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WEIGHTED SOBOLEV-TYPE EMBEDDING THEOREMS FOR FUNCTIONS WITH SYMMETRIES

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Abstract. It is well known that Sobolev embeddings can be refined in the presence of symmetries. Hebey and Vaugon (1997) studied this phenomena in the context of an arbitrary Riemannian manifold $\mathcal{M}$ and a compact group of isometries G. They showed that the limit Sobolev exponent increases if there are no points in $\mathcal{M}$ with discrete orbits under the action of G.

In the paper, the situation where $\mathcal{M}$ contains points with discrete orbits is considered. It is shown that the limit Sobolev exponent for $W_p^1(\mathcal{M})$ increases in the case of embeddings into weighted spaces $L_q(\mathcal{M}, w)$ instead of the usual L_q spaces, where the weight function $w(x)$ is a positive power of the distance from x to the set of points with discrete orbits. Also, embeddings of $W_p^1(\mathcal{M})$ into weighted Hölder and Orlicz spaces are treated.

Introduction

It is well known that Sobolev embeddings can be refined if we deal with subspaces of functions invariant under some group of symmetries. This phenomenon was used in some particular cases to prove the existence of solutions of various boundary value problems (see, e.g., [1]–[5]; a similar effect for trace embeddings was employed, e.g., in the recent paper [6]).

In [7], this problem was studied in the more general context of an arbitrary Riemannian manifold and a compact group of isometries. The critical embedding exponent increases if the manifold contains no points with orbits of dimension zero, due to reduction of the effective manifold dimension.

Our goal is to consider the case where the manifold contains points with zero-dimensional orbits. The conventional embedding theorem cannot be refined in this case. For example, if $p < n$, then, in general, even a radially symmetric function in $W_p^1(B_R^n)$ may fail to be integrable in a power exceeding $p^* = \frac{np}{n-p}$ (here and in the sequel, $B_R^n(X)$ stands for the ball of radius R in $\mathbb{R}^n$ centered at X, $B_R^n = B_R^n(0)$). However, the origin is the only possible singular point for radial functions, and their properties improve after multiplication by a positive power of $r = |x|$. In [8, Theorem 2.5], it was shown that, for $q \geq p^*$ and $\alpha > \frac{n}{p} - \frac{n}{q} - 1$, the set of radially symmetric functions $f \in W_p^1(B_1^n)$ is compactly embedded in $L_q(B_1^n; r^{\alpha q})$.

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We study this effect in a considerably more general situation. First, as a model case we treat the **m**-radial functions, i.e., the functions invariant under the group

$$\prod_{j=1}^{\ell} \mathcal{O}(m_j), \quad \sum_{j=1}^{\ell} m_j \le n$$

acting in $\mathbb{R}^n$ in a specific way; see the precise definition below in §1. We establish the limit exponents for the embedding of the space of **m**-radial functions $f \in W_p^1(B_1^n)$ in weighted Lebesgue and Hölder spaces. For this, we modify the well-known proofs of the Sobolev, Morrey, and John–Nirenberg inequalities; see, e.g., [9, Chapter 7]. Then, in §2, we consider the general case of a compact manifold $\mathcal{M}$ and the functions invariant under some closed subgroup $H \subset \mathrm{Iso}(\mathcal{M})$. The weight function turns out to be some power of the distance to the set of points with zero-dimensional orbits.

In what follows we assume that $n \ge 2$ and $1 \le p \le \infty$; $p' = \frac{p}{p-1}$ is the Hölder conjugate exponent to p. For $0 \le m \le n-1$, we denote by p_m^* the Sobolev conjugate exponent in $\mathbb{R}^{n-m}$:

$$p_m^* = \begin{cases} \frac{(n-m)p}{n-m-p} & \text{if } p < n-m, \\ \infty & \text{if } p \ge n-m; \end{cases}$$

$p_0^* = p^*$ is the conventional Sobolev conjugate exponent in $\mathbb{R}^n$.

For $p > n-m$, we denote by $\widehat{\gamma}_m = 1 - \frac{n-m}{p}$ the limit exponent for the embedding of the Sobolev space into the Hölder space in $\mathbb{R}^{n-m}$.

We use the letter C to denote various positive constants. To indicate that C depends on some parameters, we write $C(\dots)$.

§1. The model case

We introduce the space decomposition $\mathbb{R}^n = \bigoplus_{j=1}^{\ell} \mathbb{R}^{m_j} \oplus \mathbb{R}^k$, with $m_j \ge 2$, $k \ge 0$. The points of $\mathbb{R}^n$ are denoted by x; y_j stands for a point of $\mathbb{R}^{m_j}$, while z stands for a point of $\mathbb{R}^k$; therefore, $x = (y_1, \dots, y_\ell; z)$. The spherical coordinates of y_j are denoted by (ρ_j, ϑ_j), $\vartheta_j \in \mathbb{S}_{m_j}$ ($\mathbb{S}_m$ is the unit sphere in $\mathbb{R}^m$). The "cylindrical" coordinates of x are denoted by (r, Θ, z), where $r = \sqrt{\rho_1^2 + \dots + \rho_\ell^2}$, $\Theta \in \mathbb{S}_{n-k}$.

A function u is said to be **m**-*radial* if u depends only on ρ_j, $j = 1, \dots, \ell$, and on z.

By definition, we put $M = \min_j m_j$. We assume that $M < n$; the case of $M = n$ corresponds to the radially symmetric functions that were considered in [8, Theorem 2.5].

Theorem 1.1. I. *Let $p < n-M+1$. Let $\overset{\circ}{W}{}_{p,\mathbf{m}}^1(B_1^n)$ be the subspace of $\overset{\circ}{W}{}_p^1(B_1^n)$ consisting of* **m**-*radial functions. Then $\overset{\circ}{W}{}_{p,\mathbf{m}}^1(B_1^n)$ is embedded continuously in $L_q(B_1^n; r^{\alpha q})$ provided*

$$q \le p_{M-1}^*, \quad \alpha \ge \widehat{\alpha}_{pq} \equiv \Big(\frac{n}{p} - \frac{n}{q} - 1\Big)_+. \tag{1}$$

In other words, if (1) *is fulfilled, then the embedding*

$$r^\alpha \cdot \overset{\circ}{W}{}_{p,\mathbf{m}}^1(B_1^n) \hookrightarrow L_q(B_1^n) \tag{2}$$

is continuous. Moreover, this embedding is compact whenever both inequalities in (1) *are strict.*

II. *Let $p \ge n-M+1$. Then the embedding* (2) *is continuous for all $q < p_{M-1}^* = \infty$, $\alpha \ge \widehat{\alpha}_{pq}$. If $\alpha > \widehat{\alpha}_{pq}$, then this embedding is compact.*

Remark. For $p < n$, the limit exponent in (1) is greater than p^*. If $p \ge n$ or $q < p^*$, then the claim of Theorem 1.1 is trivial.

