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ON THE STRUCTURE OF THE SET OF PERIODS FOR PERIODIC SOLUTIONS OF SOME LINEAR INTEGRO-DIFFERENTIAL EQUATIONS ON THE MULTIDIMENSIONAL SPHERE

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Abstract. The problem of periodic solutions for the family of linear differential equations

$$(L-\lambda)u \equiv \Big(\frac{1}{i}\frac{\partial}{\partial t} - a\Delta - \lambda\Big)u(x,t) = \nu G(u-f)$$

is considered on the multidimensional sphere $x \in S^n$ under the periodicity condition $u|_{t=0} = u|_{t=b}$. Here a and λ are given reals, ν is a fixed complex number, $Gu(x,t)$ is a linear integral operator, and Δ is the Laplace operator on S^n. It is shown that the set of parameters (ν, b) for which the above problem admits a unique solution is a measurable set of full measure in $\mathbb{C}\times\mathbb{R}^+$.

§1

In [4, 5] it was discovered that, for some partial differential equations, the set of periods for which a periodic solution is unique may have an unexpectedly complicated structure. In this paper, we study this issue for a class of linear equations on the multidimensional sphere.

We consider the problem of periodic solutions for the nonlocal Schrödinger type equation

$$\Big(\frac{1}{i}\frac{\partial}{\partial t} - a\Delta - \lambda\Big)u(x,t) = \nu G(u-f) \tag{1}$$

with the t-periodicity condition

$$u|_{t=0} = u|_{t=b}. \tag{2}$$

Here $u(x,t)$ is a complex function on $S^n \times [0,b]$, where S^n is the multidimensional sphere, $n \geq 2$; $a \neq 0$, λ, and ν are given complex numbers; $f(x,t)$ is a given function.

The change of variables $t = b\tau$ reduces our problem to a problem with a fixed period, but with a new equation in which the coefficient of the τ-derivative is equal to $\frac{1}{b}$,

$$\Big(\frac{1}{i}\frac{\partial}{b\,\partial\tau} - a\Delta - \lambda\Big)u(x,b\tau) = \nu G(u(x,b\tau) - f(x,b\tau)).$$

§2

Thus, problem (1), (2) turns into a problem on periodic solutions of the equation

$$(L-\lambda)u \equiv \Big(\frac{1}{i}\frac{\partial}{b\,\partial t} - a\Delta - \lambda\Big)u(x,t) = \nu G(u-f) \tag{3}$$

with the fixed periodicity condition

$$u|_{t=0} = u|_{t=1}. \tag{4}$$

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Here $Gu(x,t) = \int_{S^n} g(x,y)u(y,t)dy$ (dy is the Lebesgue–Hausdorff measure on the sphere S^n) is an integral operator on the space $L_2(S^n \times [0,1])$ with smooth kernel $g(x,y)$ defined on $S^n \times S^n$. The differential operation $\frac{1}{i}\frac{\partial}{b\,\partial t} - a\Delta$ is assumed to be defined on the functions $u(x,t) \in C^\infty(S^n \times [0,1])$ such that $u|_{t=0} = u|_{t=1}$. Let L denote the closure of this operation $\frac{1}{i}\frac{\partial}{b\,\partial t} - a\Delta$ in $\mathcal{H} = L_2(S^n \times [0,1])$. So, an element $u \in \mathcal{H}$ belongs to the domain $\mathcal{D}(L)$ of $L = \frac{1}{i}\frac{\partial}{b\,\partial t} - a\Delta$ if and only if there is a sequence $\{u_j\} \subset C^\infty(S^n \times [0,1])$, $u_j|_{t=0} = u_j|_{t=1}$, such that $\lim u_j = u$, $\lim Lu_j = Lu$ in $\mathcal{H}$.

It is well known that the eigenvalues of the Laplace operator Δ on the sphere S^n are of the form $-k(k+n-1)$, $k \in \mathbb{Z}$, $k \geq 0$, and that Δ admits the corresponding orthonormal basis of eigenfunctions $w_k(x) \in C^\infty(S^n)$ (see, e.g., [3]).

