

ON THE RIEMANN–ROCH THEOREM WITHOUT DENOMINATORS

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ABSTRACT. A proof of the Riemann–Roch theorem without denominators is given. It is also proved that Grothendieck’s ring functor CH_{mult} is not an oriented cohomology pretheory.

The Riemann–Roch formula without denominators for a closed embedding $i : Y \hookrightarrow X$ of codimension d expresses the Chern class $c_d(i_*\mathcal{O}_Y)$ in terms of the class $[Y] \in CH^d(X)$. In the present paper, we give a proof of this formula in the spirit of Verdier [Ve] but without using local Chern classes. The proof consists of several reductions. First, we use excision of singularities and deformation to the normal cone to reduce the proof to the case of an embedding of a smooth variety Y in a projective bundle $\mathbf{P}(E)$ over Y . At the next step, we reduce the proof to the simplest case of an embedding of a rational point in a projective space. Finally, we present two simple arguments that prove the formula in this case: one involves the Koszul complex, and the other involves the ring functor CH_{mult} , which arises naturally under the study of the behavior of the Chern classes under direct images.

In the second part of the paper, we present some properties of the functor CH_{mult} and prove that this functor is not an oriented cohomology pretheory.

The authors are thankful to I. A. Panin and K. I. Pimenov for generous guidance and support and to J. L. Colliot-Thélène (Orsay University, Paris) and B. Erez (University of Bordeaux I) for inspiring discussions and hospitality.

In the present paper, we use the following notation.

- 1) X is a quasiprojective variety over a field k .
- 2) $CH^*(X) = \bigoplus CH^i(X)$ is the Chow ring of classes of rationally equivalent cycles on X .
- 3) $K'_0(X)$ is the Grothendieck K -group of the category of coherent sheaves on X . If \mathcal{F} is a coherent sheaf on X , then:
 - 4) $c_i(\mathcal{F})$ is the i th Chern class of the sheaf \mathcal{F} (see [Fu]);
 - 5) $s_i(\mathcal{F})$ is the i th Segre class of the sheaf \mathcal{F} (see [Fu]).
- If $f : Y \rightarrow X$ is a flat morphism of varieties, then:
 - 6) $f^{CH} : CH^*(X) \rightarrow CH^*(Y)$ is the inverse image homomorphism for the Chow theory (see [Fu]);
 - 7) $f^K : K'_0(X) \rightarrow K'_0(Y)$ is the inverse image homomorphism for K -theory (see [Qu]).
- If $f : Y \rightarrow X$ is a proper morphism of varieties, then:
 - 8) $f_{CH} : CH^*(X) \rightarrow CH^*(Y)$ is the direct image homomorphism for the Chow theory (see [Fu]);
 - 9) $f_K : K'_0(X) \rightarrow K'_0(Y)$ is the direct image homomorphism for K -theory (see [Qu]).

2000 *Mathematics Subject Classification*. Primary 14C40.

Key words and phrases. Riemann–Roch formula without denominators, deformation to the normal cone, Koszul complex, Chern classes, oriented cohomology pretheory.

Partially supported by CNRS, France.

§1. RIEMANN–ROCH THEOREM WITHOUT DENOMINATORS

Theorem. *Let X be a nonsingular variety over a field k . Let $i : Y \hookrightarrow X$ be a closed embedding of an irreducible subvariety Y of codimension d . Then the following relation is true in $CH^d(X)$:*

$$(1) \quad c_d(i_*\mathcal{O}_Y) = (-1)^{d-1}(d-1)![Y].$$

Proof. The proof is given in Subsections 1.1–1.3. □

1.1. Reduction to the case of the zero section of a projective bundle.

1. Reduction to the case of a nonsingular Y . From now on, we write $i_K\mathcal{O}_Y$ for $i_K[\mathcal{O}_Y] = [i_*\mathcal{O}_Y]$. Let Z be the subvariety of singularities of Y . Let $i' : Y - Z \hookrightarrow X - Z$ be the regular embedding of the nonsingular variety $Y - Z$. We consider the exact sequence of localization,

$$CH^d(Z) \rightarrow CH^d(X) \rightarrow CH^d(X - Z) \rightarrow 0,$$

where $CH^d(Z) = 0$ because $\dim Z < d$. Thus, we have an isomorphism

$$CH^d(X) \rightarrow CH^d(X - Z).$$

Under this isomorphism, the classes $c_d(i_K\mathcal{O}_Y)$ and $(-1)^{d-1}(d-1)![Y]$ are mapped to the classes $c_d(i'_K\mathcal{O}_{Y-Z})$ and $(-1)^{d-1}(d-1)![Y - Z]$, respectively. This reduces the proof to the case where Y is nonsingular.