Proof. 1. Obviously, the subspace of $L_q(B_1^n)$ consisting of $\mathbf{m}$-radial functions is isometric, up to a constant, to the weighted space $L_q(\mathcal{D}_r, w)$ with

$$\mathcal{D}_r = \Big\{(\rho_1, \dots, \rho_\ell; z) \in \mathbb{R}^{k+\ell} : \rho_j > 0, \sum_{j=1}^{\ell} \rho_j^2 + |z|^2 < 1\Big\}$$

and

$$w = \prod_{j=1}^{\ell} \rho_j^{m_j - 1}.$$

Similarly, again up to a constant, the space $\overset{\circ}{W}{}^1_{p,\mathbf{m}}(B_1^n)$ is isometric to the subspace of $W_p^1(\mathcal{D}_r, w)$ formed by the functions vanishing on the spherical part of $\partial\mathcal{D}_r$.

We split B_1^n into the sets

$$\Omega_j = B_1^n \cap \big\{\max_i \rho_i = \rho_j\big\}$$

and consider the corresponding partition of $\mathcal{D}_r$:

$$\mathcal{D}_j = \mathcal{D}_r \cap \big\{\max_i \rho_i = \rho_j\big\}.$$

Since the relation $\rho_j \asymp r$ is valid on $\mathcal{D}_j$, the space $L_q(\mathcal{D}_j, w)$ is isomorphic to the space $L_q(\mathcal{D}_j, w_j \cdot r^{m_j - 1})$ with

$$w_j = \prod_{i \neq j} \rho_i^{m_i - 1}.$$

Furthermore, $L_q(\mathcal{D}_j, w_j \cdot r^{m_j-1})$ is isometric, up to a constant, to the subspace of functions with some symmetries in the space $L_q(\widetilde{\mathcal{D}}_j, r^{m_j-1})$, where $\widetilde{\mathcal{D}}_j$ is the intersection of the ball $B_1^{n-m_j+1}$ and the wedge $\big\{y_j > \max_{i\neq j} |y_i|\big\}$.

Similarly, $W_p^1(\mathcal{D}_j, w)$ is isometric, up to a constant, to a subspace of $W_p^1(\widetilde{\mathcal{D}}_j, r^{m_j-1})$.

2. For a fixed index j, let u be a smooth $\mathbf{m}$-radial function in Ω_j vanishing on the spherical part of $\partial\Omega_j$. The "transplanted" function in $\widetilde{\mathcal{D}}_j$ will also be denoted by u. Moreover, temporarily we omit the index j.

Given $\beta > 0$, for any (Θ, z) we have

$$\int_0^{\sqrt{1-|z|^2}} |u(r,\Theta,z)| r^{n-k-m+\beta-1}\, dr \le \frac{1}{n-k-m+\beta} \int_0^{\sqrt{1-|z|^2}} |u_r(r,\Theta,z)| r^{n-k-m+\beta}\, dr.$$

Integrating this inequality with respect to (Θ, z), we obtain

$$\int_\Omega r^{\beta-m}|u| \le C \int_\Omega r^{\beta-m+1}|\nabla u|, \tag{3}$$

so that the function $v = r^{\beta-m+1}u$ satisfies

$$\int_\Omega |\nabla v| \le C \int_\Omega \big(r^{\beta-m+1}|\nabla u| + r^{\beta-m}|u|\big) \le C \int_\Omega r^{\beta-m+1}|\nabla u|.$$

The Sobolev embedding theorem in Ω yields

$$\|r^{\beta-m+1}u\|_{\frac{n}{n-1},\Omega} \le C\|r^{\beta-m+1}\nabla u\|_{1,\Omega}, \tag{4}$$

whence, by the isomorphisms described in part 1,

$$\|r^{\beta-\frac{m-1}{n}}u\|_{\frac{n}{n-1},\widetilde{\mathcal{D}}} \le C\|r^{\beta}\nabla u\|_{1,\widetilde{\mathcal{D}}}. \tag{5}$$

On the other hand, estimate (3) and the isomorphisms described in part 1 imply

$$\int_{\widetilde{\mathcal{D}}} r^{\beta-1}|u| \le C \int_{\widetilde{\mathcal{D}}} r^{\beta}|\nabla u|.$$

Therefore, the function $v = r^{\beta}u$ satisfies

$$\int_{\widetilde{\mathcal{D}}} |\nabla v| \le C \int_{\widetilde{\mathcal{D}}} \left(r^{\beta}|\nabla u| + r^{\beta-1}|u|\right) \le C \int_{\widetilde{\mathcal{D}}} r^{\beta}|\nabla u|.$$

The Sobolev embedding theorem in $\widetilde{\mathcal{D}}$ gives

$$\|r^{\beta}u\|_{\frac{n-m+1}{n-m},\widetilde{\mathcal{D}}} \le C\|r^{\beta}\nabla u\|_{1,\widetilde{\mathcal{D}}}. \tag{6}$$

By the Hölder inequality, estimates (5) and (6) imply that

$$\|r^{\beta+\frac{n-m+1-s}{s}}u\|_{s',\widetilde{\mathcal{D}}} \le C\|r^{\beta}\nabla u\|_{1,\widetilde{\mathcal{D}}} \tag{7}$$

for $s \in [n-m+1, n]$.