Lemma 1. *The functions $e_{km}(x,t) = e^{i2\pi mt}w_k(x)$, $k,m \in \mathbb{Z}$, $k \geq 0$, are eigenfunctions of the operator L in the space $\mathcal{H} = L_2(S^n \times [0,1])$ that correspond to the eigenvalues*

$$\lambda_{km} = \frac{2m\pi}{b} + ak(k+n-1) = \frac{2m\pi}{b} + \lambda_k. \tag{5}$$

These functions form an orthonormal basis in $\mathcal{H}$. The domain of L is given by the formula

$$\mathcal{D}(L) = \Big\{u = \sum u_{km}e_{km} \;\Big|\; \sum |\lambda_{km}u_{km}|^2 < \infty, \sum |u_{km}|^2 < \infty\Big\}.$$

The spectrum $\sigma(L)$ is the closure of the set $\{\lambda_{km}\}$.

Lemma 2.

$$\|G\|^2 \leq M_0^2 = \int_{S^n}\int_{S^n} |g(x,y)|^2\,dxdy.$$

Proof. We have

$$\begin{aligned}
|Gu(x,t)|^2 &= \left|\int_{S^n} g(x,y)u(y,t)\,dy\right|^2 \leq \int_{S^n} |g(x,y)|^2\,dy \int_{S^n} |u(y,t)|^2 dy,\\
\|Gu(x,t)\|^2 &= \int_0^1\int_{S^n} |Gu(x,t)|^2 dxdt\\
&\leq \int_0^1\int_{S^n}\left(\int_{S^n} |g(x,y)|^2\,dy \int_{S^n} |u(y,t)|^2 dy\right)dx\,dt,\\
\|Gu(x,t)\|^2 &\leq \int_{S^n}\int_{S^n} |g(x,y)|^2 dxdy \int_{S^n}\int_0^1 |u(y,t)|^2 dy\,dt = M_0^2\|u\|^2,\\
\|G\| &\leq M_0.
\end{aligned}$$

The lemma is proved. □

We note that the Laplace operator is formally selfadjoint relative to the scalar product $(u,v) = \int_{S^n} u(x)\overline{v(x)}dx$ on the space $C^\infty(S^n)$. The product $\Delta_x \circ G = \Delta_x G$ coincides with the integral operator with the kernel $\Delta_x g(x,y)$. We put $M = \max\{\|\Delta_x G\|, \|G\|\}$.

Lemma 3. *Let $v = Gu = \sum v_{km}e_{km}$; then*

$$|v_{km}|^2 \leq \frac{4M^2}{(k(k+n-1)+1)^2}\|u\|^2.$$

Also, for $k \neq 0$ we have

$$|v_{km}|^2 \leq \frac{4|\alpha_{km}|^2}{(k(k+n-1)+1)^2}, \tag{6}$$

where $\alpha_{km} = \langle \Delta_x Gu, e_{km}\rangle$, and $\sum |\alpha_{km}|^2 \leq M^2\|u\|^2$.

Proof. The Parseval identity $\sum |v_{km}|^2 = \|Gu\|^2$ yields

$$|v_{0m}|^2 \le \|G\|^2\|u\|^2 \le 4M^2\|u\|^2.$$

Since the Laplace operator is selfadjoint, for $k \ne 0$ we have

$$\alpha_{km} = \langle \Delta_x Gu, e_{km}\rangle = \langle Gu, \Delta_x e_{km}(x,t)\rangle = \langle Gu, -k(k+n-1)e_{km}(x,t)\rangle,$$
$$\alpha_{km} = -k(k+n-1)\langle Gu, e_{km}(x,t)\rangle = -k(k+n-1)v_{km}.$$

It follows that

$$|v_{km}|^2 = \frac{|\alpha_{km}|^2}{(k(k+n-1))^2} \le \frac{4|\alpha_{km}|^2}{(k(k+n-1)+1)^2}.$$

