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ON QUANTIZATION OF THE SEMENOV-TIAN-SHANSKY POISSON BRACKET ON SIMPLE ALGEBRAIC GROUPS

A. MUDROV

Dedicated to the memory of Joseph Donin

ABSTRACT. Let G be a simple complex factorizable Poisson algebraic group. Let $\mathcal{U}_\hbar(\mathfrak{g})$ be the corresponding quantum group. We study the $\mathcal{U}_\hbar(\mathfrak{g})$-equivariant quantization $\mathcal{C}_\hbar[G]$ of the affine coordinate ring $\mathcal{C}[G]$ along the Semenov-Tian-Shansky bracket. For a simply connected group G, we give an elementary proof for the analog of the Kostant–Richardson theorem stating that $\mathcal{C}_\hbar[G]$ is a free module over its center.

§1. INTRODUCTION

Let G be a simple complex algebraic group. Suppose G is a Poisson group relative to a quasitriangular Lie bialgebra structure on $\mathfrak{g} = \operatorname{Lie} G$. We view G as a G-manifold with respect to the conjugation action. In the present paper we study quantization of a special Poisson structure on G making it a Poisson G-manifold. The Poisson structure in question is due to Semenov-Tian-Shansky (STS); see [STS]. In fact, the STS bracket makes G a Poisson Lie manifold over $\mathfrak{D}G$, where $\mathfrak{D}G = G \times G$ is the Poisson group corresponding to the double Lie bialgebra $\mathfrak{D}\mathfrak{g} \simeq \mathfrak{g} \oplus \mathfrak{g}$.

The STS bracket is a Poisson–Lie analog of the famous Kirillov–Kostant–Souriau bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ and can be restricted to every conjugacy class of G. The problem of quantizing conjugacy classes along this bracket is analogous to the classical problem of quantizing coadjoint orbits, but rather in the quantum group setting. It is natural to realize quantum conjugacy classes as quotients of the quantized affine coordinate ring $\mathbb{C}[G]$. For semisimple classes, that was done in [M], and the solution was based on certain properties of the quantum analog of $\mathbb{C}[G]$ studied in the present paper.

The algebra $\mathbb{C}[G]$ can be quantized along the STS Poisson bracket to a $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-algebra $\mathbb{C}_\hbar[G]$. For G connected, this quantization can be realized as a subalgebra in $\mathcal{U}_\hbar(\mathfrak{g})$. The algebra $\mathbb{C}_\hbar[G]$ can also be presented as a quotient of the so-called reflection equation (RE) algebra associated with $\mathcal{U}_\hbar(\mathfrak{g})$. If G is a classical matrix group, the corresponding ideal in the RE algebra is explicitly described in the present paper.

In [K] Kostant proved that the algebras $\mathbb{C}[\mathfrak{g}^*]$ and $\mathcal{U}(\mathfrak{g})$ are free modules over their subalgebras of $\mathfrak{g}$-invariants. Richardson generalized the case of $\mathbb{C}[\mathfrak{g}]$ to the affine coordinate ring of a semisimple complex algebraic group [R]. Namely, if the subalgebra of invariants $I(G)$ (class functions) is polynomial, then $\mathbb{C}[G]$ is a free $I(G)$-module generated by a

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G-submodule in $\mathbb{C}[G]$ with finite-dimensional isotypical components. The quantum analog of this theorem was proved in [B] by using the advanced technique of crystal bases. In the present paper, we give another proof of the quantum Richardson theorem within the formal deformation setting. Our approach is elementary and employs the classical Richardson theorem as the "initial point". The quantum version of that theorem can be formulated as follows.

Theorem. *Let G be a simple complex algebraic group, and let $\mathbb{C}_\hbar[G]$ be the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantization of $\mathbb{C}[G]$ along the* STS *bracket. Then*

i) *the subalgebra $I_\hbar(G)$ of $\mathcal{U}_\hbar(\mathfrak{g})$-invariants coincides with the center of $\mathbb{C}_\hbar[G]$,*

ii) *$I_\hbar(G) \simeq I(G) \otimes \mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra.*

Suppose that $I(G)$ is a polynomial algebra. Then

iii) *$\mathbb{C}_\hbar[G]$ is a free $I_\hbar(G)$-module generated by a $\mathcal{U}_\hbar(\mathfrak{g})$-submodule $\mathcal{E} \subset \mathbb{C}_\hbar[G]$. Each isotypic component in $\mathcal{E}$ is $\mathbb{C}[[\hbar]]$-finite.*

We remark that for a connected simply connected G the algebra of invariants is a polynomial algebra generated by the characters of fundamental representations [St]. That is also true for some non-simply-connected groups, for example, for $\mathrm{SO}(2n+1)$.

§2. Quantized universal enveloping algebras

Throughout the paper, $\mathfrak{g}$ is a simple complex Lie algebra equipped with a quasitriangular Lie bialgebra structure. That is, we fix a classical solution $r \in \mathfrak{g} \otimes \mathfrak{g}$ to the Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1}$$

and normalize it so that the symmetric part $\Omega := \frac{1}{2}(r_{12}+r_{21})$ of r is the inverse (canonical element) of the Killing form on $\mathfrak{g}$. Recall that quasitriangular solutions to equation (1) are parameterized by combinatorial objects called Belavin–Drinfeld triples [BD].

By $\mathcal{U}_\hbar(\mathfrak{g})$ we denote the quantization of the Lie bialgebra $(\mathfrak{g}, r)$ [Dr1, EK]. It is a quasitriangular topological Hopf $\mathbb{C}[[\hbar]]$-algebra isomorphic (as an algebra) to the space $\mathcal{U}(\mathfrak{g})[[\hbar]]$ of formal power series in $\hbar$ with coefficients in $\mathcal{U}(\mathfrak{g})$ completed in the $\hbar$-adic topology.

Let $\mathcal{R} \in \mathcal{U}_\hbar^{\hat{\otimes}2}(\mathfrak{g})$ be the quasitriangular structure (universal R-matrix) on $\mathcal{U}_\hbar(\mathfrak{g})$, the quantization of $r \in \mathfrak{g}^{\otimes 2}$. Consider the twisted tensor square $\mathcal{U}_\hbar(\mathfrak{g}) \otimes^{\mathcal{R}} \mathcal{U}_\hbar(\mathfrak{g})$ of $\mathcal{U}_\hbar(\mathfrak{g})$ constructed as described below [RS]. The Hopf algebra $\mathcal{U}_\hbar(\mathfrak{g}) \otimes^{\mathcal{R}} \mathcal{U}_\hbar(\mathfrak{g})$ is obtained by the twist of the usual tensor square $\mathcal{U}_\hbar^{\hat{\otimes}2}(\mathfrak{g})$ by the cocycle $\mathcal{R}_{23} \in \mathcal{U}_\hbar^{\hat{\otimes}4}(\mathfrak{g})$. The symbol $\hat{\otimes}$ means the completed tensor product (in the $\hbar$-adic topology). The diagonal embedding $\Delta\colon \mathcal{U}_\hbar(\mathfrak{g}) \to \mathcal{U}_\hbar(\mathfrak{g}) \otimes^{\mathcal{R}} \mathcal{U}_\hbar(\mathfrak{g})$ via comultiplication is a homomorphism of Hopf algebras. The algebra $\mathcal{U}_\hbar(\mathfrak{g}) \otimes^{\mathcal{R}} \mathcal{U}_\hbar(\mathfrak{g})$ is a quantization of the double $\mathfrak{D}\mathfrak{g}$; in the simple quasitriangular (factorizable) case, this double is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$ as a Lie algebra. Thus, we shall use the notation $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ for $\mathcal{U}_\hbar(\mathfrak{g}) \otimes^{\mathcal{R}} \mathcal{U}_\hbar(\mathfrak{g})$.

