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DERIVATIVES OF REGULAR MEASURES

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ABSTRACT. Let μ be a positive singular measure on Euclidean space. If μ is sufficiently regular, then for any $a \in [0, +\infty]$ the set where the derivative of μ is equal to a is large in the sense of the Hausdorff dimension.

§1. INTRODUCTION

Derivatives of regular singular measures on the circle. Consider a real Borel measure μ on the unit circle $\mathbb{T} \subset \mathbb{C}$. Fix a point $\zeta \in \mathbb{T}$. For an arbitrary arc I on the circle, let $|I|$ denote its length. If the limit

$$\lim_{\zeta \in I,\ |I| \to 0} \frac{\mu(I)}{|I|}$$

exists, it is denoted by $\mathcal{D}\mu(\zeta)$; this limit is called the *derivative* of the measure μ at the point $\zeta \in \mathbb{T}$.

Now, assume that the measure μ is positive and singular. Here and in what follows, "singular" means "singular with respect to the corresponding Lebesgue measure". In the present work, our motivation was to find certain instances of the following heuristic principle:

If the singularity of the measure μ is compatible with sufficient regularity, then for any $a \in [0, +\infty]$ the set

$$E_a(\mu) = \{\zeta \in \mathbb{T} : \ \mathcal{D}\mu(\zeta) = a\}$$

is large in a sense.

Carmona and Donaire [3] obtained the following realization of the above general principle. Let $\dim_{\mathcal{H}}$ denote the Hausdorff dimension. Then

$$\dim_{\mathcal{H}} E_a(\mu) = 1$$

for all $a \in [0, +\infty]$ if the positive singular measure μ satisfies the condition

$$\lim_{|I| \to 0} \frac{\mu(I) - \mu(I')}{|I|} = 0, \tag{1.1}$$

where I and I' denote arbitrary *adjacent* arcs. By definition, two arcs $I, I' \subset \mathbb{T}$ are said to be *adjacent* if they have a common boundary point and $|I| = |I'|$. In [4], Donaire also proved a stronger assertion about the upper and lower derivatives of the measures with property (1.1).

The arguments in [3] were based on a theorem by Rohde on *inner* functions. By definition, an analytic function f in the unit disk $\mathbb{D} \subset \mathbb{C}$ is *inner* if the radial limits $f^*(\zeta)$

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satisfy the identity $|f^*(\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$. Also, an analytic function g belongs to the *little* Bloch space $\mathcal{B}_0(\mathbb{D})$ if

$$\lim_{\delta\to 0+} \sup_{|z|\geq 1-\delta} (1-|z|^2)|g'(z)| = 0.$$

Theorem 1.1 (Rohde [12])**.** *Let f be an inner function. If f is not a finite Blaschke product and $f \in \mathcal{B}_0(\mathbb{D})$, then*

$$\dim_{\mathcal{H}}\{\zeta \in \mathbb{T} : \ f^*(\zeta) = w\} = 1$$

for all $w \in \mathbb{D}$.

We start with showing that Theorem 1.1 makes it possible to obtain the desired property $\dim_{\mathcal{H}} E_a(\mu) = 1$ even if (1.1) fails for the measure μ.

So, let μ be a positive singular measure on $\mathbb{T}$. Then the *associated singular inner function* is defined by the identity

$$S[\mu](z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta)\right), \quad z \in \mathbb{D}.$$

Corollary 1.2. *Assume that*

$$S[\mu] \in \mathcal{B}_0(\mathbb{D}). \tag{1.2}$$

Then $\dim_{\mathcal{H}} E_a(\mu) = 1$ *for all* $a \in (0, +\infty)$.

Proof. The holomorphic function $S[\mu]$ is bounded. Therefore, by the classical theorem of Lindelöf, the existence of the radial limit

$$\lim_{r\to 1-} S[\mu](re^{i\phi})$$

implies the existence of the nontangential limit

$$\lim_{z \to> e^{i\phi}} S[\mu](z) = \lim_{re^{i\theta}\to e^{i\phi},\ |\theta-\phi|\leq c(1-r)} S[\mu](re^{i\theta}),$$

where c is a positive constant. Thus, Theorem 1.1 with $w = \exp(-a) \in \mathbb{D}$ guarantees that $\dim_{\mathcal{H}} E(a) = 1$, where

$$E(a) = \{\zeta \in \mathbb{T} : \ \lim_{z\to>\zeta} S[\mu](z) = \exp(-a)\}.$$

Now, consider the Poisson integral

$$P[\mu](z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2}\, d\mu(\zeta), \quad z \in \mathbb{D}.$$

Then

$$\lim_{z\to>\zeta} S[\mu](z) = \exp(-a) \ \Rightarrow \ \lim_{z\to>\zeta} P[\mu](z) = a.$$

Next, since the measure μ is positive, the Tauberian theorem in Loomis [8] says that

$$\lim_{z\to>\zeta} P[\mu](z) = a \ \Rightarrow \ \mathcal{D}\mu(\zeta) = a.$$

In other words, $E_a(\mu) \supset E(a)$; thus, $\dim_{\mathcal{H}} E_a(\mu) = 1$. □

It is well known that (1.1) $\Rightarrow$ (1.2). But the inverse implication is false. In fact, Bishop [2] obtained a complete description of the positive singular measures μ with property (1.2). For example, (1.2) follows from the identity

$$\lim_{|I|\to 0} \frac{\mu(I)}{\mu(I')} = 1, \tag{1.3}$$

where the limit is calculated with respect to all adjacent arcs I and I' (see [1] and [2]). For further reference we formulate the corresponding statement.

Corollary 1.3. *If a positive singular measure μ satisfies* (1.3), *then* $\dim_{\mathcal{H}} E_a(\mu) = 1$ *for all* $a \in (0, +\infty)$.