By the Parseval identity, we have $\sum |\alpha_{km}|^2 = \|\Delta_x Gu\|^2 \le M^2\|u\|^2$, whence

$$|v_{km}|^2 \le \frac{4M^2\|u\|^2}{(k(k+n-1)+1)^2}.$$

The lemma is proved. □

We assume that a and λ are real numbers. Then, by Lemma 1, the spectrum $\sigma(L)$ lies on the real axis. Most typical and interesting is the case where the number $ab/(2\pi)$ is irrational. The H. Weyl theorem (see, e.g., [1]) says that, in this case, the set of the numbers λ_{km} is everywhere dense on $\mathbb{R}$ and $\sigma(L) = \mathbb{R}$. Now, suppose that $\lambda \ne \lambda_{km}$ for all $k, m \in \mathbb{Z}$, $k \ge 0$. Then the inverse operator $(L-\lambda)^{-1}$ is well defined, but unbounded. The expression for this inverse operator involves small denominators:

$$(L-\lambda)^{-1}v(x,t) = \sum \frac{v_{km}}{\lambda_{km}-\lambda}e_{km}, \tag{7}$$

where the v_{km} are the Fourier coefficients of the series

$$v(x,t) = \sum_{k,m\in\mathbb{Z},k\ge 0} v_{km}e_{km}.$$

For positive numbers σ and C, let $A_\sigma(C)$ denote the set of all positive b such that

$$|\lambda_{km} - \lambda| \ge \frac{C}{(k+1)^{1+\sigma}} \tag{8}$$

for all m, $k \in \mathbb{Z}$, $k \ge 0$.

This definition shows that the sets $A_\sigma(C)$ extend as C reduces and as σ grows. Therefore, in what follows, to prove that such a set or its part is nonempty, we require that C be sufficiently small and σ sufficiently large. Let A_σ denote the union of the sets $A_\sigma(C)$ over all $C > 0$.

If inequality (8) is fulfilled for some b and all m, k, then it is fulfilled for $m = 0$; this provides a condition necessary for the nonemptiness of $A_\sigma(C)$:

$$C \le (k+1)^{1+\sigma}|ak(k+n-1) - \lambda|, \quad \forall k \ge 0. \tag{9}$$

We put $d = \min_{k\in\mathbb{Z},k\ge 0}(k+1)^{1+\sigma}|ak(k+n-1)-\lambda| > 0$.

Theorem 1. *The sets $A_\sigma(C)$ and A_σ are Borel. The set A_σ has full measure, i.e., its complement to the half-line $\mathbb{R}^+$ is of zero measure.*

Proof. Obviously, the sets $A_\sigma(C)$ are closed in $\mathbb{R}^+$. The set $A_\sigma = \bigcup_{r=1}^\infty A_\sigma(1/r)$ is Borel, being a countable union of closed sets. We show that A_σ has full measure in $\mathbb{R}^+$. Suppose b, $l > 0$ and $C \le \frac{d}{2}$; we consider the complement $(0,l)\backslash A_\sigma(C)$. This set consists of all positive numbers b for which there exist m and k such that

$$|\lambda_{km} - \lambda| < \frac{C}{(k+1)^{1+\sigma}}. \tag{10}$$

Solving this inequality for b, we see that, for m, k fixed, the numbers b form an interval $I_{k,m} = (m\alpha_k, m\beta_k)$, where $m = 1, 2, 3, \ldots$,

$$\alpha_k = \frac{2\pi}{|ak(k+n-1)-\lambda| + \frac{C}{(k+1)^{1+\sigma}}},$$

$$\beta_k = \frac{2\pi}{|ak(k+n-1)-\lambda| - \frac{C}{(k+1)^{1+\sigma}}}.$$

The length of $I_{k,m}$ is $m\delta_k$, with

$$\delta_k = \frac{4\pi C(k+1)^{-1-\sigma}}{|ak(k+n-1)-\lambda|^2 - C^2(k+1)^{-2-2\sigma}}.$$

Since $C \leq \frac{d}{2}$ by assumption, we have

$$\delta_k \leq \frac{16\pi C}{3(k+1)^{1+\sigma}|ak(k+n-1)-\lambda|^2}. \tag{11}$$

For k fixed and m varying, there are only finitely many intervals $I_{k,m}$ that intersect the given segment $(0, l)$.