§3. Simple groups as Poisson Lie manifolds

Given an element $\xi \in \mathfrak{g}$, let ξ^l and ξ^r denote, respectively, the left- and right-invariant vector fields on G generated by ξ. Namely, we put

$$(\xi^l f)(g) = \frac{d}{dt} f(g e^{t\xi})|_{t=0}, \quad (\xi^r f)(g) = \frac{d}{dt} f(e^{t\xi} g)|_{t=0} \tag{2}$$

for every smooth function f on G.

There are two important Poisson structures on G. The first of them, the Drinfeld–Sklyanin (DS) Poisson bracket [Dr1], is determined by the bivector field

$$\varpi_{\mathrm{DS}} = r^{l,l} - r^{r,r}. \tag{3}$$

This bracket makes G a Poisson group.

The Semenov-Tian-Shansky (STS) Poisson structure on the group G is determined by the bivector field

$$\begin{aligned} \varpi_{\mathrm{STS}} &= r_-^{l,l} + r_-^{r,r} - r_-^{r,l} - r_-^{l,r} + \Omega^{l,l} - \Omega^{r,r} + \Omega^{r,l} - \Omega^{l,r} \\ &= r_-^{\mathrm{ad},\mathrm{ad}} + (\Omega^{r,l} - \Omega^{l,r}). \end{aligned} \tag{4}$$

Here $r_- = \frac{1}{2}(r_{12} - r_{21})$ is the skew symmetric part of r.

Consider the group G as a G-space with respect to conjugation. Then the STS bracket makes G a Poisson Lie manifold over G endowed with the Drinfeld–Sklyanin bracket [STS].

We assume that G is a linear algebraic group, i.e., a subgroup of $\mathrm{GL}(V)$, where V is a finite-dimensional G-module. Then G is an affine variety. Its irreducible (connected) component is also an affine variety [VO]. Unless otherwise explicitly stated, G is assumed to be connected.

The Lie algebra $\mathfrak{g}$ generates the left- and right-invariant vector fields on $\mathrm{End}(V)$ defined as in (2). We introduce a bivector field on $\mathrm{End}(V)$ by formula (4), where the superscripts l, r mark the left- and right-invariant vector fields on $\mathrm{End}(V)$. This bivector field is Poisson on the $G \times G$-invariant variety $\mathrm{End}(V)^\Omega$ of matrices $A \in \mathrm{End}(V)$ satisfying the quadratic equation $[A \otimes A, \Omega] = 0$. The restriction of this Poisson structure to $G \subset \mathrm{End}(V)$ coincides with (4).

In the basic representation of $\mathrm{SL}(n)$, the variety $\mathrm{End}(V)^\Omega$ is the entire matrix space. Let G be an orthogonal or symplectic group and (V, ρ) its basic representation with the invariant form $B \in V \otimes V$. Here ρ is the homomorphism $G \to \mathrm{GL}(V)$. The variety $\mathrm{End}(V)^\Omega$ coincides with the set of matrices fulfilling

$$BX^tB^{-1}X = f^2, \quad XBX^tB^{-1} = f^2. \tag{5}$$

Here f is a numeric parameter. The condition $f \neq 0$ specifies a principal open set $G^\sharp \subset \mathrm{End}(V)$, which is a group and a trivial central extension of G ($f = 1$). Such an extension can be defined for an arbitrary matrix algebraic group, and it will play a role in our consideration.

Namely, our analysis is based on some facts of commutative algebra that assume the finiteness of the $\mathbb{C}[[\hbar]]$-modules involved; cf. Subsection 4.1. To apply that machinery, we need to ensure some finiteness conditions on our modules and their classical counterparts. Such conditions may be, e.g., the finiteness of isotypical components of $\mathfrak{g}$- and $\mathcal{U}_\hbar(\mathfrak{g})$-modules or homogeneous components under some (invariant) grading. To this end, we pass from G to $G^\sharp$ because $\mathbb{C}[G^\sharp]$ is equipped with a natural grading.

§4. Quantization of the STS bracket on the group

By *quantization* of a Poisson affine variety M we understand a $\mathbb{C}[[\hbar]]$-free $\mathbb{C}[[\hbar]]$-algebra $\mathbb{C}_\hbar[M]$ such that $\mathbb{C}_\hbar[M]/\hbar\mathbb{C}_\hbar[M] \simeq \mathbb{C}[M]$. The quantization is called *equivariant* if it is equipped with an action of a quantum group $\mathcal{U}_\hbar(\mathfrak{g})$ that is compatible with the multiplication:

$$x \triangleright (ab) = (x^{(1)} \triangleright a)(x^{(2)} \triangleright a) \quad \text{for all } x \in \mathcal{U}_\hbar(\mathfrak{g}) \quad \text{for all } a, b \in \mathbb{C}_\hbar[M].$$

For an equivariant quantization to exist, M must be a Poisson manifold over the Poisson group G corresponding to $\mathcal{U}_\hbar(\mathfrak{g})$.

4.1. Some commutative algebra. For the reader's convenience, in the present subsection we collect some standard facts about $\mathbb{C}[[\hbar]]$-modules that we use in what follows.

Lemma 4.1. *Let E be a free finite $\mathbb{C}[[h]]$-module. Then every $\mathbb{C}[[h]]$-submodule of E is finite and free.*

This assertion holds true for modules over principal ideal domains (see, e.g., [Jac]).

Given a $\mathbb{C}[[h]]$-module E, we denote by E_0 its "classical limit", the complex vector space $E/\hbar E$. A $\mathbb{C}[[\hbar]]$-linear map $\Psi\colon E \to F$ induces a $\mathbb{C}$-linear map $E_0 \to F_0$, which will be denoted by Ψ_0.

Lemma 4.2. *Let E be a finite and W an arbitrary $\mathbb{C}[[h]]$-module. A $\mathbb{C}[[h]]$-linear map $W \to E$ is an epimorphism if the induced map $W_0 \to E_0$ is an epimorphism of vector spaces.*

This is a particular case of the Nakayama lemma for modules over local rings; see, e.g., [GH].

We say that a $\mathbb{C}[[\hbar]]$-module E has no torsion (is torsion free) if $\hbar x = 0 \implies x = 0$ for $x \in E$.

Lemma 4.3. *A finitely generated $\mathbb{C}[[h]]$-module is free if it is torsion free.*

The latter assertion easily follows from the Nakayama lemma.

Lemma 4.4. *Every submodule and quotient module of a finite $\mathbb{C}[[\hbar]]$-module is finite.*

This statement is obvious for quotient modules. For submodules, it follows from Lemma 4.1.

Lemma 4.5. *Let $\Psi\colon E \to F$ be a morphism of free finite $\mathbb{C}[[\hbar]]$-modules such that the induced map $\Psi_0\colon E_0 \to F_0$ is an isomorphism of $\mathbb{C}$-vector spaces. Then Ψ is an isomorphism.*

By using Lemma 4.1, the latter assertion can be reduced to the case where $E = F$ and Ψ is an endomorphism of E. An endomorphism of a free module is invertible if and only if its residue $\bmod \hbar$ is invertible.

