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OVERGROUPS OF $\mathrm{EO}(n, R)$

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Abstract. Let R be a commutative ring with 1, n a natural number, and let $l = [n/2]$. Suppose that $2 \in R^*$ and $l \geq 3$. We describe the subgroups of the general linear group $\mathrm{GL}(n, R)$ that contain the elementary orthogonal group $\mathrm{EO}(n, R)$. The main result of the paper says that, for every intermediate subgroup H, there exists a largest ideal $A \trianglelefteq R$ such that $\mathrm{EEO}(n, R, A) = \mathrm{EO}(n, R)E(n, R, A) \trianglelefteq H$. Another important result is an explicit calculation of the normalizer of the group $\mathrm{EEO}(n, R, A)$. If $R = K$ is a field, similar results were obtained earlier by Dye, King, Shang Zhi Li, and Bashkirov. For overgroups of the *even* split elementary orthogonal group $\mathrm{EO}(2l, R)$ and the elementary symplectic group $E_p(2l, R)$, analogous results appeared in previous papers by the authors (Zapiski Nauchn. Semin. POMI, 2000, v. 272; Algebra i Analiz, 2003, v. 15, no. 3).

The present work, together with [19], constitutes a final and fully developed version of the paper "Overgroups of classical groups over commutative rings", which was predicted in §11 of the survey [92]. Together with our papers [18, 19] it *completely* describes the subgroups of the general linear group $G = \mathrm{GL}(n, R)$ over a commutative ring R, containing the elementary subgroup of a *split* classical group in the vector representation. In [78], the second-named author generalized these results to the *even* unitary groups.

Namely, in [18] we described the overgroups of $\mathrm{EO}(2l, R)$ under the assumption that $l \geq 3$, $2 \in R^*$, and in [19] we described the overgroups of $\mathrm{Ep}(2l, R)$ under the assumption that either $l \geq 3$, or $l = 2$ and the ring R does not have the residue field $\mathbb{F}_2$. In the present paper, we consider the last remaining case; namely, we describe the overgroups of $\mathrm{EO}(n, R)$, where $n = 2l + 1$. As in the papers [18, 19], it turns out that when $n \geq 7$ and $2 \in R^*$, for every such subgroup H there exists a unique ideal A in R such that H lies between the group

$$\mathrm{EEO}(n, R, A) = \mathrm{EO}(n, R)E(n, R, A)$$

and its normalizer in $\mathrm{GL}(n, R)$. Since the even-dimensional case is very similar to the odd-dimensional one, and is in fact slightly easier, in the present paper we also give another proof, and a refinement of the results of [18]. The principal result of the present paper can be stated as follows.

Theorem 1. *Let R be a commutative ring. Suppose that $2 \in R^*$ and $n \geq 7$. Then for every subgroup H in $G = \mathrm{GL}(n, R)$ containing the elementary orthogonal group $\mathrm{EO}(n, R)$ there exists a unique ideal $A \trianglelefteq R$ such that*

$$\mathrm{EEO}(n, R, A) \leq H \leq N_G(\mathrm{EEO}(n, R, A)).$$

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An important pendant to Theorem 1 is the following result, in which we explicitly calculate the normalizer of $\mathrm{EEO}(n, R, A)$. Namely, consider the reduction homomorphism $\rho_A : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/A)$ and denote by $\mathrm{CGO}(n, R, A)$ the full preimage of the group $\mathrm{GO}(n, R)$ with respect to ρ_A. The condition for a matrix to belong to $\mathrm{CGO}(n, R, A)$ is described by obvious quadratic congruences on its entries. Now we are in a position to state the second major result of the present paper.

Theorem 2. *Under the assumptions of Theorem* 1, *for any ideal $A \trianglelefteq R$ we have*

$$N_G(\mathrm{EEO}(n, R, A)) = \mathrm{CGO}(n, R, A).$$

Thus, combining Theorems 1 and 2, we see that for any subgroup H, $\mathrm{EO}(n, R) \leq H \leq \mathrm{GL}(n, R)$, there exists a unique ideal such that

$$\mathrm{EO}(n, R, A) \leq H \leq \mathrm{CGO}(n, R, A).$$

In the sequel such a description of the overgroups of $\mathrm{EO}(n, R)$ will be called *standard*.

Let us confront our theorem with the known results pertaining to the case of a field $R = K$, of characteristic not 2. There are only two possibilities for A, namely, $A = 0$ or $A = K$. Next, $\mathrm{EEO}(n, K, 0) = \mathrm{EO}(n, K) = \Omega(n, K)$ coincides with the kernel of the spinor norm (see [1, 25, 47, 56, 57, 66, 75, 78]), whereas $\mathrm{CGO}(n, K, 0) = \mathrm{GO}(n, K)$. On the other hand, $\mathrm{EEO}(n, K, K) = E(n, K) = \mathrm{SL}(n, K)$, $\mathrm{CGO}(n, K, K) = \mathrm{GL}(n, K)$. Thus, in this special case Theorem 1 asserts that any overgroup of $\Omega(n, K)$ in $\mathrm{GL}(n, K)$ is either contained in $\mathrm{GO}(n, K)$, or contains $\mathrm{SL}(n, K)$. Thus, for a field our result is (in the case $n \geq 6$) a petty refinement of a result due to Oliver King [63, Theorem 1], which described the overgroups of $\mathrm{SO}(n, R)$.

Of course, in the case of a field, King proved his theorem for the isotropic orthogonal groups $\mathrm{EO}(n, R, f)$. More precisely, let $\nu(f)$ be the Witt index of the quadratic form f. In Theorem 1 of [63] it is assumed that $n \geq 3$ and $\nu(f) \geq 1$. Theorem 1 of the present paper remains valid (and can be proven by the methods of the present paper!) under the assumption that $\nu(f) \geq 3$. By the methods of [19] one can prove a similar result under the assumption that $\nu(f) \geq 1$ and $\nu(f_M) \geq 3$ for all maximal ideals $M \in \mathrm{Max}(R)$. The authors plan to return to these and similar generalizations in a subsequent paper.

Observe that Roger Dye, Oliver King, and Shang Zhi Li obtained similar results for overgroups of all classical groups, not necessarily split, but isotropic enough [53]–[55], [63]–[65], [71]–[73]. In most cases the normalizers of classical groups are maximal in $\mathrm{GL}(n, K)$ (or in characteristic 2, in another classical group). From the viewpoint of the *Maximal Subgroup Classification Project* they form the Aschbacher class $\mathcal{C}_8$, and for an algebraically closed or a finite field they were studied in the context of classification of the maximal subgroups; see, for example, [37, 66, 75].

A possible approach to the proof of these results in the field case is based on the fact that the isotropic classical groups in vector representations contain unipotent elements of small residues, namely, elements of residue 1 in the symplectic case and of residue 2 (products of two commuting linear transvections) in the orthogonal case. Thus, in the field case, a description of their overgroups follows also from MacLaughlin type theorems (description of irreducible subgroups generated by small dimensional elements). It seems that the most powerful specific results in this direction were obtained by Evgeny Bashkirov [2]–[6]; in fact, he described the subgroups of $\mathrm{GL}(n, K)$ that contain a classical group not over the field K itself, but over its subfield $L \leq K$ such that the extension K/L is algebraic. In the survey [93], one can find a detailed exposition of these results. Similar results should follow also from the general theorems of Franz Timmesfeld on quadratic action subgroups [85].

The possibility of generalizing theorems of Dye–King–Li type to arbitrary commutative rings by using *decomposition of unipotents* was suggested in [92, 83]. However, before

[18, 19], detailed proofs had never been published. In fact, it was even claimed there that by the same method one can describe the subgroups of $\mathrm{GL}(n, R)$ *normalized* by the elementary classical group. This result would be a vast generalization not only of the above-mentioned results by R. Dye, J. King, and Shang Zhi Li on overgroups of classical groups, but also of the theorems on the normal structure of classical groups, such as the Wilson–Golubchik theorem for GL_n and the theorems of Golubchik, Abe–Suzuki and Abe–Vaserstein–Fuan Li for Sp_{2l} and SO_n; see references in [92, 83, 44] and further discussion in §15.

The core of our proof of Theorem 1 is the following alternative (as above, we assume that $2 \in R^*$, $l \geq 3$).

Main lemma. *Let H be a subgroup of $\mathrm{GL}(n, R)$ containing $\mathrm{EO}(n, R)$. Then either $H \leq \mathrm{GO}(n, R)$, or H contains a nontrivial elementary linear transvection $t_{ij}(\xi)$, $\xi \in R$, $\xi \neq 0$, where $i \neq -j$, $i, j \neq 0$.*

This lemma is established in §§7–12, after which Theorem 1 follows from it and Theorem 2 by an application of the standard device known as level reduction. The proof of the main lemma itself is based on reduction to the standard parabolic subgroups. For the key step of this reduction, here, as in [18], we deploy decomposition of unipotents rather than localization. In this respect, the present paper is considerably easier than [19], because now all calculations occur at the level of the ground ring R itself. On the other hand, technically the reduction to the groups of smaller rank is *distinctly* harder than in the symplectic case. This is due to the fact that in $\mathrm{SO}(n, R)$ there are no elements of residue 1. As a result, we must implement reduction not to the maximal parabolic subgroup P_1 itself, but rather to intersections of two such subgroups, which can be placed differently with respect to $\mathrm{SO}(n, R)$.

Yet another *patent* distinction of the orthogonal case from the symplectic one is that, describing overgroups of $\mathrm{EO}(n, R)$, it is downright *impossible* to renounce some form of the condition $2 \in R^*$, at least if we hope to get the *standard* answer. It suffices to look at the case when K is a field of characteristic 2. In that case $\mathrm{EO}(2l, K)$ is contained in $\mathrm{Sp}(2l, K)$, so that the description of the overgroups of $\mathrm{EO}(2l, K)$ in $\mathrm{GL}(2l, K)$ is blatantly nonstandard. Worse than that, if the field K is not perfect, there even exist *nonalgebraic* subgroups of $\mathrm{Sp}(2l, K)$ containing $\mathrm{SO}(2l, K)$; see, for example, [53].

A conceptual explanation of this phenomenon is that, from the viewpoint of Bak's theory, $\mathrm{EO}(2l, R)$ corresponds to the *minimal* form parameter, whereas $\mathrm{Ep}(2l, R)$ corresponds to the *maximal* one [40, 41, 44, 46, 57]. This means that while for the symplectic case all complications related to the noninvertibility of 2 are concealed in the description of the *normal subgroups* of $\mathrm{Sp}(2l, R)$ itself, in the orthogonal case they become apparent already in the description of *overgroups*. In other words, without some version of the condition $2 \in R^*$, to describe overgroups of $\mathrm{EO}(n, R)$ is at least as hard as to describe the normal subgroups of $\mathrm{Sp}(2l, R)$. This last problem is notoriously difficult. For *even* groups of rank $l \geq 4$ a right statement of such a description without *any* form of the condition $2 \in R^*$ was obtained in the recent paper [78] of the second-named author.

Even for ranks $l \geq 3$, without the invertibility of 2 one has to state the answer in terms of admissible pairs [34]–[36], form ideals [40, 44], quasi-ideals [87, 90], or Jordan ideals [51]. For ranks $l \leq 2$, the situation gets out of hand completely. It is known that over an arbitrary commutative ring any sensible classification of the overgroups of $\mathrm{EO}(3, R)$ or $\mathrm{EO}(4, R)$ is bluntly impossible. From the paper [51] by Costa and Keller it follows that the classification of the normal subgroups of $\mathrm{Sp}(4, R)$ over an arbitrary commutative ring R is sort of possible. However, it should be stated very differently from the generic case, and in a *terribly* much more complicated way. Thus, it might be possible to classify the overgroups of $\mathrm{EO}(5, R)$, but this is an *extremely* arduous problem.

Ultimately, we are interested in the description of the subgroups of $\mathrm{GL}(n, R)$ *normalized* by an elementary classical group. The proof of such a result (of course, only for the case when 2 is invertible!) will be completed in our next paper. The above implies that *some* version of the condition $2 \in R^*$ is *absolutely* unavoidable in the description of subgroups normalized by an elementary classical group, *both* in the orthogonal and the symplectic cases, but *for different reasons*.

As has already been mentioned, from the viewpoint of Aschbacher's *subgroup structure theorem* we are describing overgroups of groups from the class $\mathcal{C}_8$. Ideologically, our paper closely follows the works of Zenon Borewicz, the first-named author, Alexei Stepanov, and Igor Golubchik on overgroups of subsystem subgroups (Aschbacher classes $\mathcal{C}_1 + \mathcal{C}_2$) in the classical groups over commutative rings; see [7, 8], [10]–[16], [23, 30], and further references in [83, 93, 95]. Until quite recently, very little has been known concerning the description of overgroups for other Aschbacher classes, at least in the case of rings. Presently, the situation is changing rapidly. Quite recently Alexei Stepanov [82] obtained striking results on subring subgroups (class $\mathcal{C}_5$). It turned out that such a description is, *as a rule*, not standard (roughly speaking, it is standard only for algebraic extensions and for rings of dimension 1). As another recent result one could mention the classification of overgroups of tensored subgroups (classes $\mathcal{C}_4 + \mathcal{C}_7$), currently under way. Such a classification can be obtained under the following assumptions: the ring R is commutative and the degrees of all factors are at least 3. Observe that for rings it does not make sense to consider the class $\mathcal{C}_6$, whereas the description of overgroups for groups of class $\mathcal{C}_3$ is blocked by serious technical obstructions, and it seems that such a description is possible only in some very special cases.