Such intervals arise for the values of $m = 1, 2, \ldots$ satisfying $m\alpha_k < l$, i.e.,

$$0 < m < \frac{l}{2\pi}(|ak(k+n-1)-\lambda| + C(k+1)^{-1-\sigma}).$$

Since $C(k+1)^{-1-\sigma} \leq \frac{1}{2}|ak(k+n-1)-\lambda|$, we can write simpler restrictions on m:

$$0 < m < \frac{l}{2\pi}\frac{3}{2}|ak(k+n-1)-\lambda| < \frac{l}{\pi}|ak(k+n-1)-\lambda|. \tag{12}$$

The measure of the intervals indicated (for k fixed) is dominated by $\delta_k\widetilde{S}_k$, where $\widetilde{S}_k = \widetilde{S}_k(l)$ is the sum of all integers m satisfying (12). Summing the arithmetic progression, we obtain

$$\widetilde{S}_k \leq \frac{l}{2\pi^2}|ak(k+n-1)-\lambda|\{l|ak(k+n-1)-\lambda| + \pi\}. \tag{13}$$

Passing to the union of the intervals in question over k and m and using (11), we see that

$$\mu((0,l)\backslash A_\sigma(C)) \leq \sum_{k=0}^{\infty} \delta_k\widetilde{S}_k \leq CS(l),$$

where

$$S = S(l) = \sum_{k=0}^{\infty} \frac{8l\{l|ak(k+n-1)-\lambda| + \pi\}}{3\pi(k+1)^{1+\sigma}|ak(k+n-1)-\lambda|}.$$

Observe that the quantity

$$\frac{l|ak(k+n-1)-\lambda| + \pi}{\pi|ak(k+n-1)-\lambda|}$$

is dominated by a constant D; therefore,

$$S(l) \leq \frac{8}{3}lD\sum_{k=0}^{\infty}\frac{1}{(k+1)^{1+\sigma}} < \infty.$$

We have

$$\mu((0,l) \setminus A_\sigma) \leq \mu((0,l) \setminus A_\sigma(C)) \leq CS(l)$$

for all $C > 0$. It follows that $\mu((0,l) \setminus A_\sigma) = 0$ for all $l > 0$. Thus, $\mu((0,\infty) \setminus A_\sigma) = 0$ and A_σ has full measure. The theorem is proved. □

Theorem 2. *Suppose $g(x,y)$ is a function defined on $S^n\times S^n$ and such that the function $\Delta_x g(x,y)$ is continuous on $S^n\times S^n$. Let $0<\sigma<1$, and let $b\in A_\sigma(C)$. Then the inverse operator $(L-\lambda)^{-1}$ is well defined, and the operator $(L-\lambda)^{-1}\circ G$ is compact.*

Proof. Since $b\in A_\sigma(C)$, we have $\lambda_{km}\neq\lambda$ for all $k,m\in\mathbb{Z}$, $k\geq 0$, so that $(L-\lambda)^{-1}$ is well defined and looks like the expression in (7). Observe that $\lim\frac{(k+1)^{2+2\sigma}}{(k(k+n-1)+1)^2}=0$ as $k\to\infty$. Therefore, given $\varepsilon>0$, we can find an integer $k_0>0$ such that

$$\frac{(k+1)^{2+2\sigma}}{(k(k+n-1)+1)^2}<\frac{(\varepsilon C)^2}{(2M)^2}$$

for all $k>k_0$.

