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NEW VERSION OF THE LADYZHENSKAYA–PRODI–SERRIN CONDITION

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Abstract. A new local version of the Ladyzhenskaya–Prodi–Serrin regularity condition for weak solutions of the nonstationary 3-dimensional Navier–Stokes system is proved. The novelty is in that the energy of the solution is not assumed to be finite.

§1. Introduction

In this paper we present a new local version of the Ladyzhenskaya–Prodi–Serrin condition (LPS condition), which ensures the regularity of weak solutions of the nonstationary 3-dimensional Navier–Stokes system. It is known that, in the regularity theory for this system, there is a considerable distinction between the local and the global version. What is usually meant by global regularity is the differentiability properties of solutions of initial-boundary value problems for the Navier–Stokes equations, treated in dependence on the initial and boundary data and the right-hand sides of the equations. In contrast, the local theory describes the property of the Navier–Stokes equations to smooth out their solutions under the assumption that these solutions possess a certain regularity from the outset.

Usually, the distinction mentioned above is illustrated by Serrin's example in which the velocity field $v(x,t) = c(t)\nabla h(x)$ and the pressure field $p(x,t) = -c'(t)h(x) - \frac{1}{2}c^2(t)|\nabla h(x)|^2$ satisfy the Navier–Stokes equations

$$\partial_t v + v\cdot\nabla v - \triangle v = -\nabla p, \quad \operatorname{div} v = 0 \tag{1.1}$$

in $Q_T = \Omega\times]0,T[$, where $\Omega\subset\mathbb{R}^3$ and T is a positive parameter, provided h is a function harmonic in Ω. It is seen that this solution is not smooth in t if the function c is not. Formally, this is explained by the fact that, locally, $\partial_t v$ and ∇p may compensate each other and cannot be estimated separately. However, in the case of initial-boundary value problems, one usually starts with attempts to estimate $\partial_t v$ (globally), after which the properties of ∇p can easily be obtained from the equations.

In the present paper we shall not touch on the global regularity, referring the reader to [1]–[4].

In its turn, in the local theory, two cases are distinguished: interior and boundary. First, we discuss the simpler case of interior regularity. The first result in this direction was established by Serrin in [5]; by the way, this was the starting point of the local theory of nonstationary Navier–Stokes equations. Later, it was generalized by Struwe in [6]. The next theorem is stated in Struwe's wording.

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Theorem 1.1. *Let v be a divergence free velocity field defined in the unit parabolic cylinder $Q = B \times]-1, 0[$, where B is the unit ball in $\mathbb{R}^3$ centered at the origin. Suppose that v satisfies the following three conditions*:

$$v \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q), \tag{1.2}$$

$$\int_Q \{-v \cdot \partial_t u - v \otimes v : \nabla u + \nabla v : \nabla u\}\, dz = 0 \tag{1.3}$$

for all $u \in C_0^\infty(Q)$ such that $\operatorname{div} u = 0$, *and*

$$v \in L_{s,l}(Q), \quad \frac{3}{s} + \frac{2}{l} = 1, \quad s > 3. \tag{1.4}$$

Then $z = (x, t) = 0 = (0, 0)$ is a regular point[1] *of the velocity field v.*

Before commenting on Theorem 1.1, we explain our notation:

$$v \cdot u = v_i u_i, \quad v \otimes u = (v_i u_j), \quad A : B = A_{vj} B_{ij},$$

and summation is assumed over the repeated Latin indices ranging from 1 to 3; $L_{m,n}(Q) = L_n(-1, 0; L_m(B))$ is the Lebesgue space with the norm

$$\|v\|_{m,n,Q} \equiv \|v\|_{L_{m,n,Q}} = \begin{cases} \left(\int_{-1}^0 \|v(\cdot,t)\|_{m,B}^n\, dt\right)^{1/n}, & 1 \le n < +\infty, \\ \operatorname*{ess\,sup}_{t \in]-1,0[} \|v(\cdot,t)\|_{m,B}, & n = +\infty, \end{cases}$$

$$\|v\|_{m,B} = \left(\int_B |v|^m\, dx\right)^{1/m} \quad \text{for } m < \infty,$$

$$\|v\|_{\infty,B} = \operatorname*{ess\,sup}_B |v| \quad \text{for } m = \infty,$$

$$L_m(Q) = L_{m,m}(Q), \quad \|v\|_{m,m,Q} = \|v\|_{m,Q};$$

$W_{m,n}^{1,0}(Q)$, $W_{m,n}^{2,1}(Q)$ are the Sobolev spaces with mixed norm,

$$W_{m,n}^{1,0}(Q) = \{v, \nabla v \in L_{m,n}(Q)\},$$
$$W_{m,n}^{2,1}(Q) = \{v, \nabla v, \nabla^2 v, \partial_t v \in L_{m,n}(Q)\},$$
$$W_m^{1,0}(Q) = W_{m,m}^{1,0}(Q), \quad W_m^{2,1}(Q) = W_{m,m}^{2,1}(Q).$$

Definition 1.1. The point $z = 0$ is called a regular point of the velocity field v if there exists a number $0 < r \le 1$ such that $v \in L_\infty(Q(r))$, where $Q(r) = B(r) \times]-r^2, 0[$ is the corresponding parabolic cylinder, $B(r)$ being the ball of radius r in $\mathbb{R}^3$ centered at the origin.

This definition was introduced in [7] by Caffarelli, Kohn, and Nirenberg, and is the most commonly accepted. In principle, the space $L_\infty(Q(r))$ can be replaced with $C(\overline{Q}(r))$; see Scheffer's papers [8]–[11], or even with $C^\alpha(\overline{Q}(r))$ for some positive α, see [12] and also [13].

Condition (1.4) is called the Ladyzhenskaya–Prodi–Serrin condition (LPS condition).

Unfortunately, in general, finite energy velocity fields (i.e., those satisfying (1.2)) fail to fulfil the LPS condition. Indeed, by applying the well-known multiplicative inequalities (see, e.g., [12]), it is easy to show that if v has finite energy, then

$$v \in L_{s,l}(Q), \quad \frac{3}{s} + \frac{2}{l} \ge \frac{3}{2}, \quad s \le 6.$$

[1]See Definition 1.1 below.