Lemma 4.6. *Let $\Psi\colon E \to F$ be a morphism of $\mathbb{C}[[\hbar]]$-modules. Suppose that E is finite, F is torsion free, and $\Psi_0\colon E_0 \to F_0$ is injective. Then E is free and Ψ is injective.*

Proof. First, we prove that Ψ is an embedding, assuming E to be free. In this case the image $\operatorname{Im}\Psi$ is finite and has no torsion. Therefore it is free, by Lemma 4.3. The map Ψ_0 factors through the composition $E_0 \to (\operatorname{Im}\Psi)_0 \to F_0$, and the left arrow is surjective by construction. Since Ψ_0 is injective, the map $E_0 \to (\operatorname{Im}\Psi)_0$ is also injective and hence an isomorphism. Thus, by Lemma 4.5, we have $E \simeq \operatorname{Im}\Psi$.

Now let E be arbitrary and let $\{e_i\}$ be a set of generators such that their projections mod $\hbar$ form a base in E_0. Such generators do exist by the Nakayama lemma. Let $\hat{E}$ be the $\mathbb{C}[[\hbar]]$-free covering of E generated by $\{e_i\}$. The composite map $\hat{\Psi}\colon \hat{E} \to E \to F$ satisfies the hypotheses of the lemma with free $\hat{E}$. We conclude that $\hat{\Psi}$ is injective. This implies that $E = \hat{E}$, i.e., E is free, and that Ψ is injective. $\square$

4.2. Quantization of the DS **and** STS **brackets.** Let G be a simple complex algebraic group and (V,ρ) its faithful representation. The affine ring $\mathbb{C}[G]$ is realized as a quotient of $\mathbb{C}[\operatorname{End}(V)]$ by the ideal generated by a finite system of polynomials $\{p_i\}$. We do not require G to be connected but assume the inclusion $G \subset \operatorname{End}(V)^{\Omega}$. Recall that $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ stands for the split Casimir.

Let $G^\sharp$ denote the smooth affine variety $G \times \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of the field $\mathbb{C}$. The variety $G^\sharp$ is an algebraic group; however, we shall not use this fact until §6.

The affine coordinate ring $\mathbb{C}[G^\sharp]$ is isomorphic to the tensor product $\mathbb{C}[G] \otimes \mathbb{C}[f, f^{-1}]$. It can be realized as the quotient of $\mathbb{C}[\mathrm{End}(V)] \otimes \mathbb{C}[f, f^{-1}]$ by the ideal $(p_i^\sharp)$, where $p_i^\sharp(f, X) = f^{k_i} p_i(f^{-1}X)$ and k_i is the degree of the polynomial p_i.

The algebra $\mathbb{C}[\mathrm{End}(V)] \otimes \mathbb{C}[f, f^{-1}]$ can be equipped with a $\mathbb{Z}$-grading by setting $\deg \mathrm{End}^*(V) = 1$, $\deg f = 1$, and $\deg f^{-1} = -1$. The polynomials $\{p_i^\sharp\}$ are homogeneous; hence $\mathbb{C}[G^\sharp]$ is a $\mathbb{Z}$-graded algebra. In $\mathbb{C}[G^\sharp]$, we select the subalgebra that is the quotient of $\mathbb{C}[\mathrm{End}(V)][f]$ by the ideal $(p_i^\sharp)$. This subalgebra is identified with the affine ring of the Zariski closure $\bar{G}^\sharp$ in $\mathrm{End}(V) \times \mathbb{C}$. It is graded, with finite-dimensional homogeneous components. Clearly, $\mathbb{C}[G^\sharp]$ is generated by $\mathbb{C}[\bar{G}^\sharp]$ over $\mathbb{C}[f^{-1}]$.

We define a two-sided G-action on $\mathbb{C}[G^\sharp]$ by setting it to be trivial on $\mathbb{C}[f, f^{-1}]$. This makes $\mathbb{C}[G^\sharp]$ a $\mathcal{U}(\mathfrak{g})$-bimodule algebra. The action preserves the grading and preserves the subalgebra $\mathbb{C}[\bar{G}^\sharp]$. The DS and STS Poisson brackets (3) and (4) are defined naturally on $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar{G}^\sharp]$ via the right and left $\mathfrak{g}$-actions on $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar{G}^\sharp]$. They make both $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar{G}^\sharp]$ Poisson algebras over the Lie bialgebras $\mathfrak{g}_{\mathrm{op}} \oplus \mathfrak{g}$ and $\mathfrak{D}\mathfrak{g}$, respectively (both summands of the former are the same as Lie coalgebras; the subscript *op* designates the opposite Lie structure). The Poisson varieties G_{DS} and G_{STS} are subvarieties in $G^\sharp_{\mathrm{DS}}$ and $G^\sharp_{\mathrm{STS}}$ (as well as in $\bar{G}^\sharp_{\mathrm{DS}}$ and $\bar{G}^\sharp_{\mathrm{STS}}$) specified by the equation $f = 1$.

Recall the Takhtajan quantization [T] of the DS Poisson structure on G. Consider the quasitriangular quasi-Hopf algebra $\big(\mathcal{U}(\mathfrak{g})[[\hbar]], \Phi, \mathcal{R}_0\big)$, where $\mathcal{U}(\mathfrak{g})[[\hbar]]$ is equipped with the standard comultiplication, Φ is a $\mathfrak{g}$-invariant associator, and $\mathcal{R}_0 = e^{\frac{\hbar}{2}\Omega}$ is the universal R-matrix. Since Φ and $\mathcal{R}_0$ are G-invariant, $\mathbb{C}[G] \otimes \mathbb{C}[[\hbar]]$ is a commutative algebra in the quasitensor category of $\mathcal{U}(\mathfrak{g})_{\mathrm{op}}[[\hbar]] \hat{\otimes} \mathcal{U}(\mathfrak{g})[[\hbar]]$-modules. The latter is a quasi-Hopf algebra with the associator $(\Phi^{-1})'\Phi''$ and the universal R-matrix $(\mathcal{R}_0^{-1})'\mathcal{R}_0''$, [Dr3]. Here the prime is relative to the $\mathcal{U}(\mathfrak{g})_{\mathrm{op}}[[\hbar]]$-factor while the double prime is relative to the $\mathcal{U}(\mathfrak{g})[[\hbar]]$-factor.

Let $\mathcal{J} \in \mathcal{U}(\mathfrak{g})^{\hat{\otimes} 2}[[\hbar]]$ be a twist converting $\mathcal{U}(\mathfrak{g})[[\hbar]]$ to the quasitriangular Hopf algebra $\mathcal{U}_\hbar(\mathfrak{g})$. Such twists were constructed in [EK] for all quasitriangular Lie bialgebra structures on $\mathfrak{g}$, and every quasitriangular quantization of $\mathcal{U}(\mathfrak{g})$ can be obtained in this way. The twist $(\mathcal{J}^{-1})'\mathcal{J}''$ converts $\mathcal{U}(\mathfrak{g})_{\mathrm{op}}[[\hbar]] \hat{\otimes} \mathcal{U}(\mathfrak{g})[[\hbar]]$ into the Hopf algebra $\mathcal{U}_\hbar(\mathfrak{g})_{\mathrm{op}} \hat{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$. Applied to $\mathbb{C}[G] \otimes \mathbb{C}[[\hbar]]$, this twist makes it a $\mathcal{U}_\hbar(\mathfrak{g})_{\mathrm{op}} \hat{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$-module algebra, $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$. This algebra is a quantization of the DS Poisson bracket on G.

