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AUTOMORPHISMS OF A FREE GROUP OF INFINITE RANK

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Abstract. The problem of classifying the automorphisms of a free group of infinite countable rank is investigated. Quite a reasonable generating set for the group $\operatorname{Aut} F_\infty$ is described. Some new subgroups of this group and structural results for them are presented. The main result says that the group of all automorphisms is generated (modulo the IA-automorphisms) by strings and lower triangular automorphisms.

§1. Introduction

The automorphism group of a free group of finite rank has been investigated intensively. In [24], Nielsen obtained its presentation in terms of "elementary" automorphisms, which are called the Nielsen automorphisms nowadays. His powerful method initiated extensive research and systematic study in this area. Nice surveys of the results can be found in [22] and [19].

Unlike the case of finite rank, the situation for the automorphism group $\operatorname{Aut} F_\infty$ of a free group F_∞ of infinite countable rank is not well understood. The problem of classification of all its subgroups seems to be very difficult. Only a few natural subgroups and isolated results are known. The group $\operatorname{Aut} F_\infty$ is "very big"; for example, the group $\operatorname{Sym}(\mathbb{N})$ of all permutations on the natural numbers can be viewed as its subgroup. Moreover, since $\operatorname{Sym}(\mathbb{N})$ is known to contain a free group of rank $2^{\aleph_0}$ [11], we obtain one more subgroup of $\operatorname{Aut} F_\infty$.

Some nice properties of $\operatorname{Aut} F_\infty$ are related to the properties of the free group F_∞ on free generators $x_1, x_2, \dots$. For example, F_∞ has the *basis cofinality property* [19]; i.e., for every $\alpha \in \operatorname{Aut} F_\infty$ and every $n \in \mathbb{N}$ there exists $r \in \mathbb{N}$ and $\beta \in \operatorname{Aut}(\langle x_1, \dots, x_r\rangle)$ such that $r \geq n$ and $\beta(x_i) = \alpha(x_i)$ for $i = 1, \dots, n$ (here $\langle x_1, \dots, x_r\rangle$ denotes the subgroup generated by $x_1, \dots, x_r$). The group F_∞ has also the *small index property*; i.e., for every subgroup Λ in $\operatorname{Aut} F_\infty$ of index less than $2^{\aleph_0}$ there exists a finite subset Y of F_∞ such that the pointwise stabilizer of Y in $\operatorname{Aut} F_\infty$ is contained in Λ (the converse statement is obvious). In particular, these results imply that $\operatorname{Aut} F_\infty$ is not the union of an ascending chain of countably many proper subgroups [1].

The subgroup lattice of $\operatorname{Aut} F_\infty$ is far from being understood. Some natural subgroups consisting of inner, finitary, bounded, triangular, permutational, or diagonal automorphisms have been studied (the definitions are given in the next section). However, even for the group of bounded automorphisms no satisfying generating set is known. In [9], Solitar conjectured that this group can be generated by infinite elementary simultaneous

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Nielsen automorphisms, but this problem still remains open. We note that, in the infinite case, the usual Nielsen automorphisms generate the group $\mathrm{Aut}_{\mathrm{fin}}\, F_\infty$ consisting of all automorphisms that act nontrivially only on finitely many generators (see [19]).

In our paper we construct some new subgroups of $\mathrm{Aut}\, F_\infty$ and give structural results for them.

The standard notion of boundedness for an automorphism $\alpha \in \mathrm{Aut}\, F_\infty$ requires a fixed upper bound n on the length of each freely reduced word of the form $\alpha(x_i)$ or $\alpha^{-1}(x_i)$. We shall consider boundedness from a different point of view. We require only that each x_j occurring in $\alpha(x_i)$ be "not far" from x_i in the sense of $|i-j|$ being sufficiently small. Note that the exponent sum of x_j in $\alpha(x_i)$ can be arbitrary. It is very far from the notion of boundedness introduced in [9]. In §2 we also introduce a notion of a string, which realizes this idea. The strings are analogs of infinite block-diagonal matrices with an infinite number of finite blocks on the main diagonal. A special kind of strings in the group of upper triangular automorphisms was considered in [9]. Our fundamental observation is that the set of finite compositions of strings forms a group $\mathcal{H}$, called the group of strings. This group plays a crucial role in our considerations. In §3 we prove that $\mathcal{H}$ contains the group generated by the permutational and the upper triangular automorphisms and investigate some properties of parabolic subgroups of $\mathcal{H}$. We also describe a large sublattice of subgroups of $\mathcal{H}$ associated with growths. The main result in §4 says that $\mathrm{Aut}\, F_\infty$ is generated (modulo the IA-automorphisms) by strings and lower triangular automorphisms. In the last section we formulate similar results for automorphisms of relatively free groups for some varieties of groups.

