

## SOME FUNCTIONAL-DIFFERENCE EQUATIONS SOLVABLE IN FINITARY FUNCTIONS

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*Dedicated to the 100th anniversary of B. Ya. Levin's birth*

ABSTRACT. The following equation is considered:  $q(-i\partial/\partial x)u(x) = (f * u)(Ax)$ , where  $q$  is a polynomial with complex coefficients,  $f$  is a compactly supported distribution, and  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a linear operator whose complexification has no spectrum in the closed unit disk. It turns out that this equation has a (smooth) solution  $u(x)$  with compact support. In the one-dimensional case, this problem was treated earlier in detail by V. A. Rvachev and V. L. Rvachev and their numerous students.

### 1. Equations of the form

$$(1) \quad \alpha_0 u(x) + \alpha_1 u'(x) + \cdots + \alpha_k u^{(k)}(x) \\ = \beta_0 u(ax - b_0) + \beta_1 u(ax - b_1) + \cdots + \beta_l u(ax - b_l)$$

arise in probability theory and in some sections of physics. Here  $u = u(x)$  is an unknown function of the real variable  $x$ , the numbers  $k$  and  $l$  are assumed usually to be natural,<sup>1</sup>  $a$  and  $b_r$  are real numbers, and  $\alpha_r, \beta_r$  may be complex.

In particular, we mention the Ambartsumyan equation in astrophysics, which can be written as follows:

$$u(x) + u'(x) = au(ax)$$

(see, e.g., Minin's article in [1, p. 346]).

If  $a > 1$  and the other parameters are chosen duly, equation (1) may have a nontrivial finitary (i.e., compactly supported) solution. Sometimes, such solutions are called *atomic functions*.

A systematic treatment of such functions was started in the 1970s by V. A. Rvachev and V. L. Rvachev (and their numerous colleagues). To a certain extent, these studies were summed up in Chapter 3 of the monograph [2]. In particular, a fairly simple criterion for the solvability of equation (1) in finitary functions can be found there. In this Introduction we do not repeat detailed statements because more general results will be presented in the sequel; however, below we describe a nice and fundamental example. A survey of subsequent results together with an extensive list of problems can be found in [3, Chapter 2] and in [4]. It should be noted that the study of the subject in question continues nowadays in a variety of fashions (see, e.g., [5]), but multidimensional examples (below we shall explain what is meant) are rare; in particular, nobody seems to have rediscovered the results listed in the author's note [6].

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<sup>1</sup>We want to indicate that the case where one of them equals 0 is not senseless (nor the case where both are zero).

The example mentioned above looks like this. Consider the equation

$$(2) \quad u'(x) = \lambda \cdot (u(ax + 1) - u(ax - 1)),$$

where  $a > 1$  is a fixed number and  $\lambda$  is an additional parameter. It is required to learn whether equation (2) has finitary solutions and how many, and what “characteristic properties” of these solutions should be expected.

It turns out that finitary solutions exist if and only if  $\lambda = a^2/2$ , the space of such solutions is one-dimensional, and precisely one among them satisfies  $\int_{\mathbb{R}} u(x) dx = 1$ . This solution is supported on the segment with the endpoints  $\pm(a - 1)^{-1}$ . In particular, for  $a = 2$  an ideal “cap” arises, and for  $a = 3$  we obtain a  $C^\infty$ -function that coincides with a polynomial of degree  $r - 1$  on each rank  $r$  contiguous interval of the classical Cantor set placed to be symmetric with respect to the point 0. Surely, here and below  $\mathbb{R}$  is the field of reals (and  $\mathbb{C}$  is the complex field).

By using the Fourier transformation, it is easy to show that these functions are infinite convolutions similar to those occurring quite often in probability theory.

**2.** In this subsection, we fix the main notation, introduce the principal notions, and describe the object to be dealt with (equation (1) is included as quite a particular case).

Parentheses will be employed for various purposes (I strongly believe this will not lead to confusion). In particular, if  $z, w \in \mathbb{C}^n$ , then  $(z, w)$  is the usual scalar product of  $z$  and  $w$  (in terms of Cartesian coordinates).

The symbol  $L^p(\mathbb{R}^n)$ ,  $p > 0$ , stands for the standard Lebesgue space (as usual, we do not distinguish between functions and equivalence classes).

For a function  $f \in L^1(\mathbb{R}^n)$ , its Fourier transform is a function of  $\xi \in \mathbb{R}^n$  (the dual copy of the Euclidean space) defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{i(\xi, x)} f(x) dx.$$

If  $f$  is continuous and  $\widehat{f} \in L^1(\mathbb{R}^n)$ , we have the inversion formula

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(x, \xi)} \widehat{f}(\xi) d\xi.$$

We work with *tempered* distributions, i.e., continuous linear functionals on the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  of rapidly decaying functions  $\varphi$ . The value of a functional  $f$  at a test function  $\varphi$  is denoted by  $(f, \varphi)$  or even by  $(f(x), \varphi(x))$ . The latter notation may horrify a consistent formalizer. It is used not because it is closer to physical origins but because the nature of the problems in question often implies an essential dependence of  $f$  and  $\varphi$  on some supplementary parameters.

If  $g$  is the Fourier transform of a distribution  $f$ , then

$$(2\pi)^n(g, \psi) = (f, \varphi),$$

where  $\psi = \widehat{\varphi}$ .

If  $f$  is a distribution with compact support, it is of finite order and extends uniquely up to a continuous functional on the space  $\mathcal{E}(\mathbb{R}^n)$  of all infinitely differentiable functions (endowed with the topology of uniform convergence for functions and all derivatives on compact sets). In particular, the Fourier transforms of distributions of this (and only of this) class are identifiable with entire functions of exponential type and of power growth on  $\mathbb{R}^n$ , and for all  $z \in \mathbb{C}^n$  such an entire function  $g = g(z)$  admits an “explicit” representation:

$$g(z) = (f(x), e^{i(z, x)}).$$

The left-hand side of (1) can be written in the form  $(f * u)(ax)$ , where  $f$  is a linear combination of shifts of the Dirac  $\delta$ -function:

$$f(x) = \beta_1 \delta(x - b_1) + \beta_2 \delta(x - b_2) + \cdots + \beta_l \delta(x - b_l).$$

Our aim in this paper is to learn whether the equation

$$(ME) \quad q(-i\partial/\partial x)u(x) = (f * u)(Ax),$$

is solvable in finitary functions. Here  $x \in \mathbb{R}^n$ ,  $q = q(z)$  is a polynomial in  $z \in \mathbb{C}^n$ ,  $f$  is a compactly supported distribution, and  $A$  is a linear transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  such that the spectrum of its natural complexification lies entirely *out of the disk*  $|\lambda| \leq 1$ .

Thus, even for  $n = 1$ , the equation we consider is more general than equation (1) for atomic functions.

Certainly, the Fourier transformation suggests itself for the treatment of the problem in question; then the matters reduce to polynomials and entire functions of exponential type. If  $n > 1$ , the issue is complicated (i.e., becomes more interesting), in particular, because only for  $n = 1$  do the irreducible polynomials have degree 1. Next, only for  $n = 1$  are the zeros of analytic functions isolated and can be described locally in terms of usual (rather than Weierstrass) polynomials: it suffices to consider the functions  $z_1 + \sin z_2$ .

From this viewpoint, the following conditions (*necessary*, as we shall see, for the solvability of (ME) in finitary functions) are substantial for  $n > 1$ , whereas their one-dimensional analogs are trivial. If (ME) has a finitary solution and  $\hat{f}(0) = 0$ , then  $\hat{f} = g_0 \cdot g_1$ , where  $g_0$  is a polynomial and  $g_1$  is an entire function with  $g_1(0) \neq 0$ . The Fourier transform of the solution has a similar property. If, moreover,  $f$  is a linear combination of derivatives of shifts of the  $\delta$ -function, then the primitive divisors of the polynomial  $q$  are of degree 1.