The algebra $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ can be generated by the matrix elements of any faithful finite-dimensional representation of G. In particular, the generators satisfy the so-called RTT relations; see formula (9) below. To put it differently, $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ is commutative in the braided category of $\mathcal{U}_\hbar(\mathfrak{g})$-bimodules or, which is the same, of modules over $\mathcal{U}_\hbar(\mathfrak{g})_{\mathrm{op}} \otimes \mathcal{U}_\hbar(\mathfrak{g})$.

The above quantization extends to the algebras $\mathbb{C}_\hbar[G^\sharp_{\mathrm{DS}}]$ and $\mathbb{C}_\hbar[\bar{G}^\sharp_{\mathrm{DS}}]$; the construction is literally the same. Since the two-sided action of $\mathfrak{g}$ preserves the grading, the algebras $\mathbb{C}_\hbar[G^\sharp_{\mathrm{DS}}]$ and $\mathbb{C}_\hbar[\bar{G}^\sharp_{\mathrm{DS}}]$ are $\mathbb{Z}$-graded. The algebra $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ is obtained from $\mathbb{C}_\hbar[G^\sharp_{\mathrm{DS}}]$ or from $\mathbb{C}_\hbar[\bar{G}^\sharp_{\mathrm{DS}}]$ as the quotient by the ideal $(f - 1)$.

Now we regard $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$, $\mathbb{C}_\hbar[G^\sharp_{\mathrm{DS}}]$, and $\mathbb{C}_\hbar[\bar{G}^\sharp_{\mathrm{DS}}]$ as $\mathcal{U}_\hbar(\mathfrak{g})^{\mathrm{op}} \hat{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$-algebras, using the identification between $\mathcal{U}_\hbar(\mathfrak{g})^{\mathrm{op}}$ and $\mathcal{U}_\hbar(\mathfrak{g})_{\mathrm{op}}$ via the antipode. We perform the twist from $\mathcal{U}_\hbar(\mathfrak{g})^{\mathrm{op}} \hat{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$ to $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ and consider the corresponding transformation of the algebras $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$, $\mathbb{C}_\hbar[G^\sharp_{\mathrm{DS}}]$, and $\mathbb{C}_\hbar[\bar{G}^\sharp_{\mathrm{DS}}]$. The resulting algebras $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$, $\mathbb{C}_\hbar[G^\sharp_{\mathrm{STS}}]$, and $\mathbb{C}_\hbar[\bar{G}^\sharp_{\mathrm{STS}}]$ are $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantizations along the STS bracket [DM].

The algebras $\mathbb{C}_\hbar[G^\sharp_{\mathrm{STS}}]$ and $\mathbb{C}_\hbar[\bar G^\sharp_{\mathrm{STS}}]$ are $\mathbb{Z}$-graded, and $\mathbb{C}_\hbar[G^\sharp_{\mathrm{STS}}]=\mathbb{C}_\hbar[\bar G_{\mathrm{STS}}][f^{-1}]$. The homogeneous components in $\mathbb{C}_\hbar[\bar G^\sharp_{\mathrm{STS}}]$ are $\mathbb{C}[[\hbar]]$-finite and vanish for negative degrees. The algebra $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ is obtained from $\mathbb{C}_\hbar[G^\sharp_{\mathrm{STS}}]$ (or from $\mathbb{C}_\hbar[\bar G^\sharp_{\mathrm{STS}}]$) by factoring out the ideal $(f-1)$.

Like $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$, the algebra $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ can be generated by the matrix elements of any faithful finite-dimensional representation of G. This time the generators satisfy the RE; see formulas (10) below. In other words, $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ is commutative in the braided category of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-modules. The Hopf algebra twist from $\mathcal{U}^{\mathrm{op}}_\hbar(\mathfrak{g})\hat\otimes\mathcal{U}_\hbar(\mathfrak{g})$ to $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ is called the RE twist.

§5. The algebra $\mathcal{C}_\hbar[G_{\mathrm{STS}}]$ as a module over its center

In the present section, G is connected and $\mathbb{C}_\hbar[G]$ stands for $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$, that is, for the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantization of $\mathbb{C}[G]$ along the STS bracket. The action of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathbb{C}_\hbar[G]$ is induced by the diagonal embedding $\Delta\colon\mathcal{U}_\hbar(\mathfrak{g})\to\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ and can be expressed in terms of the left and right coregular actions of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ as follows:

$$x(a)=x^{(2)}\triangleright a\triangleleft\gamma(x^{(1)}).$$

Here γ designates the antipode in $\mathcal{U}_\hbar(\mathfrak{g})$; the coregular actions are defined by $\xi\triangleright a=\xi^l(a)$ and $a\triangleleft\xi=\xi^r(a)$ for $\xi\in\mathfrak{g}$; cf. (2). We use the fact that $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ and $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ coincide as $\mathcal{U}_\hbar(\mathfrak{g})$-bimodules (but not algebras) and the $\mathcal{U}_\hbar(\mathfrak{g})$-actions are the actions of $\mathcal{U}(\mathfrak{g})[[\hbar]]$.

Proposition 5.1. *Let G be a simple complex algebraic group equipped with the* STS *bracket. Let $\mathfrak{g}$ be its Lie bialgebra, $\mathfrak{D}\mathfrak{g}$ the double of $\mathfrak{g}$, and let $\mathbb{C}_\hbar[G]$ be the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantization of the affine ring $\mathbb{C}[G]$ along the* STS *bracket. Then the subalgebra $I_\hbar(G)$ of $\mathcal{U}_\hbar(\mathfrak{g})$-invariants in $\mathbb{C}_\hbar[G]$ coincides with the center.*

Proof. The statement holds true for $\bar G^\sharp$ as well, so we prove it for $\bar G^\sharp$ first. The case of G will be obtained by factoring out the ideal $(f-1)$.

The subalgebra $I_\hbar(\bar G^\sharp)$ lies in the center of $\mathbb{C}_\hbar[\bar G^\sharp]$. Indeed, let $\hat{\mathcal{R}}$ be the universal R-matrix of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. It is expressed in terms of the universal R-matrix $\mathcal{R}\in\mathcal{U}_\hbar^{\hat\otimes 2}(\mathfrak{g})$ by $\hat{\mathcal{R}}=\mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}$; therefore $\hat{\mathcal{R}}\in\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})\hat\otimes\mathcal{U}_\hbar(\mathfrak{g})$. The algebra $\mathbb{C}_\hbar[\bar G^\sharp]$ is commutative in the category of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-modules; hence $(\hat{\mathcal{R}}_2\triangleright a)(\hat{\mathcal{R}}_1\triangleright b)=ba$ for any $a,b\in\mathbb{C}_\hbar[\bar G^\sharp]$. Hence $ab=ba$ for $a\in I_\hbar(\bar G^\sharp)$.