In the proof of their statements, Serrin and Struwe used the following vorticity equation:

$$\partial_t w + v \cdot \nabla \omega - \triangle \omega = w \cdot \nabla v, \quad \omega = \nabla \wedge v.$$

It is precisely for this reason that the pressure p is absent in Theorem 1.1. The limit case where $s = 3$ and $l = +\infty$ was studied in [6] and [14] under the condition that the $L_{3,\infty}$-norm is small. We also mention the papers [15]–[17], in which the boundedness in the mixed Lebesgue spaces was replaced in (1.4) by the smallness of the norms in the mixed Lorentz spaces. Basically, in all the cited publications the method consists in a thorough analysis of the equation for ω. However, so far it is unclear how to use this method when dealing with regularity up to the boundary.

New methods were invoked to lift the smallness condition in the limit case mentioned above. In particular, this required the unique solvability theorem for the inverse heat conduction equation with lower terms in the half-space (see [18]–[22]).

In the present paper we describe a new method allowing us to investigate whether the LPS condition is sufficient for the local regularity of the weak solution of the Navier–Stokes equations. This method has the advantage that, with some modifications and supplements, it works near the spatial boundary. Our version of Theorem 1.1 reads like this.

Theorem 1.2. *Suppose that two functions v and p are given in Ω, and that v satisfies the LPS condition* (1.4). *Also, assume that*

$$v \in W^{1,0}_{m,n}(Q), \quad p \in L_{m,n}(Q) \tag{1.5}$$

for some m, n such that

$$1 < m < s, \quad 1 < n < l. \tag{1.6}$$

Next, let

$$v \in L_{d,r}(Q) \tag{1.7}$$

with

$$d = \frac{sm}{s-m}, \quad r = \frac{ln}{l-n}.$$

Finally, assume that v and p satisfy the Navier–Stokes equations (1.1) *in the sense of distributions. Then $z = 0$ is a regular point of v.*

In contrast to Theorem 1.1, in Theorem 1.2 we do not assume that v has finite energy. The assumptions of Theorem 1.2 are reasonable in the sense that if, in addition to (1.4), we require that the energy be bounded, then these assumptions will be met automatically, at least for the solutions of initial-boundary value problems. Indeed, as has already been mentioned, the fact that the energy is finite implies that

$$v \in L_{d',r'}(Q) \quad \text{with } \frac{3}{d'} + \frac{2}{r'} \geq \frac{3}{2} \tag{1.8}$$

and, moreover,

$$|v|\,|\nabla v| \in L_{m',n'}(Q) \quad \text{with } \frac{1}{m'} = \frac{1}{d'} + \frac{1}{2},\ \frac{1}{n'} = \frac{1}{r'} + \frac{1}{2}. \tag{1.9}$$

In its turn, at least for solutions of initial-boundary value problems, (1.9) implies that v and p belong to $W^{2,1}_{m',n'}$ and $W^{1,0}_{m',n'}$, respectively. As to condition (1.7), it is fulfilled for $d = \frac{sm'}{s-m'}$, $r = \frac{ln'}{l-n'}$, because $d < d'$, $r \leq r'$.

We pass to discussing the results on regularity up to the boundary. As has already been mentioned, it is unknown whether Theorem 1.1 is true near the boundary. However, a certain analog of Theorem 1.2 can be stated as follows.

Theorem 1.3. *Let $Q^+ = B^+ \times]-1,0[$, where $B^+ = \{x = (x_1, x_2, x_3) \in B \mid x_3 > 0\}$, and let v and p be two functions with the following properties*:

$$v \in W^{2,1}_{m,n}(Q^+), \quad p \in W^{1,0}_{m,n}(Q^+), \tag{1.10}$$

$$v \in L_{s,l}(Q^+), \quad \frac{3}{s} + \frac{2}{l} = 1, \quad s > 3. \tag{1.11}$$

Here the numbers m, n, s, and l satisfy (1.6).

Next, suppose that v and p solve the Navier–Stokes equations,

$$\partial_t v + v \cdot \nabla v - \triangle v = -\nabla p, \quad \operatorname{div} v = 0 \text{ in } Q^+ \tag{1.12}$$

with the boundary condition

$$v|_{x_3=0} = 0. \tag{1.13}$$

Then the point $z = 0$ is a regular point for the velocity field v in the sense that there exists a radius $r \in]0,1]$ such that v is Hölder continuous in the closure of the set $Q^+(r) = B^+(r) \times]-r^2, 0[$. Here $B^+(r) = \{x \in B(r) \mid x_3 > 0\}$.

The main distinction between the assumptions of Theorems 1.2 and 1.3 is in the original regularity. In the boundary case it is higher (compare (1.5) and (1.10)). This fact was mentioned earlier in the author's paper [23], where the Stokes system was treated locally near the boundary. Another distinction, implied by the first, consists in the fact that now condition (1.7) is fulfilled automatically, because the space $W^{2,1}_{m,n}(Q^+)$ is embedded in the space $L_{d,r}(Q^+)$ for the parameters m, n, s, l, d, and r as in Theorem 1.3.

Quite a particular case of Theorem 1.3 was proved in [23]. Under some additional restrictions on the data, in [24] Solonnikov proved a similar result even in the case of a curvilinear boundary. In our notation, those additional restrictions are as follows: $3/m + 2/n = 4$, $v \in L_2(Q^+)$, $2/n - 1 < 1 - 3/s = 2/l$. In the context of solutions of initial-boundary value problems, we mention also the papers [25]–[28].

As to technicalities, in the proof of Theorems 1.2 and 1.3 we follow the author's paper [23] where the Stokes system was studied locally near the boundary. There, it was explained how the solutions of the Stokes system become more smooth locally at the expense of the spatial variables. Unfortunately, in local considerations the presence of pressure prevents us from tracing the smoothing in time, which may fail to exist, as Serrin's example shows. Unlike [23], in the present paper we consider a perturbed Stokes system,

$$\partial_t v + u \cdot \nabla v - \triangle v = f - \nabla p, \quad \operatorname{div} v = 0,$$

and the $L_{s,l}$-norm of the perturbation u is assumed to be sufficiently small. For this perturbed system, we establish local estimates of the higher derivatives. For this, we use coercive $L_{m,n}$-estimates for solutions of initial-boundary value problems for the Stokes system. The fact that the quantity $\|u\|_{s,l}$ is small allows us not to incorporate the entire "convection term" $u \cdot \nabla v$ in the right-hand side, but rather to take only its "weak" part arising when we differentiate the cut-off function. The statements of Theorems 1.2 and 1.3 are obtained by iteration of estimates for solutions of the perturbed system.