The present paper is organized as follows. The first three sections are of an introductory nature; we recall the main notation and necessary facts on the groups $\mathrm{GL}(n, R)$ and $\mathrm{GO}(n, R)$, and on elementary orthogonal transvections. The three following sections are devoted to the proof of Theorem 2. In §4 we calculate the normalizer of $\mathrm{EO}(n, R)$ in $\mathrm{GL}(n, R)$. In §5 we start the study of transvections in subgroups containing $\mathrm{EO}(n, R)$. Finally, in §6 we characterize matrices in $\mathrm{CGO}(n, R)$ and prove Theorem 2. From a technical viewpoint, the core of the paper is §§7–14, directly devoted to the proof of Theorem 1. In §7 we recall the version of decomposition of unipotents used in the sequel. This method implements a reduction to the parabolic subgroups of type P_1. There we also state Proposition 4, asserting the possibility of extracting transvections in P_1. After that we should — in the style of "Sickle and Hammer — Karacharovo" (*and immediately calculated*)[1] — declare that this proposition is *obvious* and pass directly to §14. Instead, in §§8–13 we reproduce an excruciatingly detailed proof of this fact. Both in style and mood, the exposition in these sections corresponds to the *inner chapters* of the poem, *here is my cherished lemma, in two words*. The partitioning into subcases in these sections is essentially the same as in [18], but the lemmas themselves are entirely reconfigured. We believe that this makes the layout of the proof much more transparent. Furthermore, we had to add the analysis of additional subcases, cropping up for an odd n. The proof of Theorem 1 is finished in §14. Finally, in §15 we state some open problems.

§1. Principal notation

Our notation is for the most part fairly standard and coincides with the notation used in [8, 14, 82, 18, 19]. A canonical reference, where one can find *all* the definitions

[1]For the sake of a non-Russian reader, we mention that this text, as well as some other fragments, and the titles of §§8–14, refer to the most important Russian novel ever, "Moscow — Petushki" by Venedikt Erofeev, designated by the author himself as a *poem*. The inner chapters of that novel are characterized by a peculiar blend of compulsory reiterations with supreme ingenuity.

and facts we need, is the remarkable monograph by A. Hahn and O. T. O'Meara [57]. Nevertheless, to set the scene and enable the reading of the present paper *independently* of [19], below we reproduce §1 of [19], pertaining to subgroups of $\mathrm{GL}(n, R)$.

First, let G be an arbitrary group. By a commutator of two elements $x, y \in G$ we always understand the left-normed commutator $[x, y] = xyx^{-1}y^{-1}$. By ${}^{x}y = xyx^{-1}$ and $y^x = x^{-1}yx$ we denote, respectively, the left and right conjugates of y by x. The notation $H \leq G$ means that H is a subgroup in G, while $H \trianglelefteq G$ means that H is a normal subgroup. For a subset $X \subseteq G$ we denote by $\langle X \rangle$ the subgroup it generates. For $H \leq G$, we denote by $\langle X \rangle^H$ the smallest subgroup in G containing X and normalized by H. For two subgroups $F, H \leq G$, we denote by $[F, H]$ their mutual commutator subgroup generated by all commutators $[f, h]$, $f \in F$, $h \in H$. Multiple commutators are also left-normed; in particular, $[E, F, H] = [[E, F], H]$.

Now, let R be an arbitrary associative ring with 1; by default it is assumed to be commutative. Let $M(m, n, R)$ be the R-bimodule of $(m \times n)$-matrices with entries in R, and let $M(n, R) = M(n, n, R)$ be the full matrix ring of degree n over R. Further, let $R^\bullet = R \backslash \{0\}$ be the set of nonzero elements of the ring R, and R^* its multiplicative group. We denote by $G = \mathrm{GL}(n, R) = M(n, R)^*$ the general linear group of degree n over R. When R is commutative, $\mathrm{SL}(n, R)$ is the special linear group of degree n over R. As usual, a_{ij} denotes the entry of a matrix $a \in G$ at the position (i, j), i.e., $a = (a_{ij})$, $1 \leq i, j \leq n$. Further, $a^{-1} = (a'_{ij})$ is the inverse of a and a^t is its transpose. By $a_{*j} = (a_{1j}, \dots, a_{nj})^t$ we denote the jth column of the matrix a, and by $a_{i*} = (a_{i1}, \dots, a_{in})$ its ith row.

As usual, e is the identity matrix, and e_{ij} is a standard matrix unit, i.e., the matrix that has 1 at the position (i, j) and zeros elsewhere. By $t_{ij}(\xi) = e + \xi e_{ij}$, $\xi \in R$ and $1 \leq i \neq j \leq n$, we denote an *elementary transvection*. By $X_{ij} = \{t_{ij}(\xi), \xi \in R\}$, $i \neq j$, we denote the corresponding *root subgroup*. In the sequel we use (without any special reference) standard relations among elementary transvections, such as the additivity of $t_{ij}(\xi)t_{ij}(\zeta) = t_{ij}(\xi + \zeta)$ and the Chevalley commutator formula $[t_{ij}(\xi), t_{jh}(\zeta)] = t_{ih}(\xi\zeta)$; see, for example, [29, 49].

Now, let $I \trianglelefteq R$ be an ideal in R. Denote by $E(n, I)$ the subgroup in G generated by all elementary transvections of level I:

$$E(n, I) = \langle t_{ij}(\xi), \xi \in I, 1 \leq i \neq j \leq n \rangle .$$

In the most important case where $I = R$, the group $E(n, R)$ generated by all elementary transvections is called the (absolute) *elementary group*. In the sequel a major role is played by the *relative elementary group* $E(n, R, I)$. Recall that the group $E(n, R, I)$ is the normal closure of $E(n, I)$ in $E(n, R)$:

$$E(n, R, I) = \langle t_{ij}(\xi), \xi \in I, 1 \leq i \neq j \leq n \rangle^{E(n,R)}.$$

In what follows, we shall use a bunch of classical facts on elementary groups (often without any special reference). The following fact, first proved in [28], is cited as *Suslin's theorem*.

Lemma 1. *If R is commutative and $n \geq 3$, then the group $E(n, R, I)$ is normal in* $\mathrm{GL}(n, R)$.

The following statement was proved by Vaserstein and Suslin [19] and, in the context of Chevalley groups, by Tits [86].

Lemma 2. *For $n \geq 3$ the relative elementary subgroup $E(n, R, I)$ is generated by all transvections of the form $z_{ij}(\xi, \zeta) = t_{ji}(\zeta)t_{ij}(\xi)t_{ji}(-\zeta)$, $\xi \in I$, $\zeta \in R$, $1 \leq i \neq j \leq n$.*

Let, as above, $I \trianglelefteq R$, and let R/I be the factor-ring of R modulo I. Denote by $\rho_I : R \longrightarrow R/I$ the canonical projection sending $\lambda \in R$ to $\overline{\lambda} = \lambda + I \in R/I$. Applying the projection to all entries of a matrix, we get the reduction homomorphism

$\rho_I : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/I)$, which assigns to a matrix $a = (a_{ij})$ its class $\overline{a} = (\overline{a}_{ij})$ modulo I. The kernel of the homomorphism ρ_I is denoted by $\mathrm{GL}(n, R, I)$ and is called the *principal congruence-subgroup* in G of level I.

Now, let $C(n, R)$ be the center of the group $\mathrm{GL}(n, R)$, consisting of the scalar matrices εe, $\varepsilon \in R^*$. The full preimage of the center of $\mathrm{GL}(n, R/I)$ is denoted by $C(n, R, I)$ and is called the *full congruence-subgroup* of level I. The group $C(n, R, I)$ consists of all matrices congruent to a scalar matrix modulo I.

A key point in reduction modulo an ideal is the following *standard commutator formula*, proved by Vaserstein [87] and Borevich and Vavilov [8].

Lemma 3. *If R is commutative and $n \geq 3$, then*

$$[E(n, R), C(n, R, I)] = E(n, R, I).$$

In the sequel we consider the left action of $\mathrm{GL}(n, R)$ on a *right* R-module $V = R^n$ of rank n; V consists of all columns of height n over the ring R. Following Paul Cohn, we denote by ${}^nR$ the *left* R-module V^* of rank n, which can be identified with the set of all rows of length n over R. The standard bases of R^n and ${}^nR$ will be denoted by $e_1, \dots, e_n$ and $e^1, \dots, e^n$, respectively. In other words, $e_i = e_{*i}$ is the ith column of the identity matrix, whereas $e^i = e_{i*}$ is its ith row. In this notation, $a_{*j} = ae_j$, $a_{i*} = e^i a$, and $a_{ij} = e^i a e_j$. These obvious formulas turn useful for cross-cultural communication: where in linear algebra one speaks of rows or columns of a matrix in the group G, experts in K-theory and algebraic groups usually speak of orbits of e^1 or e_1 under the action of G ("orbit of the highest weight vector").

A column $v = (v_1, \dots, v_n)^t \in R^n$ is called *unimodular* if $Rv_1 + \dots + Rv_n = R$; i.e., the *left* ideal generated by the elements $v_1, \dots, v_n$ coincides with R. Similarly, a row $u = (u_1, \dots, u_n) \in {}^nR$ is called *unimodular* if $u_1R + \dots + u_nR = R$; i.e., the *right* ideal generated by the elements $u_1, \dots, u_n$ coincides with R. Unimodularity of rows/columns is a *necessary* condition for this column/row to be a column/row of an invertible matrix. However, except for some very special classes of rings, this condition is, generally speaking, not sufficient: as we have seen, to actually be a column of an invertible matrix, a column should lie in the orbit of e_1.

We need the following generalization of elementary transvections. A *transvection* (or, in the context where other classical groups are considered, a *linear transvection*) is a matrix of the form $t_{uv}(\xi) = e + u\xi v$, where $u \in R^n$, $v \in {}^nR$ are a column and a row such that $vu = 0$, whereas $\xi \in R$. Clearly, an elementary transvection $t_{ij}(\xi)$ is indeed a transvection (to see that, it suffices to set $u = e_i$, $v = e^j$). The identity $gt_{uv}(\xi)g^{-1} = t_{gu, vg^{-1}}(\xi)$, which holds for all $g \in \mathrm{GL}(n, R)$, shows that the class of transvections is stable with respect to conjugation.

By P_i we denote the ith standard *maximal parabolic subgroup* in $G = \mathrm{GL}(n, R)$. From a geometric viewpoint, the subgroup P_i, $i = 1, \dots, n-1$, is precisely the stabilizer of the submodule V_i generated by $e_1, \dots, e_i$ in V. In matrix form, P_i is realised as a group of upper block triangular matrices,

$$P_i = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, x \in \mathrm{GL}(i, R), y \in M(i, n-i, R), z \in \mathrm{GL}(n-i, R) \right\}.$$

From the viewpoint of the dual module V^*, the group P_i is the stabiliser of the submodule $V^{i+1} = V_i^{\perp}$ and should be denoted by ${}_{i+1}P$. Any subgroup conjugate to P_i is called a *parabolic subgroup of type P_i*. For reduction to smaller rank we also need the submaximal parabolic subgroups ${}_{j+1}P_i = P_{ij} = P_i \cap P_j$, where $1 \leq i < j \leq n-1$, stabilizing the flag $V_i < V_j$.

We are most interested in the group P_1 (this is precisely the stabilizer of the submodule $e_1R \leq V$, which consists of all matrices whose *first* column is proportional to e_1), and the group ${}_nP = P_{n-1}$ (which is the stabilizer of $Re^n \leq V^*$, and consists of all matrices whose *last* row is proportional to e^n). In fact we deal fairly often with the conjugates of P_1. Let $w_{ij} = e - e_{ii} - e_{jj} + e_{ij} + e_{ji}$. Consider the subgroup $Q_i = w_{1i}P_1w_{1i}$ that coincides with the stabilizer of the submodule $e_iR \leq V$. The group Q_i consists of the matrices whose ith row is proportional to e_i. In particular, $Q_1 = P_1$.

The group P_i can be decomposed as the semidirect product $P_i = L_i \leftthreetimes U_i$ of its *Levi subgroup* L_i and its *unipotent radical* U_i, where

$$L_i = \left\{ \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}, x \in \mathrm{GL}(i, R), z \in \mathrm{GL}(n-i, R) \right\},$$

$$U_i = \left\{ \begin{pmatrix} e & y \\ 0 & e \end{pmatrix}, y \in M(i, n-i, R) \right\}.$$

Alongside the subgroup P_i, we consider its *opposite* subgroup P_i^-, stabilizing the submodule in V generated by $e_{i+1}, \dots, e_n$ (thus, P_i^- is a subgroup of type P_{n-i}, rather than that of type P_i), In matrices, P_i^- is realized as

$$P_i^- = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}, x \in \mathrm{GL}(i, R), y \in M(n-i, i, R), z \in \mathrm{GL}(n-i, R) \right\}.$$

In other words, $P_i^- = P_i^t$ and the unipotent radical U_i^- of the group P_i^- equals U_i^t.

§2. Split orthogonal group

In the present section we briefly recall the principal notation, pertaining directly to the orthogonal case; see also [1, 12, 13, 16, 25, 47, 56, 57, 76, 79, 89]. Orthogonal groups $\mathrm{SO}(n, R)$ are adjoint Chevalley groups of type B_l (for odd $n = 2l + 1$) and intermediate Chevalley groups of type D_l (for even $n = 2l$) and are discussed in this respect in [34]–[36], [59, 60, 81, 84, 88, 92]. On the other hand, even orthogonal groups form a very special example of Bak hyperbolic unitary groups $U^\lambda(2l, R, \Lambda)$ over a form ring (R, Λ), which arises in the case when the involution on R is trivial, $\lambda = 1$, and $\Lambda = \Lambda_{\min}$. Thus, all results of the papers [40], [41], [44]–[46], [58], [91] are applicable to these groups. In fact, usually a proof for an orthogonal group requires only a small part of the calculations necessary in the general case. Moreover, in [67] orthogonal groups were studied in the language of associative algebras with involution. In the following sections, whenever we are not aware of a convenient reference (directly in the orthogonal case) to a result we need, without any further fuss we specialize results obtained in any of these more general contexts.

Starting with this section, we always assume that $2 \in R^*$. As we explained in the Introduction, this condition is absolutely unavoidable for the standard description in Theorem 1 to be valid. Moreover, when 2 is not invertible, orthogonal groups should be defined differently. However, the true reason why we do not consider the general case here is that for odd groups even most of our auxiliary results cannot be proved (as stated here!) without the assumption $2 \in R^*$. In fact, without this assumption, the *natural* answers to the problems we consider here can be given *only* in the context of the theory of unitary groups over form rings, see [40, 41], [44]–[46], [57, 58], and these answers are far beyond the scope of the present paper.