Surely, we do not restrict ourselves to necessary conditions. Besides discussing the existence of finitary solutions, we show that they form a finite-dimensional space and examine their smoothness. This question did not arise in the work of V. A. Rvachev and V. L. Rvachev on atomic functions, because these authors assumed tacitly that  $\deg(q) \geq 1$ . However, the Dirac  $\delta$ -function is a solution of the equation  $u(x) = 2u(2x)$  (and the other finitary solutions are multiples of this one). A more substantial example is given by the equation

$$u(x) = u(2x + 1) + u(2x - 1),$$

to be treated in detail below.

Finally, it should be noted that the referee (I am grateful to him or her very much) indicated a series of misprints (they are corrected) and gave an extensive commentary on the history and the development of the issue, some recent studies, and bibliography. I put the references suggested by the referee at the end of the citation list. Moreover, I have decided to reproduce the referee's commentary almost entirely.

**Along the lines of the referee's remarks.** The case where the left-hand side of equation (ME) involves no (nontrivial) differential operator has been studied vastly ... In the first place, we mean scaling or refinement equations involved in the construction of wavelets and frames with compact support, as well as in the treatment of subdivision algorithms in approximation theory and surface design. For such equations (with  $q = 1$ ),  $f$  is a linear combination of shifts of the  $\delta$ -function in  $\mathbb{R}^n$ , and the operator  $A$  is defined via an integral matrix. Refinement equations have been studied since the 1990s, and now the relevant bibliography counts hundreds of sources, of which [21] (the one-dimensional case) and [22] (the multidimensional case) are worthy of immediate notice. The discussion of some important particular cases in [23] was motivated by a suggestion by V. M. Borok and Ya. I. Zhitomirskii. Equations of the form  $2u(x) = \lambda(u(\lambda x - 1) + u(\lambda x + 1))$ ,  $x \in \mathbb{R}$ , have

been studied in connection with the Erdős problem pertaining to Bernoulli convolutions (see, in particular, [24]). Finally, in [25] the equation  $d \cdot u(x) = 2 \sum_{k=0}^d u(2x - k)$  was applied to number theory for the study of the asymptotics for the Euler binary splitting function.

§1. PRELIMINARY INFORMATION

1. For  $n > 1$ , there are many “natural” equivalent norms in  $\mathbb{C}^n$ , which makes it impossible to define an entire function of exponential type in a natural way. However, the character of the problem enables us to fix any norm. In what follows, we tacitly assume that  $\mathbb{C}^n$  is endowed with the *Euclidean norm* and the entire functions of exponential type are defined accordingly; however, sometimes we use some other norm better adjusted to the operator  $A$ .

We retain the notation introduced above. Putting  $B = (A^*)^{-1}$  and using the Gelfand formula and our assumptions about  $A$ , we see that  $\lim_{m \rightarrow \infty} \|B^m\|^{1/m} < 1$ .

The Fourier transformation reduces (ME) to the equation

$$(FE) \quad q(z)v(z) = g(z)v(Bz),$$

where  $v(z) = \widehat{u}(z)$  and  $g(z) = \det(B) \cdot \widehat{f}(Bz)$ .

General arguments imply that *equation (ME) is solvable in finitary functions (which may happen to be distributions) if and only if (FE) is solvable in functions  $v$  of exponential type and of power-like growth on the real part of  $\mathbb{C}^n$ .*

At first glance, equation (FE) seems to be trivial. Indeed, if we assume (to begin with) that  $v(0) = 1$ , then the “explicit representation”

$$v(z) = \prod_{k=0}^{\infty} g(B^k z)/q(B^k z)$$

follows; it remains to understand this formula in precise terms and to find out when the resulting function  $v$  possesses the properties mentioned above. Surely, this phrase communicates something useful, but not too specific. So, we must analyze the situation more carefully.

2. We shall need several fairly simple facts of multidimensional complex analysis. All of them are classical and can be uncovered (in an explicit or almost explicit form) in monographs and textbooks on the theory of analytic functions of several variables. However, for references, we state some of these facts as lemmas and supply them with drafts of the proofs. The details can be found, e.g., in [7]–[9] and [10].

If  $h$  is an analytic function (in some domain), we denote by  $V_h$  the hypersurface  $\{z \mid h(z) = 0\}$ .

**Lemma 1.** *Suppose  $g$  is an entire function and  $p$  is an irreducible polynomial. If  $g|_{V_p} = 0$ , then  $p \mid g$ , i.e.,  $g = ph$ , where  $h$  is an entire function.*

**A sketch of the proof.** The function  $g/p$  is analytic outside of the hypersurface  $V_p$ . If  $z_0 \in V_p$  is a regular point, then near  $z_0$  the “division problem” reduces to a one-dimensional problem; therefore  $g/p$  extends up to an analytic function in a neighborhood of  $x_0$ . The remaining points of  $V_p$  constitute an algebraic set of complex codimension at least 2 in  $\mathbb{C}^n$ ; consequently, at such points only reducible singularities may occur.  $\square$

We start the next lemma with a brief description of an approach to its proof. The main tool is the Weierstrass preparation theorem. That theorem makes it possible to show (not immediately, alas!) that the ring of germs of functions analytic in a neighborhood of a given point is Noetherian and factorial. In the long run, this leads to the understanding of the fact that the local structure of analytic sets (in particular, of hypersurfaces) is not

much more complicated than that of algebraic sets (i.e., it is fairly simple). V. Ya. Levin called my attention to a general statement in [10, p. 49], which, apparently, is the most appropriate for deducing the lemma below.

**Lemma 2.** *Suppose  $g$  is a function analytic in a neighborhood of 0 and  $g(0) = 0$ . Then there exists a neighborhood  $U$  of 0 and a finite collection  $\{W_1, W_2, \dots, W_r\}$  of (connected) analytic submanifolds of (complex) dimension  $n - 1$  such that*

$$\{z \in U \mid g(z) = 0\} = \overline{W}_1 \cup \overline{W}_2 \cup \dots \cup \overline{W}_r$$

(the bar stands for closure),  $0 \in \overline{W}_k$ ,  $\dim(\overline{W}_k \setminus W_k) \leq 2(n - 2)$  for every  $k$ , and  $\dim(\overline{W}_l \cap \overline{W}_k) \leq 2(n - 2)$  for  $l \neq k$ .

**Lemma 3.** *Suppose  $g$  is a function analytic in a neighborhood of 0 and  $g(0) = 0$ . Then there exists a neighborhood  $U$  of 0 such that if  $h$  is analytic in  $U$  and*

$$\emptyset \neq \{z \in U \mid h(z) = 0\} \subset \{z \in U \mid g(z) = 0\},$$

then  $h(0) = 0$ .