The paper is organized as follows. In §2 we prove estimates for the smoothing of solutions of the perturbed system in interior domains, and then prove Theorem 1.2. In §3 we study the perturbed problem near the boundary and prove Theorem 1.3.

All constants that depend only on unessential parameters will be denoted by c.

§2. Proof of Theorem 1.2

For the reader's convenience, we formulate some known facts to be used actively in what follows.

Proposition 2.1. *Let Ω be a bounded domain with Lipschitz boundary, let T be a positive parameter, and let $Q_T = \Omega \times]0,T[$.*

Suppose $m,n,s,l,d,r \in [1,\infty]$ and put $\alpha = 2 - \frac{3}{m} - \frac{2}{n} + \frac{3}{s} + \frac{2}{l}$, $\beta = 1 - \frac{3}{m} - \frac{2}{n} + \frac{3}{d} + \frac{2}{r}$.

The space $W^{2,1}_{m,n}(Q_T)$ is embedded continuously in $L_{s,l}(Q_T)$ if $m \le s$, $n \le l$, and $\alpha > 0$, or $1 \le m \le s < +\infty$, $1 < n \le l < +\infty$, and $\alpha = 0$. The former space is embedded continuously in $W^{1,0}_{d,r}(Q_T)$ if $m \le d$, $n \le r$, and $\beta > 0$, or $1 \le m \le d < +\infty$, $1 < n \le r < +\infty$, and $\beta = 0$.

Proposition 2.2. *Let Ω be a bounded domain with a sufficiently smooth boundary, and let T be a positive parameter.*

Suppose

$$f \in L_{m,n}(Q_T), \quad 1 < m < +\infty, \ 1 < n < +\infty.$$

There exists a unique pair of functions $v \in W^{2,1}_{m,n}(Q_T)$ and $p \in W^{1,0}_{m,n}(Q_T)$ such that

$$\partial_t v - \triangle v = f - \nabla p, \quad \operatorname{div} v = 0 \text{ in } Q_T,$$
$$\int_\Omega p(x,t)\,dx = 0, \ 0 \le t \le T, \quad v|_{\partial' Q_T} = 0,$$

where $\partial' Q_T$ is the parabolic boundary of the cylinder Q_T. Moreover, we have the coercive estimate

$$\|v\|_{W^{2,1}_{m,n}(Q_T)} + \|p\|_{W^{1,0}_{m,n}(Q_T)} \le c(m,n,Q_T)\|f\|_{m,n,Q_T}.$$

See [29], [30] for the proof of Proposition 2.1, and [31], [32] for the proof of Proposition 2.2.

The next statement is a consequence of the two propositions above.

Proposition 2.3. *Let Ω be a bounded domain with a sufficiently smooth boundary, and let T be a positive parameter.*

Suppose

$$1 < m < s, \quad 1 < n < l, \quad \frac{3}{s} + \frac{2}{l} = 1, \tag{2.1}$$

$$f \in L_{m,n}(Q_T), \quad u \in L_{s,l}(Q_T). \tag{2.2}$$

There exists a positive number $\varepsilon = \varepsilon(m,n,s,l,Q_T)$ such that if

$$\|u\|_{s,l,Q_T} < \varepsilon, \tag{2.3}$$

then the initial-boundary value problem

$$\begin{gathered} \partial_t w + u \cdot \nabla w - \triangle w = f - \nabla q, \quad \operatorname{div} w = 0 \text{ in } Q_T, \\ \int_\Omega q(x,t)\,dx = 0, \quad 0 \le t \le T, \quad w|_{\partial' Q_T} = 0 \end{gathered} \tag{2.4}$$

has a unique solution, which satisfies the estimate

$$\|w\|_{W^{2,1}_{m,n}(Q_T)} + \|q\|_{W^{1,0}_{m,n}(Q_T)} \le c(m,n,s,l,Q_T)\|f\|_{m,n,Q_T}. \tag{2.5}$$

Proposition 2.3 is known; nevertheless, we present the proof of it, since the arguments are typical of the present work.

Proof of Proposition 2.3. The proof is based on two inequalities. The first is the usual Hölder inequality

$$\|u\cdot\nabla w\|_{m,n,Q_T}\le\|u\|_{s,l,Q_T}\|\nabla w\|_{d,r,Q_T}<\varepsilon\|\nabla w\|_{d,r,Q_T},\qquad d=\frac{sm}{s-m},\quad r=\frac{ln}{l-n}. \tag{2.6}$$

The second expresses the boundedness of the operator that embeds $W^{2,1}_{m,n}(Q_T)$ in the space $W^{1,0}_{d,r}(Q_T)$:

$$\|\nabla w\|_{d,r,Q_T}\le c(m,n,s,l,Q_T)\|w\|_{W^{2,1}_{m,n}(Q_T)}. \tag{2.7}$$

This inequality is valid because

$$1-\frac{3}{m}-\frac{2}{n}+\frac{3}{d}+\frac{2}{r}=1-\frac{3}{s}-\frac{2}{l}=0;$$

see condition (2.1) and Proposition 2.1. Thus, we have

$$\|u\cdot\nabla w\|_{m,n,Q_T}\le\varepsilon c(m,n,s,l,Q_T)\|w\|_{W^{2,1}_{m,n}(Q_T)}. \tag{2.8}$$

Estimate (2.8) allows us to introduce an operator $\mathcal{L}:\widehat{W}^{2,1}_{m,n}(Q_T)\to\widehat{W}^{2,1}_{m,n}(Q_T)=\{v\in W^{2,1}_{m,n}(Q_T)\mid v|_{\partial' Q_T}=0\}$ as follows: $\mathcal{L}w=v$ means that, for a given $w\in\widehat{W}^{2,1}_{m,n}(Q_T)$, v is a unique solution of the initial-boundary value problem

$$\partial_t v+u\cdot\nabla w-\triangle v=f-\nabla p,\quad \operatorname{div}v=0\text{ in }Q_T,$$
$$\int_\Omega p(x,t)\,dx=0,\quad 0\le t\le T,\quad v|_{\partial' Q_T}=0.$$

Estimate (2.8) and Proposition 2.2 show that $\mathcal{L}$ is a contraction operator if ε is sufficiently small. This proves Proposition 2.3. □

Now we state and prove the main statement of this section.