Our results remain valid also for a free group of uncountable rank. However, we need to assume that the set of free generators is well ordered. This condition is not surprising, since even the proof of the Nielsen–Schreier theorem (saying that every subgroup of a free group is free) requires such an ordering in the case of infinite rank (see [7, p. 89]).

§2. Definitions and examples

Let F_∞ be the free group on a countable set of free generators $X=\{x_i, i \in \mathbb{N}\}$, and let $\mathrm{Aut}\, F_\infty$ be the group of its automorphisms. Let $\alpha \in \mathrm{Aut}\, F_\infty$. We say that an automorphism α is

upper triangular if $\alpha(x_i) \in \langle x_{i+k} : k \in \mathbb{N} \cup \{0\}\rangle$ for all $i \in \mathbb{N}$;
lower triangular if $\alpha(x_1) = x_1^{\pm 1}$ and $\alpha(x_i) \in \langle x_1, \dots, x_i\rangle$ for all $i > 1$;
diagonal if $\alpha(x_i) = x_i^{\epsilon_i}$ for $\epsilon_i \in \{1, -1\}$ and all $i \in \mathbb{N}$;
a *permutational automorphism* if $\alpha(x_i) = x_{\pi(i)}$ for some fixed permutation π on the natural numbers;
a *column-finite automorphism* if for every $i \in \mathbb{N}$, x_i occurs only in a finite number of freely reduced words $\alpha(x_j)$, $j \in \mathbb{N}$.

The following results of R. Cohen characterize the upper and lower triangular automorphisms.

Lemma 1 [9, Theorem 2.4]. *If an automorphism α is upper triangular, then, for every $i = 1, 2, \dots,$ $\alpha|_{\langle x_i, x_{i+1}, \dots\rangle}$ is an automorphism of $\langle x_i, x_{i+1}, \dots\rangle$ and $\alpha(x_i) = A_i x_i^{\epsilon_i} B_i$, where $\epsilon_i \in \{1, -1\}$ and $A_i, B_i \in \langle x_{i+1}, x_{i+2}, \dots\rangle$.*

Lemma 2 [9, Corollary 2.3]. *If an automorphism α is lower triangular, then, for every $i = 1, 2, \dots,$ $\alpha(x_i) = A_i x_i^{\epsilon_i} B_i$, where $\epsilon_i \in \{1, -1\}$, $A_1 = B_1 = 1$, and $A_i, B_i \in \langle x_1, \dots, x_{i-1}\rangle$ for $i > 1$.*

We say that an automorphism $\alpha \in \mathrm{Aut}\, F_\infty$ is *n-bounded* if the words $\alpha(x_i)$ and $\alpha^{-1}(x_i)$ have length at most n for all i, and α is *bounded* if α is n-bounded for some n.

Let $\mathcal{B}_n$ denote the set of all n-bounded automorphisms, and let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. Clearly, $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$ for all n.

Let $\mathcal{T}^+$, $\mathcal{T}^-$, $\mathcal{D}$, $\mathcal{S}$ denote the sets of all upper triangular, lower triangular, diagonal, and permutational automorphisms, respectively, and let $\mathcal{K}$ be the set of all automorphisms α such that both α and α^{-1} are column-finite. Clearly, $\mathcal{D}, \mathcal{S} \subseteq \mathcal{B}_1$, $\mathcal{T}^+ \cap \mathcal{T}^- = \mathcal{D}$, and the intersections $\mathcal{T}^+ \cap \mathcal{S}$ and $\mathcal{T}^- \cap \mathcal{S}$ are trivial. From now on we shall not distinguish a permutational automorphism and the corresponding permutation on the natural numbers.

The following result describes some natural subgroups of $\operatorname{Aut} F_\infty$.