Measures on Euclidean spaces. Since conditions (1.1) and (1.3) are of pure-real variable nature, it is reasonable to ask whether complex analysis methods are essential in the proofs of the Carmona–Donaire theorem and Corollary 1.3. In the present paper we answer this question in the positive, obtaining the corresponding theorems for measures defined on Euclidean spaces.

Organization of the paper. In §2 we give definitions, state the desired results (Theorems 2.1 and 2.4), and give proofs, assuming the corresponding assertions for radial limits of functions in the little Bloch spaces (Theorems 2.2 and 2.5). The remaining part of the paper is devoted to Theorems 2.2 and 2.5. Some auxiliary results are collected in §3. Theorems 2.2 and 2.5 are proved with the help of N. G. Makarov's stopping time technique [9] in §§4 and 5, respectively. Also, in §5 we discuss some generalizations of Theorems 2.1 and 2.4.

Comments and remarks. 1. The method of Makarov mentioned above can be used to give an alternative proof of Theorem 1.1 (see [11]) and to generalize Theorem 1.1 (see [4]). Also, the same approach was used in [10] to investigate the harmonic functions that are defined on $\mathbb{R}^{n+1}_+$ and have no finite radial limits almost everywhere.

2. The symmetry property (1.3) can be viewed as a multiplicative version of the classical additive smoothness property (1.1); therefore, many arguments for symmetric measures are similar to those for smooth measures (cf. [5]). Since the technical details related to property (1.1) are more standard, below we primarily consider symmetric measures.

3. If the measure μ under consideration is singular, then the relation $\dim_{\mathcal{H}} E_a(\mu) = 1$ is trivial for $a = 0$, because $\mathcal{D}\mu = 0$ almost everywhere. Also, the case where $a = +\infty$ usually follows from known results, so we concentrate our attention on the sets $E_a(\mu)$ with $a \in (0, +\infty)$.

4. In what follows, if μ is a positive measure on $\mathbb{R}^n$, the case where $\mu(\mathbb{R}^n) = +\infty$ is not excluded.

§2. Main results

Let $|E|$ denote the Lebesgue measure of a set $E \subset \mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $h > 0$, the set

$$Q_h(x) = \{\xi \in \mathbb{R}^n : \ |x_j - \xi_j| < h/2, \ j = 1, 2, \dots, n\}$$

is the cube of side length $h > 0$ and with center x. We denote by Q_h a cube with nonspecified center.

Let μ be a real Borel measure on $\mathbb{R}^n$. The derivative of the measure μ at a point x is defined by the identity

$$\mathcal{D}\mu(x) = \lim_{x \in Q_h,\, h \to 0} \frac{\mu(Q_h)}{|Q_h|},$$

under the assumption that the above limit exists. We put

$$E_a(\mu) = \{x \in \mathbb{R}^n : \ \mathcal{D}\mu(x) = a\}.$$

2.1. Symmetric measures. Two cubes $Q_h(x)$ and $Q_h(x')$ are *adjacent* if $|x_p - x'_p| = h$ for exactly one index p and $x_j = x'_j$ for $j \neq p$.

Definition. A positive measure μ on $\mathbb{R}^n$ is *symmetric* if

$$\lim_{h \to 0} \frac{\mu(Q)}{\mu(Q')} = 1,$$

where Q and Q' are arbitrary adjacent cubes of side length h.

The first result is a direct analog of Corollary 1.3 in the case of several real variables.

Theorem 2.1. *Let μ be a singular symmetric measure on $\mathbb{R}^n$. Then $\dim_{\mathcal{H}} E_a(\mu) = n$ for all $a \in [0, +\infty]$.*

In the proof of the above theorem, the main technical step is the corresponding assertion pertaining to radial limits of functions in the little logarithmic Bloch set $\mathcal{B}_0^{\log}(\mathbb{R}^{n+1}_+)$. By definition, $\mathcal{B}_0^{\log}(\mathbb{R}^{n+1}_+)$ consists of all harmonic functions $u : \mathbb{R}^{n+1}_+ \to (0, +\infty)$ for which

$$\lim_{t\to 0+} \sup_{0<y\le t} y|\nabla \log u(x,y)| = 0.$$

Theorem 2.2. *Consider a function $u \in \mathcal{B}_0^{\log}(\mathbb{R}^{n+1}_+)$ and a cube $Q^0 \subset \mathbb{R}^n$. Assume that*

$$\lim_{y\to 0+} u(\xi, y) = 0 \quad \textit{for almost all} \quad \xi \in Q^0 \tag{2.1}$$

and

$$\lim_{y\to 0+} u(\xi', y) = +\infty \quad \textit{for a point} \quad \xi' \in Q^0. \tag{2.2}$$

Then for any $a \in (0, +\infty)$ we have $\dim_{\mathcal{H}} E(a) = n$, where

$$E(a) = \{x \in Q^0 : \lim_{y\to 0+} u(x, y) = a\}.$$

In the present section we show that Theorem 2.2 implies Theorem 2.1. Theorem 2.2 will be proved later, in §4.

Recall that the harmonic extension (the Poisson integral) of a measure μ is defined on the half-space $\mathbb{R}^{n+1}_+$ by the identity

$$P[\mu](x,y) = c_n \int_{\mathbb{R}^n} \frac{y}{(\|x-t\|^2 + y^2)^{\frac{n+1}{2}}}\, d\mu(t), \quad x \in \mathbb{R}^n,\ y > 0,$$

where c_n is a positive normalizing constant.

Also, we use a description of the ω-symmetric measures, obtained in [5].

By definition, a *regular gauge function* is a monotone nondecreasing bounded function $\omega : (0, +\infty) \to (0, +\infty)$ such that the ratio $\omega(t)/t^{1-\varepsilon}$ is monotone decreasing for some $\varepsilon > 0$. A positive measure μ on $\mathbb{R}^n$ is said to be ω-symmetric if there exists a positive constant C such that

$$\left|\frac{\mu(Q)}{\mu(Q')} - 1\right| \le C\omega\left(|Q|^{1/n}\right)$$

for all pairs of adjacent cubes $Q, Q' \subset \mathbb{R}^n$.