Set $l = [n/2]$, i.e., $n = 2l$ or $n = 2l+1$, depending on the parity of n. It is convenient to index the rows and columns of matrices in G as follows: $1, \dots, l, -l, \dots, -1$ when $n = 2l$ is even, and $1, \dots, l, 0, -l, \dots, -1$ when $n = 2l + 1$ is odd. By I we denote the set of indices from 1 to -1. Thus, $I = I^+ \cup I^-$, where $I^+ = \{1, \dots, l\}$ and $I^- = \{-l, \dots, -1\}$ when n is even, and $I = I^+ \cup \{0\} \cup I^-$, when n is odd. In the sequel we denote the sign of the index i by ε_i. Thus, $\varepsilon_i = +1$ for $i = 1, \dots, l$, $\varepsilon_i = -1$ for $i = -l, \dots, -1$, and $\varepsilon_0 = 0$. The same indexing is applied to all other objects. Thus, the base of the module $V = R^n$ will be indexed as $e_1, \dots, e_l, e_{-l}, \dots, e_{-1}$ or $e_1, \dots, e_l, e_0, e_{-l}, \dots, e_{-1}$, depending on the parity of n, while the base of the module $V^* = {}^nR$ will be indexed as $e^1, \dots, e^l, e^{-l}, \dots, e^{-1}$ or $e^1, \dots, e^l, e^0, e^{-l}, \dots, e^{-1}$, respectively.

Denote by $\mathrm{sdiag}(\lambda_1, \dots, \lambda_n)$ the matrix that has $\lambda_1, \dots, \lambda_n$ at the second diagonal in the Northeast to Southwest direction and zeros elsewhere. Consider the *superidentity* matrix $f = f_n = \mathrm{sdiag}(1, \dots, 1)$ of degree n, which has 1's alongside the second diagonal, and zeros elsewhere. Under the indexing agreement adopted in the preceding paragraph, the entries of $f = (f_{ij})$ are described as $f_{ij} = \delta_{i,-j}$.

By definition the *split special orthogonal group* $\Gamma = \mathrm{SO}(n, R)$ — in the present paper we do not consider other forms of orthogonal groups, and therefore usually omit the specification "split" — consists of all matrices $g \in \mathrm{SL}(n, R)$ in the general linear group that preserve the bilinear inner product B with the Gram matrix $f = f_n$. In other words, $\mathrm{SO}(n, R)$ consists of those matrices $g \in G$ for which $\det g = 1$ and $gfg^t = f$. The orthogonality condition of $g = (g_{ij})$ is most conveniently expressed in the form $g'_{ij} = g_{-j,-i}$.

Apart from the special orthogonal group, we consider also the *general orthogonal group*, which preserves the corresponding form B *up to similarity*. In other words, $\mathrm{GO}(n, R)$ consists of the matrices $g \in G$ for which $gfg^t = \lambda f$ for an appropriate *multiplier* $\lambda = \lambda(g) \in R^*$. Thus, the matrix entries of a matrix in the group $\mathrm{GO}(n, R)$ satisfy the following condition:

$$\lambda(g)g'_{ij} = g_{-j,-i} \quad \text{for all } i, j = 1, \dots, -1.$$

The subgroup in $\mathrm{GO}(n, R)$ consisting of all matrices a such that $\lambda = 1$ is the usual *orthogonal group* $\mathrm{O}(n, R)$, whereas the subgroup $\mathrm{SGO}(n, R)$ consisting of all matrices with determinant 1 is the *special general orthogonal group*. Thus, the usual special orthogonal group

$$\mathrm{SO}(n, R) = \mathrm{O}(n, R) \cap \mathrm{SGO}(n, R)$$

consists of the matrices for which *both* the multiplier and the determinant are equal to 1.

We translate these definitions to the geometric language. Let $V \cong R^n$ be the free right R-module of rank n with the standard base $e_1, \dots, e_{-1}$. On V, we fix the bilinear form with values in R whose matrix in the base $e_1, \dots, e_{-1}$ coincides with $f = f_n$. In that case the base $e_1, \dots, e_{-1}$ itself is usually called a *Witt base*. In other words, $B(u, v) = u^t f v$. Thus, the inner products of the base vectors are determined by the formula $B(e_i, e_j) = \delta_{i,-j}$. Clearly, B is *symmetric*, i.e., $B(u, v) = B(v, u)$, and *nondegenerate*, i.e., the map $u \mapsto \phi_u$ from V to the dual module $V^* = \mathrm{Hom}(V, R)$ defined by the formula $\phi_u(v) = B(u, v)$ is an isomorphism between V and V^*.

By definition, the *orthogonal group* $\mathrm{O}(n, R)$ is precisely the *isometry* group of B, i.e., consists of the matrices g in the general linear group $\mathrm{GL}(V) = \mathrm{GL}(n, R)$ that preserve the inner product, $B(gu, gv) = B(u, v)$. This can be expressed slightly differently. Namely, $\mathrm{SO}(n, R)$ preserves the quadratic form $x_1x_{-1} + \ldots + x_lx_{-l}$ when n is even, and the form $x_0^2/2 + x_1x_{-1} + \ldots + x_lx_{-l}$ when n is odd. The general orthogonal group $\mathrm{GO}(n, R)$ consists of all *similarities*, i.e., of $g \in G$ such that $B(gu, gv) = \lambda B(u, v)$ for some $\lambda \in R^*$.

The groups we have defined are *split* in the sense that they preserve a form of the maximal Witt index, and, consequently, contain a *split maximal torus*. From the viewpoint of the theory of algebraic groups, $\mathrm{SO}(n, R)$ is a Chevalley group of type D_l when $n = 2l$ is even, and of type B_l when $n = 2l + 1$ is odd (*Ree–Dieudonné theorem*). A detailed exposition of this framework, and a modern proof of the Ree–Dieudonné theorem (also for characteristic 2) can be found in the talk [60] by J.-Y. Hée.

Remark. In the preceding papers [11]–[14], [16] we usually considered a slightly different realization of the orthogonal group (described in the language of Lie algebras in [9, pp. 243–255]). Namely, for an *odd* $n = 2l + 1$, define a matrix $F = F_n$ as follows:

$$F = \operatorname{sdiag}(1, \dots, 1, 2, 1, \dots, 1),$$

where both series of 1's consist of l terms. In other words, F_{2l+1} is a block matrix whose blocks at the second diagonal are equal to $f_l, 2, f_l$, whereas all other blocks are zeros. In other words, now our orthogonal group preserves the form $x_0^2 + x_1x_{-1} + \dots + x_lx_{-l}$. This is a proper choice of the base from the viewpoint of the theory of Chevalley groups, since it generates an admissible $\mathbb{Z}$-form of the vector representation. This choice gives us proper formulas for root unipotents. However, in the present paper we are hardly interested in the short root elements, but equations arise over and over. This induced us to choose a wrong base, to avoid any reference to the parity of n, as long as possible.

In the sequel we need to write out explicitly the isomorphism between $V = R^n$ and $V^* = {}^nR$ determined by the inner product B. This isomorphism, usually referred to as the (orthogonal) *polarity*, to a column $v = (v_1, \dots, v_{-1})^t \in R^n$ assigns the row $\tilde{v} = (v_{-1}, \dots, v_1) = v^t f \in {}^{2l}R$, and, respectively, to a row $u = (u_1, \dots, u_{-1}) \in {}^{2l}R$ it assigns the column $\tilde{u} = (u_{-1}, \dots, u_1) = fu^t \in R^{2l}$. Clearly, the map $v \mapsto \tilde{v}$ is linear, $\tilde{\tilde{v}} = v$, and for any $u, v \in R^n$ one has $B(u, v) = \tilde{u}v$. A straightforward calculation shows that $\widetilde{gv} = \tilde{v}g^{-1}$ for all $v \in V$ and $g \in \mathrm{SO}(n, R)$ (see, for example, [44, Lemma 2.5]). In fact, this is merely another form of the equations defining the orthogonal group.

§3. Orthogonal transvections

A most important tool for the study of the above groups is provided by elementary root unipotent elements, or, in the classical language, elementary orthogonal transvections. An *elementary orthogonal transvection* is one of the matrices $T_{ij}(\xi)$, $\xi \in R$, $i \neq \pm j$, of the form

$$T_{ij}(\xi) = T_{-j,-i}(-\xi) = e + \xi e_{ij} - \xi e_{-j,-i}$$

when $i, j \neq 0$ (from the viewpoint of Chevalley groups, this is a *long root unipotent*), and of the form

$$T_{i0}(\xi) = T_{0,-i}(-\xi) = e + \xi e_{i0} - \xi e_{0,-i} - \xi^2/2 e_{i,-i}$$

when exactly one of the indices i, j equals 0 (this is a *short root unipotent*). Obviously, the elements of the second kind arise only for an odd n (in the root system of type D_l, all roots are long). Sometimes it is convenient to assume that $T_{i,-i}(\xi) = e$.

The elementary orthogonal transvections satisfy well-known commutation relations, which are special cases of the *Chevalley commutator formula* (see, for example, [29, 49]). These relations are explicitly listed, for example, in [12, 13, 16]. We reproduce specifically some of them. When all six indices $\pm i$, $\pm j$, $\pm h$ are pairwise distinct, we have

$$\big[T_{ij}(\xi), T_{jh}(\zeta)\big] = T_{ih}(\xi\zeta),$$

$$\big[T_{i0}(\xi), T_{0j}(\zeta)\big] = T_{ij}(2\xi\zeta),$$

$$\big[T_{ij}(\xi), T_{j0}(\zeta)\big] = T_{i0}(\xi\zeta)T_{i,-j}(\xi\zeta^2).$$

The above formulas are simply the Chevalley commutator formula for the following cases: two long roots whose sum is a long root; two short roots whose sum is a long root; a short and a long root whose sum is a root. In all other cases, except for those listed above (and those obtained from them by replacing $T_{ij}(\xi)$ with $T_{-j,-i}(\xi)$), the orthogonal transvections $T_{ij}(\xi)$ and $T_{hk}(\zeta)$ commute.

A central role in the present paper is played by the *elementary orthogonal group* $\mathrm{EO}(n,R)$, i.e., the subgroup in Γ generated by all elementary orthogonal transvections of the form $T_{ij}(\xi)$, $\xi\in R$, $i\neq j$. From the viewpoint of algebraic groups, this is the *elementary Chevalley group* $E(\Phi,R)$, where Φ is the root system of the group Γ. For fields, and, more generally, for semilocal and other similar rings, the group $\mathrm{EO}(n,K)$ coincides with the commutator subgroup of the orthogonal group, which is usually denoted by $\Gamma^0=\Omega(n,K)$ and is called the *kernel of the spinorial norm*. The group $\Omega(n,K)$ consists of all matrices with spinorial norm 1.

Many arguments in the present paper essentially depend on the following key result, first established by Suslin and Kopeĭko [33] for $n=2l$, by Kopeĭko [27] for $n=5$, and by Taddei [84] for an arbitrary odd degree (see also [20, 43, 44, 59, 70, 83, 88, 89, 92] for other proofs and further references).

Lemma 4. *Let R be a commutative ring. Then for any $n\geq 5$ the elementary group $\mathrm{EO}(n,R)$ is normal in $\mathrm{GO}(n,R)$.*

The following easy but fundamental fact is well known (see, for example, [81, Corollary 4.4]).

Lemma 5. *Let R be a commutative ring and $n\geq 5$; moreover, for $n=5$ we assume that R does not have factor-fields of 2 elements. Then the elementary group $\mathrm{EO}(n,R)$ is perfect.*

As was known to the XIX century classics, the proviso in the case of $n=5$ is necessary: the group $\Omega(5,2)=\mathrm{Sp}(4,2)$ over the field of 2 elements is isomorphic to the symmetric group S_6, and its commutator subgroup has index 2.

All elements $x=gT_{ij}(\xi)g^{-1}$, $g\in\mathrm{GO}(n,R)$, conjugate to $T_{ij}(\xi)$, $i\neq\pm j$, are also called *root unipotents*. Next, in the case of $i,j\neq 0$ the element x is said to be *long*, whereas when one of these indices equals 0, it is called a *short root element*. The following assertion is essentially Lemma 1 of [16]. The statement there is somewhat weaker, but this is exactly what results from the proof.

Lemma 6. *For any $n\geq 4$, the elementary group $\mathrm{EO}(n,R)$ is generated by the long root elements $hT_{ij}(\xi)h^{-1}$, where $i\neq\pm j$, $i,j\neq 0$, $\xi\in R$, and $h\in\mathrm{EO}(n,R)$.*

Proof. For an even n there is nothing to prove, while for an odd n the commutator formula for a long and a short root can be rewritten as follows:

$$T_{i0}(\xi)=T_{ij}(1)(T_{j0}(\xi)T_{ij}(-1)T_{j0}(-\xi))T_{i,-j}(-\xi^2).\qquad\square$$

Observe that Lemmas 1 and 3 immediately imply the following result.

Corollary. *For $n\geq 5$, the group $\mathrm{EO}(n,R)$ coincides with the group generated by the long root elements.*

Denote by $W=W(C_l)\cong\mathrm{Oct}_l$ the Weyl group of the group Γ. Let $n=2l$, and let the indices run as usual, $1,\dots,l,-l,\dots,-1$. Recall that the octahedral group Oct_l consists of all permutations in the symmetric group S_{2l} commuting with the sign change, i.e., such that $w(-i)=-w(i)$ for all i. For a permutation $w\in W$, we denote by the same letter any symplectic monomial matrix of the form $\sum\lambda_i e_{w(i),j}$, where $\lambda_i\in R^*$.

In extraction of transvections, we have to make an explicit use of semisimple elements contained in $\mathrm{EO}(n,R)$. Namely, for any $i=1,\dots,-1$, $\varepsilon\in R^*$, we denote by $D_i(\varepsilon)$ the *long root semisimple element*

$$D_i(\varepsilon)=e+(\varepsilon-1)e_{ii}+(\varepsilon^{-1}-1)e_{-i,-i}.$$

If, moreover, $i \neq \pm j$, then $D_{ij}(\varepsilon)$ denotes the *short root semisimple element*

$$D_{ij}(\varepsilon) = e + (\varepsilon - 1)e_{ii} + (\varepsilon^{-1} - 1)e_{jj} + (\varepsilon^{-1} - 1)e_{-i,-i} + (\varepsilon - 1)e_{-j,-j}.$$

It is very well known (see, for example, [29]) that the matrices $D_i(\varepsilon)$ and $D_{ij}(\varepsilon)$ are expressed as products of elementary orthogonal transvections, i.e., belong to $\mathrm{EO}(n, R)$.

