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ADMISSIBLE CONDITIONS FOR PARABOLIC EQUATIONS DEGENERATING AT INFINITY

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ABSTRACT. Well-posedness in $L^\infty(\mathbb{R}^n)$ $(n \geq 3)$ of the Cauchy problem is studied for a class of linear parabolic equations with variable density. In view of degeneracy at infinity, some *conditions at infinity* are possibly needed to make the problem well-posed. Existence and uniqueness results are proved for bounded solutions that satisfy either *Dirichlet* or *Neumann conditions* at infinity.

§1. Introduction

We investigate existence and uniqueness for bounded solutions of the parabolic Cauchy problem

$$\begin{cases} \rho\, \partial_t u = \Delta u & \text{in } \mathbb{R}^n \times \mathbb{R}_+ =: S, \\ u = u_0 & \text{in } \mathbb{R}^n \times \{0\} \quad (n \geq 3). \end{cases} \tag{1.1}$$

Concerning the coefficient $\rho = \rho(x)$ and the initial data u_0 in (1.1), we always assume the following:

$$\begin{cases} \text{(i)} \quad \rho \in C^{1+\alpha}_{\text{loc}}(\mathbb{R}^n), \quad \rho > 0 \quad \text{in } \mathbb{R}^n; \\ \text{(ii)} \quad u_0 \in C^{\alpha}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad (\alpha \in (0,1)). \end{cases} \tag{H_0}$$

The solutions of problem (1.1) are always meant in the classical sense.

As is well known, problem (1.1) can be ill posed in the set of bounded solutions, depending on the behavior of ρ as $r := |x| \to \infty$ and on the space dimension n. To be specific, it is well posed in the above class for any smooth, positive ρ if $n \leq 2$ whereas, if $n \geq 3$ and ρ tends to 0 sufficiently fast as $r \to \infty$ (depending on n), some *conditions at infinity* are needed to restore well-posedness (see [3]–[11], [14]–[15], [17] and the references therein). In the papers mentioned above, such conditions are *of Dirichlet type* and *homogeneous*, for they require the unknown to vanish at infinity in a suitable sense.

However, it is quite natural to consider conditions of a different type. In this paper we address the uniqueness of bounded solutions of (1.1) that satisfy at infinity either *inhomogeneous* conditions of Dirichlet type (see Subsection 1.1), or conditions of *Neumann type* (see Subsection 1.2). Remarkably, the results obtained for the former enable us to address the latter (see the proof of Theorem 1.5). It is also worthy of mention that the solutions satisfying conditions of Dirichlet type are uniquely determined by the limiting value at infinity of their spherical means (see Theorems 1.1, 1.3).

In the present paper we only deal with the model equation in (1.1); however, similar results should be valid for a wider class of quasilinear parabolic equations.

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1.1. Dirichlet conditions. Besides condition (H_0), to deal with this case we assume that

$$\Gamma * \rho \in L^\infty(\mathbb{R}^n), \tag{H$_1$}$$

where Γ denotes the fundamental solution of the Laplace equation in $\mathbb{R}^n$. As was observed in [1], condition (H_1) is satisfied if and only if the equation

$$-\Delta v = \rho \quad \text{in } \mathbb{R}^n \tag{1.2}$$

has a bounded solution.

Set $B_R := \{x \in \mathbb{R}^n \mid |x| < R\},\ R > 0$. We denote by

$$⨍_{\partial B_R} v\, d\sigma := \frac{1}{|\partial B_R|} \int_{\partial B_R} v\, d\sigma$$

the mean of a function v on the sphere ∂B_R.

Our first result shows that any bounded solution of problem (1.1) has a *trace at infinity* in a suitable sense.