**A sketch of the proof.** Let  $\{W_k\}$  be a system of manifolds that is provided by Lemma 2 for  $g$ . Lemma 2 is also applicable to  $h$ , whence it follows easily that  $h(z) = 0$  on an open subset of some  $W_k$ . Then  $h|_{W_k} = 0$ , and therefore  $h(0) = 0$ .  $\square$

**Lemma 4.** *Let  $g$  be a nontrivial entire function,  $Q$  a compact subset of  $\mathbb{C}^n$ , and  $\{p\}$  a family of pairwise nonequivalent irreducible polynomials. Then*

$$\sum_p \{\nu_p(g) \mid Q \cap V_p \neq \emptyset\} < \infty.$$

**A sketch of the proof.** Suppose the contrary. Since  $\nu_p(g) < \infty$  for every  $p$ , we may assume the existence of an infinite sequence  $\{p_i\}$  of pairwise nonequivalent irreducible polynomials such that  $g|_{V_{p_i}} = 0$  and  $Q \cap V_{p_i} \neq \emptyset$ . There is no loss of generality in assuming that some sequence of points  $z_i \in V_{p_i} \cap Q$  converges to 0 and  $g(0) = 0$ . By Lemma 3, we may also assume that  $p_i(0) = 0$  for all members of the sequence. Let  $z_0 \in \mathbb{C}^n$  be a point such that  $g(z_0) \neq 0$ . Consider the function  $t \mapsto g(tz_0)$ . This nontrivial entire function has a zero of infinite order at  $t = 0$ , a contradiction.  $\square$

The next lemma (which can be made local) follows by a simple combination of Lemmas 1 and 2.

**Lemma 5.** *Suppose  $g$  is an entire function with  $g(0) = 0$ . Then either  $g$  is not divisible by any polynomial  $p$  with  $p(0) = 0$ , or  $g = g_0g_1$ , where  $g_0$  is a polynomial all irreducible divisors  $p$  of which satisfy  $p(0) = 0$ , and  $g_1$  is an entire function not divisible by any polynomial  $p$  with  $p(0) = 0$ . The factors in this representation are determined uniquely modulo a multiplicative constant.*

**3.** In the sequel, we fix the linear transformation  $B$  discussed above (so that the spectrum of  $B$  lies in the punctured disk  $0 < |\lambda| < 1$ ).

If  $p$  is an irreducible polynomial, we put  $p^{(k)}(z) = p(B^{-k}z)$ ,  $k \in \mathbb{Z}$  (here and below  $\mathbb{Z}$  is the group of integers). Clearly,  $V_{p^{(k)}} = B^kV_p$ .

If  $g$  is an entire function, we put  $\nu_p^{(k)}(g) \stackrel{\text{def}}{=} \nu_{p^{(k)}}(g)$ . Clearly, for  $h(z) = g(Bz)$  we have  $\nu_p^{(k)}(h) = \nu_p^{(k+1)}(g)$ .

We return to equation (FE). Since  $\nu_p(g_1g_2) = \nu_p(g_1) + \nu_p(g_2)$ , it follows that if (FE) is solvable in entire functions, then for all irreducible polynomials  $p$  the system of equations

$$(DE) \quad \nu_p^{(k)}(q) + \nu_p^{(k)}(v) = \nu_p^{(k)}(g) + \nu_p^{(k+1)}(v), \quad k \in \mathbb{Z},$$

is solvable for the unknowns  $\nu_p^{(k)}(v)$  in nonnegative integers.

The mapping  $B^{-1}$  takes the zero sets of irreducible polynomials to sets of the same kind. A polynomial  $p$  is said to be  $B$ -periodic with  $B$ -period equal to a positive integer  $r > 0$  if  $B^{-r}V_p = V_p$ , and no similar identity is available for  $0 < m < r$ . For example, for  $n = 1$  and  $p(z) = z$  we have  $r = 1$ , and there is no periodicity for  $p(z) = z - c$ , where  $c \neq 0$  (because  $B$  reduces to a number in the interval  $(-1, 1)$ ). The next lemma follows immediately from the definitions.

**Lemma 6.** *If an (irreducible) polynomial  $p$  is  $B$ -periodic with period  $r$ , then  $p(0) = 0$ . Moreover, for every (nontrivial) entire function  $g$ , the function  $k \rightarrow \nu_p^{(k)}(g)$  is periodic with the same period  $r$ .*

Finally, the following lemma is a consequence of Lemma 4.

**Lemma 7.** *Suppose an (irreducible) polynomial  $p$  is not  $B$ -periodic, and let  $g$  be an entire function. Then (a) if  $p(0) \neq 0$ , then  $\nu_p^{(k)}(g) = 0$  for  $k \geq k_0 = k_0(p, g)$ ; (b) if  $p(0) = 0$ , then  $\nu_p^{(k)}(g) = 0$  for  $|k| \geq k_1 = k_1(p, g)$ .*

Clearly, for  $n = 1$ , we have  $r = 1$  forcedly in the periodic case, and we have  $p(0) \neq 0$  forcedly in the nonperiodic case.

4. The system (DE) will be called periodic, semiinfinite, or finitary in accordance with Lemmas 6 and 7. A similar terminology will be applied in the general case of systems of the form

$$(3) \quad \alpha_k + \xi_k = \beta_k + \xi_{k+1} \quad (k \in \mathbb{Z}),$$

where  $\{\alpha_k\}$  and  $\{\beta_k\}$  are given sequences, and  $\{\xi_k\}$  is to be found; moreover, the three sequences are assumed to be of the same type (for instance, to be periodic with one and the same period).

In the next lemma all parameters and unknowns are assumed to be nonnegative integers, as dictated by our initial problem. For the proof it suffices (in essence) to add the equations (within certain summation limits), and we omit the details.

**Lemma 8.** (a) *A periodic system (3) is solvable if and only if*

$$\sum_{k=0}^{r-1} \alpha_k = \sum_{k=0}^{r-1} \beta_k,$$

where  $r$  is the period. In this case a solution is not unique, but the difference of two solutions is a vector with equal coordinates. (b) *A finitary system is solvable if and only if similar sums extended (formally) to all  $k \in \mathbb{Z}$  coincide. In this case a solution is unique (and can easily be expressed in terms of sums of coefficients).* (c) *A semiinfinite system is solvable if and only if*

$$\sum_{k=m}^{\infty} \alpha_k \leq \sum_{k=m}^{\infty} \beta_k$$

for all  $m$  (the sums are finite in fact). In this case, a solution is also unique.

In the sequel, we shall need only cases (a) and (b). Surely, we shall not search through all irreducible polynomials one-by-one.

## §2. SPLITTING AND SOLVABILITY OF EQUATION (FE)

1. All functions involved in (FE) (known and unknown) can be factored by using Lemma 8. We observe that a factorization is determined up to a multiplicative constant and, until some point, we shall mean *an arbitrary (but fixed) factorization*. It

should be noted that this problem does not occur if  $n = 1$ , because a pure power of  $z$  is chosen “tacitly” as the first factor in this case.

By uniqueness (see the same Lemma 8), equation (FE) splits into a pair of similar equations one of which is related to polynomials vanishing at  $z = 0$ , and, on the contrary, the other is free of such polynomials in all terms. The first factor is called the *algebraic 0-component* of the function in question, and the second is called the *transcendental component*.<sup>2</sup> It should be mentioned that a good strategy is to consider the two equations simultaneously, but we shall not choose this option, to avoid deviation from a consecutive way of presentation. At first glance, even the assumption  $v(0) = 1$  seems to be unessential (and to a great extent this will be justified); so, we begin with the transcendental component.

**2.** Recall that  $|B|$  is the spectral radius of a linear transformation  $B: \mathbb{C}^n \rightarrow \mathbb{C}^n$ , and that  $0 < |B| < 1$ . In this subsection, it is convenient to replace the Euclidean norm in  $\mathbb{C}^n$  with an (equivalent) norm  $\|z\|$  so as to ensure the inequality  $\|B\| < 1$  for the corresponding operator norm. Such a renorming can be constructed in plenty of ways (see, e.g., [11, Theorem 2.1.2]). If  $|B| < b < \|B\|$ , we may put

$$(4) \quad \|z\| \stackrel{\text{def}}{=} \sup_{k \geq 0} b^{-k} \|B^k z\|.$$

Then  $\|B\| = b$ . This procedure is also applicable to infinite-dimensional Banach spaces (and was used in [13] for other purposes). In the sequel, for definiteness, we mean the norm (4) when speaking about a “new” norm.