We write

$$(L-\lambda)^{-1}v(x,t)=Q_{k_0 1}v+Q_{k_0 2}v,\quad v=Gu,$$

where

$$Q_{k_0 1}v=\sum_{0\leq k\leq k_0}\frac{v_{km}}{\lambda_{km}-\lambda}e_{km},\quad Q_{k_0 2}v=\sum_{k>k_0}\frac{v_{km}}{\lambda_{km}-\lambda}e_{km}.$$

For the operator $Q_{k_0 1}$ we have

$$\|Q_{k_0 1}v\|^2=\sum_{0\leq k\leq k_0}\frac{|v_{km}|^2}{|\lambda_{km}-\lambda|^2}.$$

Observe that if $0\leq k\leq k_0$, then

$$\lim\frac{1}{|\frac{2m\pi}{b}+ak(k+n-1)-\lambda|^2}=0$$

as $|m|\to\infty$. Therefore, the quantity $\frac{1}{|\frac{2m\pi}{b}+ak(k+n-1)-\lambda|^2}$ is dominated by a constant $C(k_0)$. Then

$$\|Q_{k_0 1}v\|^2\leq\sum|v_{km}|^2C(k_0)\leq C(k_0)\|v\|^2,$$

which means that $Q_{k_0 1}$ is a bounded operator.

Consider the operator $Q_{k_0 2}\circ G$. By 3 and Lemma (8), we have

$$\begin{aligned}\|Q_{k_0 2}v\|^2=\|Q_{k_0 2}\circ Gu\|^2&=\sum_{k>k_0}\frac{|v_{km}|^2}{|\lambda_{km}-\lambda|^2}\\&\leq\sum_{k>k_0}\frac{4\alpha_{km}^2}{(k(k+n-1)+1)^2}\Big(\frac{1}{C}\Big)^2(k+1)^{2+2\sigma}\\&\leq\Big(\frac{1}{C}\Big)^2\Big(\frac{\varepsilon C}{2M}\Big)^2\sum_{k>k_0}4|\alpha_{km}|^2\leq\varepsilon^2\|u\|^2.\end{aligned}$$

Consequently, $\|Q_{k_0 2}\circ G\|\leq\varepsilon$.

Since G is compact and $Q_{k_0 1}$ is bounded, $Q_{k_0 1}\circ G$ is compact. Next, we have

$$\|(L-\lambda)^{-1}\circ G-Q_{k_0 1}\circ G\|=\|Q_{k_0 2}\circ G\|<\varepsilon.$$

Thus, the operator $(L-\lambda)^{-1}\circ G$ is the limit of a sequence of compact operators. Therefore, it is compact itself. The theorem is proved. □

We denote $K=K_b=(L-\lambda)^{-1}\circ G$.

Theorem 3. *Suppose $b\in A_\sigma(C)$. Then problem* (1), (2) *admits a unique periodic solution with period b for all $\nu\in\mathbb{C}$ except, possibily, an at most countable discrete set of values of ν.*

Proof. Equation (1) reduces to

$$\Big((L-\lambda)^{-1}\circ G-\frac{1}{\nu}\Big)u=(L-\lambda)^{-1}\circ G(f).$$

We write $(L-\lambda)^{-1}\circ G-\frac{1}{\nu}=K-\frac{1}{\nu}$.

Since K is a compact operator, its spectrum $\sigma(K)$ is at most countable, and the limit point of $\sigma(K)$ (if any) can only be zero. Therefore, the set $S=\{\nu\neq 0\mid \frac{1}{\nu}\in\sigma(K)\}$ is at most countable and discrete, and if $\nu\neq 0$ and $\nu\notin S$, then the operator $(K-\frac{1}{\nu})$ is invertible, i.e., equation (1) is uniquely solvable. The theorem is proved. □

We pass to the question of the solvability of problem (1), (2) for fixed ν. We need to study the structure of the set $E\subset\mathbb{C}\times\mathbb{R}^+$ that consists of all pairs (ν,b) with $\nu\neq 0$ and $\frac{1}{\nu}\notin\sigma(K_b)$, where $K_b=(L-\lambda)^{-1}\circ G$.