Theorem 1.1. *Let assumptions* (H_0) *and* (H_1) *be satisfied, and let u be any bounded solution of problem* (1.1)*. Set*

$$U(x,t) := \int_0^t u(x,\tau)\, d\tau \quad ((x,t) \in S). \tag{1.3}$$

Then there exists $A \in \mathrm{Lip}(\overline{\mathbb{R}}_+)$ with $A(0) = 0$ such that

$$\lim_{R\to\infty} ⨍_{\partial B_R} U(x,t)\, d\sigma = A(t). \tag{1.4}$$

Moreover, we have

$$\lim_{R\to\infty} ⨍_{\partial B_R} \big|U(x,t) - A(t)\big| d\sigma = 0 \tag{1.5}$$

uniformly with respect to $t \in [0,T]$ for any $T > 0$.

Conversely, it is natural to regard (1.5) as a (possibly inhomogeneous) *Dirichlet condition at infinity* for any given $A \in \mathrm{Lip}(\overline{\mathbb{R}}_+)$. The existence of a bounded solution of problem (1.1) satisfying such a condition is established in the following theorem.

Theorem 1.2. *Let assumptions* (H_0), (H_1) *be satisfied, and let $A \in \mathrm{Lip}(\overline{\mathbb{R}}_+)$ with $A(0) = 0$. Then there exists a bounded solution u of problem* (1.1) *satisfying* (1.5), *with U defined by* (1.3).

Imposing condition (1.5) also implies uniqueness, as the following theorem shows.

Theorem 1.3. *Let assumptions* (H_0), (H_1) *be satisfied, and let $A \in \mathrm{Lip}(\overline{\mathbb{R}}_+)$ with $A(0) = 0$. Then there exists at most one bounded solution of* (1.1) *satisfying* (1.5).

We note that a similar uniqueness result concerning *positive* solutions of the quasilinear equation

$$\rho\, \partial_t u = \Delta G(u) \quad \text{in } S$$

(here $G : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function satisfying suitable assumptions) was proved in [5] in the particular case where $A \equiv 0$.

1.2. Neumann conditions. When dealing with conditions of Neumann type, besides (H_0) we assume that

$$(\mathrm{H}_2) \qquad \rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Observe that condition (H_2) is stronger than (H_1). In fact, for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\{\Gamma * \rho\}(x) &= C \int_{|x-y|>1} \frac{\rho(y)}{|x-y|^{n-2}}\, dy + C \int_{|x-y|\le 1} \frac{\rho(y)}{|x-y|^{n-2}}\, dy \\ &\le C \int_{|x-y|>1} \rho(y)\, dy + C \int_{|x-y|\le 1} \frac{\rho(y)}{|x-y|^{n-2}}\, dy \\ &\le C\Big\{ \|\rho\|_1 + \|\rho\|_\infty \frac{|\partial B_1|}{2} \Big\}, \quad C>0,\end{aligned}$$

and the claim follows.

If u is a bounded solution of problem (1.1), then

$$\int_{B_R} \rho\, u(x,t)\, dx - \int_{B_R} \rho\, u_0\, dx = \int_0^t \int_{\partial B_R} \partial_\nu u\, d\sigma d\tau \quad \text{for any } R>0,\ t>0;$$

here ∂_ν denotes the outer normal derivative at any point of the sphere ∂B_R. By (H_2), we have

$$\lim_{R\to\infty} \Big\{ \int_{B_R} \rho\, u(x,t)\, dx - \int_{B_R} \rho\, u_0\, dx \Big\} = \int_{\mathbb{R}^n} \rho\, u(x,t)\, dx - \int_{\mathbb{R}^n} \rho\, u_0\, dx.$$

Hence the *conservation law*

$$(1.6) \qquad \int_{\mathbb{R}^n} \rho\, u(x,t)\, dx = \int_{\mathbb{R}^n} \rho\, u_0\, dx \quad (t>0)$$

is fulfilled if and only if the *Neumann condition at infinity*

$$\lim_{R\to\infty} \int_0^t \int_{\partial B_R} \partial_\nu u\, d\sigma d\tau = 0 \quad (t>0)$$

is satisfied. This motivates the following definition (where, obviously, the letter $\mathcal{N}$ stands for "Neumann").

Definition 1.1. A solution of problem (1.1) belongs to the *class* $\mathcal{N}$ if it satisfies the *conservation law* (1.6) for any $t>0$.