The diagonal subgroup contained in $\mathrm{GO}(n, R)$ will be denoted by $T(n, R)$.[2] It is generated by the long semisimple root elements $D_i(\varepsilon)$, where $i = 1, \dots, l$, and $\varepsilon \in R^*$, and the following matrices:

- matrices $h(\varepsilon) = \mathrm{diag}(\varepsilon, \dots, \varepsilon, 1 \dots, 1)$,
- matrices $h(\varepsilon) = \mathrm{diag}(\varepsilon^2, \dots, \varepsilon^2, \varepsilon, 1 \dots, 1)$,

where the number of 1's is equal to l. From the viewpoint of algebraic groups, these are weight elements of weight ϖ_l.

§4. The normalizer of EO(n, R)

The Suslin–Kopeĭko–Taddei theorem (Lemma 4) tells us that the normalizer of the group $\mathrm{EO}(n, R)$ in $\mathrm{GL}(n, R)$ *contains* $\mathrm{GO}(n, R)$. In the present section we show that in fact it *coincides* with $\mathrm{GO}(n, R)$. It seems incredible that such a result could be unknown to experts. However, as we observed in [52, 19], we could not find its proof — or even an explcit statement — in the literature *even for the case of fields*. In [65] this result was stated for *finite* fields. We start with a more convenient form of equations determining when a matrix belongs to the group $\mathrm{GO}(n, R)$. Observe that the multiplier does not enter into these equations.

Denote by $\mathrm{Max}(R)$ the maximal spectrum of the ring R. For any $M \in \mathrm{Max}(R)$, we can define the localization $R_M = (R \setminus M)^{-1}R$ at the ideal M. Denote by $F_M : R \longrightarrow R_M$ the corresponding localization homomorphism. For any covariant functor G from the category of commutative rings to the category of groups, the map F_M induces a homomorphism $G(F_M) : G(R) \longrightarrow G(R_M)$, which is usually denoted simply by F_M. Since $\mathrm{GO}(n, R)$ is the group of points of an affine group scheme, the property of a matrix to belong to $\mathrm{GO}(n, R)$ is *local* (for *obvious* reasons, which we recalled once more in §3 of [19], see the paragraph preceding Lemma 8 there). In other words, equations need to be verified only modulo maximal ideals.

Lemma 7. *Let $g \in \mathrm{GL}(n, R)$, and let $F_M(g) \in \mathrm{GO}(n, R_M)$ for all $M \in \mathrm{Max}(R)$. Then $g \in \mathrm{GO}(n, R)$.*

Proposition 1. *A matrix $g = (g_{ij}) \in \mathrm{GL}(n, R)$ belongs to the group $\mathrm{GO}(n, R)$ if and only if*

$$g_{ir} g'_{sj} = g_{-j,-s} g'_{-r,-i}$$

for all $i, j, r, s = 1, \dots, -1$. These equations for all i, j, r, s are equivalent to the same equations for $r \neq \pm s$.

Proof. Let G be an affine scheme over $\mathbb{Z}\left[\frac{1}{2}\right]$ defined by these equations. Obviously, $\mathrm{GO}(n, R) \subseteq G(R)$. By Lemma 7, it suffices to establish the inverse inclusion for the case of a local ring R. Let M be a maximal ideal of R.

First, we show that if $g = (g_{ij}) \in G(R)$, then there exists a pair (i, r) such that $g_{ir} g'_{-r,-i} \in R^*$. Indeed, assume that, contrary to expectations, $g_{ir} g'_{-r,-i} \in M$ for all i, r. Since the matrix g is invertible, for any i there exists r such that $g_{ir} \notin M$ and for any j there exists s such that $g'_{sj} \notin M$. Then $g_{ir} g'_{sj} \in R^*$, but $g_{-j,-s} g'_{-r,-i} \in M$, so that the matrix g cannot belong to $G(R)$.

[2] In the papers devoted to overgroups of the maximal tori, see references in [14, 92, 94], this subgroup was usually denoted by $\overline{T}(n, R)$, whereas the symbol $T(n, R)$ denoted the diagonal subgroup of $\mathrm{SO}(n, R)$.

Now, fix a pair (i, r) such that $g_{ir}g'_{-r,-i} \in R^*$. Set $\lambda = g_{ir}(g'_{-r,-i})^{-1} \in R^*$. Then the equations on g take the form $\lambda g'_{sj} = g_{-j,-s}$ for all j, s. But this means precisely that the matrix g belongs to $\mathrm{GO}(n, R)$.

It remains to show that the above equations for $r = \pm s$ and $i = -j$ follow from the rest. To this end, choose $r \neq s$ and employ the habitual trick (see, for example, [8, §3]):

$$g_{ir}g'_{\pm r,j} = \sum g_{ir}g'_{sh}g_{hs}g'_{\pm r,j} = \sum g_{-h,-s}g'_{\pm r,-i}g_{-j,\mp r}g'_{-s,-h} = g_{-j,\mp r}g'_{-r,-i}. \qquad \square$$

Let E, F be two subgroups of the group G. Recall that the *transporter* of the subgroup E into the subgroup F is the set

$$\mathrm{Tran}_G(E, F) = \{g \in G \mid E^g \leq F\}.$$

Mostly, we use this notation when $E \leq F$, and then

$$\mathrm{Tran}_G(E, F) = \{g \in G \mid [g, E] \leq F\}.$$

The proof of the following theorem mimics the proof of Proposition 2 in [15] or, in the orthogonal case, of Proposition 2 in [13].

Theorem 3. *Suppose R is any commutative ring, $n \geq 5$, $E = \mathrm{EO}(n, R)$, $\Gamma = \mathrm{SO}(n, R)$, and $G = \mathrm{GL}(n, R)$. Then*

$$N_G(E) = N_G(\Gamma) = \mathrm{Tran}_G(E, \Gamma) = \mathrm{GO}(n, R).$$

Proof. Obviously, $\mathrm{GO}(n, R) \leq N_G(\Gamma)$. This immediately follows from Proposition 1. Indeed, let $x \in \mathrm{SO}(n, R)$, $g \in \mathrm{GO}(n, R)$. We wish to show that $y = gxg^{-1} \in \Gamma$. For this, it suffices to verify the equations $y'_{ij} = y_{-j,-i}$ for all i, j. Indeed, $y'_{ij} = \sum g_{ih}x'_{hk}g'_{kj}$, where the sum is taken over all h, k. Using the orthogonality condition for the matrix x and equations from the preceding proposition for the matrix g, we see that $y'_{ij} = \sum g_{-j,-k}x_{-k,-h}g'_{-h,-j} = y_{-j,-i}$, as expected. Now, Lemma 4 asserts that $\mathrm{GO}(n, R) \leq N_G(E)$.

Conversely, it is clear that $N_G(E), N_G(\Gamma) \leq \mathrm{Tran}_G(E, \Gamma)$. Thus, to prove the theorem, it suffices to verify the inclusion $\mathrm{Tran}_G(E, \Gamma) \leq \mathrm{GO}(n, R)$. Take any matrix $g = (g_{ij}) \in \mathrm{Tran}_G(E, \Gamma)$.

1) First, let $r \neq \pm s$, $r, s \neq 0$. Then, writing out the orthogonality condition for the matrix $y = gT_{rs}(1)g^{-1}$, we get

$$g_{ir}g'_{s,j} - g_{i,-s}g'_{-r,j} = g_{-j,r}g'_{s,-i} - g_{-j,-s}g'_{-r,-i}.$$

2) Let $s = 0$ and $r \neq 0$. Then, comparing the orthogonality conditions for the matrices $gT_{r0}(1)g^{-1}$ and $gT_{r0}(1)g^{-1}$, we see that $g_{ir}g'_{-r,j} = g_{-j,r}g'_{-r,-i}$ and

$$g_{ir}g'_{0j} - g_{i0}g'_{-r,j} = g_{-j,r}g'_{0,-i} - g_{-j,0}g'_{-r,-i}.$$

3) The case where $r = 0$, $s \neq 0$, is treated similarly. Here we can consider the matrices $gT_{r0}(1)g^{-1}$ and $gT_{r0}(2)g^{-1}$; however, this gives no new equations on the matrix g.

4) For $n \geq 6$ all equations for $r \neq s$, $r, s \neq 0$, easily follow from the above. Indeed, let q be an index such that all six indices $\pm r, \pm s, \pm q$ are pairwise distinct. Writing the orthogonality condition for the matrix $y = gT_{rq}(1)T_{qs}g^{-1}$ and cancelling the terms obtained at step 1 (for the pairs r, q and q, s), we can conclude that $g_{ir}g'_{s,j} = g_{-j,-s}g'_{-r,-i}$.

5) Now, assume that $n \geq 5$. Writing the orthogonality condition for the matrices $y = gT_{r,-s}(1)T_{-s,0}(1)g^{-1}$ and $y = gT_{r,-s}(1)T_{-s,0}(2)g^{-1}$ and cancelling the terms obtained at step 1 (for the pair $r, -s$) and at step 2 (for the pair $-s, 0$), we again get the same equations $g_{ir}g'_{s,j} = g_{-j,-s}g'_{-r,-i}$ as at step 4, and, moreover, the equation $g_{ir}g'_{0j} = g_{-j0}g'_{-r,-i}$.

6) The equations $g_{i0}g'_{sj} = g_{-js}g'_{0,-i}$ can be deduced in exactly the same way as at step 5. However, they follow from the equations obtained at steps 2 and 5.

7) As we know from the proposition, we do not need to verify the equations for $r = s$ since they automatically follow from the rest. However, in principle we could do that in exactly the same way as above, by considering the matrices $gT_{r0}(1)T_{0r}(1)g^{-1}$ and $gT_{r0}(1)T_{0r}(2)g^{-1}$. □

Usually, we apply this result in the following form.

Corollary. *Under the conditions of Lemma* 5*, we have*

$$\mathrm{Tran}_G(\mathrm{EO}(n,R),\mathrm{GO}(n,R)) = \mathrm{GO}(n,R).$$

Proof. We wish to show that for any $g \in \mathrm{GL}(n,R)$ the inclusion $[g,\mathrm{EO}(n,R)] \leq \mathrm{GO}(n,R)$ implies $g \in \mathrm{GO}(n,R)$. Indeed, by Lemma 4 we have

$$[g,\mathrm{EO}(n,R),\mathrm{EO}(n,R)] \leq \mathrm{EO}(n,R).$$

Since the group $\mathrm{EO}(n,R)$ is perfect, from the three subgroup lemma it follows that $g \in N_G(\mathrm{EO}(n,R)) = \mathrm{GO}(n,R)$. □

§5. Elementary transvections in subgroups normalized by EO(n, R)

In the present section, we assume that $n \geq 5$. Observe that in the orthogonal case the proof of the following two lemmas is harder than in the symplectic case, because in the latter situation all linear transvections $t_{i,-i}(\xi)$ belong already to the elementary symplectic group $\mathrm{Ep}(2l,R)$ itself. As a result, in the symplectic case we could skip the *only* difficult argument (pertaining to $i = -j$) in the next lemma. In [18] we adopted the worst possible strategy: there we sketched calculations, but did not supply all the details. The correct solution in all such cases is always one of the extremes: *either* to say that the following lemma is *obvious* [21], alias *follows from the Chevalley commutator formula* [86], *or* to painstakingly reproduce *all* details of the calculations, as we did in [44, Proposition 5.1]. In the present paper *for diversity* we take the second route.

Lemma 8. *For any ideal $A \trianglelefteq R$, we have*

$$E(n,A)^{\mathrm{EO}(n,R)} = E(n,R,A).$$

Proof. Clearly, the left-hand side is contained in the right-hand side. By Lemma 2, to prove the reverse inclusion, it suffices to check that for any $\xi \in A$, $\zeta \in R$, $i \neq j$, the matrix $z_{ij}(\xi,\zeta) = {}^{t_{ji}(\zeta)}t_{ij}(\xi)$ belongs to $H = E(2l,A)^{\mathrm{EO}(2l,R)}$.