Theorem 4. *E is a measurable set of full measure in $\mathbb{C}\times\mathbb{R}^+$.*

For the proof, we need several auxiliary statements.

Lemma 4. *For any $\varepsilon>0$, there exists an integer k_0 such that $\|K_b-\widetilde{K_b}\|<\varepsilon$ for all $b\in A_\sigma(\frac{1}{r})$, $0<\sigma<1$, where $r=1,2,\dots$,*

$$K_bu=(L_b-\lambda)^{-1}v=\sum\frac{v_{km}}{\lambda_{km}(b)-\lambda}e_{km},\quad \widetilde{K_b}u=\sum_{0\le k\le k_0}\frac{v_{km}}{\lambda_{km}(b)-\lambda}e_{km}.$$

Proof. Observe that for any $\varepsilon>0$ there is an integer k_0 such that

$$\frac{(k+1)^{2+2\sigma}}{(k(k+n-1)+1)^2}\le\Big(\frac{\varepsilon}{2rM}\Big)^2$$

for all $k\ge k_0$, $0<\sigma<1$. We have

$$(K_b-\widetilde{K_b})u=K_{k_0b}u=\sum_{k>k_0}\frac{v_{km}}{\lambda_{km}(b)-\lambda}e_{km},$$

$$\begin{aligned}\|(K_b-\widetilde{K_b})u\|^2=\|K_{k_0b}u\|^2&=\sum_{k>k_0}\Big|\frac{v_{km}}{\lambda_{km}(b)-\lambda}\Big|^2\\ &\le\sum_{k>k_0}\frac{4r^2\alpha_{km}^2(k+1)^{2+2\sigma}}{(k(k+n-1)+1)^2}\\ &\le r^2\Big(\frac{\varepsilon}{2rM}\Big)^2 4\sum_{k>k_0}|\alpha_{km}|^2\\ &\le r^2\Big(\frac{\varepsilon}{2rM}\Big)^2 4M^2\|u\|^2=\varepsilon^2\|u\|^2.\end{aligned}$$

Thus, $\|K_b-\widetilde{K_b}\|=\|K_{k_0b}\|<\varepsilon$, as required. □

Lemma 5. *The operator-valued function $b\mapsto K_b$ is continuous for $b\in A_\sigma(\frac{1}{r})$.*

Proof. Suppose $b,b+\Delta b\in A_\sigma(\frac{1}{r})$ and $\varepsilon>0$. By Lemma 4, for some k_0 (independent of b and $b+\Delta b$), we have $\|K_b-\widetilde{K_b}\|=\|K_{k_0b}\|<\varepsilon$ and $\|K_{b+\Delta b}-\widetilde{K_{b+\Delta b}}\|=\|K_{k_0(b+\Delta b)}\|<\varepsilon$. Next,

$$K_{b+\Delta b}-K_b=(\widetilde{K_{b+\Delta b}}+K_{k_0(b+\Delta b)})-(\widetilde{K_b}+K_{k_0b}),$$

whence we obtain

$$\|K_{b+\Delta b}-K_b\|\le\|\widetilde{K_{b+\Delta b}}-\widetilde{K_b}\|+\|K_{k_0(b+\Delta b)}\|+\|K_{k_0b}\|.$$

Consider the operators $\widetilde{K_{b+\Delta b}}$ and $\widetilde{K_b}$. We have

$$(\widetilde{K_{b+\Delta b}} - \widetilde{K_b})u = \sum_{0\le k\le k_0}\Big(\frac{1}{\lambda_{km}(b+\Delta b)-\lambda} - \frac{1}{\lambda_{km}(b)-\lambda}\Big)v_{km}e_{km},$$

$$\|\widetilde{K_b}u - \widetilde{K_{b+\Delta b}}u\|^2 = \frac{|\Delta b|^2}{|b(b+\Delta b)|^2}\sum_{0\le k\le k_0}\frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b)-\lambda|^2}\frac{4m^2\pi^2}{|\lambda_{km}(b)-\lambda|^2}. \tag{14}$$