The existence of solutions in the class $\mathcal{N}$ is the content of the following theorem, where by $L^2_\rho(\mathbb{R}^n)$ we denote the weighted Lebesgue space with the norm

$$\|h\|_{L^2_\rho} := \Big(\int_{\mathbb{R}^n} |h|^2 \rho\, dx \Big)^{\frac{1}{2}}.$$

Theorem 1.4. *Let assumptions* (H_0) *and* (H_2) *be satisfied. Then there exists a bounded solution* $\hat{u} \in \mathcal{N}$ *of problem* (1.1)*; moreover,* $\hat{u} \in L^\infty(\mathbb{R}_+; L^2_\rho(\mathbb{R}^n))$ *and* $|\nabla \hat{u}| \in L^2(S)$.

We shall also prove a uniqueness result.

Theorem 1.5. *Let assumptions* (H_0) *and* (H_2) *be satisfied. Then in the class* $\mathcal{N}$ *there exists at most one bounded solution of problem* (1.1).

Theorems 1.4 and 1.5 immediately imply the following.

Corollary 1.6. *Let assumptions* (H_0) *and* (H_2) *be satisfied. Then any bounded solution* $u \in \mathcal{N}$ *of problem* (1.1) *belongs to* $L^\infty(\mathbb{R}_+; L^2_\rho(\mathbb{R}^n))$*; moreover,* $|\nabla u| \in L^2(S)$.

In view of the conservation law (1.6), it is natural to expect any bounded solution of class $\mathcal{N}$ to converge to the weighted mean of the initial data as $t \to \infty$. This is established in the following theorem.

Theorem 1.7. *Let assumptions* (H_0) *and* (H_2) *be satisfied. Let* u *be any bounded solution of class* $\mathcal{N}$ *to problem* (1.1). *Then*

$$\lim_{t\to\infty} u(\cdot,t) = \frac{\int_{\mathbb{R}^n} \rho\, u_0\, dx}{\int_{\mathbb{R}^n} \rho\, dx} \tag{1.7}$$

uniformly on compact subsets of $\mathbb{R}^n$.

Note that the asymptotic behavior of solutions of (1.1), in dependence on the behavior of ρ at infinity, was investigated in [2] for $n = 1$ (see also [6]–[8]).

§2. Dirichlet conditions: Proofs

Lemma 2.1. *Suppose* $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $\Gamma * |f| \in L^\infty(\mathbb{R}^n)$. *Let* v *be a bounded solution of the equation*

$$-\Delta v = f \quad \text{in } \mathbb{R}^n. \tag{2.1}$$

Then there exists $A \in \mathbb{R}$ *such that*

$$\lim_{R\to\infty} ⨍_{\partial B_R} v\, d\sigma = A. \tag{2.2}$$

Moreover,

$$\lim_{R\to\infty} ⨍_{\partial B_R} \big|v - A\big|\, d\sigma = 0. \tag{2.3}$$

Proof. We set $f^\pm := \max\{\pm f, 0\}$ and

$$\tilde v_\pm := \Gamma * f^\pm \quad \text{in } \mathbb{R}^n, \tag{2.4}$$

where Γ denotes the fundamental solution of the Laplace equation in $\mathbb{R}^n$; then

$$-\Delta \tilde v_\pm = f^\pm \quad \text{in } \mathbb{R}^n.$$

Moreover, since

$$0 \le \tilde v_\pm \le \Gamma * |f| \quad \text{in } \mathbb{R}^n,$$

we have $\tilde v_\pm \in L^\infty(\mathbb{R}^n)$. By [1, Lemma A.4],

$$\lim_{R\to\infty} ⨍_{\partial B_R} \tilde v_\pm\, d\sigma = 0. \tag{2.5}$$

Define $\tilde v := \tilde v_+ - \tilde v_-$. Then

$$-\Delta \tilde v = f \quad \text{in } \mathbb{R}^n,$$

$\tilde v \in L^\infty(\mathbb{R}^n)$, and

$$\lim_{R\to\infty} ⨍_{\partial B_R} \tilde v\, d\sigma = 0. \tag{2.6}$$

Being a bounded harmonic function in $\mathbb{R}^n$, the difference $v - \tilde v$ is a constant; namely,