Proposition 1. *The sets $\mathcal{T}^+$, $\mathcal{T}^-$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{K}$, and $\mathcal{B}$ form subgroups of* $\operatorname{Aut} F_\infty$.

Proof. For $\mathcal{D}$ and $\mathcal{S}$, the statement is obvious. Since both $\mathcal{K}$ and $\mathcal{B}$ are closed under taking compositions, they are subgroups of $\operatorname{Aut} F_\infty$. The results for $\mathcal{T}^+$ and $\mathcal{T}^-$ can be deduced from Lemmas 1 and 2. □

Remark. $\mathcal{B}$ is called the group of bounded automorphisms. By analogy with infinite matrices, we call $\mathcal{K}$ the group of column-finite automorphisms.

The group $\mathcal{S}$ contains a free subgroup of rank $2^{\aleph_0}$ (see [11]). However, for $\mathcal{T}^+$ we can only prove the following statement.

Theorem 1. *The group $\mathcal{T}^+$ contains a free subgroup of countable rank.*

Proof. Let α_1, α_2 be two automorphisms of F_∞ defined as follows: for all $k \in \mathbb{N}$,

$\alpha_1(x_{2k-1}) = x_{2k-1}x_{2k}, \quad \alpha_1(x_{2k}) = x_{2k},$

$\alpha_2(x_{2k-1}) = x_{2k-1}, \quad \alpha_2(x_{2k}) = x_{2k}x_{2k+1}.$

Let A_∞ denote a free Abelian group of countable rank, and let $\chi : \operatorname{Aut} F_\infty \to \operatorname{Aut} A_\infty$ be the homomorphism induced by the natural mapping $\bar{\chi} : F_\infty \to A_\infty$. Since $\operatorname{Aut} A_\infty$ is isomorphic to $GL(\infty, \mathbb{Z})$ (this is the group of invertible row-finite countably infinite matrices over $\mathbb{Z}$), we shall identify $\chi(f)$ with the corresponding matrix in $GL(\infty, \mathbb{Z})$. In [16] it was proved that the matrices $\chi(\alpha_1)$, $\chi(\alpha_2)$ generate a free subgroup of rank two in $GL(\infty, \mathbb{Z})$. This means that $\langle \alpha_1, \alpha_2 \rangle$ is also a free group of rank two contained in $\mathcal{T}^+ \cap \mathcal{B}$. Its commutator subgroup is a required free group of countable rank. □

Now we define a notion fundamental to our paper.

Definition 1. Let α be an automorphism of F_∞. If the following two conditions are fulfilled:

(1) there is a partition $\{X_j | j \in \mathbb{N}\}$ of the set of generators $\{x_1, x_2, \dots\}$ such that $X_1 = \{x_1, \dots, x_{n_1}\}$ and for $j > 1$ we have $X_j = \{x_{n_j+1}, \dots, x_{n_{j+1}}\}$, where $n_1 < n_2 < \cdots$ is a strictly increasing sequence of natural numbers;

(2) for $n_j + 1 \le k \le n_{j+1}$ we have $\alpha(x_k) \in \langle x_{n_j+1}, \dots, x_{n_{j+1}} \rangle$,

then α will be called a *string*.

Note that if α is a string, we can write $F_\infty = \prod^{\star}_{j\in\mathbb{N}} \langle X_j \rangle$, where * denotes a free product, and define automorphisms α_j to act as α on $\langle X_j \rangle$ (this means that $\alpha_j = \alpha|_{\langle X_j \rangle}$ on $\langle X_j \rangle$) and to act trivially on $\langle X_i \rangle$ with $i \ne j$. Thus, α can be expressed as a product, $\alpha = \prod_{j=1}^{\infty} \alpha_j$, and there is no ambiguity in this representation, because the α_j commute and all but a finite number of these automorphisms act trivially on each particular freely reduced word in F_∞.

If α is a string and $\alpha = \prod \alpha_j$, then α^{-1} is also a string (because $\alpha^{-1} = \prod \alpha_j^{-1}$). However, a composition of two strings may fail to be a string, as the following example shows.

Example. Let α_1 and α_2 be defined as in the proof of Theorem 1. Then α_1, α_2 are strings and $\alpha_2 \circ \alpha_1$ is not a string, because $\alpha_2 \circ \alpha_1(x_{2k-1}) = x_{2k-1}x_{2k}$ and $\alpha_2 \circ \alpha_1(x_{2k}) = x_{2k}x_{2k+1}x_{2k+2}$ for all k.