If $b+\Delta b\in A_\sigma(\frac{1}{r})$, $0\le k\le k_0$, and $0<\sigma<1$, then

$$\frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b)-\lambda|^2}\le |v_{km}|^2r^2(k+1)^{2+2\sigma}\le r^2(k_0+1)^4|v_{km}|^2.$$

The relation $\lim_{m\to\infty}\frac{4m^2\pi^2}{|\lambda_{km}(b)-\lambda|^2}=b^2$ and the condition $0\le k\le k_0$ imply that the quantity

$$\frac{4m^2\pi^2}{|\lambda_{km}(b)-\lambda|^2|} = \frac{4m^2\pi^2}{\big|\frac{2m\pi}{b}+ak(k+n-1)-\lambda|^2}$$

is dominated by a constant $C(k_0)$ depending on k_0. Therefore,

$$\begin{aligned}\frac{|\Delta b|^2}{|b(b+\Delta b)|^2}&\sum_{0<k\le k_0}\frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b)-\lambda|^2}\frac{4m^2\pi^2}{|\lambda_{km}(b)-\lambda|^2}\\ &\le \frac{|\Delta b|^2}{|b(b+\Delta b)|^2}\sum_{0\le k\le k_0}r^2(k_0+1)^4C(k_0)|v_{km}|^2\\ &\le \frac{|\Delta b|^2}{|b(b+\Delta b)|^2}r^2(k_0+1)^4C(k_0)\sum_{0\le k\le k_0}|v_{km}|^2.\end{aligned}$$

Since

$$\sum_{0\le k\le k_0}|v_{km}|^2\le\|v\|^2\le M^2\|u\|^2,$$

we arrive at the estimate

$$\|\widetilde{K_{b+\Delta b}}-\widetilde{K_b}\|^2\le\frac{|\Delta b|^2}{|b(b+\Delta b)|^2}M^2r^2(k_0+1)^4C(k_0).$$

We choose Δb so as to satisfy the condition

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2}M^2r^2(k_0+1)^4C(k_0)<\varepsilon.$$

Then $\|K_{b+\Delta b}-K_b\|<3\varepsilon$. This shows that the operator-valued function $b\mapsto K_b$ is continuous on $A_\sigma(\frac{1}{r})$. The lemma is proved. □

Lemma 6. *The spectrum $\sigma(K)$ of the compact operator K depends continuously on K in the space* Comp($\mathcal{H}$) *of compact operators on $\mathcal{H}$, in the sense that for any $\varepsilon>0$ there exists $\delta>0$ such that for all compact* (*and even bounded*) *operators B with $\|B-K\|<\delta$ we have*

$$\sigma(B)\subset\sigma(K)+V_\varepsilon(0),\quad \sigma(K)\subset\sigma(B)+V_\varepsilon(0). \tag{15}$$

Here $V_\varepsilon(0)=\{\lambda\in\mathbb{C}\mid|\lambda|<\varepsilon\}$ is the ε-neighborhood of the point 0 *in* $\mathbb{C}$.

Proof. Let K be a compact operator; we fix $\varepsilon>0$. The structure of the spectrum of a compact operator shows that there exists $\varepsilon_1<\varepsilon/2$ such that $\varepsilon_1\ne|\lambda|$ for all $\lambda\in\sigma(K)$. Let $S=\{\lambda_1,\dots,\lambda_k\}$ be the set of all spectrum points λ with $|\lambda|>\varepsilon_1$, and let $V=\bigcup_{\lambda\in S\cup\{0\}}V_{\varepsilon_1}(\lambda)$. Then V is a neighborhood of $\sigma(K)$, and $V\subset\sigma(K)+V_\varepsilon(0)$. By the well-known property of spectra (see, e.g., [2, Theorem 10.20]), there exists $\delta>0$

such that $\sigma(B) \subset V$ for any bounded operator B with $\|B - K\| < \delta$. Moreover (see, e.g., [2, p. 293, Exercise 20]), the number $\delta > 0$ can be chosen so that $\sigma(B) \cap V_{\varepsilon_1}(\lambda) \neq \varnothing$ for all $\lambda \in S \cup \{0\}$. Then for all bounded operators B with $\|B - K\| < \delta$ the required inclusions $\sigma(K) \subset \sigma(B) + V_{2\varepsilon_1}(0) \subset \sigma(B) + V_\varepsilon(0)$ and $\sigma(B) \subset V \subset \sigma(K) + V_\varepsilon(0)$ are fulfilled. The lemma is proved. □

It is easy to deduce the following statement from Lemma 6.