Proposition 2.4. *Assume conditions* (2.1), *let* $\tau_0\in\,]0,1]$ *be a fixed number, and let* d *and* r *be given by*

$$d=\frac{sm}{s-m},\quad l=\frac{ln}{l-n}. \tag{2.9}$$

Suppose that four functions $f\in L_{m,n}(Q(\tau_0))$, $u\in L_{s,l}(Q(\tau_0))$, $p\in L_{m,n}(Q(\tau_0))$, *and* $v\in W^{1,0}_{m,n}(Q(\tau_0))\cap L_{d,r}(Q(\tau_0))$ *satisfy the equations*

$$\partial_t v+u\cdot\nabla v-\triangle v=f-\nabla p,\quad \operatorname{div}v=\operatorname{div}u=0\ \text{ in }Q(\tau_0), \tag{2.10}$$

and that

$$\|u\|_{s,l,Q(\tau_0)}<\varepsilon_0=\varepsilon(m,n,s,l,Q). \tag{2.11}$$

Here $\varepsilon(m,n,l,s,Q)$ *is the number described in Proposition* 2.3 *for the unit cylinder* $Q=B\times\,]-1,0[$. *Then for any* $\tau_1\in\,]0,\tau_0[$ *the functions* v *and* p *belong to* $W^{2,1}_{m,n}(Q(\tau_1))$ *and* $W^{1,0}_{m,n}(Q(\tau_1))$, *respectively, and*

$$\begin{aligned}&\|\partial_t v\|_{m,n,Q(\tau_1)}+\|\nabla^2 v\|_{m,n,Q(\tau_1)}+\|\nabla p\|_{m,n,Q(\tau_1)}\\&\quad\le c(m,n,s,l,\tau_1,\tau_0)[\|f\|_{m,n,Q(\tau_0)}+\varepsilon\|v\|_{d,r,Q(\tau_0)}\\&\qquad+\|v\|_{m,n,Q(\tau_0)}+\|\nabla v\|_{m,n,Q(\tau_0)}+\|p\|_{m,n,Q(\tau_0)}].\end{aligned} \tag{2.12}$$

Proof. We extend the functions f, u, v, and p by zero off $Q(\tau_0)$ and denote by A the expression in brackets on the right-hand side of (2.12). Let φ be a cut-off function of class $C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ such that

$$0 \le \varphi \le 1, \quad \varphi \equiv 1 \quad \text{in } B(\tau_1) \times]-\tau_1^2, \tau_1^2[,$$
$$\varphi \equiv 0 \quad \text{off } B(\tau_0) \times]-\tau_0^2, \tau_0^2[.$$

We have

$$\left.\begin{aligned} &\partial_t(\varphi v) + u \cdot \nabla(\varphi v) - \triangle(\varphi v) = \mathcal{F} - \nabla(\varphi p), \\ &\operatorname{div}(\varphi v) = v \cdot \nabla \varphi \end{aligned}\right\} \quad \text{in } Q, \tag{2.13}$$
$$\varphi v|_{\partial' Q} = 0.$$

Here

$$\mathcal{F} = v\partial_t \varphi + vu \cdot \nabla \varphi - 2\nabla v \nabla \varphi - v\triangle\varphi + p\nabla\varphi + \varphi f.$$

Applying the Hölder inequality

$$\||\nabla\varphi||v||u|\|_{m,n,Q} \le c(\tau_0, \tau_1)\|u\|_{s,l,Q(\tau_0)}\|v\|_{d,r,Q(r_0)} < \varepsilon c(\tau_0, \tau_1)\|v\|_{d,r,Q(\tau_0)},$$

we obtain

$$\|\mathcal{F}\|_{m,n,Q} \le c(\tau_0, \tau_1)A. \tag{2.14}$$

In order to eliminate the nonhomogeneity in the second equation in (2.13), we consider the following boundary value problem depending on the parameter t:

$$\left.\begin{aligned} &-\triangle w + \nabla q = 0, \\ &\operatorname{div} w = v \cdot \nabla \varphi \end{aligned}\right\} \quad \text{in } B, \tag{2.15}$$
$$w|_{\partial B} = 0, \quad \int_B q(x,t)\,dx = 0, \quad -1 \le t \le 0.$$

It is known that the solution of (2.15) satisfies the estimates

$$\begin{aligned} &\|w\|_{m,n,Q} + \|\nabla w\|_{m,n,Q} + \|\nabla^2 w\|_{m,n,Q} + \|q\|_{m,n,Q} + \|\nabla q\|_{m,n,Q} \\ &\quad \le c(m,n)\|\nabla(v \cdot \nabla\varphi)\|_{m,n,Q} \le c(m,n,\tau_0,\tau_1)A, \end{aligned} \tag{2.16}$$

$$\|\nabla w\|_{d,r,Q} \le c(d,r)\|v \cdot \nabla\varphi\|_{d,r,Q}. \tag{2.17}$$

By (2.13) and (2.15), the functions

$$V = \varphi v - w, \quad P = \varphi p - q$$

solve the equations

$$\left.\begin{aligned} &\partial_t V + u \cdot \nabla V - \triangle V + \nabla P = G \equiv \mathcal{F} - \partial_t w - u \cdot \nabla w, \\ &\operatorname{div} V = 0 \end{aligned}\right\} \quad \text{in } Q, \tag{2.18}$$
$$V|_{\partial' Q} = 0.$$

Now, we need to estimate the right-hand side in (2.18). We start with the function $\partial_t w$, which is a solution of the following problem:

$$\left.\begin{aligned} &\triangle\partial_t w - \nabla\partial_t q = 0, \\ &\operatorname{div}\partial_t w = \nabla\varphi \cdot \partial_t v + v \cdot \nabla\partial_t\varphi \end{aligned}\right\} \quad \text{in } B, \tag{2.19}$$
$$\partial_t w|_{\partial B} = 0.$$

We are going to apply duality arguments; see [23, 24]. Namely, for an arbitrary $g \in L_{m'}(B)$ with $m' = \frac{m}{m-1}$ we find $\hat{u} \in W^2_{m'}(B)$ such that

$$\begin{aligned} &-\triangle \hat{u} + \nabla r = g, \quad \operatorname{div} \hat{u} = 0 \text{ in } B, \\ &\int_B r(x,t)\,dx = 0, \quad \hat{u}|_{\partial B} = 0. \end{aligned} \tag{2.20}$$