Theorem 2.3 ([5])**.** *Consider a finite positive measure μ on $\mathbb{R}^n$ and a regular gauge function ω such that $\omega(0+) = 0$. Let u denote the harmonic extension of the measure μ. Then the following properties are equivalent*:

$$\mu \textit{ is an } \omega\textit{-symmetric measure;} \tag{2.3}$$

there exists a positive constant $C > 0$ such that

$$y\left|\frac{\nabla u(x,y)}{u(x,y)}\right| \le C\omega(y) \quad \textit{for all points } (x,y) \in \mathbb{R}^{n+1}_+. \tag{2.4}$$

Proof of Theorem 2.1. We fix $a \in (0, +\infty)$ and a singular symmetric measure μ on $\mathbb{R}^n$. Consider an auxiliary continuous function $f : [0, +\infty) \to (0, +\infty)$ such that $f\big|_{[0,1]} = 1$ and

$$f(r) = \exp(C_f(1 - r))$$

for $r \ge 1$, where C_f is a positive constant. Let $\nu = F\mu$, where $F(x) = f(\|x\|)$ for $x \in \mathbb{R}^n$. Observe that $\mu(Q_h(0)) \le A(n)\exp(Bh)$ for some positive constants $A(n)$ and

B. Therefore, choosing a sufficiently large constant C_f, we have $\nu(\mathbb{R}^n) < +\infty$. Also, note that ν is a symmetric measure. Hence, by Lemma 4 in [6], the measure ν is ω-symmetric for some regular gauge function ω with $\omega(0+) = 0$. Thus, Theorem 2.3 guarantees that the Poisson integral $u = P[\nu]$ belongs to $\mathcal{B}_0^{\log}(\mathbb{R}^{n+1}_+)$.

Next, we fix a cube $Q^0 \subset \mathbb{R}^n$ such that $F\big|_{Q^0} = 1$. Since the measure ν is singular and positive, properties (2.1) and (2.2) hold. Consider the set

$$E(a) = \{x \in Q^0 : \lim_{y\to 0+} u(x,y) = a\}.$$

We have $E_\nu(a) \cap Q^0 = E_\mu(a) \cap Q^0$; hence, by Theorem 2.2, it suffices to show that

$$E_\nu(a) \cap Q^0 \supset E(a). \tag{2.5}$$

Let $x_0 \in E(a)$. Consider a cube $Q = Q_h(\xi)$ such that $x_0 \in Q$. Applying Green's formula to the pair of functions y and $u(x,y)$ on the set $Q \times (0,h)$ and using property (2.4), we obtain

$$\left|\mu(Q) - \int_Q u(x,h)\,dx\right| \le C(n)\omega(h)u(\xi,h)|Q|$$

(the details of the corresponding argument can be found in [5]). Next, since $x_0 \in Q$, condition (2.4) guarantees that

$$|\log u(x,h) - \log(x_0,h)| \le C(n)\omega(h)$$

for all $x \in Q$. Therefore,

$$\lim_{h\to 0} \frac{\mu(Q_h)}{|Q_h| u(x_0,h)} = 1,$$

where the limit is calculated with respect to all cubes Q_h such that $x_0 \in Q_h$. In other words, we have (2.5). Note that the above proof of property (2.5) is valid for $a = +\infty$ and for $a = 0$.

Finally, using the above notation, consider the case where $a = +\infty$. We recall that ν is a singular symmetric measure on $\mathbb{R}^n$. It is well known that $\nu(E) = 0$ if $\dim_{\mathcal{H}}(E) < n$. On the other hand, we have $\nu(E(+\infty)) = \nu(Q^0) > 0$, because ν is a positive singular measure. Therefore, $\dim_{\mathcal{H}}(E(+\infty)) = n$. To finish the proof, we apply inclusion (2.5) for $a = +\infty$. □

Comments. The argument used in the proof of property (2.5) shows that

$$\lim_{y\to 0+} P[\nu](x,y) = a \;\Rightarrow\; \mathcal{D}\nu(x) = a$$

for all $a \in [0,+\infty]$ if ν is a finite symmetric measure on $\mathbb{R}^n$. Statements of this type are called converse Fatou theorems. Recall that, by the Tauberian theorem of Rudin [13], for $a \in [0,+\infty)$ we have a weaker implication, namely,

$$\lim_{y\to 0+} P[\mu](x,y) = a \;\Rightarrow\; \mathcal{D}_{\mathrm{sym}}\mu(x) = a$$

if μ is a finite positive measure on $\mathbb{R}^n$. Here

$$\mathcal{D}_{\mathrm{sym}}\mu(x) = \lim_{h\to 0} \frac{\mu(Q_h(x))}{|Q_h|},$$

the symmetric derivative at the point $x \in \mathbb{R}^n$. On the other hand, Rudin [13] constructed a positive measure μ such that $\lim_{y\to 0+} P[\mu](x,y) = +\infty$, but $\mathcal{D}_{\mathrm{sym}}\mu(0)$ does not exist.

2.2. Smooth measures.

Definition. A real Borel measure μ on $\mathbb{R}^n$ is said to be *smooth* if

$$\lim_{h\to 0} \frac{\mu(Q) - \mu(Q')}{|Q|} = 0,$$

where Q and Q' are arbitrary adjacent cubes of side length h. The term "small Zygmund measures" is also used for the smooth measures (cf. [3, 5]).

An analog of the Carmona–Donaire theorem for measures defined on $\mathbb{R}^n$ has the following form.