Remark. In [9], a string in $\mathcal{T}^+$ was called a *splitting automorphism.*

Let $\mathcal{H}$ be the set of all finite compositions of strings (for various partitions, in general). The following observation is clear, but we state it explicitly because of its great importance.

Proposition 2. *The set $\mathcal{H}$ is a subgroup of* $\operatorname{Aut} F_\infty$.

The group $\mathcal{H}$, which we call the *group of strings*, plays a crucial role in further considerations.

Example. Let $\beta(x_1) = x_1$ and $\beta(x_k) = x_1 \cdot x_k$ for all $k > 1$. Clearly, β is an automorphism belonging to $\mathcal{T}^- \cap \mathcal{B}$, and $\beta \notin \mathcal{H}$.

§3. The structure of the group of strings

In this section we consider the subgroup structure of the group $\mathcal{H}$ of strings.

Proposition 3. *The group $\mathcal{H}$ contains the subgroup generated by the upper triangular and the permutational automorphisms, i.e., $\mathcal{H} \supseteq \langle \mathcal{T}^+, \mathcal{S} \rangle$.*

Proof. The inclusion $\mathcal{T}^+ \subseteq \mathcal{H}$ follows from Theorem 3.3 in [9]. Lemma 2.3 in [27] (see also [25]) implies that $\mathcal{S} \subseteq \mathcal{H}$, which finishes the proof. □

It seems plausible that $\mathcal{H} = \langle \mathcal{T}^+, \mathcal{S} \rangle$; however, neither a proof nor a counterexample is known.

The above result on the structure of the group $\mathcal{H}$ can be used for describing its subgroups. We say that a subgroup G of $\mathcal{H}$ is *parabolic* if it contains $\mathcal{T}^+$. It is clear that the description of the parabolic subgroups in $\mathcal{H}$ depends heavily on the subgroup structure of the symmetric group $\operatorname{Sym}(\mathbb{N})$. Moreover, for every subgroup S_1 of $\mathcal{S}$ the subgroup $\langle \mathcal{T}^+, S_1 \rangle$ is parabolic. If S_2 is a proper subgroup of S_1, then $\langle S_2, \mathcal{T}^+ \rangle$ is a proper subgroup of $\langle S_1, \mathcal{T}^+ \rangle$. Therefore, it seems that the minimal parabolic subgroups in $\mathcal{H}$ can be found by using such arguments. A more detailed discussion on this topic will appear elsewhere.

Two subgroups A, B are said to be *comparable* if $A \subseteq B$ or $B \subseteq A$, and *incomparable* otherwise. The known properties of subgroups in $S(\mathbb{N})$ imply the following statement.

Theorem 2. *There exists an uncountable family of pairwise incomparable parabolic subgroups in $\mathcal{H}$.*

Proof. First, we recall a classical result of W. Sierpiński, proved in 1928 (see [26]). A subset Γ of an infinite set Δ is called a *moiety* if $|\Gamma| = |\Delta \setminus \Gamma|$. Let $\Delta = \mathbb{Q}$. For each real number r, we can take an infinite sequence Γ_r of distinct rationals that converges to r. Then the set $\{\Gamma_r : r \in \mathbb{R}\}$ forms an uncountable family of moieties such that, for any two distinct members Γ, Γ' of this family, the set $\Gamma \cap \Gamma'$ is finite.

Now, let $\phi : \mathbb{Q} \to \mathbb{N}$ be any bijection, and let $\bar{\Gamma}_r := \phi(\Gamma_r)$. We denote by $S(\bar{\Gamma}_r)$ the pointwise stabilizer of the set $\mathbb{N} \setminus \bar{\Gamma}_r$ in $S(\mathbb{N})$. It is easily seen that the family $\{\langle S(\bar{\Gamma}_r), \mathcal{T}^+ \rangle : r \in \mathbb{R}\}$ consists of pairwise incomparable parabolic subgroups. □

In other words, the above theorem states that a lattice of parabolic subgroups of $\mathcal{H}$ contains an uncountable family of chains.