3. Let $1 \le p < n$. Application of inequality (7) to the function $v = u^{\gamma}$, $\gamma > 1$, yields

$$\begin{aligned} \|r^{\beta+\frac{n-m+1-s}{s}}u^{\gamma}\|_{s',\widetilde{\mathcal{D}}} &\le C\|r^{\beta}u^{\gamma-1}\nabla u\|_{1,\widetilde{\mathcal{D}}} \\ &\le C\|r^{\frac{m-1}{p}}\nabla u\|_{p,\widetilde{\mathcal{D}}} \cdot \|r^{\beta-\frac{m-1}{p}}u^{\gamma-1}\|_{p',\widetilde{\mathcal{D}}}. \end{aligned} \tag{8}$$

For $s > p$, we can set $\gamma = \frac{p'}{p'-s'}$ and $\beta = m-1+\frac{(p-1)(n-s)}{s-p}$. Then the last factor on the right-hand side of (8) is a power of the left-hand side. Therefore, setting $q = \gamma s'$, we get

$$\|r^{\widehat{\alpha}_{pq}+\frac{m-1}{q}}u\|_{q,\widetilde{\mathcal{D}}} \le C\|r^{\frac{m-1}{p}}\nabla u\|_{p,\widetilde{\mathcal{D}}},$$

or, by the isomorphisms described in part 1,

$$\|r^{\widehat{\alpha}_{pq}}u\|_{q,\Omega} \le C\|\nabla u\|_{p,\Omega}. \tag{9}$$

By approximation, inequality (9) can be extended to any function in $\overset{\circ}{W}{}^1_{p,\mathbf{m}}(B^n_1)$.

4. Since $s \ge n-m+1$ and $s > p$, inequality (9) is true with $j = 1, \dots, \ell$ for $q \le p^*_{M-1}$ if $p < n-M+1$, and for $q < p^*_{M-1} = \infty$ if $p \ge n-M+1$. Summing these inequalities over j, we obtain

$$\|r^{\widehat{\alpha}_{pq}}u\|_{q,B^n_1} \le C\|\nabla u\|_{p,B^n_1}.$$

Furthermore, for $q < p^*_{M-1}$ and $\alpha > \widehat{\alpha}_{pq}$, in place of (4), the estimate

$$\|r^{\beta-m+1}u\|_{\widehat{s}',\Omega} \le C\|r^{\beta-m+1}\nabla u\|_{1,\Omega} \tag{4'}$$

with suitable $\widehat{s} > n$ should be applied. Then the above argument can be repeated and the compactness of the embedding (4′) can be used. This completes the proof of both statements. □

Remark. In the simplest case, the claim of Theorem 1.1 was proved in the term paper of S.B. Kolonitskiĭ (St.-Petersburg State University).

We need the following technical lemma.

Lemma 1.2. *Let G be a convex domain in $\mathbb{R}^n$, and let $v \in \mathcal{C}^1(G) \cap \mathcal{C}(\overline{G})$. Then for any $X \in \overline{G}$ we have*

$$\frac{1}{|G|}\int_G |v(x)-v(X)|\,dx \le C\int_G \frac{|\nabla v(x)|}{|x-X|^{n-1}}\,dx, \tag{10}$$

where the constant $C = C(G)$ is invariant under dilations.

Proof. In fact, the proof repeats that of Lemma 7.16 in [9]. □

Theorem 1.3. I. *Let $n-M+1<p<n$. Then the embedding*

$$r^\alpha \cdot \overset{\circ}{W}{}^1_{p,\mathbf{m}}(B^n_1) \hookrightarrow \mathcal{C}^\gamma(B^n_1) \tag{11}$$

is continuous provided

$$\gamma \le \widehat{\gamma}_{M-1}, \quad \alpha \ge \gamma + \frac{n}{p} - 1. \tag{12}$$

If both inequalities in (12) *are strict, then this embedding is compact.*

II. *Let $p=n$. Then the embedding* (11) *is continuous and compact for all $\gamma<\widehat{\gamma}_{M-1}$, $\alpha>\gamma$.*

III. *Let $p>n$. Then the embedding* (11) *is continuous provided*

$$\gamma \le \widehat{\gamma}_{M-1}, \quad \alpha \ge \gamma. \tag{13}$$

If the first inequality in (13) *is strict, then this embedding is compact.*

Remark. Statement III of this theorem is meaningful only for $\gamma>\widehat{\gamma}_0$.

Proof. 1. We fix some index j (and temporarily omit it in writing). Let u be a smooth $\mathbf{m}$-radial function in Ω vanishing on the spherical part of $\partial\Omega$. The "transplanted" function on $\widetilde{\mathcal{D}}$ will also be denoted by u.

2. Let $n-m+1<p<n$. We set $v=r^\alpha u$, where $\frac{n}{p}-1<\alpha\le\frac{m-1}{p}$.

Let $X_1,X_2\in\widetilde{\mathcal{D}}$, $\mathfrak{r}=|X_1-X_2|$. For the domain $G=B^{n-m+1}_{\mathfrak{r}}(X_1)\cap B^{n-m+1}_{\mathfrak{r}}(X_2)\cap\widetilde{\mathcal{D}}$, we have

$$|v(X_1)-v(X_2)| \le \frac{1}{|G|}\int_G \left(|v(x)-v(X_1)|+|v(x)-v(X_2)|\right)dx. \tag{14}$$

By Lemma 1.2, if $X=X_1$ or $X=X_2$, then

$$\frac{1}{|G|}\int_G |v(x)-v(X)|\,dx \le C\int_G \frac{|\nabla v(x)|}{|x-X|^{n-m}}\,dx \le C\int_G \frac{r^\alpha|\nabla u(x)|+r^{\alpha-1}|u|}{|x-X|^{n-m}}\,dx. \tag{15}$$

We estimate the first term on the right in (15):

$$\int_G \frac{r^\alpha|\nabla u(x)|}{|x-X|^{n-m}}\,dx \le \|r^{\frac{m-1}{p}}\nabla u\|_{p,G}\cdot\left(\int_{B^{n-m+1}_{\mathfrak{r}}(X)} \frac{r^{\alpha p'-\frac{m-1}{p-1}}}{|x-X|^{(n-m)p'}}\,dx\right)^{1/p'}.$$

Since $\alpha\le\frac{m-1}{p}$, the exponent $\alpha p'-\frac{m-1}{p-1}$ is nonpositive. Hence, the last-written integral is maximal if the point X is situated in the subspace $\{(0,\dots,0;z): z\in\mathbb{R}^k\}$, i.e., its r-coordinate equals zero. In this case, the integral in question does not exceed

$$\begin{aligned}&\int_0^{\mathfrak{r}}\int_0^\infty \frac{r^{n-m-k+\alpha p'-\frac{m-1}{p-1}}|z|^{k-1}}{(r^2+|z|^2)^{\frac{(n-m)p'}{2}}}\,d|z|dr\\ &\quad=\int_0^{\mathfrak{r}} r^{\frac{\alpha p-n+1}{p-1}}\,dr\cdot\int_0^\infty \frac{t^{k-1}}{(t^2+1)^{\frac{(n-m)p'}{2}}}\,dt \le C\mathfrak{r}^{(\alpha-\frac{n}{p}+1)p'}\end{aligned}$$

(we have used the assumption $\alpha>\frac{n}{p}-1$).