The function $\hat{u}$ depends on the parameter t, and for the function r we have the estimate

$$\int_B (|r|^{m'} + |\nabla r|^{m'})\,dx \le c(m) \int_B |g|^{m'}\,dx. \tag{2.21}$$

Recalling (2.19) and (2.20) and expressing $\partial_t v$ from equation (2.10), after integration by parts we arrive at

(2.22)

$$\begin{aligned} \int_B g \cdot \partial_t w\,dx &= \int_B (-\triangle \hat{u} + \nabla r) \cdot \partial_t w\,dx = -\int_B r \operatorname{div} \partial_t w\,dx \\ &= -\int_B r\big(\nabla\varphi \cdot (-u \cdot \nabla v + \triangle v - \nabla p + f) + v \cdot \partial_t \nabla\varphi\big)\,dx \\ &= \int_{B(\tau_0)} \big[-\nabla(r\nabla\varphi) : v \otimes u + \nabla(r\nabla\varphi) : \nabla v - \operatorname{div}(r\nabla\varphi)p - r\nabla\varphi \cdot f - v \cdot \partial_t \nabla\varphi\big]\,dx. \end{aligned}$$

When integrating by parts, in the first summand we have used the fact that u is a solenoidal vector in $B(\tau_0)$. Next, from (2.21) and (2.22) we deduce the inequality

$$\begin{aligned} \|\partial_t w\|_B \le c(m,\tau_0,\tau_1)[&\|u\|_{s,B(\tau_0)}\|v\|_{m,B(\tau_0)} + \|v\|_{m,B(\tau_0)} \\ &+ \|\nabla v\|_{m,B(\tau_0)} + \|p\|_{m,B(\tau_0)} + \|f\|_{m,B(\tau_0)}], \end{aligned}$$

which implies

$$\|\partial_t w\|_{m,n,Q} \le c(m,n,s,l,\tau_0,\tau_1)A. \tag{2.23}$$

Now we estimate $u \cdot \nabla w$. The Hölder inequality and (2.17) yield

$$\begin{aligned} \|u \cdot \nabla w\|_{m,n,Q} &\le \|u\|_{s,l,Q(\tau_0)}\|\nabla w\|_{d,r,Q} \\ &\le \varepsilon c(m,n,s,l,\tau_0,\tau_1)\|v\|_{d,r,Q(\tau_0)} \le c(m,n,s,l,\tau_0,\tau_1)A. \end{aligned}$$

We arrive at the final estimate for G:

$$\|G\|_{m,n,Q} \le c(m,n,s,l,\tau_0,\tau_1)A.$$

Now, assume condition (2.11). Then, by Proposition 2.3, we have

$$\begin{aligned} &\|\partial_t V\|_{m,n,Q} + \|\nabla^2 V\|_{m,n,Q} + \|\nabla P\|_{m,n,Q} \\ &\quad \le c(m,n,s,l,\tau_0,\tau_1)\|G\|_{m,n,Q} \le c(m,n,s,l,\tau_0,\tau_1)A. \end{aligned}$$

To deduce the required inequality (2.12), it remains to use the properties of the cut-off function φ and estimates (2.16) and (2.23). Proposition 2.4 is proved. □

Proof of Theorem 1.2. Let $\varepsilon_0 = \varepsilon(m,n,s,l,Q)$ be the number defined in Proposition 2.4 (see (2.11)). Since $s > 3$, we can find a number $\tau_0 \in]0,1]$ such that

$$\|v\|_{s,l,Q(\tau_0)} < \varepsilon_0.$$

This means that the functions $u = v$ and $f = 0$ satisfy all the assumptions of Proposition 2.4 in $Q(\tau_0)$. Thus, putting $\tau_1 = \tau_0/2$, we have

$$v \in W^{2,1}_{m,n}(Q(\tau_1)), \quad p \in W^{1,0}_{m,n}(Q(\tau_1)). \tag{2.24}$$

We consider two cases: $m \geq 3$ and $m < 3$. In the first case, there is no loss of generality in assuming that

$$v \in W^{2,1}_{3,n}(Q(\tau_1)), \quad p \in W^{1,0}_{3,n}(Q(\tau_1)), \quad 1 < n < 2. \tag{2.25}$$

We put

$$l_1 = \frac{2n}{2-n}\delta, \tag{2.26}$$

and suppose that $\delta \in]0,1[$ is subject to the restriction

$$n > \frac{2}{1+\delta}. \tag{2.27}$$

It is easy to check that, with such l_1 and δ, we have

$$l_1 > 2, \quad s_1 = \frac{3n\delta}{n(1+\delta)-2} > \frac{3n}{2(n-1)} > 3 \tag{2.28}$$

and

$$\frac{3}{s_1} + \frac{2}{l_1} = 1. \tag{2.29}$$

We fix m_1 so that

$$\frac{3n}{2(n-1)} < m_1 < s_1,$$

and calculate

$$d_1 = \frac{s_1 m}{s_1 - m}, \quad r_1 = \frac{l_1 n}{l_1 - n}. \tag{2.30}$$

By the space-type embedding theorem,

$$v \in W^{1,0}_{m_1,n}(Q(\tau_1)), \quad p \in L_{m_1,n}(Q(\tau_1)).$$

Applying Proposition 2.1, we conclude that $v \in L_{s_1,l_1}(Q(\tau_1))$, because

$$2 - \frac{3}{3} - \frac{2}{n} + \frac{3}{s_1} + \frac{2}{l_1} = 2 - \frac{2}{n} > 0,$$

and $v \in L_{d_1,r_1}(Q(\tau_1))$, because

$$2 - \frac{3}{3} - \frac{2}{n} + \frac{3}{d_1} + \frac{2}{r_1} = \frac{3}{m_1} > 0.$$

Next, reducing the number τ_1 if necessary, we obtain

$$\|v\|_{s_1,l_1 Q(\tau_1)} < \varepsilon(m_1, n, s_1, l_1, Q).$$

Consequently, all the assumptions of Proposition 2.4 are satisfied in $Q(\tau_1)$ with $u = v$ and $f = 0$. But then we have

$$v \in W^{2,1}_{m_1,n}(Q(\tau_2)), \quad p \in W^{1,0}_{m_1,n}(Q(\tau_2)), \quad \tau_2 = \tau_1/2. \tag{2.31}$$

Since

$$\mu = 2 - \frac{3}{m_1} - \frac{2}{n} > 0,$$

we deduce that there exists $r \in]0,1[$ such that v is Hölder continuous in the closure of $Q(r)$ (see, e.g., [13] or [23]).