To describe another large family of subgroups of $\mathcal{H}$, we need the notion of a *growth.* Let $\mathbb{N}$ be the set of positive integers with natural order. We extend this order to the set

$\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, assuming that $n < \infty$ for all $n \in \mathbb{N}$. The set $P(\mathbb{N})$ of all functions $f : \mathbb{N}_\infty \to \mathbb{N}_\infty$, equipped with the operation of composition of functions, forms a semigroup. We denote by Ω_∞ the subsemigroup of all functions $f \in P(\mathbb{N})$ satisfying the conditions $f(\infty) = \infty$ and $f(n+1) \geq f(n) > n$ for all $n \in \mathbb{N}$. We introduce an order on Ω_∞: for every $f, g \in \Omega_\infty$ we write

$$f \prec g \quad \text{if and only if there exists } n_0 \quad \text{such that } f(n) < g(n) \quad \text{for all } n > n_0.$$

As usual, $f < g$ means that $f(n) < g(n)$ for all natural n.

We write $f \ll g$ if for all $k \in \mathbb{N}$ we have $f^k \prec g$, where f^k denotes the composition of k copies of f. We take $M(f) := \{h \in \Omega_\infty : f \ll h\}$ and define an equivalence relation $\sim$ on Ω_∞:

$$f \sim g \quad \text{if and only if } M(f) = M(g).$$

Let $\Omega^\star$ denote the set of equivalence classes of $\sim$. Each element of $\Omega^\star$ is called a *growth.*

For example, consider two special functions in Ω_∞ : $f_\infty(n) = \infty$ and $f_0(n) = n + 1$; their growths are $\omega_\infty = [f_\infty]$ and $\omega_0 = [f_0]$. Clearly, $f \in \omega_\infty$ if there exists $n \in \mathbb{N}$ such that $f(n) = \infty$, and $g \in \omega_0$ if there exists $d \in \mathbb{N}$ such that $g(n) \leq n + d$ for all $n \in \mathbb{N}$.

We define a partial order $\leq$ in $\Omega^\star$:

$$\omega_1 \leq \omega_2 \quad \text{if and only if } M(f) \supseteq M(g) \quad \text{for some } f \in \omega_1,\ g \in \omega_2.$$

Clearly, for all $\omega \in \Omega^\star$ we have $\omega_0 \leq \omega \leq \omega_\infty$. If $\omega_1 = [f]$ and $\omega_2 = [g]$, then we can define two new growths:

$$\omega_1 \vee \omega_2 = [\max\{f, g\}],$$
$$\omega_1 \wedge \omega_2 = [\min\{f, g\}].$$

The main result of [17] reads as follows.

Theorem 3. *Equipped with the operations $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$, the set $\Omega^\star$ forms a lattice with the following properties*:

a) $\Omega^\star$ *has a unique minimal element ω_0 and a unique maximal element ω_∞*;

b) $\Omega^\star$ *is dense*, *i.e.*, *if for some $\omega_1, \omega_2 \in \Omega^\star$ we have $\omega_1 < \omega_2$, then there exists $\omega_3 \in \Omega^\star$ such that $\omega_1 < \omega_3 < \omega_2$*;

c) $\Omega^\star$ *has neither atoms nor coatoms*;

d) *for every ω such that $\omega_0 < \omega < \omega_\infty$, there exists an uncountable family* (*an uncountable antichain*) *of pairwise incomparable growths that are not comparable with ω*;

e) $\Omega^\star$ *is distributive and modular.*

Now we define some subgroups of $\mathcal{H}$ associated with growths. Let α be a string corresponding to a sequence $n_1 < n_2 < \cdots$. We say that α is bounded by a function f of class ω if $f(1) \geq n_1$ and $f(n_k + 1) \geq n_{k+1}$ for each natural k. We say that an automorphism β in $\operatorname{Aut} F_\infty$ is bounded by $f \in \omega$ (or β is $[f]$-bounded) if for every natural n there exists $k = k(n)$ depending only on n such that $\beta(x_n)$ and $\beta^{-1}(x_n)$ are reduced words in the variables $x_{n-k(n)}, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots, x_{n+k(n)}$ and $f(n) \geq n + k(n)$.

It is easily seen that if α is $[f]$-bounded and β is $[g]$-bounded, then $\alpha \circ \beta$ is $[f \circ g]$-bounded. Hence, the following definition makes sense.