Lemma 1 (Properties of the classes $c_d(i_K\mathcal{O}_Y)$ and $[Y] \in CH^d(X)$). *1. The classes $c_d(i_K\mathcal{O}_Y)$ and $[Y]$ are functorial with respect to transversal squares (see [Pa]); i.e., for a transversal square*

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ i' \uparrow & & \uparrow i \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

we have

$$\begin{aligned} \phi^{CH}(c_d(i_K\mathcal{O}_Y)) &= c_d(i'_K\mathcal{O}_{Y'}), \\ \phi^{CH}([Y]) &= [Y']. \end{aligned}$$

2. The classes $c_d(i_K\mathcal{O}_Y)$ and $[Y]$ lie in the kernel of the homomorphism j_{CH} , where j is the open embedding $X - Y \hookrightarrow X$.

Proof. 1. From the commutative diagrams

$$\begin{array}{ccc} K_0(X') & \xleftarrow{\phi^K} & K_0(X) \\ i'_K \uparrow & & \uparrow i_K \\ K_0(Y') & \xleftarrow{\psi^K} & K_0(Y) \end{array}$$

and

$$\begin{array}{ccc} CH^d(X') & \xleftarrow{\phi^{CH}} & CH^d(X) \\ i'_{CH} \uparrow & & \uparrow i_{CH} \\ CH^0(Y') & \xleftarrow{\psi^{CH}} & CH^0(Y), \end{array}$$

we obtain

$$\begin{aligned} \phi^{CH}(c_d(i_K \mathcal{O}_Y)) &= c_d(\phi^K(i_K \mathcal{O}_Y)) \\ &= c_d(\phi^K(i_K(1))) = c_d(i'_K(\psi^K(1))) = c_d(i'_K(1)) = c_d(i'_K \mathcal{O}_{Y'}) \end{aligned}$$

and

$$\phi^{CH}([Y]) = \phi^{CH}(i_{CH}(1)) = i'_{CH}(\psi^{CH}(1)) = i'_{CH}(1) = [Y'].$$

2. This is a special case of item 1, where we put $X' = X - Y$, $Y' = \emptyset$, and $\phi = j$.

The lemma is proved. □

2. Deformation to the normal cone. In the case of a nonsingular variety Y , we have the following commutative diagram of deformation to the normal cone [Pa]:

$$\begin{array}{ccccc} \mathbf{P}(1 \oplus N) & \xrightarrow{j_0} & X_t & \xleftarrow{j_1} & X \\ s=i_0 \uparrow & & i_t \uparrow & & \uparrow i=i_1 \\ Y & \xrightarrow{k_0} & Y_t & \xleftarrow{k_1} & Y, \end{array}$$

where both squares are transversal. For the Chow groups, we obtain the commutative diagram

$$\begin{array}{ccccc} CH^d(\mathbf{P}(1 \oplus N)) & \xleftarrow{j_0^{CH}} & CH^d(X_t) & \xrightarrow{j_1^{CH}} & CH^d(X) \\ s_{CH} \uparrow & & (i_t)_{CH} \uparrow & & \uparrow i_{CH} \\ CH^0(Y) & \xleftarrow{k_0^{CH}} & CH^0(Y_t) & \xrightarrow{k_1^{CH}} & CH^0(Y). \end{array}$$

Let j_t be the open embedding $X_t - Y_t \rightarrow X_t$. The following statement was proved in [Pa, Lemma 1.4.2].

Lemma 2. $\text{Ker} j_t^{CH} \cap \text{Ker} j_0^{CH} = 0$.

In our setting, we see that, since the classes occurring in formula (1) are functorial with respect to transversal squares (see Lemma 1), it suffices to prove (1) for the embedding $Y_t \rightarrow X_t$. By Lemma 1, the elements in question lie in $\text{Ker} j_t^{CH}$. Lemma 2 shows that, to prove that the elements are equal in $CH^d(X_t)$, it suffices to check that their images under j_0^{CH} are equal, i.e., by Lemma 1, that the corresponding classes for the embedding $s : Y \rightarrow \mathbf{P}(1 \oplus N)$ are equal (the zero section).

Thus, we may assume that $X = \mathbf{P}(E)$ for a vector bundle E/X and that $i = s$ is the zero section.

