^2-1})^4}{16a^2(a^2-1)}\right) = 1,$$

the monotonicity established above implies that for every $1 < \alpha < 4$ there is exactly one solution $a = a(\alpha)$ of equation (1.3).

To evaluate the minimal area $A(\alpha) := \operatorname{area}(D_\alpha)$, we apply the standard line integral formula:

$$\begin{aligned}
\operatorname{area}(D_\alpha) &= \frac{1}{2}\operatorname{Im}\int_{\partial D_f(d)} \bar{w}\,dw \\
&= \frac{1}{2}\operatorname{Im}\int_{L_{\mathrm{fr}}} \bar{w}\,dw = \frac{1}{2}\operatorname{Re}\int_{-\theta_0}^{\theta_0} \overline{f_\alpha(e^{i\theta})}e^{i\theta}f_\alpha'(e^{i\theta})\,d\theta \\
&= \frac{\beta^2}{2}\operatorname{Re}\int_{-\pi}^{\pi} \frac{f_\alpha(e^{i\theta})}{e^{i\theta}f_\alpha'(e^{i\theta})}\,d\theta = \frac{\beta^2}{2}\operatorname{Im}\int_{|z|=1}\frac{f_\alpha(z)}{z^2 f_\alpha'(z)}\,dz \\
&= \pi\beta^2 \operatorname{Re}\operatorname{Res}\left[\frac{f_\alpha}{z^2 f_\alpha'}, 0\right] = \pi\beta^2\operatorname{Re}\left[(f_\alpha/f_\alpha')'\Big|_{z=0}\right] \\
&= \pi\beta^2(1 - \alpha^{-2}f_\alpha''(0)).
\end{aligned} \tag{4.21}$$

To find $f_\alpha''(0)$, we differentiate equation (1.4) with g_α and $\zeta = \varphi_1(z)$ defined by (1.6) and (1.5). Then we obtain

$$\alpha^{-2}f_\alpha''(0) = \frac{2}{\alpha a^2}\sqrt{a^2-1}(a+\sqrt{a^2-1}). \tag{4.22}$$

Combining (4.21) and (4.22), we find the minimal area:

$$\operatorname{area}(D_\alpha) = \pi\alpha a^2(a+\sqrt{a^2-1})^2(\alpha a^2 - 2\sqrt{a^2-1}(a+\sqrt{a^2-1})),$$

which gives the right-hand side of inequality (1.2).

References

[1] D. Aharonov, C. Bénéteau, D. Khavinson, and H. Shapiro, *Extremal problems for nonvanishing functions in Bergman spaces*, Selected Topics in Complex Analysis. S. Ya. Khavinson Memorial Volume (V. Eiderman and M. Samokhin, eds.), Oper. Theory Adv. Appl., vol. 158, Birkhäuser, Basel, 2005, pp. 59–86. MR2147588 (2006i:30047)

[2] D. Aharonov, H. S. Shapiro, and A. Yu. Solynin, *A minimal area problem in conformal mapping*, J. Anal. Math. **78** (1999), 157–176. MR1714449 (2000j:30040)

[3] ———, *Minimal area problems for functions with integral representation*, J. Anal. Math. **98** (2006) (to appear).

[4] R. W. Barnard, C. Richardson, and A. Yu. Solynin, *Concentration of area in half-planes*, Proc. Amer. Math. Soc. **133** (2005), no. 7, 2091–2099. MR2137876 (2006b:30048)

[5] R. W. Barnard and A. Yu. Solynin, *Local variations and minimal area problem for Carathéodory functions*, Indiana Univ. Math. J. **53** (2004), no. 1, 135–167. MR2048187 (2005a:30028)

[6] V. N. Dubinin, *Symmetrization in the geometric theory of functions of a complex variable*, Uspekhi Mat. Nauk **49** (1994), no. 1, 3–76; English transl., Russian Math. Surveys **49** (1994), no. 1, 1–79. MR1307130 (96b:30054)

[7] P. Duren, *Univalent functions*, Grundlehren Math. Wiss., vol. 259, Springer-Verlag, New York, 1983. MR0708494 (85j:30034)
[8] W. K. Hayman, *Multivalent functions*, 2nd ed., Cambridge Tracts in Math., vol. 110, Cambridge Univ. Press, Cambridge, 1994. MR1310776 (96f:30003)
[9] W. K. Hayman and P. B. Kennedy, *Subharmonic functions*. Vol. I, London Math. Soc. Monogr., No. 9, Academic Press, London–New York, 1976. MR0460672 (57:665)
[10] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren Math. Wiss., vol. 299, Springer-Verlag, Berlin, 1992. MR1217706 (95b:30008)

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