For $i \neq \pm j$, $i, j \neq 0$, this is obvious, because in this case,

$${}^{t_{ji}(\zeta)}t_{ij}(\xi) = {}^{T_{ji}(\zeta)}t_{ij}(\xi).$$

It is almost as easy to account for the case of $i \neq 0$, $j = 0$ (the case where $i = 0$, $j \neq 0$ can be considered similarly). To do this, it suffices to notice that

$${}^{t_{0i}(\zeta)}t_{i0}(\xi) = {}^{T_{0i}(\zeta)}t_{i0}(\xi) \cdot t_{-i,0}(\xi\zeta^2/2)t_{-i,i}(-\xi\zeta^3/2) \in H.$$

Thus, we are left with the inclusion $z_{i,-i}(\xi,\zeta) \in H$. We take an index j such that $j \neq \pm i$ and express $t_{i,-i}(\xi)$ as a commutator of elementary transvections, $t_{i,-i}(\xi) = [t_{ij}(\xi), t_{j,-i}(1)]$. Then

$$z_{i,-i}(\xi,\zeta) = {}^{t_{-i,i}(\zeta)}t_{i,-i}(\xi) = {}^{t_{-i,i}(\zeta)}[t_{ij}(\xi), t_{j,-i}(1)].$$

Conjugating the arguments of the commutator by $t_{-i,i}(\zeta)$, we get

$$z_{i,-i}(\xi,\zeta) = [t_{ij}(\xi)t_{-i,j}(\zeta\xi), t_{ji}(-\zeta)t_{j,-i}(1)] = [ab,cd].$$

Next, we decompose the right-hand side with the help of the formula

$$[ab,cd] = {}^{a}[b,c] \cdot {}^{ac}[b,d] \cdot [a,c] \cdot {}^{c}[a,d],$$

and observe that the exponent a belongs to $E(n,A)$ and can be ignored. Now a direct calculation, based upon the Chevalley commutator formula, shows that

$$\begin{aligned}
&[t_{ij}(\xi),t_{ji}(-\zeta)]=[t_{ij}(\xi),T_{ji}(-\zeta)],\\
&{}^{t_{ji}(-\zeta)}[t_{ij}(\xi),t_{j,-i}(1)]={}^{t_{ji}(-\zeta)}[t_{ij}(\xi),T_{j,-i}(1)]\\
&\qquad\qquad\qquad=t_{j,-j}(-\xi\zeta^2)t_{i,-j}(\xi\zeta)\cdot{}^{T_{ji}(-\zeta)}[t_{ij}(\xi),T_{j,-i}(1)],\\
&[t_{-i,j}(\zeta\xi),t_{ji}(-\zeta)]=[t_{-i,j}(\zeta\xi),T_{ji}(-\zeta)],\\
&{}^{t_{ji}(-\zeta)}[t_{-i,j}(\zeta\xi),t_{j,-i}(1)]={}^{t_{ji}(-\zeta)}[t_{-i,j}(\zeta\xi),T_{j,-i}(1)]\\
&\qquad\qquad=t_{-i,-j}(\xi\zeta^2(1+\xi\zeta))t_{j,-j}(\xi\zeta^2)\cdot{}^{T_{ji}(-\zeta)}[t_{-i,j}(\zeta\xi),T_{j,-i}(1)],
\end{aligned}$$

where all factors on the right-hand side belong to H. □

Lemma 9. *Let H be a subgroup in $\mathrm{GL}(n,R)$ containing $\mathrm{EO}(n,R)$. For $i\neq\pm j$, set $A_{ij}=\{\xi\in R\mid t_{ij}(\xi)\in H\}$. Then for any $h\neq\pm k$ we have $A_{hk}=A_{ij}=A$ for an ideal $A\trianglelefteq R$.*

Proof. Clearly, all subsets A_{ij}, $i\neq j$, are additive subgroups. First, we show that they are in fact ideals. Take arbitrary $\xi\in A_{ij}$ and $\zeta\in R$.

1) First, let $n\geq 6$. For $i\neq\pm j$, $i,j\neq 0$, we can use essentially the same argument as in the proof of Lemma 4 in [18]. Namely, take any index h such that the six indices $\pm i,\pm j,\pm h$ are pairwise distinct, and consider the commutator

$$t_{ih}(\xi\zeta)=[t_{ij}(\xi),t_{jh}(\zeta)]=[t_{ij}(\xi),T_{jh}(\zeta)]\in H.$$

This means that $A_{ij}R\subseteq A_{ih}$. In particular, $A_{ij}\subseteq A_{ih}$. Similarly,

$$t_{hj}(\zeta\xi)=[t_{hi}(\zeta),t_{ij}(\xi)]=[T_{hi}(\zeta),t_{ij}(\xi)]\in H,$$

so that $RA_{ij}\subseteq A_{hj}$, and, in particular, $A_{ij}\subseteq A_{hj}$. This shows that for $l\geq 3$ all subgroups A_{ij}, $i\neq\pm j$, coincide and are ideals indeed. Denote their common value by A. Observe that in this argument we have never invoked the invertibility of 2.

2) Next, let $n\geq 5$. Using the condition $2\in R^*$, we can prove that all ideals A_{ij}, $i\neq\pm j$, coincide, this time without the proviso $i,j\neq 0$. Let $\xi\in A_{ij}$, where, as always, $i\neq\pm j$. If, moreover, $i,j\neq 0$, we have

$$t_{i0}(\xi\zeta)=[t_{ij}(\xi),T_{j0}(\zeta)][t_{ij}(\xi),T_{j0}(\zeta),T_{j0}(\zeta/2)],$$

so that $A_{ij}R\subseteq A_{i0}$. On the other hand,

$$[t_{i0}(\xi),T_{0j}(\zeta/2)]=t_{ij}(\xi\zeta),$$

so that $A_{i0}R\subseteq A_{ij}$. This shows that all additive subgroups A_{ij}, $j\neq\pm i$, situated in the same row with number $i\neq 0$, coincide and are ideals. A similar argument shows that all additive subgroups A_{ij}, $i\neq\pm j$, situated in the same column with number $j\neq 0$, coincide and are ideals. It only remains to observe that, since H contains $\mathrm{EO}(n,R)$, we clearly have $A_{ij}=A_{-j,-i}$ for all $i\neq\pm j$. Thus, $A_{i0}=A_{i,-j}=A_{j,-i}=A_{j,0}$, so that again all ideals A_{ij}, $i\neq\pm j$, coincide.

3) Finally, consider the additive subgroups $A_{i,-i}$. Take any $\xi\in A$ and fix an index $j\neq\pm i$. Then for any $\zeta\in R$ we have

$$t_{i,-i}(\xi\zeta)=[t_{ij}(\xi),t_{j,-i}(\zeta)]=[t_{ij}(\xi),T_{j,-i}(\zeta)]\in H,$$

so that $A\subseteq A_{i,-i}$. We are left with the case of $\xi\in A_{i,-i}$. But then for any $\zeta\in R$ and any index $j\neq\pm i$ we have

$$t_{j,-i}(\zeta\xi)t_{i,-j}(\xi\zeta)=[T_{ji}(\zeta),t_{i,-i}(\xi)]\in H.$$

Since, moreover, $T_{j,-i}(\zeta\xi) = t_{j,-i}(\zeta\xi)t_{i,-j}(\zeta\xi) \in H$, we obtain $t_{j,-i}(2\zeta\xi) \in H$, so that $2RA_{i,-i} \subseteq A$. By the condition $2 \in R^*$, this means that $A_{i,-i} \subseteq A$. It follows that all subgroups $A_{i,-i}$ also coincide with A. $\square$

Summarizing the above two lemmas, we get the following result.

Proposition 2. *Let H be a subgroup in $\mathrm{GL}(n, R)$ containing $\mathrm{EO}(n, R)$. Then there exists a unique largest ideal $A \trianglelefteq R$ such that*

$$\mathrm{EEO}(n, R, A) = \mathrm{EO}(n, R)E(n, R, A) \leq H.$$

Namely, if $t_{ij}(\xi) \in H$ for some $i \neq \pm j$, then $\xi \in A$.

The ideal we defined in this proposition is called the *level* (or, when we have to be extremely precise, the *lower level*) of the subgroup H, and is denoted by $\mathrm{lev}(H)$. Traditionally, one assigns levels to subgroups normalized by $\mathrm{E}(n, R)$. What we have shown in the present section is precisely that the level can be assigned unambiguously already to a subgroup normalized by $\mathrm{EO}(n, R)$.

The following easy, but very important observation simplifies the proofs of some auxiliary results. Strictly speaking, we have not yet verified that $H \cdot \mathrm{GO}(n, R)$ is a subgroup. Thus, we define the level $\mathrm{lev}(X)$ of a subset $X \leq \mathrm{GL}(n, R)$ as the set of all $\xi \in R$ for which there *exists* a pair of indices (i, j), $i \neq \pm j$, such that $t_{ij}(\xi) \in X$. Clearly, as usual $\mathrm{lev}(X) \trianglelefteq R$.

Lemma 10. *If $H \geq \mathrm{EO}(n, R)$, then $\mathrm{lev}(H \cdot \mathrm{GO}(n, R)) = \mathrm{lev}(H)$.*

Proof. Clearly, $\mathrm{lev}(H) \leq \mathrm{lev}(H \cdot \mathrm{GO}(n, R))$. To establish the reverse inclusion, we consider a transvection $t_{ij}(\xi)$ belonging to $H \cdot \mathrm{GO}(n, R)$, and express it in the form $t_{ij}(\xi) = xy$, where $x \in H$, $y \in \mathrm{GO}(n, R)$. Since by Lemma 9 the ideal A_{ij} does not depend on the choice of $i \neq j$, we can assume that $i \neq \pm j$. Now, if $h \neq \pm i, \pm j$, then

$$t_{ih}(\xi) = [t_{ij}(\xi), T_{jh}(1)] = [xy, T_{jh}(1)] = {}^{x}[y, T_{jh}(1)][x, T_{jh}(1)] \in H.$$

Thus, indeed, $\mathrm{lev}(H \cdot \mathrm{GO}(n, R)) \leq \mathrm{lev}(H)$. $\square$

§6. Proof of Theorem 2

Proposition 2 focuses our attention on the following class of subgroups. For an ideal $A \trianglelefteq R$, set

$$\mathrm{EEO}(n, R, A) = \mathrm{EO}(n, R)E(n, R, A).$$

Lemma 8 asserts precisely that $\mathrm{EEO}(n, R, A)$ is generated as a subgroup by all elementary symplectic transvections $T_{ij}(\xi)$, $i \neq j$, $\xi \in R$, and all elementary linear transvections $t_{ij}(\zeta)$, $i \neq \pm j$, $\zeta \in A$, of level A. As always, we assume that $n \geq 5$ and $2 \in R^*$.

Lemma 11. *The group $\mathrm{EEO}(n, R, A)$ is perfect for any ideal $A \trianglelefteq R$.*

Proof. It suffices to verify that all generators of the group $\mathrm{EEO}(n, R, A)$ lie in its commutator subgroup, which will be denoted by H. For the orthogonal transvections $T_{ij}(\zeta)$ this follows from Lemma 5. On the other hand, for the linear transvections $t_{ij}(\xi)$ with $i \neq \pm j$ and $\xi \in A$ this is obvious. Indeed, if $j \neq \pm i$ and $i, j \neq 0$, then $t_{i,-i}(\xi) = [t_{ij}(\xi), T_{j,-i}(1)] \in H$. On the other hand,

$$[t_{i,-i}(\xi/2), T_{j,-i}(1)] = t_{ij}(\xi/2)t_{-j,-i}(\xi/2)t_{-j,j}(-\xi/2).$$

Since we already know that the element $t_{-j,j}(-\xi/2)$ belongs to H, we have

$$y = t_{ij}(\xi/2)t_{-j,-i}(\xi/2) \in H.$$

But then $t_{ij}(\xi) = yT_{ij}(\xi/2) \in H$. It only remains to consider the cases when one of the indices i, j equals 0. Let, say, $j = 0$. Then $t_{i0}(\xi) = [t_{ij}(\xi), T_{j0}(1)]t_{i,-j}(-\xi) \in H$. The case of $i = 0$ is treated similarly. $\square$

In the rest of the section we calculate the normalizer of the group $\mathrm{EEO}(n,R,A)$ in $\mathrm{GL}(n,R,A)$. Namely, we return to the reduction homomorphism $\rho_A : \mathrm{GL}(n,R) \longrightarrow \mathrm{GL}(n,R/A)$ and concentrate on the study of the full preimage of the group $\mathrm{GO}(n,R/A)$ relative to this reduction:

$$\mathrm{CGO}(n,R,A) = \rho_A^{-1}(\mathrm{GO}(n,R/A)).$$

Remark. Obviously, $\mathrm{CGO}(n,R,A) \geq \mathrm{GO}(n,R)\,\mathrm{GL}(n,R,A)$. However, the subgroup $\mathrm{CGO}(n,R,A)$, as defined above, is in general *strictly larger* than $\mathrm{GO}(n,R)\,\mathrm{GL}(n,R,A)$. In fact, it is easily seen that the multiplier of any matrix in $\rho_A(\mathrm{GO}(n,R))$ belongs to $\rho_A(R^*)$. At the same time, any element of $(R/A)^*$ can serve as the multiplier of a matrix in $\mathrm{GO}(n,R/A)$. It is well known that for rings of dimension ≥ 1 the homomorphism $\rho_A : R^* \longrightarrow (R/A)^*$ may fail to be surjective.

From Proposition 1 it follows that the group $\mathrm{CGO}(n,R,A)$ is determined by quadratic congruences on matrix entries.

Proposition 3. *Let A be an ideal in R. For a matrix $g = (g_{ij}) \in \mathrm{GL}(n,R)$ to belong to $\mathrm{CGO}(n,R,A)$, it is necessary and sufficient that*

$$g_{ir}g'_{sj} \equiv g_{-j,-s}g'_{-r,-i} \pmod{A}$$

for all i,j,r,s.

This proposition explains the abbreviation $\mathrm{CGO}(n,R,A)$ (the congruence general orthogonal group). Now we are in a position to finish the proof of Theorem 2. It suffices to repeat the proof of Theorem 2 of [19] word for word, replacing references to the auxiliary results of [19] by the references to their counterparts in the present paper.

Proof of Theorem 2. Since $E(n,R,A)$ and $\mathrm{GL}(n,R,A)$ are normal subgroups in $\mathrm{GL}(n,R)$, the homomorphism theorem allows us to rewrite Theorem 2 as follows:

$$N_G(\mathrm{EEO}(n,R,A)) \leq N_G(\mathrm{EEO}(n,R,A)\,\mathrm{GL}(n,R,A)) = \mathrm{CGO}(n,R,A).$$

In particular,

$$[\mathrm{CGO}(n,R,A), \mathrm{EEO}(n,R,A)] \leq \mathrm{EEO}(n,R,A)\,\mathrm{GL}(n,R,A).$$

On the other hand, it is completely clear that $\mathrm{EEO}(n,R,A)$ is normal in the right-hand side of this inclusion. Indeed, it is easy to prove the following stronger inclusion:

$$[\mathrm{GO}(n,R,A)\,\mathrm{GL}(n,R,A), \mathrm{EEO}(n,R,A)] \leq \mathrm{EEO}(n,R,A).$$

To check this, we consider a commutator of the form $[xy, hg]$, where $x \in \mathrm{GO}(n,R)$, $y \in \mathrm{GL}(n,R,A)$, $h \in \mathrm{EO}(n,R)$, and $g \in E(n,R,A)$. Then $[xy,hg] = {}^x[y,h] \cdot [x,h] \cdot {}^h[xy,g]$, and at this point the Suslin normality theorem, standard commutator formula, and Lemma 4 immediately imply that all factors on the right-hand side belong to $\mathrm{EEO}(n,R,A)$. Summarizing the above, we see that

$$[\mathrm{CGO}(n,R,A), \mathrm{EEO}(n,R,A), \mathrm{EEO}(n,R,A)] \leq \mathrm{EEO}(n,R,A).$$

To invoke the Hall–Witt identity, we need a slightly more precise version of the last inclusion:

$$[[\mathrm{CGO}(n,R,A), \mathrm{EEO}(n,R,A)], [\mathrm{CGO}(n,R,A), \mathrm{EEO}(n,R,A)]] \leq \mathrm{EEO}(n,R,A).$$

Observe that we have already checked that the left-hand side is generated by the commutators of the form $[uv, [z,y]]$, where $u, y \in \mathrm{EEO}(n,R,A)$, $v \in \mathrm{GL}(n,R,A)$, and $z \in \mathrm{CGO}(n,R,A)$. However,

$$[uv, [z,y]] = {}^u[v,[z,y]] \cdot [u,[z,y]].$$

The second commutator belongs to $\mathrm{EEO}(n, R, A)$, whereas the first is an element of $[\mathrm{GL}(n, R, A), E(n, R)] \leq E(n, R, A)$.