So, it remains to handle the case where $m < 3$. Putting $m_1 = \min\left\{\frac{2m}{3-m}, 3\right\}$, we see that

$$v \in W^{1,0}_{m_1,n}(Q(\tau_1)), \quad p \in L_{m_1,n}(Q(\tau_1)), \quad \tau_1 = \tau_0/2.$$

Next, we put $d_1 = m_1 s/(s-m_1)$ and calculate the number $\varkappa = 2 - \frac{3}{m} - \frac{2}{n} + \frac{3}{d_1} + \frac{2}{r}$; this number is equal to $1/m_1$ if $m_1 < 3$ and to $2 - 3/m \geq 1/3$ if $m_1 = 3$. Thus, by Proposition 2.1,

$$v \in L_{d_1,r}(Q(\tau_1)).$$

We find $\varepsilon_1 = \varepsilon(m_1, n, s, l, Q)$ and reduce τ_1 so as to have

$$\|v\|_{s,l,Q(\tau_1)} < \varepsilon_1.$$

Then all the assumptions of Proposition 2.4 will be fulfilled if we replace m with m_1, d with d_1, τ_0 with τ_1, and τ_1 with $\tau_2 = \tau_1/2$, and put $u = v$ and $f = 0$. Thus, again we arrive at (2.31). If $m_1 = 3$, the proof is finished; if $m_1 < 3$, we repeat the procedure starting with the calculation of $m_2 = \min\left\{\frac{2m_1}{3-m_1}, 3\right\}$. We show that after finitely many steps we shall have $m_k = 3$. Indeed, otherwise,

$$1 < m_k < m_{k+1} = \frac{2m_k}{3-m_k} < 3.$$

The sequence m_k has a limit d. Clearly,

$$d = \frac{2d}{3-d} \geq m > 1,$$

but this is impossible. This proves Theorem 1.2. □

§3. Proof of Theorem 1.3

We fix a domain $\Omega \subset \mathbb{R}^3$ with smooth boundary and such that

$$B^+ \subset \Omega \subset B^+(2),$$

and denote by Q_Ω the cylinder $\Omega \times]-1, 0[$.

The following statement is an analog of Proposition 2.4.

Proposition 3.1. *Let $0 < \Theta \leq 1$. Suppose the following*:

$$1 < m < 3 < s, \quad 1 < n < l, \quad \frac{3}{s} + \frac{2}{l} = 1, \tag{3.1}$$

$$u \in L_{s,l}(Q^+(\Theta)), \quad \operatorname{div} u = 0 \text{ in } Q^+(\Theta), \tag{3.2}$$

$$v \in W^{2,1}_{m,n}(Q^+(\Theta)), \quad p \in W^{1,0}_{m,n}(Q^+(\Theta)), \tag{3.3}$$

$$f \in L_{m_1,n}(Q^+(\Theta)) \tag{3.4}$$

with

$$m < m_1 \leq \frac{2m}{3-m}, \quad m_1 < s,$$

$$\left.\begin{aligned} &\partial_t v + u \cdot \nabla v - \triangle v = f - \nabla p, \quad \operatorname{div} v = 0 \text{ in } Q^+(\Theta), \\ &v|_{x_3=0} = 0, \end{aligned}\right\} \tag{3.5}$$

$$\|u\|_{s,l,Q^+(\Theta)} < \varepsilon(m_1, n, s, l, Q_\Omega), \tag{3.6}$$

where $\varepsilon(m_1, n, s, l, Q_\Omega)$ is the number introduced in Proposition 2.3. Then, for any $\Theta_1 \in]0, \Theta[$, we have $v \in W^{2,1}_{m_1,n}(Q^+(\Theta_1))$ and $p \in W^{1,0}_{m_1,n}(Q^+(\Theta_1))$, and

$$\begin{aligned} &\|\partial_t v\|_{m_1,n,Q^+(\Theta_1)} + \|\nabla^2 v\|_{m_1,n,Q^+(\Theta_1)} + \|\nabla p\|_{m_1,n,Q^+(\Theta_1)} \\ &\quad \leq c(m, m_1, n, s, l, \Theta, \Theta_1)\big(\|v\|_{W^{2,1}_{m,n}(Q^+(\Theta))} + \|p\|_{W^{1,0}_{m,n}(Q^+(\Theta))} + \|f\|_{m_1,n,Q^+(\Theta)}\big). \end{aligned} \tag{3.7}$$

Proof. We fix a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ so that

$$0 \le \varphi \le 1, \quad \varphi \equiv 1 \quad \text{in } B(\Theta_1) \times]-\Theta_1^2, \Theta_1^2[,$$
$$\varphi \equiv 0 \quad \text{off } B(\Theta) \times]-\Theta^2, \Theta^2[.$$

All functions are extended by zero off $Q^+(\Theta)$.

We repeat the arguments of Proposition 2.4 with the replacement of the ball B by the domain Ω. This results in the following relations:

$$\left.\begin{aligned} &\partial_t V + u \cdot \nabla V - \triangle V = G - \nabla P, \\ &\operatorname{div} V = 0 \end{aligned}\right\} \quad \text{in } Q_\Omega, \qquad (3.8)$$
$$V|_{\partial' Q_\Omega} = 0.$$

Here

$$V = \varphi v - w, \quad P = \varphi p - q,$$
$$G = \mathcal{F} - \partial_t w - u \cdot \nabla w,$$
$$\mathcal{F} = v \partial_t \varphi + v u \cdot \nabla \varphi - 2 \nabla v \nabla \varphi - v \triangle \varphi + p \nabla \varphi + \varphi f,$$

and w is a unique solution of the boundary value problem

$$-\triangle w + \nabla q = 0, \quad \operatorname{div} w = v \cdot \nabla \varphi \text{ in } \Omega,$$
$$\int_\Omega q(x,t)\, dx = 0, \quad -1 \le t \le 0, \quad w|_{\partial\Omega} = 0,$$

in which t plays the role of a parameter.