Theorem 2.4. *Let μ be a positive singular smooth measure on $\mathbb{R}^n$. Then* $\dim_{\mathcal{H}} E_a(\mu) = n$ *for all $a \in [0, +\infty]$.*

The little Bloch space $\mathcal{B}_0(\mathbb{R}^{n+1}_+)$ corresponds to smooth measures; it consists of all harmonic functions $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ such that

$$\lim_{t\to 0+} \sup_{0<y\le t} y|\nabla u(x,y)| = 0.$$

Theorem 2.5. *Consider a positive function $u \in \mathcal{B}_0(\mathbb{R}^{n+1}_+)$ and a cube $Q^0 \subset \mathbb{R}^n$. Suppose that*

$$\lim_{y\to 0+} u(\xi, y) = 0 \quad \textit{for almost all} \quad \xi \in Q^0 \tag{2.6}$$

and

$$\lim_{y\to 0+} u(\xi', y) = +\infty \quad \textit{for a point} \quad \xi' \in Q^0. \tag{2.7}$$

Then for any $a \in (0, +\infty)$ we have $\dim_{\mathcal{H}} E(a) = n$, *where*

$$E(a) = \{x \in Q^0 : \lim_{y\to 0+} u(x,y) = a\}.$$

Now we deduce Theorem 2.4 from Theorem 2.5.

Proof of Theorem 2.4. Let $a \in (0, +\infty)$, and let μ be a positive singular smooth measure on $\mathbb{R}^n$. Fixing a cube $Q^0 \subset \mathbb{R}^n$ such that $\mu(Q^0) > 0$, we consider an auxiliary measure $\nu = f\mu$, where f is a nonnegative C^1-function with compact support and $f\big|_{Q^0} = 1$. Since the measure ν is finite and smooth, the Poisson integral $u = P[\nu]$ is in the space $\mathcal{B}_0(\mathbb{R}^{n+1}_+)$ (in [6] it was shown that this implication is a consequence of the results obtained in [5]). Furthermore, properties (2.6) and (2.7) hold, because ν is a singular positive measure and $\nu(Q^0) > 0$. Therefore, with the help of Theorem 2.5, we obtain a set $E(a)$ such that $\dim_{\mathcal{H}} E(a) = n$. By Proposition 2 in [6], we have $E_\nu(a) \cap Q^0 \supset E(a)$ for $a \in [0, +\infty]$. Note that $E_\nu(a) \cap Q^0 = E_\mu(a) \cap Q^0$, so finally we have $\dim_{\mathcal{H}} E_\mu(a) = n$.

Recall that ν is a positive singular smooth measure on $\mathbb{R}^n$, so that $\nu(E) = 0$ if $\dim_{\mathcal{H}}(E) < n$ (see [9], where sharper quantitative results were obtained). Therefore, as in the proof of Theorem 2.1, for $a = +\infty$ we may apply the property $E_\nu(+\infty) \cap Q^0 \supset E(+\infty)$. $\square$

The remaining part of the present paper is devoted to the proofs of Theorems 2.2 and 2.5.

§3. Auxiliary results

Logarithmic Bloch functions. Let $Q = Q_h \subset \mathbb{R}^n$. By definition, $\widehat{Q} = Q \times (0, h] \subset \mathbb{R}^{n+1}_+$. If x_Q is the center of the cube Q, then we put $z_Q = (x_Q, h)$. Given a harmonic function $u : \mathbb{R}^{n+1}_+ \to (0, +\infty)$, we define

$$\|u\|_{\mathcal{B}^{\log}(\widehat{Q})} = \sup\{y|\nabla \log u(x, y)| : (x, y) \in \widehat{Q}\}.$$

The first lemma follows from the definition of the quantity $\|u\|_{\mathcal{B}^{\log}(\widehat{Q})}$ and from the properties of the exponential function.

Lemma 3.1. *Consider a cube $Q = Q_h \subset \mathbb{R}^n$. Assume that*

$$\|u\|_{\mathcal{B}^{\log}(\widehat{Q})} \le 1.$$

Then there exists a constant $C_{3.1}(n)$ such that

$$\left|\frac{u(z)}{u(z')} - 1\right| < C_{3.1}(n)\|u\|_{\mathcal{B}^{\log}(\widehat{Q})}$$

for all $z, z' \in Q \times [h/2, h]$.

Let $Q = Q_h \subset \mathbb{R}^n$, and let $k \in \mathbb{N}$. The standard decomposition of the cube Q into 2^{kn} pairwise disjoint cubes of side length $2^{-k}h$ is called the kth dyadic generation. The dyadic decomposition of the cube Q is the collection of all dyadic generations.

Lemma 3.2. *Fix a cube $Q = Q_h \subset \mathbb{R}^n$ and consider its dyadic decomposition. Suppose that a family $\mathcal{F} = \{P\}$ consists of pairwise disjoint dyadic cubes $P \subset Q$ and that*

$$\sum_{P\in\mathcal{F}} |P| = |Q|.$$

Assume that a positive function u is harmonic in $\mathbb{R}^{n+1}_+$ and

$$\frac{u(z)}{u(z')} \le 8 \quad \text{for all } z, z' \in E, \quad \text{where } E = \widehat{Q} \setminus \bigcup_{P\in\mathcal{F}} \widehat{P}.$$

Then there exists a positive constant $C_{3.2}(n)$ such that

$$\left|1 - \sum_{P\in\mathcal{F}} \frac{u(z_P)|P|}{u(z_Q)|Q|}\right| \le C_{3.2}(n)\|u\|_{\mathcal{B}^{\log}(\widehat{Q})}.$$

Proof. For a cube P, let $\ell(P)$ denote its side length. Put

$$\mathcal{F}_k = \{P \in \mathcal{F} : \ell(P) \ge 2^{-k}h\}$$

(the cubes of the first k generations in the dyadic decomposition of the cube Q_h). If k is sufficiently large, then

$$\sum_{P\in\mathcal{F}\setminus\mathcal{F}_k} \frac{u(z_P)|P|}{u(z_Q)|Q|} \le \|u\|_{\mathcal{B}^{\log}(\widehat{Q})}.$$

We fix such a number k. To the family $\mathcal{F}_k$, we add all dyadic cubes of side length $2^{-k-1}h$ contained in the set $\bigcup_{P\in\mathcal{F}\setminus\mathcal{F}_k} P$. Denote by $\mathcal{F}_0$ the resulting *finite* family. Observe that $\sum_{P\in\mathcal{F}_0} |P| = |Q|$.