Proposition 1. *The function* $\rho(\lambda, K) = \operatorname{dist}(\lambda, \sigma(K))$ *is continuous on* $\mathbb{C} \times \operatorname{Comp}(\mathcal{H})$.

Proof. Suppose $\lambda \in \mathbb{C}$, $K \in \operatorname{Comp}(\mathcal{H})$, and $\varepsilon > 0$. By Lemma 6, there exists $\delta > 0$ such that for any operator H lying in the δ-neighborhood of K, $\|H - K\| < \delta$, the inclusions (15) are fulfilled; these inclusions directly imply the estimate $|\rho(\lambda, K) - \rho(\lambda, H)| < \varepsilon$. Then for all $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \varepsilon$ and all H with $\|H - K\| < \delta$ we have

$$|\rho(\mu, K) - \rho(\lambda, H)| \leq |\rho(\mu, K) - \rho(\lambda, K)| + |\rho(\lambda, K) - \rho(\lambda, H)| < |\mu - \lambda| + \varepsilon < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the function $\rho(\lambda, K)$ is continuous. The proposition is proved. □

Combining Proposition 5 and Lemma 5, we obtain the following fact.

Corollary 1. *The function* $\rho(\lambda, b) = \operatorname{dist}(\lambda, \sigma(K_b))$ *is continuous on* $(\lambda, b) \in \mathbb{C} \times A_\sigma(\frac{1}{r})$.

Now we are ready to prove Theorem 4.

Proof of Theorem 4. By Corollary 1, the function $\rho(1/\nu, b)$ is continuous with respect to the variables $(\nu, b) \in (\mathbb{C} \setminus \{0\}) \times A_\sigma(\frac{1}{r})$. Consequently, the set

$$B_r = \left\{(\nu, b) \mid \rho(1/\nu, b) \neq 0, \;\; b \in A_\sigma\left(\frac{1}{r}\right)\right\}$$

is measurable, and so is the set $B = \bigcup_r B_r$. Clearly, $B \subset E$ and $E = B \cup B_0$, where $B_0 = E \setminus B$. Obviously, B_0 lies in the set $\mathbb{C} \times (\mathbb{R}^+ \setminus A_\sigma)$ of zero measure (recall that, by Theorem 1, A_σ has full measure in $\mathbb{R}^+$). Since the Lebesgue measure is complete, B_0 is measurable. Thus, the set E is measurable, being the union of two measurable sets. Next, by Theorem 3, for $b \in A_\sigma$ the section $E^b = \{\nu \in \mathbb{C} \mid (\nu, b) \in E\}$ has full measure, because its complement $\{1/\nu \mid \nu \in \sigma(K_b)\}$ is at most countable. Therefore, the set E is of full plane Lebesgue measure. The theorem is proved. □

The following important statement is a consequence of Theorem 4.

Corollary 2. *For a.e.* $\nu \in \mathbb{C}$, *problem* (1), (2) *has a unique periodic solution with almost every period* $b \in \mathbb{R}^+$.

Proof. Since the set E is measurable and has full measure, for a.e. $\nu \in \mathbb{C}$ the section $E_\nu = \{b \in \mathbb{R}^+ \mid (\nu, b) \in E\} = \{b \in \mathbb{R}^+ \mid 1/\nu \notin \sigma(K_b)\}$ has full measure, and for such b's, problem (1), (2) has a unique periodic solution with period b. The corollary is proved. □

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