1.2. Reduction to the embedding of a k -rational point $pt \rightarrow \mathbf{P}^d$. Let $K = k(Y)$ be the field of rational functions on Y . We consider the transversal square

$$\begin{array}{ccc} \mathbf{P}_K^d & \xrightarrow{\phi} & \mathbf{P}(E) \\ s' \uparrow & & \uparrow s \\ \text{Spec}(K) & \xrightarrow{\psi} & Y, \end{array}$$

where ψ is the embedding of the generic point and ϕ is the embedding of the generic fiber. Passing to the Chow groups, we obtain the following commutative diagram:

$$\begin{array}{ccc} CH^d(\mathbf{P}_K^d) & \xleftarrow{\phi^{CH}} & CH^d(\mathbf{P}(E)) \\ (s')_{CH} \uparrow & & \uparrow s_{CH} \\ CH^0(\text{Spec}(K)) & \xleftarrow{\psi^{CH}} & CH^0(Y). \end{array}$$

The lower horizontal arrow ψ^{CH} is the isomorphism $\mathbf{Z} \rightarrow \mathbf{Z}$.

We prove that the classes $[Y]$ and $c_d(\mathcal{O}_Y)$ lie in $\text{Im } s_{CH}$. For the class $[Y]$ this is obvious because

$$[Y] = s_{CH}(1).$$

Consider the exact sequence of localization,

$$0 \rightarrow CH^0(Y) \xrightarrow{s_{CH}} CH^d(\mathbf{P}(E)) \xrightarrow{j^{CH}} CH^d(\mathbf{P}(E) - Y) \rightarrow 0,$$

where the first arrow is injective because $p_{CH} \circ s_{CH} = \text{id}$.

By Lemma 1, the class $c_d(i_K \mathcal{O}_Y)$ belongs to $\text{Ker } j^{CH}$. Consequently, there exists a unique element $a \in CH^0(Y)$ such that

$$c_d(\mathcal{O}_Y) = s_{CH}(a).$$

Now, the proof of formula (1) reduces to checking the identity

$$a = (-1)^{d-1}(d-1)!$$

in $CH^0(Y) = \mathbf{Z}$. Applying the isomorphism ψ^{CH} and introducing the notation

$$a' = \psi^{CH}(a),$$

we obtain the equivalent identity

$$a' = (-1)^{d-1}(d-1)!$$

in $CH^0(pt) = \mathbf{Z}$, where $pt = \text{Spec } K(Y)$. Applying the homomorphism s'_{CH} to the latter identity and taking into account the relation

$$\begin{aligned} s'_{CH}(a') &= s'_{CH}(\psi^{CH}(a)) = \phi^{CH}(s_{CH}(a)) = \phi^{CH}(c_d(i_K \mathcal{O}_Y)) \\ &= c_d(\phi^{CH}(s_K(\mathcal{O}_Y))) = c_d(s'_{CH}(\psi^{CH}(\mathcal{O}_Y))) = c_d(s'_{CH}(\mathcal{O}_{pt})), \end{aligned}$$

we obtain the equivalent identity

$$c_d(s_K \mathcal{O}_{pt}) = (-1)^{d-1}(d-1)! [pt]$$

in $CH^d(\mathbf{P}^d)$, which is precisely formula (1) for the embedding $pt \rightarrow \mathbf{P}^d$.

1.3. The case of an embedding $i : pt \rightarrow \mathbf{P}^n_{k(pt)}$. Now, we prove formula (1) in the case where $Y = pt$ is a k -rational point, $X = \mathbf{P}^n_k$, and $i : pt \hookrightarrow \mathbf{P}^n_k$ is an embedding. By the projective bundle theorem, we have

$$CH(\mathbf{P}^n) = \mathbf{Z}[\zeta]/(\zeta^{n+1}), \quad \zeta = c_1(\mathcal{O}(1)) \in CH(\mathbf{P}^n).$$

Moreover,

$$s_{CH}(1) = [pt] = \zeta^n.$$

Therefore, in this case, (1) is equivalent to the relation

$$c(i_* \mathcal{O}_{pt}) = 1 + (-1)^{n-1}(n-1)! \zeta^n.$$

Consider the Koszul complex

$$0 \rightarrow \wedge^n Q^* \rightarrow \wedge^{n-1} Q^* \rightarrow \dots \rightarrow Q^* \rightarrow \mathcal{O}_{pt} \rightarrow 0.$$