Let $E_0 = \widehat{Q} \setminus \bigcup_{P\in\mathcal{F}_0} \widehat{P}$. We have $E_0 \subset E$; in particular, if $P \in \mathcal{F}_0 \setminus \mathcal{F}_k$, then $z_P \in E$. Thus,

$$\sum_{P\in\mathcal{F}_0\setminus\mathcal{F}_k} \frac{u(z_P)|P|}{u(z_Q)|Q|} \le 8 \sum_{P\in\mathcal{F}\setminus\mathcal{F}_k} \frac{u(z_P)|P|}{u(z_Q)|Q|} \le 8\|u\|_{\mathcal{B}^{\log}(\widehat{Q})}.$$

Therefore, without loss of generality we may assume that the family $\mathcal{F}$ is finite.

Applying Green's formula to the pair of functions y and $u(x,y)$ on the set E, we obtain

$$\int_{\partial E} y\partial_n u\, d\Sigma = \int_{\partial E} u\partial_n y\, d\Sigma = \int_{Q_h} u(x,h)\, dx - \sum_{\mathcal{F}} \int_P u(x,\ell(P))\, dx,$$

where $d\Sigma$ denotes the corresponding surface measure on ∂E. We divide both sides of the above identity by $|Q|u(z_Q)$ and estimate the summands obtained.

Since $\Sigma(\partial E) \le C(n)|Q|$, we have

$$\int_{\partial E} y\frac{|\partial_n u|}{|Q|u(z_Q)}\, d\Sigma \le \frac{8}{|Q|}\int_{\partial E} y\frac{|\partial_n u|}{u}\, d\Sigma \le C(n)\|u\|_{\mathcal{B}^{\log}(\widehat{Q})}.$$

Next, Lemma 3.1 guarantees that

$$\Big|1 - \int_{Q_h} \frac{u(x,h)\, dx}{|Q|u(z_Q)}\Big| \le C(n)\|u\|_{\mathcal{B}^{\log}(\widehat{Q})};$$

also, we have

$$\Big|\sum_{\mathcal{F}} |P|u(z_P) - \sum_{\mathcal{F}}\int_P u(x,\ell(P))\, dx\Big| \le C(n)\|u\|_{\mathcal{B}^{\log}(\widehat{Q})}\sum_{\mathcal{F}} |P|u(z_P),$$

whence

$$\Big|\sum_{\mathcal{F}} \frac{|P|u(z_P)}{|Q|u(z_Q)} - \sum_{\mathcal{F}} \frac{\int_P u(x,\ell(P))\, dx}{|Q|u(z_Q)}\Big| \le 8C(n)\|u\|_{\mathcal{B}^{\log}(\widehat{Q})}.$$

Applying the above estimates, we obtain the required inequality. □

Harmonic Bloch functions. Let $Q = Q_h \subset \mathbb{R}^n$. Given a harmonic function $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$, we define

$$\|u\|_{\mathcal{B}(\widehat{Q})} = \sup\{y|\nabla u(x,y)| : (x,y) \in \widehat{Q}\}.$$

As in the logarithmic case, the first lemma follows directly from the definition of the quantity $\|u\|_{\mathcal{B}(\widehat{Q})}$.

Lemma 3.3. *Consider a cube $Q = Q_h \subset \mathbb{R}^n$. Assume that $\|u\|_{\mathcal{B}(\widehat{Q})} < \infty$. Then there exists a constant $C_{3.3}(n)$ such that*

$$|u(z) - u(z')| < C_{3.3}(n)\|u\|_{\mathcal{B}(\widehat{Q})}$$

for all $z, z' \in Q \times [h/2, h]$.

Lemma 3.4. *Fix a cube $Q \subset \mathbb{R}^n$. Consider a collection $\mathcal{F} = \{P\}$ that consists of pairwise disjoint dyadic cubes $P \subset Q$ and satisfies the condition*

$$\sum_{P\in\mathcal{F}} |P| = |Q|.$$

Suppose that a function u is harmonic on $\mathbb{R}^{n+1}_+$ and bounded on $\widehat{Q}\setminus\bigcup_{P\in\mathcal{F}}\widehat{P}$. Then there exists a positive constant $C_{3.4}(n)$ such that

$$\Big|\sum_{P\in\mathcal{F}} (u(z_Q) - u(z_P))\frac{|P|}{|Q|}\Big| \le C_{3.4}(n)\|u\|_{\mathcal{B}(\widehat{Q})}.$$

Proof. Applying Green's formula to the pair of functions y and $u(x,y)$, we can argue by analogy with the proof of Lemma 3.2 (cf. [10]). □

An estimate for the Hausdorff dimension.

Lemma 3.5 (Hungerford [7], Makarov [9]). *Fix constants $0 < \varepsilon < C < 1$. Assume that every finite family $\mathcal{A}_j$, $j \in \mathbb{N}$, consists of paiwise disjoint cubes in $\mathbb{R}^n$. Suppose that the following two conditions are fulfilled.*
(i) *For every cube $Q = Q_h \in \mathcal{A}_j$, $j \geq 2$, there exists a cube $R = R_H \in \mathcal{A}_{j-1}$ such that $Q \subset R$; moreover, $h < \varepsilon H$.*
(ii) *If $R \in \mathcal{A}_{j-1}$, then*

$$\sum_{Q \in \mathcal{A}_j,\ Q \subset R} |Q| \geq C|R|.$$

Then

$$\dim_{\mathcal{H}}\Big(\bigcap_j \bigcup_{Q \in \mathcal{A}_j} Q\Big) \geq n\left(1 - \frac{\log C}{\log \varepsilon}\right).$$

§4. Proof of Theorem 2.2

We shall argue by induction, on the basis of the following lemma.