Now we are all set to finish the proof. By the previous lemma, the group $\mathrm{EEO}(n, R)$ is perfect, and thus, it suffices to show that $[z, [x, y]] \in \mathrm{EEO}(n, R, A)$ for all $x, y \in \mathrm{EEO}(n, R, A)$, $z \in \mathrm{CGO}(n, R, A)$. Indeed, the Hall–Witt identity yields

$$[z, [x, y]] = {}^{xz}[[z^{-1}, x^{-1}], y] \cdot {}^{xy}[[y^{-1}, z], x^{-1}],$$

where the second commutator belongs to $\mathrm{EEO}(n, R, A)$ by the above. Removing the conjugation by x in the first commutator and carrying the conjugation by z inside the commutator, we see that it only remains to prove the relation $[[x^{-1}, z], [z, y]y] \in \mathrm{EEO}(n, R, A)$. Indeed,

$$[[x^{-1}, z], [z, y]y] = [[x^{-1}, z], [z, y]] \cdot {}^{[z,y]}[[x^{-1}, z], y],$$

where both commutators on the right belong to $\mathrm{EEO}(n, R, A)$ by the above, and moreover, the conjugating element in the second commutator is an element of the group $\mathrm{EEO}(n, R, A)\,\mathrm{GL}(n, R, A)$, and thus, normalizes $\mathrm{EEO}(n, R, A)$. □

§7. Decomposition of unipotents

The following result is a variation of the theme of "decomposition of unipotents". This is a trick, which allows us from the outset to reduce all calculations to groups of smaller rank. More precisely, in this particular case we reduce to the maximal parabolic subgroup P_1 of the group $\mathrm{GL}(n, R)$. The following result was first published in [20]; somewhat more detailed expositions appeared later in [92, §11] and [83, §13].

Decompostion of unipotents. *Let R be a commutative ring such that $2 \in R^*$. Then for any matrix $g \in \mathrm{GL}(n, R)$, $n \geq 6$, the elementary orthogonal group $\mathrm{EO}(n, R)$ is generated by orthogonal transvections $T_{uv}(\xi)$ such that $T_{uv}(\xi)g_{*i} = g_{*i}$ for some i.*

Corollary. *Let H be a subgroup of $\mathrm{GL}(n, R)$ containing $\mathrm{EO}(n, R)$. If $H \nleq \mathrm{GO}(n, R)$, then there exists an index i such that $H \cap Q_i \nleq \mathrm{GO}(n, R)$.*

Proof. Let $g \in H \setminus \mathrm{GO}(n, R)$. By Lemma 3 and the corollary to Theorem 3, there exists a long root element $hT_{i,-j}(\xi)h^{-1}$, $i \neq \pm j$, $i, j \neq 0$, $\xi \in R$, $h \in \mathrm{EO}(n, R)$, such that $[g, hT_{i,-j}(\xi)h^{-1}] \notin \mathrm{GO}(n, R)$. Then $[h^{-1}gh, T_{ij}(\xi)] \notin \mathrm{GO}(n, R)$, where we still have $h^{-1}gh \in H \setminus \mathrm{GO}(n, R)$. Thus, replacing, if necessary, g by $h^{-1}gh$, we can assume from the outset that $[g, T_{ij}(\xi)] \notin \mathrm{GO}(n, R)$. Pick an index h such that all six indices $\pm i, \pm j, \pm h$ are pairwise distinct, and present $T_{i,-j}(\xi)$ in the form $T_{i,-j}(\xi) = \prod x_k$, $1 \leq k \leq -1$, where

$$x_k = T_{i,-j}(g_{k,-i}\xi g'_{-h,k})T_{i,-h}(g_{k,-i}\xi g'_{-j,k})T_{j,-h}(g_{k,-i}\xi g'_{-i,k}) \in \mathrm{EO}(n, R).$$

Since $[g, T_{i,-j}(\xi)] \notin \mathrm{GO}(n, R)$, there exists an index k with $[g, x_k] \notin \mathrm{GO}(n, R)$, and thus $gx_kg^{-1} \notin \mathrm{GO}(n, R)$. However, a straightforward calculation (see [92] or [83]) shows that the kth column of the matrix x_kg^{-1} coincides with the kth column of g^{-1}. But then the kth column of the matrix $x = gx_kg^{-1} \in H \setminus \mathrm{GO}(n, R)$ coincides with the kth column of the identity matrix. □

This makes our imminent goal the proof of the following result, which would enable the proof of Theorem 1 by level reduction.

Proposition 4. *Let H be a subgroup of $\mathrm{GL}(n, R)$ containing $\mathrm{EO}(n, R)$, $n \geq 6$. Assume that $H \cap Q_i \nleq \mathrm{GO}(n, R)$. Then H contains a nontrivial transvection.*

Conjugating by an element of the Weyl group W, we can, without loss of generality, assume that H contains a matrix $g \notin \mathrm{GO}(n,R)$ whose *first* or *zeroth* column coincides with the corresponding column of the identity matrix.

• As the first step we reduce the second of these cases to the first, i.e., show that if $H \cap Q_0 \not\leq \mathrm{GO}(n,R)$, then $H \cap Q_1 \not\leq \mathrm{GO}(n,R)$.

After that we consider a matrix $g \in H \cap Q_1 \setminus \mathrm{GO}(n,R)$ and consecutively show that:

• H contains a matrix, two of whose columns coincide with the corresponding columns of the identity matrix, $H \cap Q_1 \cap Q_i \not\leq \mathrm{GO}(n,R)$;

• H contains a matrix whose first column and last row are proportional to the corresponding column and row of the identity matrix, $H \cap {}_{-1}P_1 \not\leq \mathrm{GO}(n,R)$;

• H contains a nontrivial transvection.

Remark. For $l \geq 4$ one could avoid a separate analysis of these cases, by presenting $T_{i,-j}(\xi)$ in the form $T_{i,-j}(\xi) = \prod x_{rs}$, $1 \leq r < s \leq -1$, in such a way that *two* columns of the matrix $gx_{rs}g^{-1}$, namely, the rth and sth columns, coincide with the corresponding columns of the identity matrix. One can express such an x_{rs} as the product of *six* orthogonal transvections $T_{i,-j}(*)$, $T_{i,-h}(*)$, $T_{i,-k}(*)$, $T_{j,-h}(*)$, $T_{j,-k}(*)$, $T_{h,-k}(*)$, where all eight indices $\pm i, \pm j, \pm h, \pm k$ are pairwise distinct, while the parameters are expressed via the second order minors of the matrices g and g^{-1}; see [83]. Here we do essentially the same, but in two strokes. This allows us to avoid unwieldy formulas. What is more important, by a separate analysis of the case when the matrix obtained at the first step commutes with elementary orthogonal transvections, this strategy allows us to cover also the case of $l \geq 3$.

§8. 33rd kilometer — Electro-coals: From Q_0 to Q_1

> I laughed: "Why on earth is this a lemma if it is universal?".
> The decembrist also laughed: "If universal, why is it a lemma?".
> — "A lemma indeed, for it discards woman. This lemma applies to an abstract human rather than to a woman. When a woman appears any symmetry disappears. Should woman not be woman, the lemma would not have been a lemma. The lemma is universal as long as there is no woman. Her presense calls off the lemma, especially if she is bad and the lemma is good."
> Everybody started to speak at once: "Oh, but what is a lemma on the whole?" — "And what is a bad woman?" — "There are no bad women, only lemmas may be bad..."
>
> Venedikt Erofeev

We persevere with our study of subgroups $H \leq \mathrm{GL}(n,R)$ including $\mathrm{EO}(n,R)$. In the present section we consider the case of an *odd* n. Observe that in this case the group $\mathrm{SO}(n-1,R)$ embeds naturally into $\mathrm{SO}(n,R)$. Namely, map a matrix $g \in \mathrm{SO}(n-1,R)$ to the matrix $\tilde{g} \in \mathrm{SO}(n,R)$ such that $\tilde{g}_{ij} = g_{ij}$ for $i,j \neq 0$ and $\tilde{g}_{ij} = \delta_{ij}$ if at least one of the indices i,j equals 0. In the sequel we routinely identify an element of the group $\mathrm{SO}(n-1,R)$ with its image in $\mathrm{SO}(n,R)$ under this embedding.

Proposition 5. *If $H \cap Q_0 \not\leq \mathrm{GO}(n,R)$, then $H \cap Q_1 \not\leq \mathrm{GO}(n,R)$.*

We start with the following auxiliary result.

Lemma 12. *If a matrix $g \in H \cap Q_0$ is such that $[g, \mathrm{EO}(n-1,R)] \leq \mathrm{GO}(n,R)$, then $g \in \mathrm{GO}(n,R)$.*

Proof. By assumption, $[g, T_{ij}(\xi)] \in \mathrm{GO}(n, R)$ for all $i \neq \pm j$, $i, j \neq 0$, $\xi \in R$. Thus, $gT_{ij}(\xi)g^{-1} \in Q_0 \cap \mathrm{GO}(n, R)$. By orthogonality, $gT_{ij}(\xi)g^{-1} \in {}_0Q$. Hence, $g_{0i}g'_{jh} - g_{0,-j}g'_{-i,h} = 0$. Since g'_{j*} and g'_{-i*} are distinct columns of an invertible matrix, they are linearly independent, so that $g_{0i} = 0$ for all $i \neq 0$. Thus, $g \in Q_0 \cap {}_0Q$. Denote by $\overline{g}$ the matrix obtained from g by deleting the column and the row with index 0. We can invoke the corollary to Theorem 3 to conclude that $\overline{g} \in \mathrm{GO}(n-1, R)$. Clearly, $g \in \mathrm{GO}(n, R)$ is a matrix with the same multiplier as $\overline{g}$. □

Proof of Proposition 5. Take some $g \in H \cap Q_0$, $g \notin \mathrm{GO}(n, R)$. By the lemma, there exists $T_{ij}(\xi)$, $i \neq \pm j$, $i, j \neq 0$, $\xi \in R$, such that $[g, T_{ij}(\xi)] \notin \mathrm{GO}(n, R)$. Thus, in the decomposition of unipotents there exists a factor x_k such that $gx_kg^{-1} \notin \mathrm{GO}(n, R)$. Then gx_kg^{-1} is the desired element of $H \cap P_k \setminus \mathrm{GO}(n, R)$. □

Now we approach the technical core of the proof of Theorem 1: reduction to the proper parabolic subgroups.

Proposition 6. *If $H \cap P_1 \nleq \mathrm{GO}(n, R)$, then already $H \cdot \mathrm{GO}(n, R) \cap {}_{-1}P_1 \nleq \mathrm{GO}(n, R)$.*

This proposition will be established in the next four sections. Take a matrix $g \in (H \cap P_1) \setminus \mathrm{GO}(n, R)$ and consider the following alternative: *either* g commutes with all elementary orthogonal transvections $T_{1h}(1)$, $h \neq \pm 1$, modulo $\mathrm{GO}(n, R)$, *or* it does not commute with some of them. In the next two sections we show that in the first case g satisfies the conclusion of Propositiion 6, whereas in the second case H contains a matrix that does not belong to $\mathrm{GO}(n, R)$ and lies in the intersection of two parabolic subgroups of type P_1.

§9. Electro-coals — 43rd kilometer: Linear transvections in $\mathrm{GO}(n, R)$

In the paper [18], the following lemma was used as obvious. Here for the sake of completeness we reproduce its proof.

Lemma 13. *Suppose $u \in R^n$, $v \in {}^nR$, where $vu = 0$ and the column u is unimodular. Consider the linear transvection $x = e + uv$. If x belongs to $\mathrm{GO}(n, R)$, $l \geq 2$, then x belongs already to $O(n, R)$.*

Proof. First, assume that the ring R is local. Clearly, for any elementary matrix $h \in \mathrm{EO}(n, R)$ the linear transvection hxh^{-1} also belongs to $\mathrm{GO}(n, R)$, and furthermore, its multiplier coincides with the multiplier of x. Since R is local, there exists $h \in \mathrm{EO}(n, R)$ with $hu = (1, 0, \dots, 0, *)^t$. This claim is a very special case of the surjective stability or the functor KU_1; see, for example, [40], or further references in [78]. But for such an h all rows of the transvection $hxh^{-1} = h(e + uv)h^{-1} = e + hu \cdot vh^{-1}$ except, maybe, the rows with indices ± 1, coincide with the corresponding rows of the identity matrix. Since $l \geq 2$, the multiplier of this matrix (and, with it, also the multiplier of x) equals 1.