Conversely, suppose that $ab=ba$ for some a and all $b\in\mathbb{C}_\hbar[\bar G^\sharp]$. Present a as $a=a_0+O(\hbar)$, where $a_0\in\mathbb{C}[\bar G^\sharp]$. We have $0=\hbar\varpi_{\mathrm{STS}}(a_0,b)+O(\hbar^2)$, and therefore $\varpi_{\mathrm{STS}}(a_0,b)=0$. The Poisson bivector field ϖ_{STS} is induced by the classical r-matrix of the double $\xi^i\otimes\xi_i\in(\mathfrak{D}\mathfrak{g})^{\otimes 2}$, where $\xi_i\in\mathfrak{g}$ and $\xi^i\in\mathfrak{g}^*$. The element $\xi\in\mathfrak{g}^*$ acts on $\bar G^\sharp$ by the vector field $r_-(\xi)^l-r_-(\xi)^r-\Omega(\xi)^l-\Omega(\xi)^r$; cf. formula (4). Here we view the elements of $\mathfrak{g}\otimes\mathfrak{g}$ as operators $\mathfrak{g}^*\to\mathfrak{g}$ by pairing with the first tensor component. Let e denote the identity of the group G. At every point $(e\otimes c)\in G\times\mathbb{C}^*=G^\sharp\subset\bar G^\sharp$, this vector field equals $-2\Omega(\xi)$. Since the Killing form is nondegenerate, the mapping $\mathbb{C}[\bar G^\sharp]\to\mathfrak{g}$, $b\mapsto\varpi_{\mathrm{STS}}(b,\cdot)$ is surjective in a neighborhood of $(e\otimes c)\in\bar G^\sharp$. Thus, $\zeta\triangleright a_0=0$ for all $\zeta\in\mathfrak{g}$ in that neighborhood and, consequently, identically on $\bar G^\sharp$; that is, a_0 is $\mathfrak{g}$-invariant.

We can assume that a is homogeneous with respect to the grading in $I_\hbar(\bar G^\sharp)$. Let a_0' be a $\mathcal{U}_\hbar(\mathfrak{g})$-invariant element such that $a_0'=a_0 \mod \hbar$. We can choose a_0' of the same degree as a (in fact, we can take $a_0'=a_0\triangleleft\theta^{-\frac12}$; see the proof of Proposition 5.2). Then $a-a_0'$ is central and divisible by $\hbar$. Acting by induction, we present a as a sum $a=\sum_{\ell=0}^\infty\hbar^\ell a_\ell'$, where each summand is $\mathcal{U}_\hbar(\mathfrak{g})$-invariant. Since all a_ℓ' have the same degree, they lie in a finite $\mathbb{C}[[\hbar]]$-module. Hence the above sum converges in the $\hbar$-adic topology. $\square$

It is clear that Proposition 5.1 is valid for $G^\sharp$.

The following proposition asserts that the subalgebra of invariants in $\mathbb{C}_\hbar[G]$ is not quantized.

Proposition 5.2. *Let $\mathbb{C}_\hbar[G]$ be the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantization of the* STS *bracket on G. Then $I_\hbar(G)$ is isomorphic to $I(G)\otimes\mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra.*

Proof. Consider two subspaces $\mathcal{I}_1$ and $\mathcal{I}_2$ in $\mathcal{A}=\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ defined by the following conditions:

$$\begin{aligned} \mathcal{I}_1 &= \{a\in\mathcal{A}\colon x\triangleright a = a\triangleleft x \ \ \forall x\in\mathcal{U}_\hbar(\mathfrak{g})\},\\ \mathcal{I}_2 &= \{a\in\mathcal{A}\colon x\triangleright a = a\triangleleft \gamma^2(x) \ \ \forall x\in\mathcal{U}_\hbar(\mathfrak{g})\}. \end{aligned} \tag{6}$$

Since $\mathcal{A}$ is a $\mathcal{U}_\hbar(\mathfrak{g})$-bimodule algebra and the square antipode γ^2 is a Hopf algebra automorphism of $\mathcal{U}_\hbar(\mathfrak{g})$, both $\mathcal{I}_1$ and $\mathcal{I}_2$ are subalgebras in $\mathcal{A}$. The algebra $\mathcal{I}_1$ is isomorphic to $I(G)\otimes\mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra. This readily follows from the Takhtajan construction of $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ rendered in Subsection 4.2.

We show that the algebra $\mathcal{I}_2$ is isomorphic to $\mathcal{I}_1$. Indeed, the fourth power of the antipode in $\mathcal{U}_\hbar(\mathfrak{g})$ is implemented by the similarity transformation with a group-like element $\theta\in\mathcal{U}_\hbar(\mathfrak{g})$ [Dr2]. This element has a group-like square root $\theta^{\frac12}=e^{\frac12\ln\theta}\in\mathcal{U}_\hbar(\mathfrak{g})$. The logarithm is well defined, because $\theta=1+O(\hbar)$. In the case of the Drinfeld–Jimbo (standard) quantization of $\mathcal{U}(\mathfrak{g})$, the element $\theta^{\frac12}$ belongs to $\mathcal{U}_\hbar(\mathfrak{h})$, where $\mathfrak{h}\subset\mathfrak{g}$ is the Cartan subalgebra. The map $a\mapsto a\triangleleft\theta^{-\frac12}$ is an automorphism of $\mathcal{A}$, and this automorphism sends $\mathcal{I}_1$ to $\mathcal{I}_2$, because γ^2 is implemented by conjugation with $\theta^{\frac12}$.

Thus, we have proved that $\mathcal{I}_2$ is isomorphic to $I(G)\otimes\mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra. Now consider the RE twist from Subsection 4.2 converting $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ into $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$. This twist relates multiplications by the formula (15), where $\mathcal{T}$ should be replaced by $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ and $\mathcal{K}$ by $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$. It is straightforward to check that these multiplications coincide on $\mathcal{I}_2$. ☐

Remark 5.3. In the proof of Proposition 5.2, we used the observation that the multiplications in $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ and $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ coincide on $\mathcal{I}_2$. In fact, formula (15) implies that $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ and $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ are the same if viewed as left $\mathcal{I}_2$-modules. Therefore, the structure of a left $\mathcal{I}_1$-module on $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ is the same as the structure of an $\mathcal{I}_\hbar(G)$-module on $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$. This assertion holds true for $G^\sharp$ and $\bar{G}^\sharp$ as well.

The subalgebra of invariants $I(G)\subset\mathbb{C}[G]$ is isomorphic to $\mathbb{C}[T]^W$, where T is a maximal torus in G and W is the Weyl group, [St]. Suppose that $I(G)$ is polynomial. For example, that is the case when G is simply connected; then $\mathbb{C}[T]^W$ is generated by the characters of the fundamental representations [St]. Under the above assumption, the coordinate ring $\mathbb{C}[G]$ is a free module over $I(G)$ [R]. There exists a G-submodule $\mathcal{E}_0\subset\mathbb{C}[G]$ such that the multiplication map $I(G)\otimes\mathcal{E}_0\to\mathbb{C}[G]$ gives an isomorphism of vector spaces. Each isotypic component in $\mathcal{E}_0$ has finite multiplicity. We prove the quantum analog of this fact.

Theorem 5.4. *Let $\mathbb{C}_\hbar[G]$ be the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantization of the* STS *bracket on G. Suppose that the subalgebra $I(G)$ of $\mathfrak{g}$-invariants is a polynomial algebra. Then*

i) *$\mathbb{C}_\hbar[G]$ is a free $I_\hbar(G)$-module generated by a $\mathcal{U}_\hbar(\mathfrak{g})$-submodule $\mathcal{E}\subset\mathbb{C}_\hbar[G]$;*

ii) *each isotypic component in $\mathcal{E}$ is $\mathbb{C}[[\hbar]]$-finite.*

Proof. Let $\mathcal{E}_0$ be the $\mathcal{U}(\mathfrak{g})$-module generating $\mathbb{C}[G]$ over $I(G)$. Regarded as a subspace in $\mathbb{C}[G^\sharp]$ in a natural way, it obviously generates $\mathbb{C}[G^\sharp]$ over $I(G^\sharp)$. Using the invertibility of f, we can make $\mathcal{E}_0$ a graded submodule in $\mathbb{C}[\bar{G}^\sharp]$.