Using the embedding theorems, condition (3.3), and the requirements imposed on m_1, we show that

(3.9)

$$\begin{aligned} A \equiv{}& \|v\|_{m_1,n,Q^+(\Theta)} + \|\nabla v\|_{m_1,n,Q^+(\Theta)} + \|p\|_{m_1,n,Q^+(\Theta)} + \varepsilon \|v\|_{d_1,r,Q^+(\Theta)} + \|f\|_{m_1,n,Q^+(\Theta)} \\ &\le c(m, m_1, n, s, l, \Theta) A_0, \end{aligned}$$

where

$$A_0 \equiv \|v\|_{W^{2,1}_{m,n}(Q^+(\Theta))} + \|p\|_{W^{1,0}_{m,n}(Q^+(\Theta))} + \|f\|_{m_1,n,Q^+(\Theta)}, \quad d_1 = \frac{s m_1}{s - m_1}.$$

Observe that $\|v\|_{d_1,n,Q^+(\Theta)}$ is dominated by a quantity proportional to A_0. This follows from the fact that

$$\begin{aligned} 2 - \frac{3}{m} - \frac{2}{n} + \frac{3}{d_1} + \frac{2}{r} &= 2 - \frac{3}{m} - \frac{2}{n} + 3\Big(\frac{1}{m_1} - \frac{1}{s}\Big) + 2\Big(\frac{1}{n} - \frac{1}{l}\Big) \\ &= 1 - \frac{3}{m} + \frac{3}{m_1} \ge 1 - \frac{3}{m} + \frac{3(2-m)}{2m} = \frac{1}{m_1} > 0 \end{aligned}$$

and from Proposition 2.1.

It is easy to check that the functions $\mathcal{F}$, w, q, and $u \cdot \nabla w$ are estimated in the same way as the corresponding expressions in the proof of Theorem 2.4. It suffices to replace m by m_1 and B by Ω. As a result, we have

$$\begin{aligned} &\|\mathcal{F}\|_{m_1,n,Q_\Omega} \le c(m_1, n, s, l, \Theta, \Theta_1) A, \\ &\|w\|_{m_1,n,Q_\Omega} + \|\nabla w\|_{m_1,n,Q_\Omega} + \|\nabla^2 w\|_{m_1,n,Q_\Omega} + \|q\|_{m_1,n,Q_\Omega} + \|\nabla q\|_{m_1,n,Q_\Omega} \\ &\qquad \le c(m_1, n, \Theta, \Theta_1) A, \qquad (3.10) \\ &\|\nabla w\|_{d_1,r,Q_\Omega} \le c(m_1, n, s, l, \Theta, \Theta_1) \|v\|_{d_1,r,Q^+(\Theta)}, \\ &\|u \cdot \nabla w\|_{m_1,n,Q_\Omega} \le c(m_1, n, s, l, \Theta, \Theta_1) A. \end{aligned}$$

An essential difference in the arguments arises only when we estimate $\partial_t w$. Here we also use duality, introducing the system

$$-\triangle \hat{u} + \nabla r = g, \quad \operatorname{div} \hat{u} = 0 \text{ in } \Omega,$$

$$\int_\Omega r(x,t)\,dx = 0, \quad -1 \le t \le 0, \quad \hat{u}|_{\partial\Omega} = 0,$$

the solution of which satisfies

$$\int_\Omega \big(|r|^{m'} + |\nabla r|^{m'}\big)\,dx \le c(m) \int_\Omega |g|^{m'}\,dx, \quad m' = \frac{m}{m-1}.$$

Recall that we do not trace the dependence of the constants on the fixed domain Ω. In place of (2.22), we shall have

$$\begin{aligned}
&\int_\Omega g \cdot \partial_t w\,dx \\
&\quad = \int_{\Gamma(\Theta)} v_{i,3} r \varphi_{,i}\,dx' - \int_{\Gamma(\Theta)} \varphi_{,3} r p\,dx' \\
&\qquad + \int_{B^+(\Theta)} \big[-\nabla(r\nabla\varphi) : v \otimes u + \nabla(r\nabla\varphi) : \nabla v - \operatorname{div}(r\nabla\varphi) p - r\nabla\varphi \cdot f - r v \cdot \partial_t \nabla\varphi \big]\,dx,
\end{aligned}$$

where $x' = (x_1, x_2)$ and $\Gamma(\Theta) = \{x = (x', 0) \mid |x| < \Theta\}$. The two last-written relations imply that

$$\|\partial_t w\|_{m_1,n,Q_\Omega} \le c(m_1, n, s, l, \Theta, \Theta_1) \bigg[A + \bigg(\int_{-\Theta^2}^0 I_1^n\,dt \bigg)^{1/n} \bigg], \tag{3.11}$$

where

$$I_1 \equiv \bigg(\int_{\Gamma(\Theta)} |\nabla\varphi|^{m_1} (|\nabla v|^{m_1} + |p|^{m_1})\,dx' \bigg)^{1/m_1}.$$

Now our task is to estimate the integral I_1. The known embedding theorems and the Hölder inequality yield

$$\begin{aligned}
I_1^{m_1} &\le c(m_1) \int_{B^+(\Theta)} \big[|\nabla\varphi|^{m_1} (|\nabla v|^{m_1-1} |\nabla^2 v| + |p|^{m_1-1} |\nabla p|) \\
&\qquad\qquad + (|\nabla\varphi|^{m_1-1} |\nabla^2\varphi| + |\nabla\varphi|^{m_1})(|\nabla v|^{m_1} + |p|^{m_1}) \big]\,dx \\
&\le c(m_1, m, \Theta, \Theta_1) \bigg[\bigg(\int_{B^+(\Theta)} (|\nabla v|^{m_1^*} + |p|^{m_1^*})\,dx \bigg)^{\frac{1}{m'}} \bigg(\int_{B^+(\Theta)} (|\nabla^2 v|^m + |\nabla p|^m)\,dx \bigg)^{\frac{1}{m}} \bigg], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m_1^* = (m_1 - 1) m'.
\end{aligned}$$

Observe that, in our case,

$$m_1^* \le \Big(\frac{2m}{3-m} - 1 \Big) \frac{m}{m-1} = \frac{3m}{3-m} < +\infty. \tag{3.12}$$

