CORRECTIONS TO “ON THE RIEMANN–HILBERT–BIRKHOFF INVERSE MONODROMY PROBLEM AND THE PAINLEVÉ EQUATIONS”

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In our joint paper with A. A. Bolibrukh \[1\] there is a technical mistake in the proof of Lemma 2. The lemma itself is correct, while its proof should be modified. First, in the definition of the norm of the space $H_+$ on page 142 [English page 124], $|\lambda f(\lambda, x)|$ should be replaced with $|(1 + |\lambda|) f(\lambda, x)|$. Second, the part of the text containing Proposition 1 and its proof should be altered as follows.

**Proposition 1.** The operators $P_{\pm} \left[ \hat{g} \right]$, where $\hat{g}(\lambda, x) \equiv g(\lambda, x) - I$, are bounded operators from $H$ to $H_{\pm}$.

**Proof of the proposition** (an adaptation of the technique of \[2\]). Each $\omega_k$ can be represented as the union of the domains $\omega_k^+$ and $\omega_k^-$, in accordance with Figure 6. Assume that $\lambda \in \Omega_+ \setminus \bigcup_k \omega_k^+$; then

$$|\lambda|_{\text{dist}} \{\lambda, \gamma_+\} \leq C_\epsilon$$

for some positive constant $C_\epsilon$. We have

$$\lambda P_+ \left[ f \hat{g} \right](\lambda, x) = \frac{\lambda}{2\pi i} \int_{\gamma_+} \frac{f(\mu, x) \hat{g}(\mu, x)}{\mu - \lambda} \, d\mu,$$

$$|\lambda P_+ \left[ f \hat{g} \right](\lambda, x)| \leq \frac{|\lambda|}{2\pi} \|f\|_H \|\hat{g}\|_H \int_{\gamma_+} \frac{|d\mu|}{|\mu|^2 |\mu - \lambda|} \leq \frac{C_\epsilon}{2\pi} \|f\|_H \|\hat{g}\|_H \int_{\gamma_+} \frac{|d\mu|}{|\mu|^2} \equiv C_+ \cdot \|f\|_H.$$

Similarly, if $\lambda \in \Omega_- \setminus \bigcup_k \omega_k^-$, then

$$|\lambda P_- \left[ f \hat{g} \right](\lambda, x)| \leq C_- \cdot \|f\|_H.$$

Suppose now that $\lambda \in \omega_k^+$. Then we can use identity (53) and rewrite $\lambda P_+ \left[ f \hat{g} \right](\lambda, x)$ as

$$\lambda P_+ \left[ f \hat{g} \right](\lambda, x) = \lambda f(\lambda, x) \hat{g}(\lambda, x) - \lambda P_- \left[ f \hat{g} \right](\lambda, x).$$

On the other hand, $\lambda \in \omega_k^+ \Rightarrow \lambda \in \Omega_- \setminus \bigcup_k \omega_k^-$, so that we can use (54) on the right hand side of (55). Therefore (cf. \[2\] and the proof of Lemma A.1 in \[1\] Appendix 1), we obtain

$$|\lambda P_+ \left[ f \hat{g} \right](\lambda, x)| \leq C'_+ \cdot \|f\|_H$$

if $\lambda \in \omega_k^+$. In other words, we have the inequality

$$|\lambda P_+ \left[ f \hat{g} \right](\lambda, x)| \leq C'' \cdot \|f\|_H, \quad C'' := \max\{C_+, C'_+\},$$
for all \((\lambda, x) \in \overline{\Omega_+} \times \mathcal{K}\). Together with the estimate
\[
\left| \left( P_\pm f \right)(\lambda, x) \right| \leq \frac{1}{2\pi} \| f \|_H \| \hat{g} \|_H \int_{\gamma_-} \frac{|d\mu|}{|\mu|^2|\mu - \lambda|}
\leq \frac{1}{2\pi} \text{dist}\{\lambda, \gamma_+\} \| f \|_H \| \hat{g} \|_H \int_{\gamma_-} \frac{|d\mu|}{|\mu|^2} \equiv \tilde{C}_+ \cdot \| f \|_H, \quad |\lambda| \leq \frac{\rho}{2},
\]
this implies
\[
\| P_+ \left[ f \hat{g} \right] \|_H \leq C \cdot \| f \|_H, \quad C := \max\{C', \tilde{C}_+\}.
\]
Similarly,
\[
\| P_- \left[ f \hat{g} \right] \|_H \leq C' \cdot \| f \|_H
\]
and the proposition is proved. \qed

The rest of the proof of Lemma 2 is unchanged. It is though worth noticing that
\( P_+ \hat{g} \in H_+ \). This fact follows from the existence of the asymptotic expansion of the function \( \hat{g}(\lambda, x) \) over \( \lambda^{-1} \) as \( \lambda \to \infty, \lambda \in \Omega_0 \).

We have also noticed the following misprints in the text of the paper.

(1) In the definition of \( M(\lambda, x) \) on the top of page 138 [English page 121], the matrix product \( S_{k-1} \cdots S_1 \) should be replaced by \( S_{k-1}^{-1} \cdots S_1^{-1} \).
(2) In Figure 3, the shaded domain includes the circular hole in the middle of the picture (there should be no hole).
(3) Equation (48) on page 141 [English page 123] is, in fact, the equation for the matrix \( g^{-1}(\lambda, x) \). Accordingly, in equation (49) on same page, the matrix \( G_k(\lambda, x) \) should be replaced by its inverse.
(4) In Figure 7, the symbol \( g_k^{-1}(\lambda, x) \) on the right part of the picture should be replaced by \( g_{k-1}^{-1}(\lambda, x) \).

References