*Remark.* The nature of our analysis dictates that we view polynomials as entire functions of a special kind. We observe that there is no natural definition of the degree of a polynomial for  $n > 1$ . “Natural” should mean “invariant under polynomial changes of variables” (or under polynomial automorphisms). But things are better if we restrict ourselves to nondegenerate *linear* changes of variables. We define the degree of a monomial  $z_1^{k_1} \cdot z_2^{k_2} \cdots z_n^{k_n}$  to be  $k_1 + k_2 + \cdots + k_n$ , and the degree of a polynomial  $q$  to be the largest degree of the monomials involved in this polynomial. Clearly, this degree (denoted by  $\deg(q)$ ) is invariant under nondegenerate *linear* substitutions. Another notion of degree invariant under automorphisms will also be useful. Specifically, we denote by  $\text{dg}(q)$  the number of irreducible polynomials (counted with multiplicity) involved in the factorization of  $q$ . Surely, the two numbers coincide for  $n = 1$ ; in general,  $\text{dg}(q) \leq \deg(q)$ .

**Lemma 9.** *Let  $\{\beta_1, \beta_2, \dots, \beta_n\}$  be the collection of eigenvalues of the transformation  $B$ . In the ball  $\|z\| < r$ ,  $r > 0$ , consider the equation*

$$(5) \quad w(Bz) = \lambda w(z)$$

*with the unknown analytic function  $w = w(z)$  and with the parameter  $\lambda \in \mathbb{C}$  (so, we look for eigenfunctions). (a) If equation (5) has a solution, then this solution  $w$  is a polynomial (more precisely, the restriction of a polynomial to the ball  $\|z\| < r$ ). (b) A solution exists if and only if*

$$\lambda = \beta_1^{k_1} \cdot \beta_2^{k_2} \cdots \beta_n^{k_n},$$

*where  $\{k_1, k_2, \dots, k_n\}$  is a collection of positive integers. (c) For such  $\lambda$ , the space  $E_\lambda$  of solutions is finite-dimensional. In particular,  $E_\lambda$  is 1-dimensional if the numbers  $\log \beta_1, \log \beta_2, \dots, \log \beta_n$  are linearly independent over the field  $\mathbb{Q}$  of rational numbers (for instance, this is so if  $n = 1$ ). (d) Either  $\lambda = 1$  and  $w(z) = \text{const}$ , or  $w(0) = 0$ .*

<sup>2</sup>Similar terminology applies to the pair of equations that arise after splitting the initial one.

*Proof.* (a) We may assume that  $\lambda \neq 0$ . Then a solution extends forcedly up to an entire function. It is easily seen that this function is of at most power growth. So, the solution is a polynomial.

(b) Suppose first that the  $\beta_k$  are pairwise distinct. Then we may assume that the basis in  $\mathbb{C}^n$  consists of eigenvectors of  $B$ , so that (we do not change the notation for coordinates) we may write

$$w(\beta_1 z_1, \beta_2 z_2, \dots, \beta_n z_n) = \lambda w(z_1, z_2, \dots, z_n),$$

and it remains to equate the coefficients of the Taylor expansions near  $z = 0$ . The general case reduces to this one by a small perturbation of  $B$  and passage to the limit.

(c) This is proved similarly. Again, first we assume that the  $\beta_k$  are pairwise different and, moreover, the numbers  $\log \beta_k$  are linearly independent over  $\mathbb{Q}$ . Let  $\mu$  be an eigenvalue, so that  $\mu = \beta_1^{k_1} \cdot \beta_2^{k_2} \cdot \dots \cdot \beta_n^{k_n}$ . We observe that there is only one eigenfunction in the case in question (namely, the monomial with exponents  $k_r$  in the new coordinates). Clearly, the degree of this monomial satisfies

$$k_1 + k_2 + \dots + k_n \leq \log \mu / \log \|B\|.$$

It follows that, in the general case, the degrees of the polynomials corresponding to a given eigenvalue  $\lambda$  are dominated by  $\log \lambda / \log \|B\|$ , and this yields an explicit upper bound for  $\dim E_\lambda$ .

(d) This is trivial, and the lemma is proved.  $\square$

**3.** In this subsection we consider an equation (FE) all terms of which are *free of algebraic 0-components*. Then  $q(0) \neq 0$ . We may and do assume that  $q$  and  $g$  are not identically equal (otherwise  $v = \text{const}$ , and conversely).

**Lemma 10.** *If a (nontrivial) solution exists, then  $g(0) \neq 0$ .*

*Proof.* If  $v(0) \neq 0$ , the statement is obvious. Suppose that  $v(0) = 0$  (in the long run it will become clear that this case does not occur, but the proof *will employ* the present lemma). If  $g(0) = 0$ , then the maximum modulus principle is violated (near the origin: recall that  $v \neq \text{const}$ ) because  $\|B\| < 1$ .  $\square$

Introducing a supplementary scalar parameter  $c$ , in the case under study we can rewrite (FE) in the form

$$(6) \quad q(z)v(z) = cg(z)v(Bz)$$

and impose the additional assumption  $q(0) = g(0) = 1$ . Under this assumption,<sup>3</sup> the following entire functions make sense:

$$Q(z) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} q(B^k z) \quad \text{and} \quad G(z) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} g(B^k z).$$

It is easily seen that the two functions are entire functions of exponential type. Since  $Q(0) = G(0) = 1$ , the meromorphic function  $G/Q$  is holomorphic and has no zeros in a ball  $\|z\| < r$  centered at the origin. The solvability of (6) is closely related to the property of  $G/Q$  being entire (this will be explained soon). Observe that if  $G/Q$  is entire, it is of exponential type (this is quite important for the study of the initial equation (ME)).

*Remark.* If  $F_1$  and  $F_2$  are nontrivial entire functions of exponential type and  $F_1 = F_2 \cdot F_3$ , where  $F_3$  is an entire function, then  $F_3$  is also of exponential type. This is classical for  $n = 1$ , though, in my opinion, cannot be called trivial because lower estimates are needed for the proof. At the same time, the lower estimate deduced from the Cartan lemma

<sup>3</sup>Maybe, it would be simpler to say that we pass from the couple  $q, g$  to the couple  $q/q(0), g/g(0)$  and introduce the compensating factor  $c = g(0)/q(0)$ , but we want to retain the notation.

in [12, p. 33] makes it possible to immediately extend the result to entire functions of several variables.

**Lemma 11.** *For an equation (6) without algebraic 0-components (in  $q, g$ , and the unknown function), the following conditions are equivalent: (a)  $c = 1$  and  $G/Q$  is an entire function; (b) the equation has a nontrivial entire solution.*

*If these equivalent conditions are fulfilled, then  $v = G/Q$  is an entire solution of exponential type with  $v(0) = 1$ , and all other entire solutions (with trivial algebraic component) are scalar multiples of  $v$ .*

*Proof.* If (a) is fulfilled, then  $G/Q$  is an entire solution equal to 1 at  $z = 0$ .