Here, Q^* is the vector bundle dual to the universal quotient bundle over \mathbf{P}^n . We have

$$i_K([\mathcal{O}_{pt}]) = \sum_{k=1}^n (-1)^k [\wedge^k Q^*]$$

in $K_0(\mathbf{P}^n)$. Thus,

$$c(i_K([\mathcal{O}_{pt}])) = \prod_{k=1}^n c(\wedge^k Q^*)^{(-1)^k} = \prod_{k=1}^n (1 - k\zeta)^{(-1)^k \binom{n}{k}},$$

where $\zeta = c_1(\mathcal{O}(1))$. For brevity, we denote the latter product by Π . Taking the logarithm of Π , we obtain (the calculation below makes sense because ζ is nilpotent)

$$\begin{aligned} \log(\Pi) &= \sum_{k=1}^n (-1)^k \binom{n}{k} \log(1 - k\zeta) \\ &= \sum_{k=1}^n (-1)^k \binom{n}{k} (-1) \sum_{j=1}^{\infty} \frac{(k\zeta)^j}{j} = \sum_{j=1}^{\infty} \left(\sum_{k=1}^n (-1)^{k-1} \binom{n}{k} k^j \right) \frac{\zeta^j}{j}. \end{aligned}$$

Finally,

$$(2) \quad \log(\Pi) = \sum_{j=1}^{\infty} S_{n,j} \frac{\zeta^j}{j},$$

where

$$S_{n,j} = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} k^j, \quad j \geq 1.$$

Since the $S_{n,j}$ differ from the second kind Stirling numbers by the factor $(-1)^{n-1}n!$, we have

$$S_{n,j} = 0, \quad j < n, \quad S_{n,n} = (-1)^{n-1}n!.$$

Using (2), finally we obtain

$$c([\mathcal{O}_{pt}]) = \Pi = \exp(\log(\Pi)) = 1 + \frac{S_{n,n}}{n} \zeta^n = 1 + (-1)^{n-1} (n-1)! \zeta^n.$$

In the paper [Gr], Grothendieck introduced the functor

$$CH_{\text{mult}}(X) = \mathbf{Z} \times \{1 + a_0 + a_1 + \dots, \text{ where } a_i \in CH^i(X)\}$$

(in [Gr], it was denoted by \tilde{A}) as the direct product of groups (the second factor is regarded as the multiplicative subgroup of invertible elements in $CH(X)$). The product in $CH_{\text{mult}}(X)$ is defined so that the natural transformation of functors $c_{\text{mult}} : K_0 \rightarrow CH_{\text{mult}}$,

$$c_{\text{mult}}(X) : E \rightarrow (rk(E), c(E)),$$

is a homomorphism of rings. In more detail,

$$1 = (1, 1) \in CH_{\text{mult}}(X),$$

and multiplication is given by the formulas

$$\begin{aligned} (1, 1 + x) \times (1, 1 + y) &= (1, 1 + x + y), \\ (n, 1 + x_1 + \dots + x_n) \times (m, 1 + y_1 + \dots + y_m) &= (nm, F_{n,m}(x_1, \dots, x_n; y_1, \dots, y_m)), \\ F_{n,m}(x_1, \dots, x_n; y_1, \dots, y_m) &= \prod_{i,j} (1 + \xi_i + \eta_j), \\ 1 + x_1 + \dots + x_n &= (1 + \xi_1) \cdots (1 + \xi_n), \\ 1 + y_1 + \dots + y_m &= (1 + \eta_1) \cdots (1 + \eta_m). \end{aligned}$$

In that way, CH_{mult} becomes a ring functor. The functor CH_{mult} is not an oriented cohomology pretheory in the sense of [PS], because the projective bundle theorem fails: though the ring $CH_{\text{mult}}(\mathbf{P}^n)$ is a free $\mathbf{Z} = CH_{\text{mult}}(pt)$ -module of rank $n + 1$, it has no generators of the form $1, \zeta, \zeta^2, \dots, \zeta^{n+1}$ (see §2).

The ring CH_{mult} allows us to calculate $c(\mathcal{O}_{pt}) \in CH(\mathbf{P}^n)$ without using the Koszul complex. Specifically,

$$(0, c(\mathcal{O}_{pt})) = c_{\text{mult}}(\mathcal{O}_{pt}) = c_{\text{mult}}([\mathcal{O}_{\mathbf{P}^{n-1}}]^n) = c_{\text{mult}}(\mathcal{O}_{\mathbf{P}^{n-1}})^n = \zeta_{\text{mult}}^n,$$

where

$$\begin{aligned} \zeta_{\text{mult}} &= c_{\text{mult}}(\mathcal{O}_{\mathbf{P}^{n-1}}) = c_{\text{mult}}(\mathbf{1} - [\mathcal{O}(-1)]) \\ &= 1 - c_{\text{mult}}(\mathcal{O}(-1)) = -(0, 1 - \zeta). \end{aligned}$$