Applying the Hölder inequality once again, we get

$$\begin{aligned}
&\bigg(\int_{-\Theta^2}^0 I_1^n\,dt \bigg)^{1/n} \\
&\le c(m, m_1, n, \Theta, \Theta_1) \\
&\quad \times \bigg[\|\nabla v\|_{m_1,n,Q^+(\Theta)} + \|p\|_{m_1,n,Q^+(\Theta)} \\
&\qquad + \bigg(\int_{-\Theta^2}^0 dt \bigg(\int_{B^+(\Theta)} (|\nabla v|^{m_1^*} + |p|^{m_1^*})\,dx \bigg)^{\frac{n}{m' m_1}} \bigg(\int_{B^+(\Theta)} (|\nabla^2 v|^m + |\nabla p|^m)\,dx \bigg)^{\frac{n}{m m_1}} \bigg)^{\frac{1}{n}} \bigg] \\
&\le c(m, m_1, n, \Theta, \Theta_1) \\
&\quad \times \big[\cdots + (\|\nabla v\|_{m_1^*,n,Q^+(\Theta)} + \|p\|_{m_1^*,n,Q^+(\Theta)})^{1/m_1'} (\|\nabla^2 v\|_{m,n,Q^+(\Theta)} + \|\nabla p\|_{m,n,Q^+(\Theta)})^{1/m_1} \big].
\end{aligned}$$

Now we employ (3.12) and the usual space-type embedding theorem to conclude that

$$\left(\int_{-\Theta^2}^{0} I_1^n\,dt\right)^{1/n} \le c(m,m_1,n,\Theta,\Theta_1)A_0. \tag{3.13}$$

Collecting estimates (3.9)–(3.11) and (3.13), we obtain

$$\|G\|_{m_1,n,Q_\Omega} \le c(m,m_1,n,s,l,\Theta,\Theta_1)A_0.$$

But then condition (3.6), Proposition (2.3), and the inequality above lead to the following estimate for the solution of the initial-boundary value problem (3.8):

$$\begin{aligned}&\|\partial_t V\|_{m_1,n,Q^+(\Theta_1)} + \|\nabla^2 V\|_{m_1,n,Q^+(\Theta_1)} + \|\nabla P\|_{m_1,n,Q^+(\Theta_1)}\\ &\quad\le c(m,m_1,n,s,l,\Theta,\Theta_1)A_0.\end{aligned}$$

Combined with (3.10), (3.11), and (3.13), this allows us to deduce (3.7). Proposition 3.1 is proved. □

Much in the same spirit, we can prove the next statement.

Lemma 3.2. *Suppose* $0<\Theta\le 1$ *and* $1<n<2$. *Let* v, p, *and* f *be functions satisfying*

$$v\in W^{2,1}_{3,n}(Q^+(\Theta)),\quad p\in W^{1,0}_{3,n}(Q^+(\Theta)), \tag{3.14}$$

$$f\in L_{\infty,n}(Q^+(\Theta)), \tag{3.15}$$

$$\partial_t v + v\cdot\nabla v - \triangle v = f - \nabla p,\quad \operatorname{div} v = 0 \text{ in } Q^+(\Theta), \tag{3.16}$$

$$v|_{x_3=0} = 0. \tag{3.17}$$

Then $z=0$ *is a regular point of the velocity field* v.

Proof. We introduce the numbers l_1,δ,m_1,s_1,d_1, and r_1 by formulas (2.26), (2.28), and (2.30), assuming that $\delta\in\,]0,1[$ satisfies (2.27). The numbers s_1 and l_1 are related to each other as in (2.29) and satisfy (2.28).

By the embedding theorem,

$$v\in W^{1,0}_{m_1,n}(Q^+(\Theta)),\quad p\in L_{m_1,n}(Q^+(\Theta)).$$

Moreover, as in the proof of Theorem 1.2 we can show that

$$v\in L_{s_1,l_1}(Q^+(\Theta)).$$

Reducing Θ if necessary, we may assume that

$$\|v\|_{s_1,l_1Q^+(\Theta)} < \varepsilon(m_1,n,s_1,l_1,Q_\Omega).$$

After that, we simply repeat the proof of Proposition 3.1, putting $u=v$ and $m=3$ and replacing s by s_1, l by l_1, d by d_1, and r by r_2. Observe that

$$m_1^* = (m_1-1)\frac{3}{2} < +\infty$$

so that

$$\|\nabla v\|_{m_1^*,n,Q^+(\Theta)} + \|p\|_{m_1^*,n,Q^+(\Theta)} \le c(m_1,n,\Theta)(\|v\|_{W^{2,1}_{3,n}(Q^+(\Theta))} + \|p\|_{W^{1,0}_{3,n}(Q^+(\Theta))}).$$

Thus, estimate (3.13) remains valid, entailing the statements of Proposition 3.13 for this particular case, namely,

$$v\in W^{2,1}_{m_1,n}(Q^+(\Theta/2)),\quad p\in W^{1,0}_{m_1,n}(Q^+(\Theta/2)).$$

This implies the Hölder continuity of the velocity field in the intersection of some neighborhood of the point $z=0$ and the closure of the cylinder Q^+. Lemma 3.2 is proved. □

Proof of Theorem 1.3. By Lemma 3.2, we may assume from the outset that $m < 3$. We put $m_1 = \min\{\frac{2m}{3-m}, 3\}$ and choose $\Theta \in]0,1]$ so that $\|v\|_{s,l,Q^+(\Theta)} < \varepsilon(m,n,s,l,Q_\Omega)$, where $\varepsilon(m,n,s,l,Q_\Omega)$ is the number introduced in Proposition 2.3. Clearly, all the assumptions of Proposition 3.1 are satisfied in $Q^+(\Theta)$ with $u = v$ and $f = 0$. Consequently,

$$v \in W^{2,1}_{m_1,n}(Q^+(\Theta_1)), \quad p \in W^{1,0}_{m_1,n}(Q^+(\Theta_1)), \quad \Theta_1 = \Theta/2.$$

If $m_1 = 3$, the proof is finished. If $m_1 < 3$, we repeat the entire procedure. We reduce Θ_1 so as to have $\|v\|_{s,l,Q^+(\Theta_1)} < \varepsilon(m_1,n,s,l,Q_\Omega)$, and again apply Proposition 3.1 with Θ replaced by Θ_1 and m replaced by m_1, and with $u = v$ with $f = 0$. As in §2, it can be shown that after finitely many steps we arrive at $m_k = 3$. Now, Theorem 1.3 is implied by Lemma 3.2. □

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