Definition 2. We denote by $\mathcal{H}(\omega)$ the set of all automorphisms $\alpha \in \mathcal{H}$ that are $[f]$-bounded for some $f \in \omega$.

In fact, $\mathcal{H}(\omega)$ is a well-defined subgroup of $\mathcal{H}$.

Theorem 4. *The family $\{\mathcal{H}(\omega) \mid \omega \in \Omega^\star\}$ forms a sublattice of the lattice of subgroups of $\mathcal{H}$. Moreover, this sublattice is isomorphic to the lattice $\Omega^\star$.*

In particular, this means that the lattice $\{\mathcal{H}(\omega) \mid \omega \in \Omega^\star\}$ has all the properties listed in Theorem 3.

§4. The structure of $\operatorname{Aut} F_\infty$

In this section we prove some structural results for $\operatorname{Aut} F_\infty$ and give a description of some normal subgroups. Let $A_\infty \cong F_\infty/F'_\infty$ be a free Abelian group of infinite countable rank, and let $\chi : \operatorname{Aut} F_\infty \to \operatorname{Aut} A_\infty$ be the homomorphism induced by the natural map $\bar{\chi} : F_\infty \to A_\infty$. The kernel $\mathcal{A}$ of χ consists of the IA-automorphisms of $\operatorname{Aut} F_\infty$. As was proved in [8] and [20], χ is an epimorphism. We shall use the known isomorphism from $\operatorname{Aut} A_\infty$ onto $GL(\infty, \mathbb{Z})$, the group of invertible row-finite countably infinite matrices over $\mathbb{Z}$. We shall identify $\chi(f)$ with the corresponding matrix in $GL(\infty, \mathbb{Z})$.

Now we are able to present quite a reasonable generating set for $\operatorname{Aut} F_\infty$

Theorem 5. *Each automorphism in* $\operatorname{Aut} F_\infty$ *is the composition of some IA-automorphism and some automorphism belonging to the subgroup generated by the lower triangular and the column-finite automorphisms, i.e.,* $\operatorname{Aut} F_\infty = \langle \mathcal{T}^-, \mathcal{K}\rangle \cdot \mathcal{A}$.

Proof. First, we show that $\operatorname{Aut} A_\infty = \langle \chi(\mathcal{T}^-), \chi(\mathcal{K})\rangle$. Let $A \in GL(\infty, \mathbb{Z})$. This means that in each row of A all but a finite number of elements are zero. Since $\mathbb{Z}$ is a principal ideal domain, the invertibility of A implies that the ideal generated by the elements of any row of A is $\mathbb{Z}$. Therefore, there exists an elementary finitary matrix B_1 such that the product $A \cdot B_1$ has only one nonzero element in the first row. By a similar argument, there exists an elementary finitary matrix B_2 such that the product $A \cdot B_1 \cdot B_2$ has only one nonzero element in the second row and this element stands below one of the zeros in the first row. We continue in this way, obtaining the infinite product $A \cdot (\prod_{i=1}^\infty B_i)$. This product is well defined [10], because for every natural k and every natural $n \ge k$, in each finite product $A \cdot B_1 \cdots B_n$ the first k columns are the same and they are stabilized in $A \cdot B_1 \cdots B_k$.

Hence, the matrix $A \cdot (\prod_{i=1}^\infty B_i)$ can be transformed (by a suitably chosen permutation matrix) to a lower triangular matrix T; more precisely, $A \cdot (\prod_{i=1}^\infty B_i) \cdot P = T$. Since $(\prod_{i=1}^\infty B_i)^{-1} = \prod_{i=1}^\infty B_i^{-1}$, we have $A = T \cdot P^t \cdot (\prod_{i=1}^\infty B_i^{-1})$. Moreover, by construction we have $\prod_{i=1}^\infty B_i^{-1} \in \chi(\mathcal{K})$ and $P^t \in \chi(\mathcal{H})$, whence $\operatorname{Aut} A_\infty = \langle \chi(\mathcal{T}^-), \chi(\mathcal{K})\rangle$, as required. Now, since $\mathcal{A}$ is the kernel of χ, we can deduce the statement from the properties of the inverse image of the epimorphism χ, and the result follows. □

In fact, the proof of the above theorem implies the following statement.