By the binomial formula, we obtain

$$\begin{aligned} \zeta_{\text{mult}}^n &= (1 - c_{\text{mult}}(\mathcal{O}(-1)))^n = \sum_{k=0}^n (-1)^k \binom{n}{k} c_{\text{mult}}(\mathcal{O}(-k)) \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} (1, 1 - k\zeta) = \left(0, \prod_{k=1}^n (1 - k\zeta)^{(-1)^k \binom{n}{k}}\right). \end{aligned}$$

We have

$$(3) \quad c([\mathcal{O}_{pt}]) = \prod_{k=1}^n (1 - k\zeta)^{(-1)^k \binom{n}{k}} = \Pi.$$

Remark. In the case of the embedding of a rational point into a projective space, the Riemann–Roch formula without denominators can also be proved with the help of the Riemann–Roch–Grothendieck formula for this embedding, because the Chow ring of the projective space has no torsion.

§2. THE FUNCTOR CH_{mult}

Theorem. 1. *The ring $CH_{\text{mult}}(\mathbf{P}^n)$ is a free $CH_{\text{mult}}(pt) = \mathbf{Z}$ -module of rank $n + 1$.*

2. *For $n \geq 3$, there are no elements ζ_{mult} such that $1, \zeta_{\text{mult}}, \dots, \zeta_{\text{mult}}^n$ generate the module $CH_{\text{mult}}(\mathbf{P}^n)$.*

Proof. 1. In $CH_{\text{mult}}(X)$, there is a natural monotone decreasing filtration $CH_{\text{mult}}^{(k)}(X)$, $k \geq 0$, where

$$\begin{aligned} CH_{\text{mult}}^{(0)}(X) &= CH_{\text{mult}}(X), \\ CH_{\text{mult}}^{(1)}(X) &= \{(0, 1 + a_1 + a_2 + \dots), a_i \in CH^i(X)\}, \\ CH_{\text{mult}}^{(k)}(X) &= \{(0, 1 + 0 + \dots + 0 + a_k + \dots), a_i \in CH^i(X)\}, \quad 1 \leq k \leq n, \\ CH_{\text{mult}}^{(k)}(X) &= 0, \quad k > n. \end{aligned}$$

This filtration has the property

$$CH_{\text{mult}}^{(p)}(X)CH_{\text{mult}}^{(q)}(X) \subset CH_{\text{mult}}^{(p+q)}(X).$$

For each p , there is a natural isomorphism

$$CH_{\text{mult}}^{(p)}(X)/CH_{\text{mult}}^{(p+1)} \rightarrow CH^p(X)$$

(for $p = 0$, we assume that X is connected); i.e., the graded module $GCH_{\text{mult}}(X)$ associated with the filtration $CH_{\text{mult}}^{(p)}$ is isomorphic to $CH(X)$. Since $CH(\mathbf{P}^n)$ is a free \mathbf{Z} -module of rank $n + 1$, we see that $CH_{\text{mult}}(\mathbf{P}^n)$ is also a free \mathbf{Z} -module of rank $n + 1$.

Moreover, it is easy to show that if ζ is a generator of $CH^1(\mathbf{P}^n)$, then the elements $e_0 = 1, e_1 = (0, 1 + \zeta), e_2 = (0, 1 + \zeta^2), \dots, e_n = (0, 1 + \zeta^n)$ form an additive basis of $CH_{\text{mult}}(\mathbf{P}^n)$.

2. Now, let $\zeta_{\text{mult}} = (m, 1 + a_1\zeta + o(\zeta))$ be an arbitrary element of $CH_{\text{mult}}(\mathbf{P}^n)$. We may assume that $m = 0$. If $a_1 = 0$, i.e., $\zeta_{\text{mult}} \in CH_{\text{mult}}^{(2)}$, then the powers $\zeta_{\text{mult}}^k, k \geq 1$, also belong to $CH_{\text{mult}}^{(2)}$, and obviously, cannot form a basis of $CH_{\text{mult}}(\mathbf{P}^n)$.

Thus, we may assume that $a_1 \neq 0$. A computation similar to that in the proof of the Riemann–Roch theorem without denominators shows then that

$$\zeta_{\text{mult}}^k = (0, 1 + (-1)^s a_1 (k-1)! \zeta^k + o(\zeta^k)).$$

This means that, expanding the elements $1, \zeta_{\text{mult}}, \dots, \zeta_{\text{mult}}^n$ with respect to the basis e_0, e_1, \dots, e_n , we obtain a lower triangular matrix with entries $(-1)^s a_1 (k-1)!$ on the diagonal. Thus, $|a_1| = 1$, and $1, \zeta_{\text{mult}}, \dots, \zeta_{\text{mult}}^n$ is not a basis if $n \neq 0, 1, 2$. \square

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Received 14/JUN/2006

Translated by O. B. PODKOPEV