Put $\mathcal{E}=\mathcal{E}_0\otimes\mathbb{C}[[\hbar]]$. Let V_0 be a simple finite-dimensional $\mathfrak{g}$-module and $V=V_0\otimes\mathbb{C}[[\hbar]]$ the corresponding $\mathcal{U}_\hbar(\mathfrak{g})$-module. Let $(\mathcal{E}_0)_{V_0}$ denote the isotypic component of $\mathcal{E}_0$.

The isotypic component $\mathbb{C}_\hbar[G^\sharp]_V$ is isomorphic to $I(G^\sharp)\otimes(\mathcal{E}_0)_{V_0}\otimes\mathbb{C}[[\hbar]]$ as a $\mathcal{U}_\hbar(\mathfrak{g})$-module.

Let $\tilde m$ denote the multiplication in $\mathbb{C}_\hbar[G^\sharp]$. The map

$$\tilde m\colon I_\hbar(G^\sharp)\otimes_{\mathbb{C}[[\hbar]]}\mathcal{E}_V\to\mathbb{C}_\hbar[G^\sharp]_V \tag{7}$$

is $\mathcal{U}_\hbar(\mathfrak{g})$-equivariant and respects the grading. Let the superscript (k) denote the homogeneous component of degree k. The map (7) induces $\mathcal{U}_\hbar(\mathfrak{g})$-equivariant maps

$$\begin{gathered}\bigoplus_{i+j=k} I_\hbar(G^\sharp)^{(i)}\otimes_{\mathbb{C}[[\hbar]]}\mathcal{E}_V^{(j)}\to\mathbb{C}_\hbar[\bar G^\sharp]_V^{(k)},\\ I_\hbar(\bar G^\sharp)^{(k)}\otimes_{\mathbb{C}[[\hbar]]}\mathcal{E}_V\to\mathbb{C}_\hbar[\bar G^\sharp]_V\subset\mathbb{C}_\hbar[G^\sharp]_V.\end{gathered} \tag{8}$$

The first map has a $\mathbb{C}[[\hbar]]$-finite target, while the second has a $\mathbb{C}[[\hbar]]$-finite source. All the $\mathbb{C}[[\hbar]]$-modules in (8) are free. Modulo $\hbar$, the first map is surjective, and the second is injective. Therefore, they are surjective and injective, respectively, by Lemmas 4.2 and 4.6. Since $\mathbb{C}_\hbar[G^\sharp]=\mathbb{C}_\hbar[\bar G^\sharp][f^{-1}]$ and $I_\hbar[G^\sharp]=I_\hbar[\bar G^\sharp][f^{-1}]$, this immediately implies that the map (7) is surjective and injective and hence an isomorphism.

Now recall that $I_\hbar(G^\sharp)$ is isomorphic to $I_\hbar(G)[f,f^{-1}]$. Taking the quotient by the ideal $(f-1)$ proves the theorem for G. □

§6. Quantization in terms of generators and relations

In this section we describe the quantization of $\mathbb{C}[G]$ along the DS and STS brackets in terms of generators and relations in the case where G is a classical matrix group. We give a detailed consideration to the DS-case. The case of STS is treated similarly, upon obvious modifications. Alternatively, the defining ideal $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ can be derived from the ideal of $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ using Proposition A.1 and the twist-equivalence between $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ and $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$.

Function algebras on quantum matrix groups in the classical series were defined in terms of generators and relations in [FRT]. Here we prove that the $\mathbb{C}[q,q^{-1}]$-algebras of functions on quantum groups defined in [FRT] via generators and relations are included in free $\mathbb{C}[[\hbar]]$-algebras, $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$.

6.1. FRT and RE algebras. In this subsection we recall the definition of the FRT and RE algebras [FRT, KSkl].

Let V_0 be the basic representation of G and let V be the corresponding $\mathcal{U}_\hbar(\mathfrak{g})$-module. Let R denote the image of the universal R-matrix of $\mathcal{U}_\hbar(\mathfrak{g})$ in $\mathrm{End}(V^{\otimes 2})$. Put $N:=\dim V_0$.

The FRT algebra $\mathcal{T}$ is generated by the matrix elements $\{T^i_j\}\subset\mathrm{End}^*(V)$ subject to the relations

$$RT_1T_2=T_2T_1R, \tag{9}$$

where $T=\|T^i_j\|$. So $\mathcal{T}$ is a quotient of the free algebra $\mathbb{C}[[\hbar]]\langle T^i_j\rangle$. The latter is a $\mathcal{U}_\hbar(\mathfrak{g})$-bimodule algebra, the two-sided action being extended from the two-sided action on $\mathrm{End}^*(V)$. The ideal (9) is invariant, so $\mathcal{T}$ is also a $\mathcal{U}_\hbar(\mathfrak{g})$-bimodule algebra. It is $\mathbb{Z}$-graded with $\deg\mathrm{End}^*(V)=1$, and the grading is equivariant with respect to the two-sided $\mathcal{U}_\hbar(\mathfrak{g})$-action.

The RE algebra $\mathcal{K}$ is also generated by the matrix elements of the basic representation, this time denoted by K^i_j. Put $K=\|K^i_j\|$ to be the matrix of the generators. The RE algebra $\mathcal{K}$ is the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle K^i_j\rangle$ by the ideal generated by the relations

$$R_{21}K_1R_{12}K_2=K_2R_{21}K_1R_{12}. \tag{10}$$

The algebra $\mathcal{K}$ is a $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-module algebra. It is $\mathbb{Z}$-graded, and the grading is invariant with respect to the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-action.

Recall (see [DM]) that the RE twist of the Hopf algebra $\mathcal{U}_\hbar^{\mathrm{op}}(\mathfrak{g}) \hat{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$ to the twisted tensor square $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ converts the algebra $\mathcal{T}$ to $\mathcal{K}$ (cf. also Subsection 4.2).

6.2. Algebra $\mathbb{C}_\hbar[G_{\mathrm{DS}}]$ in generators and relations. In this section we describe the algebra $\mathbb{C}_\hbar[G] = \mathbb{C}_\hbar[G_{\mathrm{DS}}]$ in terms of generators and relations.

We shall use the group structure on $G^\sharp$, which is the trivial central extension of G. For G orthogonal and symplectic, $G^\sharp$ is defined by equation (5) with $f \neq 0$. The basic representation of G on V_0 naturally extends to a representation of $G^\sharp$ on $V_0 \oplus \mathbb{C}$, because the subgroup $\mathbb{C}^*$ acts on the module V_0 by dilations. The indeterminate f is the matrix element of the one-dimensional representation of $\mathbb{C}^*$.