Theorem 6. *The group $GL(\infty, \mathbb{Z})$ is generated by the set of all lower triangular and all block-diagonal matrices.*

For a generalization of this result from $\mathbb{Z}$ to an arbitrary associative ring, we refer the reader to [18]. It would also be of interest to know whether and how the results and methods described in [13–16] can be extended to $\operatorname{Aut} F_\infty$.

Here we note that Theorem 6 can be proved also as follows. Swan proved (see [8]) that $\operatorname{Aut} A_\infty$ is generated by the triangular automorphisms, where the word *triangular* is meant with respect to bases consisting of a fixed set of elements each, but (possibly) in a different order. Since every ordering of this set corresponds to some permutational automorphism, which in its turn is a product of strings [27], the result follows.

We denote by $\operatorname{Inn} F_\infty$ the group of inner automorphisms of F_∞.

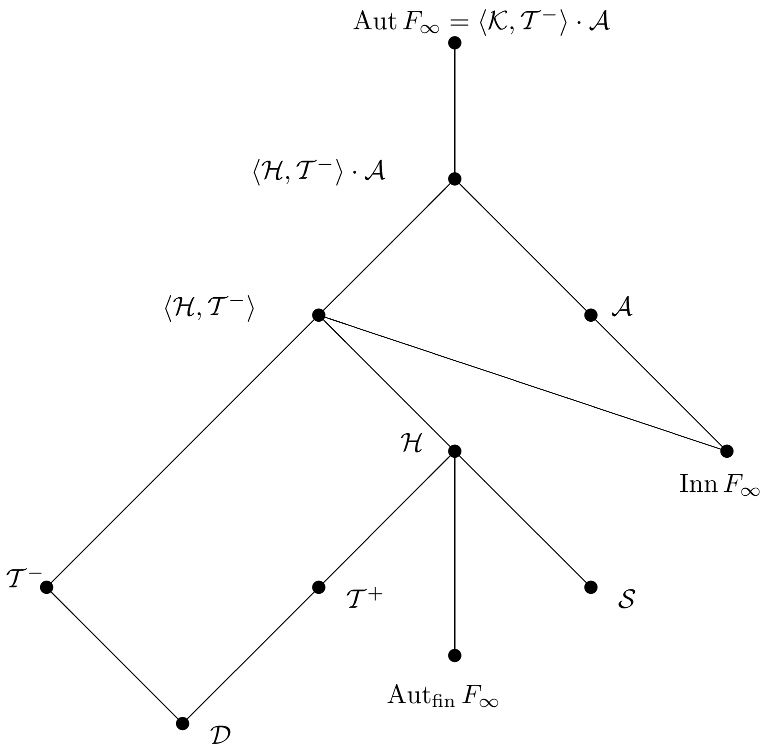
Proposition 4. a) *The group $\mathcal{K}$ contains $\mathcal{H}$.*
b) *The group $\langle \mathcal{T}^-, \mathcal{H}\rangle$ contains* $\operatorname{Inn} F_\infty$.

Proof. a) By induction on the number of strings, it can be shown that, in any finite product α of strings, every symbol x_i can occur only in finitely many words $\alpha(x_j)$. Moreover, every string belongs to $\mathcal{K}$, so that $\mathcal{H} \subset \mathcal{K}$.

b) Let $\alpha \in \operatorname{Inn} F_\infty$. Then for some $g \in \operatorname{Aut} F_\infty$ we have $\alpha(x) = g^{-1}xg$. Let n be the minimal number for which $g \in \langle x_1, \ldots, x_n \rangle$. We define two automorphisms:
$\alpha_1(x_1) = g^{-1}x_1g, \ldots, \alpha_1(x_n) = g^{-1}x_ng, \alpha_1(x_k) = x_k$ for all $k > n$,
$\alpha_2(x_1) = x_1, \ldots, \alpha_2(x_n) = x_n, \alpha_2(x_k) = g^{-1}x_kg$ for all $k > n$.
Clearly $\alpha_1 \in \mathcal{H}$, $\alpha_2 \in \mathcal{T}^-$, and $\alpha = \alpha_1 \circ \alpha_2$, which implies the statement. We note that $\alpha \neq \alpha_2 \circ \alpha_1$. □

Example. Let $\alpha_3(x_1) = x_1$, $\alpha_3(x_2) = x_2$, and let $\alpha_3(x_i) = [x_1, x_2]x_i$ for all $i > 2$. It can be checked that the mapping α_3 is an IA-automorphism that does not belong to $\mathcal{K}$.