Lemma 4.1. *Consider a cube $Q \subset \mathbb{R}^n$ and a harmonic function $u : \mathbb{R}^{n+1}_+ \to (0, +\infty)$ such that $u \in \mathcal{B}^{\log}_0(\mathbb{R}^{n+1}_+)$ and*

$$\lim_{y \to 0+} u(\xi, y) = 0 \quad \textit{for almost all} \ \ \xi \in Q.$$

Fix numbers $\varepsilon \in (0, 1/40)$ and $a \in (0, +\infty)$. Suppose that the following conditions are fulfilled for $\delta > 0$ and $k \in \mathbb{N}$, $k \geq 2$:

$$\|u\|_{\mathcal{B}^{\log}(\widehat{Q})} < \delta, \tag{4.1}$$

$$a \leq u(z_Q) \leq a(1 + 2^{-k}), \tag{4.2}$$

$$\log_2(1 + C_{3.1}\delta)\, \log_2 \frac{1}{\varepsilon} < \log_2(1 + 2^{-k-1}), \tag{4.3}$$

$$(C_{3.1} + C_{3.2})\delta < 2^{-k-2}, \tag{4.4}$$

where $C_{3.1}$ and $C_{3.2}$ are the constants provided by Lemmas 3.1 and 3.2, respectively.

Then there exists a finite collection $\mathcal{A} = \{P\}$ of pairwise disjoint cubes $P \subset Q$ with the following properties:

$$\sum_{P \in \mathcal{A}} |P| \geq |Q|/40, \tag{4.5}$$

$$|P| \leq \varepsilon^n |Q|, \tag{4.6}$$

$$a \leq u(z_P) \leq a(1 + 2^{-k-1}). \tag{4.7}$$

Proof. The required elements of the family $\mathcal{A}$ will be selected from the dyadic decomposition of the cube Q. It is convenient to perform the selection procedure in two steps.

Step 1. Consider the family $\mathcal{E}$ that consists of the maximal dyadic cubes $R \subset Q$ such that

$$\text{either} \ \ u(z_R) \geq a(1 + 2^{-k+1}), \quad \text{or} \ \ u(z_R) \leq a(1 - 2^{-k}). \tag{4.8}$$

Recall that $\lim_{y\to 0+} u(\xi, y) = 0$ for almost all $\xi \in Q$. Therefore, by Lemma 3.1, for almost every point $\xi \in Q$ there exists a dyadic cube $\widetilde{R} \subset Q$ such that $\xi \in \widetilde{R}$ and $u(z_{\widetilde{R}}) \leq a(1 - 2^{-k})$. Thus,

$$\sum_{R \in \mathcal{E}} |R| = |Q|. \tag{4.9}$$

Assume that $R \in \mathcal{E}$, $R \subset R^* \subset Q$, and $R \neq R^*$, where R^* is a dyadic cube. Since property (4.8) fails for R^* by definition, Lemma 3.1 and property (4.4) guarantee that

$$u(z) \le a(1+2^{-k+1})(1+C_{3.1}\delta) < a(1+2^{-k+2}),$$
$$u(z) \ge a(1-2^{-k})(1-C_{3.1}\delta) > a(1-2^{-k+1})$$

for all $z \in R^* \times [h^*/2, h^*]$, where h^* is the side length of the cube R^*. Therefore,

$$\frac{u(z)}{u(z')} \le 8 \quad \text{for all } z, z' \in \widehat{Q} \setminus \bigcup_{R\in\mathcal{E}} \widehat{R}.$$

For further reference we note that

$$u(z_R) < a(1+2^{-k+2}). \tag{4.10}$$

Applying Lemma 3.2 to the family $\mathcal{E}$, we obtain

$$\left| 1 - \sum_{R\in\mathcal{E}} \frac{u(z_R)|R|}{u(z_Q)|Q|} \right| < C_{3.2}\delta.$$

In particular,

$$\sum_{R\in\mathcal{E}} \frac{u(z_R)|R|}{a|Q|} > 1 - C_{3.2}\delta. \tag{4.11}$$

Now, consider the family $\mathcal{E}^+ = \{R \in \mathcal{E} : u(z_R) \ge a(1+2^{-k+1})\}$. On the one hand, if $R \in \mathcal{E}^+ \subset \mathcal{E}$, then estimate (4.10) is valid. On the other hand, if $R \in \mathcal{E} \setminus \mathcal{E}^+$, then $u(z_R) \le a(1-2^{-k})$. Therefore, with the help of (4.9) and (4.11) we obtain

$$C_{3.2}\delta > 2^{-k}\Big(1 - 5 \sum_{R\in\mathcal{E}^+} |R|/|Q|\Big).$$

Hence, by (4.4), we have

$$\sum_{R\in\mathcal{E}^+} |R| > |Q|/10. \tag{4.12}$$

Deleting all sufficiently small cubes, we obtain a finite family $\mathcal{E}^+$ with property (4.12).

Step 2. We apply the following argument to every cube $R \in \mathcal{E}^+$. Consider the family $\mathcal{F}$ that consists of the maximal dyadic cubes $P \subset R$ such that

$$\text{either} \quad u(z_P) \le a(1+2^{-k-1}) \quad \text{or} \quad u(z_P) \ge a(1+2^{-k+3}).$$

Arguing as at the first step and applying inequality (4.4), we see that

$$\sum_{P\in\mathcal{F}} |P| = |R|, \qquad \left| 1 - \sum_{P\in\mathcal{F}} \frac{u(z_P)|P|}{u(z_R)|R|} \right| < C_{3.2}\delta$$

by Lemma 3.2. The latter inequality and estimate (4.10) imply

$$\sum_{P\in\mathcal{F}} \frac{u(z_P)|P|}{a|R|} < (1+C_{3.2}\delta)(1+2^{-k+2}) < 1 + 3\cdot 2^{-k+1}. \tag{4.13}$$

Put $\mathcal{A} = \{P \in \mathcal{F} : u(z_P) \le a(1+2^{-k-1})\}$. On the one hand, Lemma 3.1 and property (4.4) show that $u(z_P) \ge a$ for $P \in \mathcal{A}$. On the other hand, $u(z_P) \ge a(1+2^{-k+3})$ for $P \in \mathcal{F} \setminus \mathcal{A}$. Therefore, (4.13) implies the inequality

$$2^{-k+3} - 3\cdot 2^{-k+1} < 2^{-k+3} \sum_{P\in\mathcal{A}} |P|/|R|.$$

In other words,

$$\sum_{P\in\mathcal{A}} |P| > |R|/4. \tag{4.14}$$

Again, deleting all sufficiently small cubes, we obtain a finite family $\mathcal{A}$ with property (4.14).