Suppose  $v$  is an entire solution (with trivial 0-component). Suppose also that  $Q(z) \neq 0$  and  $G(z) \neq 0$  for  $\|z\| < r$ . In this neighborhood of 0, we define a function  $w$  by  $w = v/(G/Q)$ . Then  $w$  is holomorphic in the neighborhood in question and satisfies  $w(z) = cw(Bz)$  there. We apply Lemma 9 to this equation. If  $c \neq 1$ , then  $w$  is a polynomial and  $w(0) = 0$ , whence it easily follows that  $v$  involves a nontrivial algebraic 0-component, a contradiction. Consequently,  $c = 1$ , and then  $w = \text{const}$ , after which the proof is finished easily.<sup>4</sup> □

**4.** In this subsection, we analyze the algebraic 0-component of equation (FE).

The initial equation can be rewritten in the form

$$(7) \quad q_0(z)v_0(z) = g_0(z)v_0(Bz).$$

An attempt at descent suggests itself, and we shall try to execute it. In doing so, we should pass from equation (7) to a similar equation of lower degree (understood in a specific sense) of the first factor. Ultimately, this will lead to an equation of the form  $w(z) = cw(Bz)$ , to which we shall apply Lemma 9 to examine solvability.

The first part of the procedure is division of  $q_0$  and  $g_0$  by their greatest common divisor  $d_0$ . If this is done *and the equation has a solution  $v_0$* , then the polynomial factorization  $v_0 = (g_0/d_0)v_1$  arises with a new unknown polynomial  $v_1$  satisfying the equation

$$(8) \quad q_1(z)v_1(z) = g_1(z)v_1(Bz),$$

where  $q_1 = q_0/d_0$  and  $g_1 = (g_0/d_0)^{(1)}$  (we remind the reader that  $h^{(k)}(z) \stackrel{\text{def}}{=} h(B^{-k}z)$ ).

A genuine simplification of the equation means that the degree of  $q_1$  is strictly smaller than that of  $q_0$ . So, under the assumption that a solution exists, we must estimate constructively the possible number of consecutive repetitions of the above procedure with trivial first operation (we may assume that  $d_0 = d_1 = \dots = 1$ , i.e., that  $q_0$  does not change).

Assume that  $m + 1$  repetitions (including (7)) occur. It is easily seen that then a chain of similar equations arises, the last having the form

$$(9) \quad q_0(z)v_m(z) = g_0(B^m z)v_m(Bz),$$

where

$$(10) \quad v_0(z) = g_0(z)g_0(Bz) \cdots g_0(B^{m-1}z)v_m(z).$$

Let  $p$  be an irreducible divisor of  $g_0$ . In our case, system (DE) (see §1) has the form

$$(11) \quad \nu_p^{(k)}(q_0) + \nu_p^{(k)}(v_0) = \nu_p^{(k)}(g_0) + \nu_p^{(k+1)}(v_0) \quad (k \in \mathbb{Z}).$$

Suppose first that  $p$  is not periodic, i.e., system (11) is finitary (see the end of the preceding section).

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<sup>4</sup>It should be noted that, in general, an equation of the form (6) with  $q(0) = g(0) = 1$  can have a solution with nontrivial algebraic 0-component (even for  $n = 1$ ). In this sense, the lemma is conditional.

Since  $p$  is an irreducible divisor of  $g_0$  (and *other, pairwise distinct* divisors arise under the action of  $B$ ), by Lemma 5 from (10) and (11) it follows that

$$1 \leq \nu_p^{(1-k)}(v_0) \leq \sum_{r \geq 0} \nu_p^{(-k-r)}(q_0)$$

for  $1 \leq k \leq m + 1$ . Summing these inequalities over  $k$  within the limits indicated, we obtain

$$m + 1 \leq \sum_{k,r=0}^{\infty} \nu_p^{(-k-r)}(q_0)$$

Clearly, the right-hand side of the above inequality does not exceed some constant  $l_1$  that can be calculated effectively and depends only on  $q_0$  and  $p$  (thus, ultimately, only on  $q_0$  and  $g_0$ ). So,  $m < l_1$  in the case in question.

It remains to treat the case where  $p$  leads to a periodic situation. This is even simpler. Let  $l_2$  denote the period length. By Lemma 8, if a solution exists, then

$$\sum_k \nu_p^{(k)}(q_0) = \sum_k \nu_p^{(k)}(g_0),$$

where  $k$  ranges over an arbitrary interval of integers of length  $l_2$ . Since  $\nu_p(g_0) > 0$ , we see among other things that the polynomial  $p(B^k z)$  is a divisor of  $q_0(z)$  for some nonnegative  $k \leq l_2 - 1$ . At the same time, if the “number of repetitions” does not exceed  $l_2$ , then in the course of the procedure an equation occurs that has the factor  $g_0(B^k z)$  on the right, and, surely, this leads to a nontrivial cancellation. This means that  $m < l_2$  in the case in question.

We put  $l = \max\{l_1, l_2\}$ . The definition of  $l$  is constructive roughly to the same extent as is the procedure of factorization of the coefficients in products of irreducible polynomials. The outcome is as follows: *either after less than  $l$  “repetitions” the initial equation simplifies in the sense that  $\text{dg}(q_0)$  reduces, or the equation has no solutions.* Clearly, the degrees of the coefficients on the left and on the right drop to zero *simultaneously*. When (and if) this happens, Lemma 9 applies.

**5.** Now it is easy to finish the study of equation (FE). First, we can specify a factorization. Indeed, keeping in mind the necessary conditions for solvability, we may assume that  $q = q_0 q_1$  and  $g = g_0 g_1$ , where  $q_1(0) = g_1(0) = 1$ , and  $q_0$  and  $g_0$  are either constants or polynomials equal to zero at  $z = 0$ . A solution  $v$  can be looked for in a similar form. We build the products  $Q_1$  and  $G_1$  (like  $Q$  and  $G$ ).

**Theorem 1.** *Equation (FE) is solvable if and only if the meromorphic function  $G_1/Q_1$  is entire and the algebraic component of the equation has a polynomial solution (this solution is uncovered in accordance with the procedure described in Subsection 4).*

### §3. SOLVABILITY OF EQUATION (ME) AND THE PROPERTIES OF SOLUTIONS

**1.** The solvability of (FE) is *necessary* for the solvability of (ME). Moreover, the above discussion shows that it suffices to consider only the case where  $q(0) = g(0) = 1$ . We assume this and also assume  $G/Q$  to be an entire function (of exponential type; see §2 for the definition of  $G$  and  $Q$ ). In other words, we assume that the requirements of the criterion presented above are fulfilled.

Yet these conditions *are not sufficient* for the existence of a (nontrivial) finitary solution for equation (ME). A simplest example illustrating this phenomenon is the equation

$$v(z) = (1 + z)v(z/2)$$

in the one-dimensional case.

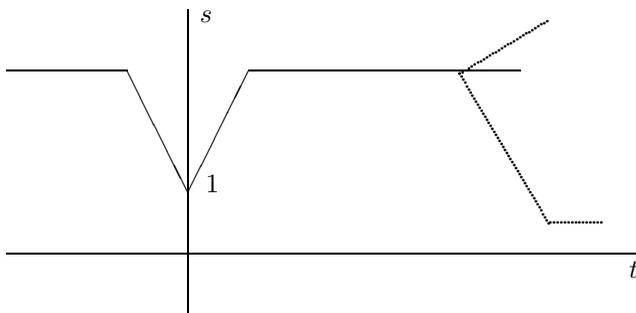


FIGURE 1. Variation of  $s = h(t)$ .

Surely, a necessary and sufficient supplementary condition is an at most power-type growth of  $G/Q$  at infinity in  $\mathbb{R}^n$ . We could have stopped at this point,<sup>5</sup> but it is interesting to learn which properties of  $q$  and  $g$  guarantee such a growth. Similarly, a criterion for the existence of a finitary  $C^\infty(\mathbb{R}^n)$ -solution  $u(x)$  is a “fast decay” of  $G/Q$  in  $\mathbb{R}^n$ .

**2.** We shall not study the problems stated above in detail. We start with an elucidating example (it reflects the essence to a great extent), and then we present sufficient conditions.