$$v = \tilde v + A \quad \text{in } \mathbb{R}^n$$

for some $A \in \mathbb{R}$. Hence, by (2.6), we have

$$\lim_{R\to\infty} ⨍_{\partial B_R} v\, d\sigma = \lim_{R\to\infty} ⨍_{\partial B_R} \tilde v\, d\sigma + A = A.$$

Clearly, $|\tilde v| \le \tilde v_+ + \tilde v_-$; now (2.3) follows immediately from (2.5). This proves the result. ◻

Proof of Theorem 1.1. Integration of the first equation in (1.1) with respect to time gives

$$-\Delta U(\cdot,t) = \rho\big[u_0 - u(\cdot,t)\big] \quad \text{in } \mathbb{R}^n \tag{2.7}$$

for any fixed $t > 0$, where U is defined by (1.3). The assumptions (H_0), (H_1) and the boundedness of u imply that for any $t > 0$ the right-hand side of (2.7) satisfies the conditions of Lemma 2.1. By this lemma, there exists $A : \overline{\mathbb{R}}_+ \to \mathbb{R}$ with $A(0) = 0$ such that relations (1.4) and (1.5) are true for any $t > 0$; in fact, these relations follow from (2.2) and (2.3), respectively.

It remains to prove the following: (i) the function $A : \overline{\mathbb{R}}_+ \to \mathbb{R}$ is Lipschitz continuous, and (ii) the convergence in (1.4), (1.5) is uniform with respect to $t \in [0,T]$ for any finite $T > 0$. Since u is bounded, we have

$$|U(x,t) - U(x,s)| \le \|u\|_\infty |t-s| \quad ((x,t),(x,s) \in S), \tag{2.8}$$

whence

$$\Big|\fint_{\partial B_R} \big[U(x,t) - U(x,s)\big]\, d\sigma\Big| \le \|u\|_\infty\, |t-s| \quad (s,t \ge 0)$$

for any $R > 0$. By (1.4), since $A(0) = 0$, we have

$$A(t) - A(s) = \lim_{R\to\infty} \fint_{\partial B_R} \big[U(x,t) - U(x,s)\big]\, d\sigma$$

for any $s,t \ge 0$, and (i) follows. As for (ii), it suffices to observe that the family of functions

$$f_R : \overline{\mathbb{R}}_+ \to \mathbb{R}, \quad f_R(t) := \fint_{\partial B_R} \big|U(x,t) - A(t)\big|\, d\sigma \quad (R > 0)$$

is uniformly equicontinuous, by inequality (2.8). This completes the proof. □

For any $A \in \mathrm{Lip}(\overline{\mathbb{R}}_+)$, the derivative $a \equiv A'$ exists almost everywhere and belongs to $L^\infty(\mathbb{R}_+)$. Let a_R denote a smooth approximation of a; we assume that $\|a_R\|_\infty \le \|a\|_\infty$ for any $R > 0$ and $a_R \to a$ in $L^1(0,T)$ for any $T > 0$ as $R \to \infty$. Also, we set $u_{0,R} := \zeta_R\, u_0 + (1-\zeta_R)\, a_R$ ($R > 0$), where $\zeta_R \in C_0^\infty(B_R)$, $0 \le \zeta_R \le 1$, $\zeta_R = 1$ in $B_{R/2}$.