We claim that the constructed family $\mathcal{A}$ has the required properties. Indeed, (4.12) and (4.14) imply (4.5). Property (4.7) is fulfilled by definition (see Step 2). Finally, Lemma 3.1 and inequality (4.3) provide the estimate $|R| \le \varepsilon^n |Q|$. Therefore, (4.6) is true as well. □

Proof of Theorem 2.2. Fix numbers $\varepsilon \in (0, 1/40)$ and $a \in (0, +\infty)$. Put $k = 2$ and fix a small number $\delta = \delta_2 > 0$ so as to ensure estimates (4.3) and (4.4).

By assumption, we are given a cube $Q^0 \subset \mathbb{R}^n$ and points $\xi, \xi' \in Q^0$ such that

$$\lim_{y\to 0+} u(\xi, y) = 0,$$
$$\lim_{y\to 0+} u(\xi', y) = +\infty.$$

Therefore, if $h > 0$ is sufficiently small, then the line segment $[\xi, \xi']$ contains a point $x_a = x_a(h)$ such that $u(x_a, h) = a$. Choosing a sufficiently small parameter $h > 0$, we obtain property (4.1) for the cube $Q = Q_h(x_a) \subset Q^0$. Note that $u(z_Q) = a$. Thus, property (4.2) is also fulfilled.

So, all assumptions of Lemma 4.1 are satisfied. With the help of Lemma 4.1 we construct a family $\mathcal{A}_1 = \mathcal{A}$. Observe that property (4.1) is inherited by all cubes in the family $\mathcal{A}_1$. Next, property (4.2) for the elements of $\mathcal{A}_1$ follows from (4.7). Therefore, Lemma 4.1 applies to the cubes in the family $\mathcal{A}_1$, and so on.

In the course of the induction, we modify the construction of the families $\mathcal{A}_j$. Namely, fix a $\delta_3 > 0$ so small that estimates (4.3) and (4.4) are true for $k = 3$ and $\delta = \delta_3$. Property (4.2) for $k = 3$ coincides with (4.7) for $k = 2$. Recall that $u \in \mathcal{B}_0^{\log}(\mathbb{R}^{n+1}_+)$. Therefore, by (4.6), inequality (4.1) is true for $\delta = \delta_3$ if the index $j = j_1$ is sufficiently large. So, we replace $k = 2$ by $k = 3$ and δ_2 by δ_3, and apply Lemma 4.1 to all cubes in the families $\mathcal{A}_{j_1}$, $\mathcal{A}_{j_1+1}$ and so on. Further, if the index $j = j_2$ is sufficiently large, then we can replace $k = 3$ by $k = 4$ and δ_3 by δ_4, and we may continue the construction with the new parameters.

By induction, we obtain families $\mathcal{A}_j$, $j \in \mathbb{N}$, such that properties (i) and (ii) from the assumption of Lemma 3.5 hold with constants $\varepsilon \in (0, 1/40)$ and $C = 1/40$. Lemma 3.1 and the maximality of the dyadic cubes selected at the first and second steps of the proof of Lemma 4.1 guarantee that the required identity

$$\lim_{y\to 0+} u(x, y) = a$$

is true for all

$$x \in \bigcap_j \bigcup_{P\in\mathcal{A}_j} P \subset Q^0.$$

Since the parameter $\varepsilon > 0$ is arbitrarily small, we have

$$\dim_{\mathcal{H}} \Big\{ x \in Q^0 : \lim_{y\to 0+} u(x, y) = a \Big\} = n$$

by Lemma 3.5. □

§5. Proof of Theorem 2.5

Lemma 5.1. *Consider a positive function $u \in \mathcal{B}_0(\mathbb{R}^{n+1}_+)$ such that*

$$\lim_{y \to 0+} u(\xi, y) = 0 \quad \textit{for almost all} \quad \xi \in \mathbb{R}^n.$$

Fix a cube $Q \subset \mathbb{R}^n$ and a number $\varepsilon > 0$. Also, fix $a \in (0, +\infty)$ and a natural number N such that $a > 2^{-N+1}$. Finally, assume that for $k \in \mathbb{N}$ and $\delta > 0$ the following conditions are fulfilled:

$$\|u\|_{\mathcal{B}(\widehat{Q})} < \delta, \tag{5.1}$$

$$a \le u(z_Q) \le a + 2^{-N-k}, \tag{5.2}$$

$$C_{3.3}\delta \log_2 \frac{1}{\varepsilon} < 2^{-N-k}, \tag{5.3}$$

$$(2C_{3.3} + C_{3.4})\delta < 2^{-N-k-1}, \tag{5.4}$$

where $C_{3.3}$ and $C_{3.4}$ are the constants provided by Lemmas 3.3 *and* 3.4, *respectively.*

Then there exists a finite collection $\mathcal{A} = \{P\}$ that consists of pairwise disjoint cubes $P \subset Q$ and has the following properties:

$$\sum_{P \in \mathcal{A}} |P| \ge |Q|/15, \tag{5.5}$$

$$|P| \le \varepsilon^n |Q|, \tag{5.6}$$

$$a \le u(z_P) \le a + 2^{-N-k-1}. \tag{5.7}$$

Proof. The construction of the family $\mathcal{A}$ is split into two steps.