The solid line in Figure 1 above shows the graph of the function  $s = h(t)$  to be dealt with. It is important that  $h(0) = 1$ . The function is linear on  $[0, 1]$  and (for definiteness)  $h(t) = 3$  for  $t \geq 1$ . Finally,  $h(-t) = h(t)$ .

For  $b \in (0, 1)$ , put

$$(12) \quad H(t) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} h(b^k t).$$

It is easily seen that, as  $t \rightarrow \infty$ , the function  $H(t)$  has an at most power growth. If, starting from some place, as shown in the figure, we replace the graph by the lower dotted line, then  $h$  decays to zero; taking the upper dotted line will lead to a growth faster than exponential (as in the example in Subsection 1).

In fact, if  $h(t) = 1 + O(t)$  as  $t \rightarrow 0$  and  $h$  is sufficiently regular on the right semiaxis, then we have weak equivalence

$$(13) \quad H(t) \leq \int_0^t \frac{\log h(\tau) d\tau}{\tau} \quad \text{as } t \rightarrow \infty.$$

In specific cases (including multidimensional) formula (13) is applied to majorants, and this leads to satisfactory growth and decay estimates.

**3.** We need certain spaces close to the Bernstein spaces  $\mathbf{B}_\sigma$  of entire functions of exponential type and bounded on the real axis. It is well known that the Phragmén–Lindelöf principle and/or Poisson summation formula unite the separate estimates in  $\mathbb{C}^n$  and  $\mathbb{R}^n$  into the inequality

$$|h(\xi + i\eta)| \leq C e^{\sigma|\eta|}.$$

A similar statement is valid in the multidimensional case. It should be noted that there exist algebraic methods that lead to such “collective” inequalities; see, e.g., [14].

We shall need functions of exponential type at most  $\sigma$  and of at most power growth on the real space. In the one-dimensional case, it is profitable to write the latter condition

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<sup>5</sup>Surely, it could have been possible to consider distributions over not quite standard test functions, but we shall not do this.

in the form  $\log |h(\xi)| \leq \alpha \log |\xi + ia|$  with a sufficiently large  $a > 0$ . Indeed, then the function  $\log |z + ia|$  is positive and harmonic in the half-plane  $\eta = \text{Im } z \geq 0$ , and again this allows us to unite the estimates almost without calculations.

We denote by  $\mathbf{B}$  the union of the classes indicated above over all positive  $\sigma$  and  $\alpha$ .  $\mathbf{B}$  is a  $\mathbb{C}$ -algebra. Lemma 1 is still valid for this class in the sense that if  $h \in \mathbf{B}$  and  $p$  is a polynomial such that the function  $h/p$  is entire, then  $h/p \in \mathbf{B}$ . For  $n = 1$  this is true simply because the zeros are isolated points. However, the following argument is also possible; after a slight modification it extends to the multidimensional case.

We can assume that

$$p(z) = z^m + c_1 z^{m-1} + \dots$$

In the strip  $0 \leq \text{Im } z \leq 2m + 1$ , we can draw a straight line  $L$  parallel to the real axis and such that  $|p(z)| \geq 1$  for  $z \in L$ . Invoking a shift, we obtain an estimate for  $|h|$  on  $L$ . This estimate is true also for  $h/p$ . Finally, to estimate  $h/p$  on  $\mathbb{R}$  we use the inverse shift.

In the multidimensional case, we may apply the method used for the proof of the existence of a fundamental solution with the help of the so-called Hörmander stairs (this method, together with the name, was invented by Treves; see, e.g., [15, §17] for this). First, we do a nondegenerate real change of variables after which the polynomial acquires the form

$$p(z) = c_0 z_1^m + c_1 z_1^{m-1} + \dots,$$

where  $c_0 = \text{const} \neq 0$ , and  $c_1, c_2, \dots$  are polynomials in the variables  $z' = (z_2, z_3, \dots)$ . Then the trick described above is applied for every fixed  $\xi' \in \mathbb{R}^{n-1}$ . As a result, we arrive at the required estimate on  $\mathbb{R}^n$ .

Similarly, an exponential estimate in  $\mathbb{C}^n$  can be united with a power *decay* in  $\mathbb{R}^n$ . If *some* decay to 0 as  $|\xi| \rightarrow \infty$  occurs on  $\mathbb{R}^n$ , then the same happens on a parallel plane. It should be noted that, to a great extent, such statements remain valid for functions analytic in the product of upper half-planes.

**4.** Now, we prove the main theorem pertaining to equation (ME). As was said at the beginning of the present section, we assume that the conditions of Theorem 1 are fulfilled and, moreover,  $q(0) = g(0) = v(0) = 1$ .

**Theorem 2.** *If the function  $g|_{\mathbb{R}^n}$  is bounded, then equation (ME) has a unique compactly supported solution  $u$  for which  $\hat{u} = v$ . If, moreover,  $g(\xi)/q(\xi) \rightarrow 0$  as  $\|\xi\| \rightarrow \infty$ ,  $\xi \in \mathbb{R}^n$ , then  $u \in C_0^{(\infty)}(\mathbb{R}^n)$ .*

*Proof.* We begin with the verification of the first statement under the additional assumption that  $q(z) = 1$  for all  $z$ . For  $t \geq 0$ , we put

$$\gamma(t) = \max\{|g(\xi)| \text{ in the ball } \|\xi\| \leq t\}.$$

Then  $\gamma(t) = 1 + O(t)$  as  $t \rightarrow 0$ ,  $\gamma(t)$  is monotone increasing, and  $\sup \gamma(t) < \infty$ . If  $\|\xi\| = r$  (in what follows we write  $r$  for  $\|\xi\|$  systematically, often without explicit stipulation), we have

$$\log |G(\xi)| = \sum_{k=0}^{\infty} \log |g(B^k \xi)| \leq \sum_{k=0}^{\infty} \log \gamma(\|B^k \xi\|) \leq c + \sum_{k=1}^{\infty} \log \gamma(b^k r),$$

where  $c$  and  $b$  are (positive) constants with  $b < 1$ . The last inequality follows from the fact that  $\gamma$  is bounded and monotone. For the same reasons

$$\log |G(\xi)| \leq c + \int_0^r \frac{\log \gamma(t) dt}{t},$$

whence it follows that  $G$  is of at most power growth on  $\mathbb{R}^n$ , and (under the assumption imposed) the first statement is proved.

Now we explain how to lift the above assumption. Since  $Q|G$ , we readily see that  $q|G$ . Hence, there exists  $m \geq 0$  such that

$$q|g_0 \cdot g_1 \cdots g_m,$$

where  $g_k(z) = g(B^k z)$ . We assume that  $m$  is chosen to be the smallest possible and use induction on  $m$ .

If  $m = 0$ , then  $q|g$  and the matter reduces to the case considered above (if we take the material of Subsection 3 into account). Suppose  $m > 0$ . In this case, we may assume that  $q$  and  $g$  are coprime (otherwise we start with the cancellation of common factors). Then

$$q|g_1 \cdot g_2 \cdots g_m,$$

so that

$$v(Bz) = q(z)w(z),$$

where  $w$  is an entire function of exponential type. Comparing this with the initial equation, we obtain

$$q(z)w(z) = g(Bz)w(Bz),$$

and we have reduced  $m$  to  $m - 1$ . The first statement is proved.

A similar induction can be applied in the case of the second statement. So, we may assume that  $q(z) = 1$  for all  $z$ , whence  $g(\xi) \rightarrow 0$  as  $\|\xi\| \rightarrow \infty$ .