Now, let R be an arbitrary ring. Since the properties of a matrix to be a transvection, to belong to $\mathrm{GO}(n, R)$, and the unimodularity of u are preserved under localization, we see that $F_M(x) \in O(n, R_M)$ for all $M \in \mathrm{Max}(R)$. It suffices to invoke Lemma 7. □

Corollary. *Under the conditions of the lemma, we have $u_iv_{-i} = 0$.*

Proof. Indeed, since the matrix x belongs to $O(n, R)$, for all $i \neq j$ we have

$$-u_iv_j = -x_{ij} = x'_{ij} = x_{-j,-i} = u_{-j}v_{-i}.$$

Observe that the resulting equality $-u_iv_j = u_{-j}v_{-i}$ is valid also for $i = j = 0$. In particular, for $j = -i$ we have $2u_iv_{-i} = 0$. Since $2 \in R^*$, it follows that $u_iv_{-i} = 0$. □

§10. 43rd kilometer — Snore Village: From Q_1 to $Q_1 \cap Q_r$

Now we are all set to accomplish yet another key step in the proof of Proposition 6.

Proposition 7. *If $[g, T_{1h}(1)] \in \mathrm{GO}(n, R)$ for all $h \neq \pm 1$, then $g \in {}_{-1}P_1$.*

Proof. Consider the orthogonal transvection

$$x = x_h = gT_{1h}(1)g^{-1} \in \mathrm{GO}(n,R), \quad h \neq \pm 1.$$

A straightforward calculation shows that

$$x = x_h = e + \sum (g_{i1}g'_{hj} - g_{i,-h}g'_{-1,j} - \frac{1}{2}\delta_{h0}g_{i1}g'_{-1,j}), \quad 1 \le i,j \le -1.$$

Since $x \in \mathrm{GO}(n,R)$ and $x_{i1} = \delta_{i1}$, we see that $x_{-1,j} = 0$ for all $j \neq -1$. Consider the matrix $\overline{x} = e - \sum g_{i,-h}g'_{-1,j}e_{ij}$, $i,j \neq \pm 1$, obtained from x by deleting the rows and columns with indices $1,-1$. This matrix is a *linear* transvection, belonging to $\mathrm{GO}(n-2,R)$, and thus, to be able to apply the preceding lemma, we only need to verify that the column $g_{i,-h}$, $h = 2,\dots,-2$, is unimodular.

Indeed, denote by $\lambda = \lambda_h = x_{-1,-1}$ the multiplier $x = x_h$. Then $g_{-1,-h}g'_{-1,j} = 0$ for all $j \neq -1$ and $g_{-1,-h}g'_{-1,-1} = 1-\lambda$. Next, we rewrite these equations in the form $g_{-1,-h}g'_{-1,j} = (1-\lambda)\delta_{-1,j}$, multiply them by g_{jk}, and sum over j. As a result, we get $(1-\lambda)g_{-1,k} = 0$ for $k \neq -1$ and $(1-\lambda)g_{-1,-1} = g_{-1,-h}$. Comparing these formulas, we see that $(1-\lambda)^2 g_{-1,k} = 0$ for all k. This means that $(1-\lambda)^2 = 0$. Thus, $1-\lambda$ and, with it, also all $g_{-1,-h}$, $h \neq \pm 1$, belong to the Jacobson radical. But then the matrix $\overline{g}$ obtained from g by deleting the rows and columns with indices ± 1 is invertible. In particular, $g_{i,-h}$, $i,j \neq \pm 1$, are unimodular, so that we can apply the above lemma to the matrix $\overline{x}$.

Now, by the corollary to Lemma 13 we can conclude that $g_{i,-h}g'_{-1,-i} = 0$ for all $i,h \neq \pm 1$. Since the matrix $\overline{g}$ is invertible, the ideal generated by $g_{i,-h}$, $i,h \neq \pm 1$, coincides with R. Thus, $g'_{-1,j} = 0$ for all $j \neq \pm 1$. But this means precisely that the matrix g is of the desired form. □

In other words, the matrix g lies in the submaximal parabolic subgroup ${}_{-1}P_1$ of the group $\mathrm{GL}(n,R)$.

Now, suppose that H contains a matrix $g \in P_1$ such that $[g, T_{1,-i}(1)] \notin \mathrm{GO}(n,R)$ for some $i \neq \pm 1$. Then we can repeat the argument from §7 *verbatim* and conclude that H contains a matrix *two* of whose columns coincide with the corresponding columns of the identity matrix. More precisely, we have the following analog of the corollary in §7.

Lemma 14. *If $g \in H \cap P_1$ is such that $[g, T_{1,-i}(1)] \notin \mathrm{GO}(n,R)$ for some $i \neq \pm 1$, then $H \cap Q_1 \cap Q_k \notin \mathrm{GO}(n,R)$.*

Proof. Choose an index $j \neq \pm 1, \pm i$, and repeat the proof of the corollary in §7, replacing i,j,h by $1,i,j$. Since $T_{1,-i}(*)$, $T_{1,-j}(*)$, $T_{i,-j}(*)$ lie in P_1, the same calculation does not bring us outside of P_1. Since $x_1 = e$ and the matrix $T_{1,-i}(1) = \prod x_k$, $1 \le k \le -1$, does not commute with g, there exists $k \neq 1$ such that x_k does not commute with g. For such a k, in the matrix $gx_kg^{-1} \in H$ the columns with indices 1 and k coincide with the corresponding columns of the identity matrix. □

Conjugating by an element of the Weyl group W, we can assume without loss of generality that one of the following three possibilities occurs: $k = 2$, $k = 0$, or $k = -1$. We consider these possibilities in §§12 and 13. To do that, we must first attend to the extraction of transvections in ${}_{-1}P_1$.

§11. Snore Village — Esino: Extraction of transvections in ${}_{-1}P_1$

In this section, we show that for any matrix $g \in {}_{-1}P_1$ the following alternative, stated in the main lemma, occurs: either $g \in \mathrm{GO}(n,R)$, or the subgroup generated by g and $\mathrm{EO}(n,R)$ contains an elementary transvection.

Proposition 8. *Let H be a subgroup in $\mathrm{GL}(n,R)$ containing $\mathrm{EO}(n,R)$, $n \ge 6$. Suppose that $H \cap {}_{-1}P_1 \not\le \mathrm{GO}(n,R)$. Then H contains a nontrivial transvection.*

Denote by Y the Heisenberg subgroup in $\mathrm{GL}(n,R)$ generated by all elementary transvections $t_{1j}(\xi)$ and $t_{i,-1}(\xi)$, where $j \neq 1$, $i \neq -1$, $\xi \in R$. Obviously, Y consists of all matrices y such that $y_{ij} = \delta_{ij}$ for all pairs (i,j), except for the pairs $(1,j)$, $j \neq 1$, and $(i,-1)$, $i \neq -1$. From the viewpoint of algebraic groups, the group $Y = {}_{-1}U_1$ is the unipotent radical of the parabolic subgroup ${}_{-1}P_1$, so that, in particular, $[{}_{-1}P_1, Y] \le Y$.

Lemma 15. *If $H \cdot \mathrm{GO}(n, R) \cap Y \not\subseteq \mathrm{GO}(n, R)$, then H contains a nontrivial elementary transvection.*

Proof. Multiplying y by the orthogonal transvections $t_{1i}(y_{-i,-1})$, $i \neq \pm 1$, we can assume from the outset that all rows of y, except the first, coincide with the corresponding rows of the identity matrix. At the same time, since $y \notin \mathrm{GO}(n, R)$, the first row must have a nonzero entry y_{1i}, $i \neq 1$. If some of the entries y_{1i}, $i \neq \pm 1$, are distinct from 0, take any index $j \neq \pm 1, \pm i$ and consider the commutator $z = [y, T_{ij}(1)] = t_{1j}(y_{1i}) t_{1,-i}(-y_{1,-j}) \in H$. Forming another commutator with an elementary orthogonal transvection $[z, T_{j,-i}(1)] = t_{1,-i}(y_{1i}) \in H$, we see that H contains a nontrivial transvection. It remains to consider the case where $y_{1i} = 0$ for all $i \neq \pm 1$, but in this case the matrix y itself is a nontrivial elementary transvection $y = t_{1,-1}(y_{1,-1}) \in H \cdot \mathrm{GO}(n, R)$. To finish the proof, we can now invoke Lemma 10. □

Lemma 16. *If $H \cdot \mathrm{GO}(n, R) \cap {}_{-1}P_1 \not\leq \mathrm{GO}(n, R)$, then $H \cdot \mathrm{GO}(n, R) \cap Y \not\subseteq \mathrm{GO}(n, R)$.*

Proof. Take an $i \neq \pm 1$ and consider the commutator $x_i = [g, T_{1i}(1)] \in [{}_{-1}P_1, Y] \leq Y$, lying in H. If for some $i \neq \pm 1$ we have $x_i \notin \mathrm{GO}(n, R)$, then $y = x_i$ is the required matrix. Thus, we can assume that *all* matrices x_i, $i \neq \pm 1$, belong to $\mathrm{GO}(n, R)$. Clearly, in this case already $x_i \in \mathrm{SO}(n, R)$. Expressing this last condition in terms of the matrix entries of the matrix g, we get $g'_{ij} = \lambda g_{-j,-i}$ for all $i, j \neq \pm 1$, where $\lambda = g'_{-1,-1}$. But this means precisely that the matrix $\overline{g}$ obtained from g by deleting the columns and rows with indices ± 1 belongs to $\mathrm{GO}(n-2, R)$ with the multiplier $g'_{-1,-1}$. Multiplying g by $\mathrm{diag}(1, \overline{g}, g'_{-1,-1})^{-1}$, we get a matrix in $Y \cap H \cdot \mathrm{GO}(n, R)$. □

§12. Esino — Fryazevo: From $Q_1 \cap Q_2$ and from $Q_1 \cap Q_0$ to ${}_{-1}P_1$

Denote by Z the group generated by all elementary transvections $t_{1i}(\xi)$ and $t_{2i}(\xi)$, where $i \neq 1, 2$, $\xi \in R$. Clearly, Z consists of all matrices y such that $y_{ij} = \delta_{ij}$ for all pairs (i, j), except for the pairs of the form $(1, i)$ and $(2, i)$, $i \neq 1, 2$. From the viewpoint of algebraic groups, Z is the unipotent radical of the maximal parabolic subgroup P_2.

Lemma 17. *If $H \cap Q_1 \cap Q_2 \not\leq \mathrm{GO}(n, R)$, but H does not contain a nontrivial elementary transvection, then $H \cap {}_{-1}P_1 \setminus \mathrm{GO}(n, R)$.*

Proof. It is easily seen that

$$y = g^{-1} T_{1,-2}(1) g = e + \sum (g'_{i1} g_{-2,j} - g'_{i2} g_{-1j}) e_{ij}, \quad 1 \leq i, j \leq -1,$$

equals $y = e + \sum (g_{-2,j} e_{1j} - g_{-1,j} e_{2j})$, $j \neq 1, 2$, and consequently, belongs to Z. Clearly, $[y, T_{12}(1)] \in Y$, while $[y, T_{21}(1)]$ belongs to a subgroup conjugate to Y by a permutation matrix. Thus, by Lemma 15, if at least one of these commutators does not belong to $\mathrm{GO}(n, R)$, then H contains a nontrivial elementary transvection. On the other hand, if both commutators belong to $\mathrm{GO}(n, R)$, then $y = T_{1,-2}(g_{-2,-2})$, so that, in particular, $g_{-1,j} = 0$ for all $j \neq -1$. □

Lemma 18. *If $H \cap Q_1 \cap Q_0 \not\leq \mathrm{GO}(n, R)$, then $H \cap {}_{-1}P_1 \not\leq \mathrm{GO}(n, R)$.*

Proof. Consider the matrix

$$x = g T_{10}(1) g^{-1} = e + \sum_h (g'_{0h} - g'_{-1,h}/2) e_{1h} - \sum_h g'_{-1,h} e_{0h}.$$

By the condition on g, we get $x \in {}_{-1}P_1$. If $x \notin \mathrm{GO}(n, R)$, we get the desired element. On the other hand, if $x \in \mathrm{GO}(n, R)$, then $g'_{1h} = 0$ for all $h \neq 1$, so that in this case already $g \in {}_{-1}P_1$. □

Now, we can assume that H contains a matrix $g \notin \mathrm{GO}(n, R)$ whose columns with indices ± 1 coincide with the corresponding columns of the identity matrix. We shall consider separately the case of an odd l, for which we have to explicitly invoke the assumption $2 \in R^*$, and the case of $l \geq 4$, when we can make use of the inductive hypothesis (since $l = 3$ is odd!).

§13. Fryazevo — 61st kilometer: From $Q_1 \cap Q_{-1}$ to ${}_{-1}P_1$

Here we denote by Z a subgroup conjugate to the subgroup Z in the preceding section by a monomial matrix, namely, the subgroup generated by all elementary transvections $t_{1i}(\xi)$ and $t_{-1,i}(\xi)$, where $i \neq \pm 1$, $\xi \in R$. Clearly, this Z consists of all matrices y such that $y_{ij} = \delta_{ij}$ for all pairs (i,j), except for the pairs $(1,i)$ and $(-1,i)$, $i \neq \pm 1$. First, we show that if H contains a nonidentity matrix from Z, then H contains a nontrivial transvection. This is almost exactly the same as what was done in the preceding section, but this subgroup Z is not conjugate to the subgroup Z considered there by an element of W. Moreover, we need matrices slightly more general than merely the matrices from Z.

Lemma 19. *If H contains a nonidentity matrix of the form $y \in t_{ij}(\xi)Z$, $i,j \neq \pm 1$, $i \neq \pm j$, $\xi \in R$, then H contains a nontrivial elementary transvection.*

Proof. The rows of the matrix y with indices $-i, -j$ coincide with the corresponding rows of the identity matrix. Applying an analog of Lemma 17 with rows replaced by columns, we see that H contains a nontrivial transvection or a matrix x such that $x \in {}_{-1}P_1 \setminus \mathrm{GO}(n,R)$. However, in the second case, the group H also contains a nontrivial transvection by Proposition 8. □

Lemma 20. *Let $l \geq 3$ be odd. If $H \cap Q_1 \cap Q_{-1} \nleq \mathrm{GO}(n,R)$, but H does not contain a nontrivial elementary transvection, then $H \cdot \mathrm{GO}(n,R) \cap {}_{-1}P_1 \nleq \mathrm{GO}(n,R)$.*

Proof. Consider the matrix $h = h_{23}h_{45}\dots h_{l-1,l} \in \mathrm{EO}(n,R)$. Clearly, h differs from the matrix $h_1 = D_1(-1)$ only by sign. This means that $y = ghg^{-1} = -gh_1g^{-1} \in H \cap Z$, and moreover, $y_{1i} = 2g_{1i}$ and $y_{-1,i} = 2g_{-1,y}$ for all $i \neq \pm 1$. By the preceding lemma, either H contains an elementary transvection, or, since 2 is invertible, $g_{1i} = g_{-1,i} = 0$ for all $i \neq \pm 1$, so that already g itself lies in ${}_{-1}P_1$. □

Observe that this lemma finishes the proof of the main lemma for the case where $l = 3$. Indeed, in §§8–12 we saw that if H contains a matrix $g \notin \mathrm{GO}(n,R)$, then H also contains a matrix $y \in {}_{-1}P_1 \setminus \mathrm{GO}(n,R)$. However, as we know from Proposition 8, in this case H must contain a nontrivial elementary transvection. It only remains to carry through the induction step. This is not too hard, and very similar in spirit to the proof of the theorem, but instead of reduction modulo an ideal, it involves reduction to a group of smaller rank.