Step 1. Consider the family $\mathcal{E}$ that consists of the maximal dyadic cubes $R \subset Q$ such that

$$|u(z_R) - a| \ge 2^{-N-k+1}.$$

On the one hand, by condition (5.2), we have $u(z_Q) \ge a > 2^{-N+1}$. On the other hand, $\lim_{y \to 0+} u(\xi, y) = 0$ for almost all $\xi \in \mathbb{R}^n$. Thus,

$$\sum_{R \in \mathcal{E}} |R| = |Q|. \tag{5.8}$$

Applying Lemma 3.4 to the family $\mathcal{E}$, we obtain

$$\left| \sum_{R \in \mathcal{E}} (u(z_R) - u(z_Q)) \frac{|R|}{|Q|} \right| < C_{3.4}\delta.$$

In particular,

$$\sum_{R \in \mathcal{E}} (u(z_R) - a) \frac{|R|}{|Q|} > -C_{3.4}\delta. \tag{5.9}$$

Consider the family $\mathcal{E}^+ = \{R \in \mathcal{E} : u(z_R) - a \ge 2^{-N-k+1}\}$. Let $R \in \mathcal{E}^+$; then the maximality property and Lemma 3.3 imply that

$$u(z_R) - a \le 2^{-N-k+1} + C_{3.3}\delta. \tag{5.10}$$

Next, if $R \in \mathcal{E} \setminus \mathcal{E}^+$, then $u(z_R) - a \le -2^{-N-k+1}$. Therefore, with the help of (5.8) and (5.9) we obtain

$$(C_{3.3} + C_{3.4})\delta > 2^{-N-k+1}\Big(1 - 2 \sum_{R \in \mathcal{E}^+} |R|/|Q|\Big).$$

Hence, by (5.4), we have

$$\sum_{R \in \mathcal{E}^+} |R| > |Q|/3. \tag{5.11}$$

Deleting all sufficiently small cubes, we obtain a finite family $\mathcal{E}^+$ with property (5.11).

Step 2. We apply the following argument to every cube $R \in \mathcal{E}^+$. Consider the family $\mathcal{F}$ that consists of the maximal dyadic cubes $P \subset R$ such that

$$\text{either}\quad u(z_P) \le a + 2^{-N-k-1} \quad\text{or}\quad u(z_P) \ge a + 3\cdot 2^{-N-k}. \tag{5.12}$$

As at the first step, we have

$$\sum_{P\in\mathcal{F}} |P| = |R|, \qquad \left|\sum_{P\in\mathcal{F}} (u(z_P) - u(z_R))\frac{|P|}{|R|}\right| < C_{3.4}\delta.$$

By (5.10), the latter inequality implies that

$$\sum_{P\in\mathcal{F}} (u(z_P) - a - 2^{-N-k+1})\frac{|P|}{|R|} < (C_{3.3} + C_{3.4})\delta.$$

Let $\mathcal{A} = \{P \in \mathcal{F} : u(z_P) \le a + 2^{-N-k-1}\}$. Applying Lemma 3.3, we obtain

$$2^{-N-k-1}\Big(2 - 5\sum_{P\in\mathcal{A}} |P|/|R|\Big) < (2C_{3.3} + C_{3.4})\delta.$$

Therefore, by (5.4),

$$\sum_{P\in\mathcal{A}} |P| > |R|/5. \tag{5.13}$$

The final part of the argument is similar to that in the logarithmic case. □

Proof of Theorem 2.5. Fix numbers $\varepsilon \in (0, 1/15)$ and $a \in (0, +\infty)$, and let $N \in \mathbb{N}$ be such that $a > 2^{-N+1}$. We put $k = 1$ and fix $\delta > 0$ so small that estimates (5.3) and (5.4) are fulfilled.

Now, replacing Lemma 4.1 by Lemma 5.1, we can repeat, practically word for word, the argument used in the proof of Theorem 2.2. □

Concluding remarks. 1. The above arguments can be repeated in somewhat more general situations. In particular, for the real Borel measures μ, the following theorem is true. In a sense, this theorem shows that the image of the derivative $\mathcal{D}\mu$ has no lacunas.

Theorem 5.2. *Consider a real smooth measure μ on $\mathbb{R}^n$ and a cube $Q \subset \mathbb{R}^n$. Let $-\infty \le A_1 < A_2 \le +\infty$, and let*

$$F_1 = \{x \in Q : \ \mathcal{D}\mu(x) \in [-\infty, A_1]\},$$
$$F_2 = \{x \in Q : \ \mathcal{D}\mu(x) \in [A_2, +\infty]\}.$$

Suppose that $F_1 \ne \varnothing$, $F_2 \ne \varnothing$ and $|F_1| + |F_2| = |Q|$. Then

$$\dim_{\mathcal{H}}\{x \in Q : \ \mathcal{D}\mu(x) = a\} = n$$

for all $a \in (A_1, A_2)$.

Since the symmetric measures are positive by definition, in a similar theorem about symmetric measures we assume that $0 \le A_1 < A_2 \le +\infty$.

2. It is possible to replace the cubes Q_h by a larger collection of sets in the definition of the derivative. For example, put

$$D\mu(x) = \lim_{x\in Q,\ |Q|\to 0} \frac{\mu(Q)}{|Q|},$$

where Q is an arbitrary rotation of the standard cube Q_h. Theorems 2.1, 2.4, and 5.2 remain valid if the derivative $\mathcal{D}\mu$ is replaced by $D\mu$.

Added in proof. After this paper had been completed, the author learned that Theorem 2.4 had been obtained previously, by a different method, in the following work: J. J. Donaire, *Conjuntos excepcionales para las classes de Zygmund*, Thesis, Barcelona, 1995.

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