The second term on the right in (15) is estimated by using the critical embedding theorem:

$$\int_G \frac{r^{\alpha-1}|u(x)|}{|x-X|^{n-m}}\,dx \le \|r^{\frac{m-1}{p^*}}u\|_{p^*,G}\cdot\left(\int_{B^{n-m+1}_{\mathfrak{r}}(X)} \frac{r^{(\alpha-1)p^{*\prime}-\frac{m-1}{p^*-1}}}{|x-X|^{(n-m)p^{*\prime}}}\,dx\right)^{1/p^{*\prime}}.$$

Since $\alpha \le \frac{m-1}{p}$, the exponent $(\alpha-1)p^{*\prime} - \frac{m-1}{p^*-1}$ is nonpositive. Hence, the last-written integral attains its maximum if the r-coordinate of the point X equals zero. In this case, the integral in question does not exceed

$$\int_0^{\mathfrak{r}} \int_0^\infty \frac{r^{n-m-k+(\alpha-1)p^{*\prime}-\frac{m-1}{p^*-1}}|z|^{k-1}}{(r^2+|z|^2)^{\frac{(n-m)p^{*\prime}}{2}}}\, d|z|dr$$
$$= \int_0^{\mathfrak{r}} r^{\frac{\alpha p^*-n}{p^*-1}-1}\, dr \cdot \int_0^\infty \frac{t^{k-1}}{(t^2+1)^{\frac{(n-m)p^{*\prime}}{2}}}\, dt \le C\mathfrak{r}^{(\alpha-\frac{n}{p}+1)p^{*\prime}}$$

(again, we have used the assumption $\alpha > \frac{n}{p} - 1$).

Thus, using the isomorphisms described in part 1 of the proof of Theorem 1.1, we see that relations (14) and (15) imply the inequality

$$\begin{aligned} |v(X_1)-v(X_2)| &\le C\mathfrak{r}^{\alpha-\frac{n}{p}+1}\cdot(\|r^{\frac{m-1}{p}}\nabla u\|_{p,G} + \|r^{\frac{m-1}{p^*}}u\|_{p^*,G}) \\ &\le C|X_1-X_2|^{\alpha-\frac{n}{p}+1}\|\nabla u\|_{p,\Omega}. \end{aligned} \tag{16}$$

By approximation, inequality (16) extends to all functions in $\overset{\circ}{W}{}^1_{p,\mathbf{m}}(B_1^n)$.

3. If $n-M+1 < p < n$, then (16) is true for all $j = 1,\dots,\ell$. This proves the embedding (11) provided (12) is fulfilled and $\gamma > 0$.

Next, setting $X_2 = 0$, from (16) we deduce that

$$|r^\alpha u(x)| \le Cr^{\alpha-\frac{n}{p}+1}\cdot\big(\|r^{\frac{m-1}{p}}\nabla u\|_{p,B^{n-m+1}_{|x|}} + \|r^{\frac{m-1}{p^*}}u\|_{p^*,B^{n-m+1}_{|x|}}\big) = o(r^{\alpha-\frac{n}{p}+1}),$$

whence $r^{\frac{n}{p}-1}u \in \mathcal{C}(B_1^n)$.

The compactness of the embedding in part I of the theorem, as well as the statement of part II, follows from the compactness of the embedding $\mathcal{C}^{\gamma_1} \hookrightarrow \mathcal{C}^{\gamma_2}$ for $\gamma_1 > \gamma_2$.

4. Now, let $p > n$. We set $v = r^\gamma u$, where $0 \le \gamma \le \widehat{\gamma}_{m-1}$.

For $X_1, X_2 \in \widetilde{\mathcal{D}}$, we put $\mathfrak{r} = |X_1 - X_2|$. Let r_k denote the r-coordinate of X_k. If $r_1 \le 2\mathfrak{r}$, then

$$|v(X_1)-v(X_2)| \le (r_1^\gamma + r_2^\gamma)\cdot\|u\|_{\mathcal{C},\Omega} \le (2^\gamma+3^\gamma)\mathfrak{r}^\gamma\cdot\|u\|_{\mathcal{C},\Omega}.$$

Otherwise, we have $r_1/2 < r_2 < 3r_1/2$. Consequently,

$$\begin{aligned} |v(X_1)-v(X_2)| &\le |r_1^\gamma - r_2^\gamma|\cdot|u(X_1)| + r_2^\gamma\cdot|u(X_1)-u(X_2)| \\ &\le \mathfrak{r}^\gamma\cdot\|u\|_{\mathcal{C},\Omega} + (3r_1/2)^\gamma\mathfrak{r}^\gamma\cdot\Big(\frac{|u(X_1)-u(X_2)|}{\mathfrak{r}^{\widehat{\gamma}_{m-1}}}\Big)^{\frac{\gamma}{\widehat{\gamma}_{m-1}}} \\ &\quad\times|u(X_1)-u(X_2)|^{1-\frac{\gamma}{\widehat{\gamma}_{m-1}}}. \end{aligned} \tag{17}$$

To estimate the second term in (17), we observe that

$$\frac{|u(X_1)-u(X_2)|}{\mathfrak{r}^{\widehat{\gamma}_{m-1}}} \le C\|\nabla u\|_{p,\widetilde{\mathcal{D}}\cap\{x:r>r_1/2\}} \le Cr_1^{-\frac{m-1}{p}}\cdot\|r^{\frac{m-1}{p}}\nabla u\|_{p,\widetilde{\mathcal{D}}}.$$

Using the isomorphisms described in part 1 of the proof of Theorem 1.1, we get

$$|v(X_1)-v(X_2)| \le \mathfrak{r}^\gamma\cdot\Big(\|u\|_{\mathcal{C},\Omega} + Cr_1^{\frac{\gamma}{\widehat{\gamma}_{m-1}}(1-\frac{n}{p})}\cdot\|\nabla u\|_{p,\Omega}^{\frac{\gamma}{\widehat{\gamma}_{m-1}}}\cdot\|u\|_{\mathcal{C},\Omega}^{1-\frac{\gamma}{\widehat{\gamma}_{m-1}}}\Big).$$

Thus, in all cases we have

$$\frac{|v(X_1)-v(X_2)|}{\mathfrak{r}^\gamma} \le C\Big(\|u\|_{\mathcal{C},\Omega} + \|\nabla u\|_{p,\Omega}^{\frac{\gamma}{\widehat{\gamma}_{m-1}}}\cdot\|u\|_{\mathcal{C},\Omega}^{1-\frac{\gamma}{\widehat{\gamma}_{m-1}}}\Big) \tag{18}$$

(we recall that $r_1 \le 1$).