We redefine the function  $\gamma(t)$ . Let  $M \stackrel{\text{def}}{=} \sup |g(\xi)|$ . It is convenient to assume that  $M > 1$ . Let  $R$  be the maximal radius of a sphere on which the value  $M$  is attained by  $|g|$ . It is convenient to assume that  $R > 1$  also. Put

$$\gamma(t) = \begin{cases} \sup\{|g(\xi)| \text{ in the ball } \|\xi\| \leq t\} & \text{if } t \leq R, \\ \sup\{|g(\xi)| \text{ outside of the ball } \|\xi\| < t\} & \text{if } t \geq R. \end{cases}$$

Clearly,  $\gamma$  is a continuous function. It is monotone on  $(0, R)$  and  $(R, \infty)$ ,  $\gamma(t) = 1 + O(t)$  as  $t \rightarrow 0$ , and  $\gamma(t) \rightarrow 0$  as  $t \rightarrow \infty$ . We may say that the graph of  $\gamma$  is quite similar to the lower version of the graph in Figure 1, but the real axis is a horizontal asymptote for the graph.

Clearly,  $|g(\xi)| \leq \gamma(r)$  if  $\|\xi\| = r$ .

Let  $T$  be the smallest value of  $t > R$  such that  $\gamma(t) = 1$ . We define two functions  $\gamma_0$  and  $\gamma_1$  on the semiaxis  $t \geq 0$ . Specifically, we put  $\gamma_0(t) = 1$  for  $t \equiv T$  and  $\gamma_0(t) = \gamma(t)$  for  $t \geq T$ ; next,  $\gamma_1(t) = \gamma(t)$  for  $t \leq R$  and  $\gamma_1(t) = M$  for  $t \geq R$ . Obviously,  $\gamma \leq \gamma_0 \cdot \gamma_1$ , whence  $|G(\xi)| \leq \Gamma_0(\xi) \cdot \Gamma_1(\xi)$ , where

$$\Gamma_j(\xi) = \prod_{k=0}^{\infty} \gamma_j(\|B^k \xi\|).$$

The proof of the first part of the theorem shows that  $\Gamma_1$  has an at most power growth. Therefore, it remains to estimate  $\Gamma_0$ . We shall present a simple constructive estimate. Below it will be explained that it leads to a refinement of the second statement.

Clearly,  $\|B^k \xi\| \geq a^k r$ , where  $a = (\|B^{-1}\|)^{-1}$ . Since  $\gamma_0$  is monotone nonincreasing, we have  $\Gamma_0(\xi) \leq \prod_{k=0}^{\infty} \gamma_0(a^k r)$ ; thus, if  $\theta \in (0, 1)$  and  $r^\theta > T$ , then

$$\begin{aligned} \log \Gamma_0(\xi) &\leq \sum_{k=0}^{\infty} \log \gamma_0(a^k r) \lesssim \int_0^{\infty} \log \gamma_0(a^t r) dt \lesssim \int_T^r \frac{\log \gamma_0(\tau) d\tau}{\tau} \\ &\leq \int_{r^\theta}^r \frac{\log \gamma_0(\tau) d\tau}{\tau} \leq (1 - \theta)(\log \gamma_0(r^\theta)) \log r, \end{aligned}$$

and the required estimate follows. This finishes the proof of the theorem. □

5. We are left with several remarks on the last theorem (more substantial examples will be given in the next section).

The following example is simpler than that in the Introduction, but it is still of some interest. Specifically, the equation

$$(14) \quad u(x) = u(2x + 1) + u(2x - 1) \quad (x \in \mathbb{R})$$

gives rise to the equation

$$v(z) = (\cos z/2)v(z/2),$$

which is related directly to the classical identity

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty} \cos \frac{z}{2^k}.$$

Viewed as a distribution, the solution  $u$  of (14) with  $\widehat{u} = v$  is equal to 1 for  $|x| < 1$  and 0 for  $|x| > 1$ . For a pointwise solution, it is most natural to take the function

$$u(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1, \\ 1/2 & \text{if } x = \pm 1. \end{cases}$$

If we retain the same values of  $u(x)$  at the rational points and change them to 0 at the irrational points, we again obtain a solution of (14). It would be interesting to study other pointwise solutions. For example, there are *no* pointwise solutions  $u \in C(\mathbb{R}) \cap L^1(\mathbb{R})$  of (14) with  $\int_{\mathbb{R}} u(x) dx = 1$ .

We return to the question of the smoothness of a compactly supported solution for the initial equation (ME). The Tarski–Seidenberg exclusion theory implies, among other things, the following. If  $q = q(\xi)$  is a polynomial and  $|q(\xi)| \rightarrow \infty$  as  $\|\xi\| \rightarrow \infty$ , then  $|q(\xi)| \geq c\|\xi\|^\rho$  with positive  $c$  and  $\rho$ , for all sufficiently large  $\|\xi\|$ . Comparing this with the end of the proof of Theorem 2, we see that if the initial equation has a compactly supported solution and  $\sup |g| < \infty$ ,  $|q| \rightarrow \infty$ , then  $|v(\xi)|$  obeys an estimate of the sort  $\leq e^{-\varepsilon \log^2 \|\xi\|}$ . This leads to an explicit upper estimate for the derivatives of  $u(x)$  in terms of a derivative’s order. For instance, if  $n = 1$ , an estimate of the sort  $\leq e^{cm^2}$  for the  $m$ th derivative is available.

#### §4. EXAMPLES

1. The preceding arguments can be simplified considerably (starting with the very beginning) if we restrict ourselves to solutions with  $v(0) = 1$ . If such a solution exists, it is unique. Note that the above somewhat bulky statements about smoothness become much more natural in this case. For clarity, we treat only such solutions in the examples in the present section.

However, it should be noted that there are equations with finitary solutions but without finitary solutions whose Fourier transform obeys the condition  $v(0) = 1$ . We begin with exhibiting such examples.

Consider equation (FE). Let  $q = q_0q_1$  and  $g = g_0g_1$  be standard factorizations (so, in particular,  $q_1(0) = g_1(0) = 1$ ). Suppose that the polynomials  $q_0$  and  $g_0$  are coprime. Moreover, we assume for simplicity that  $h = g_1/q_1$  is an entire function.

We shall seek a solution of (FE) in the form  $v = g_0w$ , where  $w$  is a new unknown function. After obvious cancellations, we arrive at

$$q_0(z)q_1(z)w(z) = g_1(z)g_0(Bz)w(Bz).$$

We assume additionally that  $q_0(z) = g_0(Bz)$ . Then

$$v(z) = Cg_0(z)h(z),$$

where  $C = \text{const}$ . The equation

$$\begin{aligned} &(1 - z_1)(1 - z_2)(2z_1 + z_2)v(z_1, z_2) \\ &= 2(\cos \pi z_1/2)(\cos \pi z_2/2)(z_1 + 2t_2)v(z_2/2, z_1/2) \end{aligned}$$

is a specific example of this theme.

**2.** In the next subsection, we shall describe a typically multidimensional obstruction to the solvability of a particular case of (ME) in finitary functions. Specifically, we consider the case where

$$f(x) = \beta_1\delta(x - b_1) + \beta_2\delta(x - b_2) + \dots + \beta_m\delta(x - b_m).$$

We shall show that if (ME) has a finitary solution (a more precise statement will be given below) with such an  $f$ , then the irreducible divisors of  $q$  are polynomials of degree 1 ( $\text{deg}(f)$  is meant). In particular, if we put the operator  $\partial^n/\partial x_1\partial x_2 \dots \partial x_n$  of mixed derivative on the left, then a nontrivial solution exists for some right-hand sides, but it never exists if the Laplacian  $\Delta$  stands on the left (this is true for  $n \geq 2$ , in distinction to the case of  $n = 1$ ).