We present a partial poset diagram for the subgroups of $\operatorname{Aut} F_\infty$ considered in this paper:



Using the properties of the epimorphism χ and the results of [5] and [12], we can describe some normal subgroups of $\operatorname{Aut} F_\infty$.

Theorem 7. *The group* $\operatorname{Aut} F_\infty$ *contains two infinite countable families: one of maximal normal subgroups and the other of normal incomparable subgroups.*

Proof. Since χ is an epimorphism, the maximal and normal subgroups in $\operatorname{Aut} A_\infty$ are lifted by χ to maximal and normal subgroups in $\operatorname{Aut} F_\infty$. Now, the first part of the statement follows from Corollary 0.7 of [5], and the second from [12] (see Theorem 7.3 and the discussion after its proof therein). □

§5. Results for some other varieties

Now we turn to certain relatively free groups. Let $\mathfrak{V}$ be a variety, let $\mathcal{V}$ be the corresponding verbal subgroup, and let $F_\infty/\mathcal{V}$ be the relatively free group of $\mathfrak{V}$ of infinite countable rank on free generators $y_1, y_2, \dots$ (see [23]). The natural homomorphism $F_\infty \to F_\infty/\mathcal{V}$ induces a homomorphism

$$\chi_{\mathfrak{V}} : \operatorname{Aut} F_\infty \to \operatorname{Aut}(F_\infty/\mathcal{V});$$

its kernel will be denoted by $\mathcal{A}_{\mathfrak{V}}$. Let $\mathfrak{A}$, $\mathfrak{A}_n$, and $\mathfrak{N}_c$ denote the varieties of Abelian groups, Abelian groups of exponent n, and nilpotent groups of class c, respectively. The following theorem summarizes the results on the surjectivity of $\chi_{\mathfrak{V}}$ obtained by many authors.

Theorem 8. *The homomorphism* $\chi_{\mathfrak{V}} : \operatorname{Aut} F_\infty \to \operatorname{Aut}(F_\infty/\mathcal{V})$ *is surjective if one of the following conditions is satisfied*:
(1) $F_\infty/\mathcal{V}$ *is nilpotent* (*see* [4]);
(2) $\mathfrak{V} = \mathfrak{A}_k\mathfrak{A}_l$ *for some natural numbers* k *and* l (*see* [2]);
(3) $\mathfrak{A}\mathfrak{A} \subseteq \mathfrak{V} \subseteq \mathfrak{N}_c\mathfrak{A}$ *for some natural number* c (*see* [3, Theorem 2]).

If in the definition of the subgroups $\mathcal{T}^+$, $\mathcal{T}^-$, $\mathcal{K}$, and $\mathcal{H}$ we replace the generators x_i by y_i, we obtain analogs of these subgroups in $\mathfrak{V}$, which we denote by $\mathcal{T}^+_{\mathfrak{V}}$, $\mathcal{T}^-_{\mathfrak{V}}$, $\mathcal{K}_{\mathfrak{V}}$, and $\mathcal{H}_{\mathfrak{V}}$, respectively.

The surjectivity of $\chi_{\mathfrak{V}}$ implies that the results of the preceding sections can easily be stated for the varieties occurring in Theorem 8.

Theorem 9. *Let* $\mathfrak{V}$ *be any of the varieties listed in Theorem* 8. *Then*:
a) $\mathcal{H}^b_{\mathfrak{V}} \supseteq \langle \mathcal{T}^+_{\mathfrak{V}}, \mathcal{S}\rangle$;
b) $\operatorname{Aut}(F_\infty/\mathcal{V}) = \langle \mathcal{T}^-_{\mathfrak{V}}, \mathcal{K}_{\mathfrak{V}}\rangle \cdot \mathcal{A}_{\mathfrak{V}}$;
c) *the group* $\mathcal{K}_{\mathfrak{V}}$ *contains* $\mathcal{H}_{\mathfrak{V}}$;
d) *the group* $\langle \mathcal{T}^-_{\mathfrak{V}}, \mathcal{H}_{\mathfrak{V}}\rangle$ *contains* $\operatorname{Inn}(F_\infty/\mathcal{V})$.

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