If $\gamma \le \widehat{\gamma}_{M-1}$, then (18) is true for all $j = 1, \dots, \ell$. This proves the embedding (11) provided (13) is fulfilled. The compactness of this embedding for $\gamma < \widehat{\gamma}_{M-1}$ follows from (18) and the compactness of the embedding $\overset{\circ}{W}{}^1_p(B^n_1) \hookrightarrow \mathcal{C}(\overline{B^n_1})$. □

Theorem 1.4. *Let $p = n - M + 1$. Then for $\alpha \ge \frac{n}{p} - 1$ the following embeddings are continuous*:

$$r^\alpha \cdot \overset{\circ}{W}{}^1_{p,\mathbf{m}}(B^n_1) \hookrightarrow BMO(B^n_1), \tag{19}$$

$$r^\alpha \cdot \overset{\circ}{W}{}^1_{p,\mathbf{m}}(B^n_1) \hookrightarrow L_\Phi(B^n_1), \quad \Phi(t) = \exp(t^q) - 1, \quad q \le p', \tag{20}$$

where L_Φ stands for the Orlicz space with N-function Φ (see [10, §3, Chapter IV]*). For $q < p'$ the embedding* (20) *is compact.*

Proof. 1. Obviously, it suffices to consider the case where $\alpha = \frac{n}{p} - 1$. We fix some index j (and temporarily omit it in writing). Let u be a smooth $\mathbf{m}$-radial function in Ω vanishing on the spherical part of $\partial\Omega$. The "transplanted" function on $\widetilde{\mathcal{D}}$ will also be denoted by u.

2. For $X \in \Omega$, let $\mathfrak{r}$ denote the r-coordinate of X. The Sobolev integral representation for u yields

$$\mathfrak{r}^\alpha |u(X)| \le \mathfrak{r}^\alpha \cdot C \int_\Omega \frac{|\nabla u(x)|}{|x - X|^{n-1}}\, dx = C \cdot \Big(\int_{\Omega^{(1)}} + \int_{\Omega^{(2)}} \Big) \frac{\mathfrak{r}^\alpha |\nabla u(x)|}{|x - X|^{n-1}}\, dx,$$

where $\Omega^{(1)} = \{x \in \Omega : 2|x - X| < \mathfrak{r}\}$, $\Omega^{(2)} = \Omega \backslash \overline{\Omega^{(1)}}$.

On the set $\Omega^{(1)}$ we have $r \asymp \mathfrak{r}$. Moreover, the diameter of $\Omega^{(1)}$ is equal to $\mathfrak{r}/2$. Therefore, since $\alpha = \frac{M-1}{p} \le \frac{m-1}{p}$, we obtain

$$\begin{aligned} \int_{\Omega^{(1)}} \frac{\mathfrak{r}^\alpha |\nabla u(x)|}{|x - X|^{n-1}}\, dx &\le C \int_{\Omega^{(1)}} \frac{\mathfrak{r}^{\frac{m-1}{p}} |\nabla u(x)|}{|x - X|^{n-1-\alpha+\frac{m-1}{p}}}\, dx \\ &\le C \int_{\widetilde{\mathcal{D}}} \frac{r^{\frac{m-1}{p}} |\nabla u(x)|}{|x - X|^{n-m-\alpha+\frac{m-1}{p}}}\, dx. \end{aligned}$$

On the set $\Omega^{(2)}$ we have $\mathfrak{r} < 2|x - X|$ and $r < 3|x - X|$. Therefore,

$$\int_{\Omega^{(2)}} \frac{\mathfrak{r}^\alpha |\nabla u(x)|}{|x - X|^{n-1}}\, dx \le C \int_{\widetilde{\mathcal{D}}} \frac{r^{m-1} \mathfrak{r}^\alpha |\nabla u(x)|}{|x - X|^{n-1}}\, dx \le C \int_{\widetilde{\mathcal{D}}} \frac{r^{\frac{m-1}{p}} |\nabla u(x)|}{|x - X|^{n-m-\alpha+\frac{m-1}{p}}}\, dx.$$

Thus, the function $v(X) = \mathfrak{r}^\alpha u(X)$ on Ω satisfies the inequality

$$|v(X)| \le C \int_{\widetilde{\mathcal{D}}} \frac{r^{\frac{m-1}{p}} |\nabla u(x)|}{|x - X|^{n-m-\alpha+\frac{m-1}{p}}}\, dx = C \int_{\widetilde{\mathcal{D}}} \frac{r^{\frac{m-1}{p}} |\nabla u(x)|}{|x - X|^{\frac{n-m+1}{p'}}}\, dx.$$

Since $\|r^{\frac{m-1}{p}} \nabla u\|_{p,\widetilde{\mathcal{D}}} \le C \|\nabla u\|_{p,\Omega}$, we can use [9, Lemma 7.13] to show that for some constant $\varkappa = \varkappa(n, m) > 0$ the inequality

$$\int_{\widetilde{\mathcal{D}}} \Phi\Big(\frac{|v(X)|}{\varkappa \|\nabla u\|_{p,\Omega}} \Big)\, dX \le C(n, m) |\widetilde{\mathcal{D}}|$$

is true with $\Phi(t) = \exp(t^{p'}) - 1$.

Summing these inequalities with $j = 1, \dots, \ell$, we obtain the embedding (20) for $q = p'$. The compactness of this embedding for $q < p'$ follows from interpolation theorems (see [11]).