The proofs will involve some facts of complex analysis that are somewhat similar to lemmas accompanying the Skolem method in number theory (see, e.g., [17, Chapter IV, §6]).

In the theory of analytic functions of one or many variables, considerable attention is paid to the representation of the ratio of two quasipolynomials (of various types) in the case where this ratio is an entire function or a not too complicated meromorphic function (a short survey of this theme can be found, e.g., in [18]).

A possible answer may be of the sort that this ratio coincides with the quotient of a quasipolynomial by an (algebraic) polynomial. For instance, it is natural to ask when a linear combination (with constant coefficients) of the exponentials of linear forms (i.e., the simplest and, maybe, most important quasipolynomial) can be divided by an irreducible polynomial, and we shall see that the denominator must be of degree at most 1 for that. The necessary solvability condition indicated above follows immediately from this fact.

**3.** Let  $R$  be a compact connected Riemann surface. An analytic function  $\varphi : R \rightarrow \mathbb{C}$  is said to be *rational* if it has finitely many singularities, each being a pole at the worst.

Let  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  and  $\{\beta_1, \beta_2, \dots, \beta_n\}$  be two collections of nonzero rational functions on  $R$  such that the functions of the second collection are pairwise distinct.

**Lemma 12.** *If*

$$(15) \quad \alpha_1 e^{\beta_1} + \alpha_2 e^{\beta_2} + \dots + \alpha_n e^{\beta_m} = 0,$$

*then  $\beta_k - \beta_l = \text{const}$  at least for one pair of different indices  $k, l$ .*

*Proof.* Naturally, we proceed by induction on the number of summands. If  $m = 2$ , formula (15) implies

$$-\alpha_1/\alpha_2 = e^{\beta_2 - \beta_1}.$$

If  $\beta_2 - \beta_1 \neq \text{const}$ , then the function on the right possesses an essential singularity, but that on the left does not, a contradiction. So, the lemma is true for  $m = 2$ .

Let  $m \geq 3$ . Equation (15) can be rewritten in the form

$$a_1 e^{b_1} + a_2 e^{b_2} + \dots + a_{m-1} e^{b_{m-1}} = -1,$$

where  $a_k = \alpha_k/\alpha_m$ ,  $b_k = \beta_k - \beta_m$ . Taking the differentials on the right and on the left, we obtain

$$(da_1 + a_1 db_1)e^{b_1} + (da_2 + a_2 db_1)e^{b_2} + \dots = 0.$$

If  $da_1 + a_1db_1 = 0$ , then  $a_1 = c_1e^{b_1}$ , where  $0 \neq c_1 = \text{const}$ . Again comparing the nature of singularities, we conclude that  $b_1 = \text{const}$ . Thus, we may assume that the differential form  $da_1 + a_1db_1$  is nonzero. The ratio of two differential forms of order 1 is a function. Therefore, after division, we reduce the claim to the case of  $m - 1$  summands.  $\square$

In the next lemma, we shall employ classical results by Lefschetz and Whitney mentioned (partly, with proofs), e.g., in [19].

**Lemma 13.** *Let  $V$  be an irreducible algebraic set in  $\mathbb{C}^n$ . If  $h|_V = 0$ , where  $h$  is a (nontrivial) quasipolynomial with constant coefficients,*

$$h(z) = c_1e^{l_1(z)} + c_2e^{l_2(z)} + \dots + c_me^{l_m(z)},$$

*then  $(l_i - l_j)|_V = \text{const}$  at least for one pair of different linear forms.*

*Proof.* We proceed by induction on the (complex, topological) dimension  $d$  of  $V$ . Formally, the claim is true if  $d = 0$ , because then  $V$  reduces to a point. But induction starts with  $d = 1$ , and in this case the claim follows from Lemma 12.

Let  $W$  be the (connected analytic) manifold of regular points of  $V$ . Assume that  $0 \in W$ . Let  $f_1, f_2, \dots, f_r$  be polynomials determining  $V$ . The rank of the matrix  $(\partial f_i / \partial z_k)$  at the point 0 is  $n - d$ , where  $d = \dim(W)$ . For almost every choice of a row to be adjoined to this matrix, its rank increases by 1. This means that a generic hyperplane intersects  $V$  by a subspace of smaller dimension; moreover, the manifold of regular points for the connected component containing 0 in the intersection is included (by the inductive assumption) in a subspace of the form  $l_i - l_j = 0$  (the constant reduces to 0 because the subspace passes through the origin). Since the set of pairs  $(i, j)$  is finite, it follows that  $(l_i - l_j)|_V = 0$  for some pair.  $\square$

**Theorem 3.** *Suppose that the function  $g$  in equation (FE) is a quasipolynomial with constant coefficients. If this equation has an entire solution  $v$  with  $v(0) = 1$ , then every irreducible divisor  $p$  of the polynomial  $q$  is a polynomial of degree 1.*

*Proof.* Let  $V_p$  be the hyperplane corresponding to the irreducible divisor  $p$ . For some  $m$ , the polynomial  $p$  divides the quasipolynomial  $h$  given by

$$h(z) = g(z) \cdot g(Bz) \cdot \dots \cdot g(B^m z).$$

By Lemma 13, there is a polynomial  $\varphi$  of degree 1 such that  $\varphi|_{V_p} = 0$ . Since  $p$  is an irreducible polynomial,  $p$  and  $\varphi$  are proportional (with a scalar factor), whence  $\deg(p) = 1$ .  $\square$

Among other things, this theorem implies the above claim pertaining to the Laplacian, but only for  $n \geq 3$ . For  $n = 2$ , the quadratic form is reducible, but its divisors are not real, so that no (ME) equation with real shifts may have a compactly supported solution. In the next subsection we shall study the measures that can be put on the right if the Laplacian occurs on the left.

**4. In conclusion,** we consider the equation

$$(16) \quad \Delta u(x) = (u * \mu)(Ax) + \lambda u(Ax),$$

where  $\mu$  is a signed Borel measure supported on the unit sphere  $S^{n-1} \subset \mathbb{R}^n$ ,  $\lambda$  is an additional parameter, and  $A$  is a linear transformation with standard properties.

The question of solvability in finitary functions with  $u(0) = 1$  in the case where  $n = 2$ ,  $A = a\mathbb{I}$ , and  $\mu$  is absolutely continuous, was mentioned as open in [3, p. 57].

On the other hand, among other things, in [2] the solution was written out explicitly (in terms of Bessel functions) in the case where  $\mu$  is a multiple of the Lebesgue measure on the sphere and  $A$  is a scalar operator.

We show that if equation (16) is solvable, then  $\mu$  is a multiple of the Lebesgue measure.

Indeed, passing to Fourier transforms, we obtain  $-z^2 v(z) = g(z)v(Bz)$ , where  $z^2$  is the sum of the squares of the coordinates. Since  $v(0) = 1$ , it follows that  $g(z)$  is divisible by  $z^2$ .

Returning to the initial variables, we obtain  $\mu + \lambda\delta = \Delta\psi$ , where  $\psi$  is a function (which may happen to be a distribution) supported on the unit ball. Consequently,

$$(\mu + \lambda\delta, f) = 0$$

for every harmonic function  $f$ .

Since  $f(0) = (\theta, f)$  for harmonic functions ( $\theta$  being the normalized Lebesgue measure on the sphere), we see that  $(\mu + \lambda\theta, f) = 0$ .

It remains to use the fact that the restriction of the space of all polynomials to the sphere coincides with that of the space of harmonic polynomials (this is well known; see, e.g., [20, p. 441]). Another way to finish the proof is to refer to the solvability of the Dirichlet problem in the ball.

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