Suppose $f \neq 0$. The group $G^\sharp$ can be identified with the $G^\sharp \times G^\sharp$-orbit in $\mathrm{End}(V_0 \oplus \mathbb{C})$, which for G orthogonal and symplectic is specified by the equation

$$B_0 T^t B_0^{-1} T = f^2, \quad T B_0 T^t B_0^{-1} = f^2, \tag{11}$$

$$\det(T) = f^N \tag{12}$$

and by (12) for $G = \mathrm{SL}(n)$. The form $B_0 \in V_0 \otimes V_0$ in the equation (11) is the classical invariant of the (orthogonal or symplectic) group G.

The ideals in $\mathbb{C}[\mathrm{End}(V_0)][f]$ generated by (11) and by (12) are radical. This is obvious for the $G = \mathrm{SL}(n)$ and follows from [We] for G orthogonal and symplectic. The corresponding quotients of $\mathbb{C}[\mathrm{End}(V_0)][f]$ are the affine coordinate rings of $\bar{G}^\sharp$.

Recall (see [FRT] and [F]) that there exists a central group-like two-sided $\mathcal{U}_\hbar(\mathfrak{g})$-invariant $\det_q(T) \in \mathcal{T}$ of degree n such that $\det_q(T) = \det(T)$ modulo $\hbar$. For G orthogonal and symplectic, let B denote the $\mathcal{U}_\hbar(\mathfrak{g})$-invariant element in $V \otimes V$ (see [FRT]).

Proposition 6.1. *Let G be a classical unimodular matrix group. The $\mathcal{U}_\hbar(\mathfrak{g})_{\mathrm{op}} \otimes \mathcal{U}_\hbar(\mathfrak{g})$-equivariant quantization $\mathbb{C}_\hbar[\bar{G}^\sharp]$ can be realized as the quotient of $\mathcal{T}[f]$ by the ideal of relations*:

$$\det_q(T) = f^N \tag{13}$$

for G special linear,

$$B T^t B^{-1} T = f^2, \quad T B T^t B^{-1} = f^2 \tag{14}$$

for G symplectic, and by (13), (14) *for G orthogonal. The quantization $\mathbb{C}_\hbar[G]$ is obtained from $\mathbb{C}_\hbar[\bar{G}^\sharp]$ by factoring out the ideal* $(f-1)$.

Proof. Denote by $\mathfrak{S}$ the algebra $\mathcal{T}[f]$, by $\mathfrak{T}$ the algebra $\mathbb{C}_\hbar[G^\sharp]$, and by $\mathfrak{J}$ the ideal in $\mathcal{T}[f]$ generated by relations (14) and (13), depending on the type of G. The algebras $\mathfrak{S}$, $\mathfrak{T}$, and $\mathfrak{J}$ are graded, and the grading is $\mathcal{U}_\hbar(\mathfrak{g}^\sharp)$-compatible. Note that homogeneous components in $\mathfrak{S}$ (and, hence, in $\mathfrak{J}$) are $\mathbb{C}[[\hbar]]$-finite. There is an obvious bialgebra structure on $\mathfrak{S}$, with f group-like.

The Takhtajan construction of the quantization, see Subsection 4.2, implies that the evaluation at the identity $\varepsilon\colon a \mapsto a(e)$ is a character of the algebra $\mathfrak{T}$. We define a pairing between $\mathfrak{T}$ and $\mathcal{U}_\hbar(\mathfrak{g}^\sharp)$ by setting $\langle a, x\rangle := \varepsilon(x \triangleright a) = \varepsilon(a \triangleleft x)$. This pairing is nondegenerate, because G is connected.

The matrix coefficients of the basic representation are viewed as elements of $\mathfrak{T}$ in a natural way. They satisfy the RTT relation, because $\mathfrak{T}$ is commutative in the category of $\mathcal{U}_\hbar(\mathfrak{g}^\sharp)$-bimodules. This determines an equivariant algebra homomorphism $\Psi\colon \mathfrak{S} \to \mathfrak{T}$. Clearly, the composition map $\varepsilon \circ \Psi$ coincides with the counit of the bialgebra $\mathfrak{S}$. From this we conclude that the invariant ideal $\mathfrak{J}$ is annihilated by $\varepsilon \circ \Psi$ (the counit returns 1 on group-like elements, including $\det_q(T)$ and f). Therefore, $\mathfrak{J}$ annihilates $\mathcal{U}_\hbar(\mathfrak{g}^\sharp)$ through the pairing $\langle \cdot, \cdot \rangle$. Since this pairing is nondegenerate, the ideal $\mathfrak{J}$ lies in the kernel of Ψ.

The homomorphism Ψ preserves grading, and it is identical on $\mathrm{End}^*(V) \oplus \mathbb{C}[[\hbar]]f$. Since the image of Ψ is $\mathbb{C}[[\hbar]]$-free, we have the direct sum decomposition $\mathfrak{S} = \ker \Psi \oplus \mathrm{Im}\, \Psi$ of $\mathbb{C}[[\hbar]]$-modules. Therefore, $(\mathrm{Im}\,\Psi)_0$ is embedded in $\mathfrak{S}_0$. We show that $\mathfrak{J} = \ker \Psi$. Since both ideals are graded and the homogeneous components are finite, it suffices to show that the map $\mathfrak{J}_0 \to (\ker \Psi)_0$ induced by the embedding $\mathfrak{J} \hookrightarrow \ker \Psi$ is surjective, but this follows readily from the construction. Then we can apply the Nakayama lemma to each homogeneous component. This proves $\mathfrak{J} = \ker \Psi$. Another consequence is that $\mathrm{Im}\,\Psi$ is a quantization of $\mathbb{C}[\bar{G}^\sharp]$ that lies in $\mathbb{C}_\hbar[\bar{G}^\sharp]$. Hence it coincides with $\mathbb{C}_\hbar[\bar{G}^\sharp]$, because that is so in the classical limit.

The quantization $\mathbb{C}_\hbar[G^\sharp]$ is isomorphic to $\mathbb{C}_\hbar[\bar{G}^\sharp][f^{-1}]$, as easily follows from the Takhtajan construction. Therefore, $\mathbb{C}_\hbar[G^\sharp]$ is realized as the quotient of the algebra $\mathcal{T}[f, f^{-1}]$ by the ideal of the relations (13), (14). On the other hand, $\mathbb{C}_\hbar[G^\sharp]$ is a free module over $\mathbb{C}[[\hbar]][f, f^{-1}]$. The quotient of $\mathbb{C}_\hbar[G^\sharp]$ by the ideal $(f-1)$ is $\mathbb{C}[[\hbar]]$-free and thus it is a quantization of $\mathbb{C}[G]$. $\square$

For the orthogonal nonunimodular group, a stronger assertion can be proved.