3. To prove the embedding (19), we consider a ball $\mathcal{B} = B^n_R(\widehat{X}) \subset \Omega$. By (10), for any $X \in \mathcal{B}$ we have

$$|v(X) - \overline{v}| \le C \int_{\mathcal{B}} \frac{|\nabla v(x)|}{|x - X|^{n-1}}\, dx$$

(here $\overline{v}$ stands for the mean value of v over $\mathcal{B}$). Therefore,

$$\int_{\mathcal{B}} |v - \overline{v}| \le CR \int_{\mathcal{B}} |\nabla v| \le CR \int_{\mathcal{B}} (r^\alpha |\nabla u| + r^{\alpha-1}|u|). \tag{21}$$

Let $\mathfrak{B}$ denote the image of $\mathcal{B}$ under the mapping of Ω into $\widetilde{\mathcal{D}}$ described in part 1 of Theorem 1.1. Then

$$\begin{aligned}\int_{\mathcal{B}} r^\alpha |\nabla u| &\le CR^{m-1} \int_{\mathfrak{B}} r^\alpha |\nabla u| \\ &\le CR^{m-1} \|r^{\frac{m-1}{p}} \nabla u\|_{p,\mathfrak{B}} \cdot \|r^{\alpha - \frac{m-1}{p}}\|_{p',\mathfrak{B}} \le CR^{n-1} \|\nabla u\|_{p,\Omega}\end{aligned}$$

(we have used the assumption $\alpha \le \frac{m-1}{p}$).

We estimate the second term on the right in (21); the critical embedding theorem yields

$$\begin{aligned}\int_{\mathcal{B}} r^{\alpha-1} |u| &\le CR^{m-1} \int_{\mathfrak{B}} r^{\alpha-1} |u| \\ &\le CR^{m-1} \|r^{\frac{m-1}{p^*}} u\|_{p^*,\mathfrak{B}} \cdot \|r^{\alpha-1-\frac{m-1}{p^*}}\|_{p^{*\prime},\mathfrak{B}} \le CR^{n-1} \|\nabla u\|_{p,\Omega}\end{aligned}$$

(again, we have used the assumption $\alpha \le \frac{m-1}{p}$).

Substituting these estimates in (21), we can divide by R^n. Taking the supremum over all balls $\mathcal{B}$, we obtain (19). □

Remark. Theorems 1.1, 1.3, and 1.4 remain valid if we replace $\overset{\circ}{W}{}^1_{p,\mathbf{m}}(B^n_1)$ by $W^1_{p,\mathbf{m}}(B^n_1)$. This follows from the extension theorem for Sobolev spaces.

§2. The general case

We use the standard properties of isometry groups; see, e.g., [12, Chapter I, §4 and Chapter VI, §3]. Let $\mathcal{M}$ be a connected compact Riemannian manifold (with or without boundary), $\dim \mathcal{M} = n$. We denote by $\mathrm{Iso}(\mathcal{M})$ the group of isometries of $\mathcal{M}$. Since $\mathcal{M}$ is compact, $\mathrm{Iso}(\mathcal{M})$ is a compact Lie group.

Let H be a closed subgroup of $\mathrm{Iso}(\mathcal{M})$. For a point $X \in \mathcal{M}$, we denote by $H(X)$ and S^H_X (respectively) the orbit and the stabilizer (isotropy group) of X under the action of H. That is, $H(X) = \{\gamma(X) : \gamma \in H\}$ and $S^H_X = \{\gamma \in H : \gamma(X) = X\}$. Clearly, S^H_X is a closed subgroup of H, and $H(X)$ is a smooth submanifold of $\mathcal{M}$ diffeomorphic to the quotient H/S^H_X.

We denote $\mathcal{M}_0 = \{X \in \mathcal{M} : \dim H(X) = 0\}$.

Let H_0 be the component of the identity of H. It is well known that H_0 is a normal subgroup of H. It is easy to check that H/S^H_X is a disjoint union of manifolds diffeomorphic to $H_0/(S^H_X \cap H_0) = H_0/S^{H_0}_X$. Hence, the orbit $H(X)$ is a disjoint union of submanifolds diffeomorphic to $H_0(X)$; in particular, $\dim H(X) = \dim H_0(X)$. Also, we note that every H-invariant function on $\mathcal{M}$ is H_0-invariant. Therefore, for our purposes it suffices to consider the case where $H = H_0$ (i.e., H is connected). In this case the orbits are also connected; in particular, the 0-dimensional orbits are singletons, so that $\mathcal{M}_0$ is the set of fixed points of the action.

Lemma 2.1. *$\mathcal{M}_0$ is a disjoint union of totally geodesic submanifolds (of possibly different dimensions). Furthermore, if $\mathcal{M}_0$ intersects $\partial\mathcal{M}$, then $\mathcal{M}_0$ is orthogonal to $\partial\mathcal{M}$ at every point of intersection.*

Proof. Let $\wp(X)$ denote the injectivity radius of the Riemannian exponential map at a point $X \in \mathcal{M} \setminus \partial\mathcal{M}$.

Suppose that $X, Y \in \mathcal{M}_0 \setminus \partial\mathcal{M}$ are sufficiently close to each other, namely, are such that $|XY| < \wp(X)$, where $|XY|$ denotes the distance with respect to the Riemannian metric on $\mathcal{M}$. Then there exists a unique geodesic s of length $|XY|$ connecting X and Y. Since the isometries send geodesics to geodesics and X and Y are fixed points of the action, all points of s are also fixed, whence $s \subset \mathcal{M}_0$. Thus, $\mathcal{M}_0 \setminus \partial\mathcal{M}$ is a totally geodesic submanifold.