Lemma 21. *Let $l \geq 4$. If $H \cap Q_1 \cap Q_{-1} \nleq \mathrm{GO}(n,R)$, then H contains a nontrivial elementary transvection, or $H \cap {}_{-1}P_1 \nleq \mathrm{GO}(n,R)$.*

Proof. Consider the map assigning to every matrix $g \in H$ whose columns with indices ± 1 coincide with the corresponding columns of the identity matrix, the matrix $\overline{g}$ obtained from g by deleting the rows and columns with indices ± 1. It is clear that this map is a homomorphism of the subgroup of all matrices in H of this form to the group $\mathrm{GL}(n-2,R)$. Denote the image of this homomorphism by $\overline{H}$. Obviously, $\overline{H} \geq \mathrm{EO}(n-2,R)$. Since the main lemma holds for $l-1 \geq 3$ by the inductive hypothesis, for $\overline{H}$ one gets the following alternative: either $\overline{H} \leq \mathrm{GO}(n-2,R)$, or $\overline{H}$ contains a nontrivial elementary transvection $t_{ij}(\xi)$, $i,j \neq \pm 1$, $i \neq \pm j$, $\xi \in R$, $\xi \neq 0$. In the second case the matrix y such that $\overline{y} = t_{ij}(\xi)$ satisfies the conditions of Lemma 19, and thus H contains a nontrivial transvection. Thus, we can assume that $\overline{g} \in \mathrm{GO}(n-2,R)$. If $g_{-1,i} = 0$ for all $i \neq \pm 1$, then the matrix g itself belongs to ${}_{-1}P_1$, and the proof is complete. If this is not the case, then fix an index $i \neq \pm 1$ such that $g_{-1,i} \neq 0$, take any index $j \neq \pm 1, \pm i$, and consider the matrix $x = gT_{ij}(1)g^{-1} \in H$ whose columns with indices ± 1 also coinside with the corresponding columns of the identity matrix. However, now by the Suslin–Kopeĭko theorem we have $\overline{x} \in \mathrm{EO}(n,R)$, so that, together with the matrix x, the group H contains a matrix $y \in Z$ that coincides with x in the columns with indices ± 1 and with the identity matrix in all other positions. Indeed, set $y = \mathrm{diag}(1, \overline{x}^{-1}, 1)x \in H$. It only remains to check that $y \neq e$. For this, observe that $y_{-1h} = g_{-1,i}g'_{jh} - g_{-1,-j}g'_{-i,h}$ for all $h \neq \pm 1$, and moreover, the linear combination of these elements with the coefficients g_{hj}, $2 \leq h \leq -2$, equals $g_{-1,i} \neq 0$. Thus, we have found a nonidentity matrix $y \in Z$ in H, and it only remains to invoke Lemma 17 once again. □

§14. 61st kilometer — 65th kilometer: The proof of Theorem 1

Proof of Proposition 4. By comparing Propositions 6 and 8, we conclude that if the group H contains a matrix $g \in P_1 \setminus \mathrm{GO}(n, R)$, then H contains a nontrivial elementary transvection, and this is exactly the claim of Proposition 4. □

Proof of the main lemma. By decomposition of unipotents, we may assume that H contains a matrix $P_1 \setminus \mathrm{GO}(n, R)$. Then, by Proposition 4, the group H contains a nontrivial elementary transvection $t_{ij}(\xi)$, $i \neq j$, $\xi \in R^\bullet$. It remains to refer to Lemma 9, which says that all ideals A_{ij} coincide. □

At this point, the proof of Theorem 1 can easily be completed by reduction modulo the largest ideal $A \trianglelefteq R$ such that $E(n, R, A) \leq H$.

Proof of Theorem 1. Let, as above, A be the largest ideal such that $\mathrm{EEO}(n, R, A) \leq H$. The existence of such an ideal was established in Proposition 2. Next, let $\overline{H} = \rho_A(H)$ be the image of the group H under the reduction homomorphism $\rho_A : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/A)$. Obviously, the group $\overline{H}$ contains $\mathrm{EO}(n, R/A)$, and, applying the main lemma, we arrive at the following alternative: either $\overline{H} \leq \mathrm{GO}(n, R/A)$, or $\overline{H}$ contains a nontrivial elementary transvection $t_{ij}(\xi + A)$, $i \neq j$, $\xi \in R \setminus A$, where without loss of generality we can assume that $i \neq -j$, $i, j \neq 0$. We show that the second possibility cannot arise.

Indeed, present $t_{ij}(\xi) \in H \cdot \mathrm{GL}(n, R, A)$ in the form $t_{ij}(\xi) = ab$, $a \in H$, $b \in \mathrm{GL}(n, R, A)$, take an index $h \neq \pm i, \pm j$, and consider the commutator $[t_{ij}(\xi), T_{jh}(1)] = t_{ih}(\xi)$. Substituting here the expression of $t_{ij}(\xi)$, we get

$$t_{ih}(\xi) = [ab, T_{jh}(1)] = {}^a[b, T_{jh}(1)][a, T_{jh}(1)].$$

The first of the commutators on the right-hand side belongs to $E(n, R, A)$ by the standard commutator formula, while the second lies in H. This means that $t_{ih}(\xi) \in H$, where $\xi \notin A$, which contradicts the maximality of A.

This shows that only the first alternative can possibly occur, $\overline{H} \leq \mathrm{GO}(n, R/A)$. But then Theorem 2 implies that

$$H \leq \mathrm{CGO}(n, R, A) = N_G(\mathrm{EEO}(n, R, A)),$$

This finishes the proof of Theorem 1. □

Thus, once again we get a "fan" description of intermediate subgroups in the sense of Zenon Borevich; see references in [8, 48, 93, 95].

§15. Concluding remarks

In the present paper we have established the first of the four most immediate generalizations mentioned in [19]. We mention three further similar results, where the description of intermediate subgroups can be obtained by the methods of [19] and the present paper.

- Description of overgroups of $\mathrm{EU}(n, R, \Lambda)$ in $\mathrm{GL}(n, R)$; see [77].
- Generalization of the main results of the present paper to overgroups of orthogonal groups $\mathrm{EO}(n, R, f)$, not necessarily split, but isotropic enough.
- Description[3] of overgroups of the elementary Chevalley group of type F_4 in the Chevalley group of type E_6.

In [28], the second-named author succeeded in generalizing Bak's theory and gave a correct definition of the odd classical groups. In this connection, it is only natural to pose the following problem. We are convinced that all the basic ingredients of its solution are already contained in [19, 78] and the present paper. On the other hand, to account for all technical details may turn extremely arduous.

Problem 1. *Describe the subgroups in* $\mathrm{GL}(n, R)$ *containing the odd elementary unitary group* $\mathrm{EU}(n, R, \mathcal{L})$ *without the assumption* $2 \in R^*$.

[3] After the publication of the Russian original, Alexander Luzgarev completely solved this problem. His proof is a technical masterpiece, skillfully blending the methods of [17, 19] and the present paper.

Now, we mention the problem whose complete solution would be of special importance, since it would be a *common* generalization of a huge number of preceding publications, *in particular*, of [96], [22]–[24], [87]–[91], [53]–[55], [63]–[65], [67]–[73], and many others.

Problem 2. *Obtain a description of the subgroups in* $\mathrm{GL}(n,R)$ *normalized by an elementary classical group.*

It is clear that, under the restriction $2 \in R^*$, the *standard* answer to this problem can be stated in terms of pairs $A \le B$ of ideals of the ring R, and the authors intend to return to this problem in the near future. The simplest examples show that for the general case, the standard description fails drastically, and this answer requires a modification in the spirit of form ideals [40, 41], [44]–[46], [57], admissible pairs [34]–[36], quasi-ideals [89]–[91], Jordan ideals or radices [50, 51]. However, at the present stage it is not completely clear what the answer should look like without the invertibility of 2.

It is interesting to gather more evidence as to the generality of the phenomenon studied in [18, 19] and the present paper. Namely, let Γ be a simple algebraic subgroup of GL_n such that for any field K every intermediate subgroup either is contained in the normalizer of $\Gamma(K)$ in $\mathrm{GL}(n,K)$, or contains $\mathrm{SL}(n,K)$. Further, let R be a commutative ring. Is it true that for every intermediate subgroup H, $\mathrm{E}_\Gamma(R) \le H \le \mathrm{GL}(n,R)$, there exists a unique ideal $A \trianglelefteq R$ such that $\mathrm{E}(n,R,A)\,\mathrm{E}_\Gamma(R) \trianglelefteq H$? Here is a model case, with which we could start the solution of this problem. In this case many of the calculations we need are already contained in [39, 92, 94, 17].

Problem 3. *Describe*[4] *the subgroups of* $\mathrm{GL}(27,R)$ *normalized by* $\mathrm{E}(\mathrm{E}_6,R)$*.*

All auxiliary results of the present paper fully generalize to this case. Moreover, in the language of roots and weights, their driving forces become much more transparent. However, neither the decomposition of unipotents nor the localization techniques from [19] generalize to this case literally. In fact, the decomposition of unipotents, as developed in [20, 91, 93], explicitly invokes the equations on the orbit of the highest weight vector in the representation $(\mathrm{E}_6,\overline{\varpi}_1)$. Thus, there is no hope to be able to stabilize an arbitrary column of a matrix in $\mathrm{GL}(27,R)$ by this method (at least in the form proposed in the above papers!). For the same reason, in this case there is no obvious analog of Proposition 1 of [19]. In other words, here we cannot refer to the surjective stability of the functor K_1 to handle the local case. Nevertheless, we strongly believe that a result parallel to Theorem 1 of [18] and Theorem 1 of the present paper is still valid. One could attempt the following strategy, which is thoroughly expounded in [62].

• Consider the field case $R = K$. Since $\mathrm{E}(\mathrm{E}_6,R)$ contains a one-parameter unipotent subgroup X_α that acts on K^{27} quadratically, such a result could be deduced from the results of [85]. For such fields, algebraically closed or finite, this would follow also from the results of Seitz and Testerman [80], or the results of Aschbacher and Kondratiev [38, 39, 26, 68]. Possibly, one could also find a much more elementary proof.

• Consider the case of local rings. Here one should implement factorization modulo the Jacobson radical. The techniques of such a factorization for similar problems are fairly standard and have repeatedly been used in the work of the first-named author.

• To treat the general case, one could use a version of the localization method, similar to that employed in [19].

For an even $n = 2l$, our results can be stated in terms of rings with involution, in exactly the same way as the results of [19, §14]. However, here we should replace Hermitian elements by anti-Hermitian ones. Namely, let A be a ring with involution, i.e., with an antiautomorphism $a \mapsto \overline{a}$ of order 2. As in the commutative case, it is natural to define the *orthogonal group* $\mathrm{O}(2,A)$ by the condition

$$\mathrm{O}(2,A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \mathrm{GL}(2,A) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \overline{d} & \overline{b} \\ \overline{c} & \overline{a} \end{pmatrix} \right\}.$$

[4]After the publication of the Russian original, Alexander Luzgarev made important progress towards the solution of this problem.

Involution on the right-hand side serves to take the noncommutativity of the ring A into account and to guarantee that $\mathrm{O}(2, A)$ is indeed a group. The most important example of a ring with involution is $A = M(l, R)$, where R is a commutative ring, while the involution is given by passage to the transpose $a \mapsto a^t$. For passage to the transpose to be an involution, the ring R *must* be commutative! If, moreover, $2 \in R^*$, then the equations defining $\mathrm{O}(2, A)$ coincide with the usual equations defining the orthogonal group:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & e \\ eps & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} 0 & e \\ eps & 0 \end{pmatrix}.$$

Thus, in this case, $\mathrm{O}(2, M(l, R)) = \mathrm{O}(2l, R)$.

We return to the general case. It is easily seen that the elementary transvections

$$t_{12}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad t_{21}(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

belong to $\mathrm{O}(2, A)$ if and only if the element a is *anti-Hermitian*, i.e., $\overline{a} = -a$. Denote the set of all anti-Hermitian elements of the ring A by $\mathrm{AH}(A)$. Then it is natural to define the *elementary orthogonal group* $\mathrm{EO}(2, A)$ by

$$\mathrm{EO}(2, A) = \langle t_{12}(a), t_{21}(a), a \in \mathrm{AH}(A) \rangle.$$

It is easy to show that $\mathrm{EO}(2, M(l, R)) = \mathrm{EO}(2l, R)$, so that this definition fully agrees with the usual definition we have used so far. From this viewpoint, [18] and the part of the present paper that treated the even case were devoted to the description of the subgroups in $\mathrm{GL}(2, A)$ containing $\mathrm{EO}(2, A)$. Here we completely solve this problem for the case where $A = M(l, R)$, the ring R is commutative, $2 \in R^*$, the involution on A is given by transposition, and $l \geq 3$. However, it is natural to pose a similar question in the general case.

Problem 4. *Describe the subgroups in* $\mathrm{GL}(2, A)$ *containing* $\mathrm{EO}(2, A)$.

In this setting, the condition on the rank could be stated as the existence of three nontrivial pairwise orthogonal idempotents in the ring A.

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