Proposition 6.2. *For $G = O(N)$, the $\mathcal{U}_\hbar(\mathfrak{g})_{\mathrm{op}} \otimes \mathcal{U}_\hbar(\mathfrak{g})$-equivariant quantization $\mathbb{C}_\hbar[\bar{G}^\sharp]$ can be realized as the quotient of $\mathcal{T}[f]$ by the ideal of relations* (14). *The quantization $\mathbb{C}_\hbar[G]$ is obtained from $\mathbb{C}_\hbar[\bar{G}^\sharp]$ by factoring out the ideal $(f-1)$.*

Proof. The group $\mathbb{Z}_2$ acts on $\mathcal{U}_\hbar(\mathfrak{g})$ by Hopf algebra automorphisms. This action is trivial for $\mathfrak{g} = so(2n+1)$ and is induced by the flip of the simple roots α_{n-1} and α_n (the automorphism of the Dynkin diagram) in the quantum Chevalley basis for $\mathfrak{g} = so(2n)$. Consider the smash product $\mathbb{Z}_2 \ltimes \mathcal{U}_\hbar(\mathfrak{g})$ with the natural structure of a Hopf algebra, which is a deformation of the Hopf algebra $\mathbb{Z}_2 \ltimes \mathcal{U}(\mathfrak{g})$. Denote by σ the involution generating the group $\mathbb{Z}_2$. The representation of $\mathcal{U}_\hbar(\mathfrak{g})$ on V extends to a representation of $\mathbb{Z}_2 \ltimes \mathcal{U}_\hbar(\mathfrak{g})$ by assigning $\sigma \mapsto -1$ for $\mathfrak{g} = so(2n+1)$ and $\sigma \mapsto 1 - e^n_n - e^{n+1}_{n+1} + e^{n+1}_n + e^n_{n+1}$ for $\mathfrak{g} = so(2n)$ (in the realization of [FRT]), where $\{e^i_j\} \subset \mathrm{End}(V)$ is the standard basis.

Now we repeat the proof of Proposition 6.1 replacing $\mathcal{U}_\hbar(\mathfrak{g}^\sharp)$ by $\mathbb{Z}_2 \ltimes \mathcal{U}_\hbar(\mathfrak{g}^\sharp)$. $\square$

6.3. Algebra $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ in generators and relations. Under the twist from $\mathcal{U}^{\mathrm{op}}_\hbar(\mathfrak{g}) \hat{\otimes}\, \mathcal{U}_\hbar(\mathfrak{g})$ to $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$, the defining relations of $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ in $\mathcal{T}$ transform to certain relations in the RE algebra $\mathcal{K}$ and generate a $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-invariant ideal in $\mathcal{K}$; see Appendix A. We compute this ideal.

The multiplications in $\mathcal{T}$ and $\mathcal{K}$ are related by the formula (see [DM])

$$m_{\mathcal{T}}(a \otimes b) = m_{\mathcal{K}}\big(\mathcal{R}_1 \triangleright a \triangleleft \mathcal{R}^{-1}_{1'} \otimes b \triangleleft \mathcal{R}^{-1}_{2'} \triangleleft \mathcal{R}_2\big). \tag{15}$$

Let G be the orthogonal or the symplectic group. For V being the defining representation of $\mathcal{U}_\hbar(\mathfrak{g})$, let $P \in \mathrm{End}(V \otimes V)$ denote the permutation operator. It is convenient to rewrite relations (14) for $f = 1$ in terms of the one-dimensional $\mathcal{U}_\hbar(\mathfrak{g})$-invariant projector κ constructed out of PR; see [I]:

$$T_1 T_2 \kappa = \kappa = \kappa T_1 T_2. \tag{16}$$

Then formula (15) applied to (16) gives

$$K_2 P R K_2 \kappa = \epsilon q^{\varepsilon - N} \kappa = \kappa K_2 P R K_2. \tag{17}$$

Here $\epsilon q^{\varepsilon-N}$ is the eigenvalue of PR corresponding to κ, and $\epsilon = 1$ for $\mathfrak{g}$ orthogonal while $\epsilon = -1$ for $\mathfrak{g}$ symplectic. The ideal generated by (17) lies in the kernel of the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant projection $\mathcal{K} \to \mathbb{C}_\hbar[G_{\mathrm{STS}}]$.

Similarly, we can express the element $\det_q(T)$ in terms of the generators K^i_j. We denote by $\det_q(K)$ the resulting form of degree n. The ideal $\big(\det_q(K) - 1\big)$ is annihilated by the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant projection $\mathcal{K} \to \mathbb{C}_\hbar[G_{\mathrm{STS}}]$.

Proposition 6.3. *Let G be a classical complex matrix group. Then the algebra $\mathbb{C}_\hbar[G_{\mathrm{STS}}]$ is isomorphic to the quotient of $\mathcal{K}$ by the ideal $\mathfrak{J}$, where*
i) $\mathfrak{J}$ *is generated by relations* (17) *and* $\det_q(K) = 1$ *for* $G = \mathrm{SO}(N)$;
ii) $\mathfrak{J} = \big(\det_q(K) - 1\big)$ *for* $G = \mathrm{SL}(n)$;
iii) $\mathfrak{J}$ *is generated by relations* (17) *for* $G = O(N)$ *or* $G = \mathrm{Sp}(n)$.

Proof. This proposition can be proved by a straightforward modification of the proof of Proposition 6.1. Another way is to start with Proposition 6.1 and use the RE twist applied to the quantization along the DS bracket; cf. Appendix A. □

A. On the twist of module algebras

In this subsection we study how the twist of Hopf algebras affects defining relations of their module algebras. Let $\mathcal{H}$ be a Hopf algebra, V a finite-dimensional left $\mathcal{H}$-module, and $T(V)$ the tensor algebra of V. Let W be an $\mathcal{H}$-submodule in $T(V)$ generating an ideal $J(W)$ in $T(V)$. Denote by $\mathcal{A}$ the quotient algebra $T(V)/J(W)$.

Let $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ be a twisting cocycle and $\tilde{\mathcal{H}}$ the corresponding twist of $\mathcal{H}$. Denote by $\tilde{\mathcal{A}}$ the twist of the module algebra $\mathcal{A}$. The multiplication in $\tilde{\mathcal{A}}$ is expressed via the multiplication in $\mathcal{A}$ by $m_{\tilde{\mathcal{A}}} = m_{\mathcal{A}} \circ \mathcal{F}$, and similarly for $\widetilde{T(V)}$ and $T(V)$.

Let Δ^k denote the k-fold comultiplications $\Delta^k\colon \mathcal{H} \to \mathcal{H}^{\otimes k}$, $\Delta^1 := \mathrm{id}$, $\Delta^2 := \Delta$, $\Delta^3 := (\Delta \otimes \mathrm{id}) \circ \Delta$, and so on. We introduce a family of automorphisms $\{\Xi_n\}_{n=0}^\infty$ of $V^{\otimes n}$ setting $\Xi_n = \mathrm{id}$ for $n = 0, 1$, and then by induction

$$\Xi_n = (\Delta^l \otimes \Delta^k)(\mathcal{F})\big(\Xi_l \otimes \Xi_k\big), \quad n = k + l, \ \ k, l \geq 1.$$

This definition does not depend on the choice of a partition $k + l = n$. The elements Ξ_n amount to a linear automorphism Ξ of $T(V)$.

Proposition A.1. *The algebra $\tilde{\mathcal{A}}$ is isomorphic to the quotient algebra*

$$T(V)/J(\Xi^{-1}W).$$

Proof. Since the ideal $J(W) \subset T(V)$ is invariant, it is also an ideal in $\widetilde{T(V)}$. It is easy to see that the quotient $\widetilde{T(V)}/J(W)$ is isomorphic to $\tilde{\mathcal{A}}$. On the other hand, the algebra $\widetilde{T(V)}$ is isomorphic to $T(V)$. The isomorphism is given by the maps $T(V) \supset m(v_1 \otimes \cdots \otimes v_n) \mapsto (\tilde{m} \circ \Xi_n)(v_1 \otimes \cdots \otimes v_n)$, $n \in \mathbb{N}$, where m and $\tilde{m}$ are multiplications in $T(V)$ and $\widetilde{T(V)}$. This implies the proposition. □

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Department of Mathematics, University of York, YO10 5DD, United Kingdom

Current address: St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia

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