Since the isometries preserve the boundary of the manifold, the same argument applied to the boundary rather than to the manifold itself shows that $\mathcal{M}_0 \cap \partial\mathcal{M}$ is a totally geodesic submanifold of the boundary. It remains to observe that, whenever the set $\mathcal{M}_0$ contains a boundary point, it also contains the geodesic orthogonal to the boundary and starting at that point. This follows from the fact that the isometries send geodesics to geodesics and preserve angles. □

Lemma 2.2. *Let $X \in \mathcal{M}_0$. Then the action of H in a neighborhood $U \ni X$ is smoothly conjugate to an action of a closed subgroup of the orthogonal group $\mathcal{O}(n)$ on the ball $B_1^n \subset \mathbb{R}^n$ (if $X \notin \partial\mathcal{M}$), or on the cylinder $B_1^{n-1} \times [0,1[\subset \mathbb{R}^{n-1} \times \overline{\mathbb{R}_+}$ (if $X \in \partial\mathcal{M}$). Under this conjugation, the submanifold $\mathcal{M}_0$ corresponds either to the intersection of B_1^n and a linear subspace of $\mathbb{R}^n$, or to a set of the form $L \times [0,1[$, where L is the intersection of the ball B_1^{n-1} and a linear subspace of $\mathbb{R}^{n-1}$.*

Proof. First, let $X \notin \partial\mathcal{M}$. We may assume that $\wp(X) > 1$ (this can be achieved by a suitable dilation of the metric). The tangent space $T_X\mathcal{M}$ is Euclidean (relative to the Riemannian structure); we can identify this space with $\mathbb{R}^n$. Let $U \subset \mathcal{M}$ be the unit Riemannian ball centered at X, and let B be the unit ball in $T_X\mathcal{M}$ centered at the origin. Then the Riemannian exponential map determines a diffeomorphism $\varphi : B \to U$ that sends the straight lines passing through the origin to geodesics passing through X. The group $H = S_X^H$ acts naturally on $T_X\mathcal{M}$ by orthogonal transformations; namely, each isometry $\gamma : \mathcal{M} \to \mathcal{M}$ (such that $\gamma(X) = X$) corresponds to the orthogonal map $d_X\gamma : T_X\mathcal{M} \to T_X\mathcal{M}$. Since the isometries send geodesics to geodesics, and a geodesic is uniquely determined by its initial velocity vector, this action commutes with φ.

Since $\mathcal{M}$ is connected, any isometry of $\mathcal{M}$ extends uniquely from U. Therefore, the action of H on B is effective (that is, distinct elements of H correspond to distinct transformations of B). Thus, this action determines an isomorphism between H and a subgroup of $\mathcal{O}(n)$.

To show that the set of fixed points of this action is a linear subspace, it suffices to apply Lemma 2.1 to $\mathbb{R}^n$ in place of $\mathcal{M}$.

In the case where $X \in \partial\mathcal{M}$, the same argument applies to the action of the group on the boundary $\partial\mathcal{M}$. This yields an action of a subgroup of $\mathcal{O}(n-1)$ on the ball B_1^{n-1}, which is identified with a neighborhood of X in $\partial\mathcal{M}$. To complete the proof, observe that a neighborhood of the boundary can be identified naturally with $\partial\mathcal{M} \times [0,1[$ by foliation into geodesics orthogonal to the boundary, and these geodesics are sent to one another by isometries. Therefore, the isometries of $\mathcal{M}$ extend uniquely from the boundary, and this agrees with the (trivial) extension of orthogonal maps from the ball B_1^{n-1} to the cylinder $B_1^{n-1} \times [0,1[$. □

Lemma 2.3. *Suppose $X \in \mathcal{M} \setminus \mathcal{M}_0$, $m = \dim H(X)$. Then a sufficiently small neighborhood of X admits a coordinate system (u, v), where $u \in \mathbb{R}^m$, $v \in \mathbb{R}^{n-m}$ ($v \in \mathbb{R}^{n-m-1} \times \overline{\mathbb{R}_+}$ if $X \in \partial\mathcal{M}$) such that any two points with equal v-coordinates belong to one orbit of the action of H.*

Proof. We only consider the case where X is an interior point; the case of a boundary point is similar.

Observe that $m = \dim H - \dim S_X^H$, because $H(X) \simeq H/S_X^H$. Let U be an arbitrary m-dimensional smooth submanifold of H containing the identity e and transversal to S_X^H. Then the map $\psi : U \to H(X) \subset \mathcal{M}$ defined by $\psi(\gamma) = \gamma(X)$ is nondegenerate (that is, has injective differential) at e, being the composition of an inclusion map and a quotient map: $U \hookrightarrow H \to H/S_X^H \simeq H(X)$.

Let V be an arbitrary $(n-m)$-dimensional submanifold of $\mathcal{M}$ containing X and transversal to the orbit $H(X)$. Consider the map $\Psi : U \times V \to \mathcal{M}$ defined by $\Psi(\gamma, y) = \gamma(y)$. Observe that Ψ is nondegenerate at $(e, X) \in U \times V$, because the restriction of Ψ to $\{e\} \times V$ equals id_V, the restriction of Ψ to $U \times \{X\}$ equals ψ, these two maps are nondegenerate, and their images are transversal submanifolds of complementary dimensions.

Therefore, Ψ is a local diffeomorphism in a neighborhood of (e, X). Introducing local coordinates u and v in U and V (respectively), we get local coordinates in a neighborhood of X. The fact that points with equal v-coordinates belong to one orbit follows immediately from the definition of Ψ. □

Theorem 2.4. *Let M be the minimal positive dimension of $H(X)$, $X \in \mathcal{M}$, and let $\mathfrak{r} = \mathfrak{r}(X)$ denote the Riemannian distance from a point X to the set $\mathcal{M}_0$.*

I. *Suppose $p < n - M$. Let $W^1_{p,H}(\mathcal{M})$ be the subspace of $W^1_p(\mathcal{M})$ consisting of H-invariant functions. Then $W^1_{p,H}(\mathcal{M})$ is continuously embedded in $L_q(\mathcal{M}; \mathfrak{r}^{\alpha q})$ provided*

$$q \le p_M^*, \quad \alpha \ge \widehat{\alpha}_{pq} \equiv \Big(\frac{n}{p} - \frac{n}{q} - 1\Big)_+. \tag{22}$$

In other words, if (22) *is fulfilled, the embedding*

$$\mathfrak{r}^\alpha \cdot W^1_{p,H}(\mathcal{M}) \hookrightarrow L_q(\mathcal{M}). \tag{23}$$

is continuous.

Moreover, this embedding is compact whenever both inequalities in (22) *are strict.*

II. *Suppose $p \ge n - M$. Then the embedding* (23) *is continuous for all $q < p_M^* = \infty$, $\alpha \ge \widehat{\alpha}_{pq}$. If $\alpha > \widehat{\alpha}_{pq}$ then this embedding is compact.*

III. *Suppose $p = n - M$. Then for $\alpha \ge \frac{n}{p} - 1$ the following embeddings are continuous*:

$$\begin{aligned} &\mathfrak{r}^\alpha \cdot W^1_{p,H}(\mathcal{M}) \hookrightarrow BMO(\mathcal{M}), \\ &\mathfrak{r}^\alpha \cdot W^1_{p,H}(\mathcal{M}) \hookrightarrow L_\Phi(\mathcal{M}), \quad \Phi(t) = \exp(t^q) - 1, \quad q \le p'. \end{aligned} \tag{24}$$

For $q < p'$ the embedding (24) *is compact.*

IV. *Suppose $n - M < p < n$. Then the embedding*

$$\mathfrak{r}^\alpha \cdot W^1_{p,H}(\mathcal{M}) \hookrightarrow \mathcal{C}^\gamma(\mathcal{M}) \tag{25}$$

is continuous provided

$$\gamma \le \widehat{\gamma}_M, \quad \alpha \ge \gamma + \frac{n}{p} - 1. \tag{26}$$

If both inequalities in (26) *are strict, then this embedding is compact.*

V. *Suppose $p = n$. Then the embedding* (25) *is continuous and compact for all $\gamma < \widehat{\gamma}_M$, $\alpha > \gamma$.*

VI. *Suppose $p > n$. Then the embedding* (25) *is continuous provided*

$$\gamma \le \widehat{\gamma}_M, \quad \alpha \ge \gamma. \tag{27}$$

If the first inequality in (27) *is strict, then this embedding is compact.*

Proof. Consider an arbitrary point $X \in \mathcal{M}$. If $X \notin \mathcal{M}_0$, we choose a neighborhood $\mathcal{U}_X$ as in Lemma 2.3. It is easily seen that $W^1_p(\mathcal{U}_X)$ is isomorphic to $W^1_p(U \times V)$, and the subspace $W^1_{p,H}(\mathcal{U}_X)$ is isomorphic to $W^1_p(V)$. Similarly, the subspace of H-invariant functions in $L_q(\mathcal{U}_X)$ is isomorphic to $L_q(V)$.

Since the function $\mathfrak{r}$ on $\mathcal{U}_X$ is bounded away from zero and infinity, for all $\alpha \in \mathbb{R}$ the space $\mathfrak{r}^\alpha \cdot W^1_{p,H}(\mathcal{U}_X)$ is isomorphic to $W^1_p(V)$. Then the Sobolev embedding theorem for $V \subset \mathbb{R}^{n-m}$ yields assertions I–II for the functions with support in $\mathcal{U}_X$. Assertions III–VI for such functions are proved in a similar way.

Now, let $X \in \mathcal{M}_0$, and let k be the dimension of the connected component of $\mathcal{M}_0$ containing X.

First, suppose that $X \notin \partial\mathcal{M}$. We choose a neighborhood $\mathcal{U}_X$ as in Lemma 2.2. It is easily seen that the space $L_q(\mathcal{U}_X)$ is isomorphic to $L_q(B^n_1)$. Furthermore, the space $W^1_{p,H}(\mathcal{U}_X)$ is isomorphic to $W^1_{p,H}(B^n_1)$, where H is regarded as a subgroup of $\mathcal{O}(n)$. Moreover, we may assume that $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ and that the set of fixed points is the coordinate subspace $\{0\} \times \mathbb{R}^k$. Then the space $\mathfrak{r}^\alpha \cdot W^1_{p,H}(\mathcal{U}_X)$ (where $\mathfrak{r} = \mathfrak{r}(x)$ is a Riemannian distance from x to the set $\mathcal{M}_0$) is isomorphic to the space $r^\alpha \cdot W^1_{p,H}(B^n_1)$ (where (r, Θ, z) are the "cylindrical" coordinates of x, $\Theta \in \mathbb{S}_{n-k}$, $z \in \mathbb{R}^k$).

Since the transformations are orthogonal and preserve the subspace $\{0\} \times \mathbb{R}^k$, they also preserve its orthogonal complement $\mathbb{R}^{n-k} \times \{0\}$. Therefore, we may assume that $H \subset \mathcal{O}(n-k)$, the action of H on $\mathbb{R}^n$ preserves the last k coordinates, and the action on $\mathbb{R}^{n-k}$ has no fixed points except 0.

Consider an arbitrary point $x \in \mathbb{S}_{n-k}$. Since H has no fixed points on the sphere, we can choose a neighborhood $\mathcal{U}_x \subset \mathbb{S}_{n-k}$ as in Lemma 2.3 (note that $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^{n-k-1-m}$). Let Ω_x denote the intersection of the ball B^n_1 and the cone based on $\mathcal{U}_x \times \mathbb{R}^k$. Then the space $W^1_{p,H}(\Omega_x)$ is isomorphic to $W^1_p(\widetilde{\mathcal{D}}_x; r^m)$, where $\widetilde{\mathcal{D}}_x$ is the intersection of the ball B^{n-m}_1 and the cone based on $V \times \mathbb{R}^k$. Similarly, the space of H-invariant functions in $L_q(\Omega_x)$ is isomorphic to $L_q(\widetilde{\mathcal{D}}_x; r^m)$.

We cover the sphere $\mathbb{S}_{n-k}$ by the above neighborhoods $\mathcal{U}_x$, pick a finite subcovering, and split the sphere into polyhedra so that every polyhedron is contained in one of the neighborhoods $\mathcal{U}_x$. The isomorphisms mentioned in the preceding paragraph allow us to prove assertions I–II for the functions with support in $\mathcal{U}_X$ in the same way as in Theorem 1.1. Similarly, assertions IV–VI for such functions are proved in the same way as in Theorem 1.3, and assertion III is proved as in Theorem 1.4.

In the case where $X \in \partial\mathcal{M}$, we use Lemma 2.2 to replace the neighborhood $\mathcal{U}_X$ by the cylinder $B^{n-1}_1 \times [0, 1[$. Then an even reflection in the boundary plane reduces the problem to functions on the cylinder $B^{n-1}_1 \times]-1, 1[$. After this we can restrict the functions to the ball B^n_1, and the remaining part of the argument applies without change.

It remains to cover $\mathcal{M}$ by neighborhoods $\mathcal{U}_X$ as above, select a finite subcovering, and merge the local estimates into a global one by using a suitable partition of unity. □

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