Fixing an arbitrary $T > 0$, for any $R > 0$ we consider the following auxiliary problem:

$$\begin{cases} \rho\, \partial_t u_R = \Delta u_R & \text{in } B_R \times (0,T) =: Q_{R,T}, \\ u_R = a_R(\cdot) & \text{in } \partial B_R \times (0,T), \\ u_R = u_{0,R} & \text{in } B_R \times \{0\}. \end{cases} \tag{2.9}$$

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Existence, uniqueness, and comparison results for solutions of problem (2.9) can be proved by standard methods. By comparison results, we have

$$|u_R| \le \|u_0\|_\infty + \|a\|_\infty \quad \text{in } Q_{R,T}. \tag{2.10}$$

The interior estimates for derivatives (see [12]) show that, in any compact set $K \subseteq \mathbb{R}^n \times [0,T]$, the first derivatives of u_R are also bounded uniformly with respect to $R > 0$. Let $R \to \infty$. By the standard compactness arguments, there exists a subsequence $\{u_{R_k}\}$ that converges uniformly in K to a bounded solution u of problem (1.1).

Let U be defined as in (1.3). Setting

$$U_R(x,t) := \int_0^t u_R(x,\tau)\, d\tau \quad ((x,t) \in Q_{R,T}),$$

we observe that $U_{R_k} \to U$ in $\mathbb{R}^n \times [0,T]$ as $k \to \infty$. The conclusion will follow if we prove that the function U satisfies (1.5) uniformly with respect to $t \in [0,T]$.

It is easily seen that for any $t \in (0,T)$ the function $U_R(\cdot,t)$ satisfies the problem

$$\begin{cases} -\Delta U_R(\cdot,t) = \rho\big[u_{0,R} - u_R(\cdot,t)\big] & \text{in } B_R, \\ U_R(\cdot,t) = A_R(t) & \text{in } \partial B_R, \end{cases} \tag{2.11}$$

with

$$A_R(t) := \int_0^t a_R(\tau)\,d\tau \quad (t > 0). \tag{2.12}$$

It follows that

$$U_R(x,t) = \int_{B_R} \Gamma_R(x-y)\rho(y)\big[u_{0,R}(y) - u_R(y,t)\big]dy + A_R(t) \quad ((x,t) \in Q_{R,T}), \tag{2.13}$$

where Γ_R denotes the Green function of the Laplace equation in B_R with the zero Dirichlet boundary conditions. We shall prove the following statement.

Claim. We have

$$U(x,t) = \big\{\Gamma * \big[\rho\,(u_0 - u(\cdot,t))\big]\big\}(x) + A(t) \quad ((x,t) \in \mathbb{R}^n \times (0,T)). \tag{2.14}$$

From this claim, the conclusion follows. Indeed, since $\Gamma * \rho \in L^\infty(\mathbb{R}^n)$ by (H_1) and $u_0, u \in L^\infty(\mathbb{R}^n)$ (see (2.10)), we have $\Gamma * [\rho|u_0 - u(\cdot,t)|] \in L^\infty(\mathbb{R}^n)$. Hence, arguing as in the proof of Lemma 2.1 we see that

$$\lim_{R\to\infty} \fint_{\partial B_R} \Gamma * [\rho(u_0 - u)]\,d\sigma = 0$$

for any $t \in [0,T]$. Combined with Theorem 1.1, this gives us (1.5).

It remains to prove the claim. Suppose $R_0 > 0$ is fixed; for any $R > 2R_0$ we rewrite (2.13) as follows:

$$\begin{aligned} U_R(x,t) = &\int_{B_{R_0}} \Gamma_R(x-y)\rho(y)\big[u_0(y) - u_R(y,t)\big]\,dy \\ &+ \int_{B_R\setminus B_{R_0}} \Gamma_R(x-y)\rho(y)\big[u_{0,R}(y) - u_R(y,t)\big]\,dy + A_R(t) \\ &\qquad ((x,t) \in Q_{R,T}) \end{aligned} \tag{2.15}$$

(observe that $u_{0,R} = u_0$ in B_{R_0} for $R > 2R_0$). Similarly, we have

$$\begin{aligned} \big\{\Gamma * \big[\rho\,(u_0 - u(\cdot,t))\big]\big\}(x) = &\int_{B_{R_0}} \Gamma(x-y)\rho(y)\big[u_0(y) - u(y,t)\big]\,dy \\ &+ \int_{\mathbb{R}^n\setminus B_{R_0}} \Gamma(x-y)\rho(y)\big[u_0(y) - u(y,t)\big]\,dy \\ &\qquad ((x,t) \in \mathbb{R}^n \times (0,T)). \end{aligned} \tag{2.16}$$

Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $\varepsilon > 0$ be fixed arbitrarily; we can choose $R_0 > 0$ such that

$$\Big|\int_{\mathbb{R}^n\setminus B_{R_0}} \Gamma(x-y)\rho(y)\big[u_0(y) - u(y,t)\big]\,dy\Big| < \varepsilon, \tag{2.17}$$

$$\Big|\int_{B_R\setminus B_{R_0}} \Gamma_R(x-y)\rho(y)\big[u_{0,R}(y) - u_R(y,t)\big]\,dy\Big| < \varepsilon \quad \text{for any } R > R_0. \tag{2.18}$$

Indeed, inequality (2.17) is true because $\Gamma * [\rho\,(u_0 - u(\cdot,t))] \in L^\infty(\mathbb{R}^n)$; on the other hand, since $\Gamma * \rho \in L^\infty(\mathbb{R}^n)$ by (H_1), $0 \le \Gamma_R \le \Gamma$ in B_R, and $u_{0,R}, u_R(\cdot,t) \in L^\infty(\mathbb{R}^n)$ uniformly for $R > 0$, it follows that inequality (2.18) is also true.

We fix R_0 such that (2.17) and (2.18) are fulfilled. By the dominated convergence theorem,

$$\lim_{k\to\infty}\int_{B_{R_0}}\Gamma_{R_k}(x-y)\rho(y)\left[u_0(y)-u_{R_k}(y,t)\right]dy=\int_{B_{R_0}}\Gamma(x-y)\rho(y)\big[u_0(y)-u(y,t)\big]\,dy.$$

Then, by (2.15), (2.16) and the above remarks, we have

$$\lim_{k\to\infty}U_{R_k}(x,t)=U(x,t)=\big\{\Gamma*\big[\rho\,(u_0-u(\cdot,t))\big]\big\}(x)+A(t)\quad ((x,t)\in\mathbb{R}^n\times(0,T)).$$

This completes the proof of the claim; the result follows. □

Proof of Theorem 1.3. Let u_1,u_2 be two bounded solutions of problem (1.1). Set

$$U_i(x,t):=\int_0^t u_i(x,\tau)\,d\tau\quad ((x,t)\in S;\ i=1,2). \tag{2.19}$$

We assume that

$$\lim_{R\to\infty}\fint_{\partial B_R}U_1(x,t)\,d\sigma=\lim_{R\to\infty}\fint_{\partial B_R}U_2(x,t)\,d\sigma=A(t) \tag{2.20}$$

for any $t>0$. Define $w:=u_1-u_2$; then w satisfies the problem

$$\begin{cases}\rho\,\partial_t w=\Delta w & \text{in } S,\\ w=0 & \text{in } \mathbb{R}^n\times\{0\}.\end{cases} \tag{2.21}$$

Setting

$$W(x,t):=\int_0^t w(x,\tau)\,d\tau=U_1(x,t)-U_2(x,t)\quad ((x,t)\in S)$$

and recalling (1.5), we see that

$$\begin{aligned}\lim_{R\to\infty}\fint_{\partial B_R}|W(x,t)|\,d\sigma&=\lim_{R\to\infty}\fint_{\partial B_R}|U_1(x,t)-U_2(x,t)|\,d\sigma\\ &\le\lim_{R\to\infty}\Big\{\fint_{\partial B_R}|U_1(x,t)-A(t)|\,d\sigma+\fint_{\partial B_R}|U_2(x,t)-A(t)|\,d\sigma\Big\}=0\end{aligned} \tag{2.22}$$

uniformly with respect to $t\in[0,T]$ for any $T>0$.

We prove that $w\equiv 0$. For this, it suffices to check that

$$\int_0^T\int_{\mathbb{R}^n}w\,F\,dx\,dt=0$$

for any $T>0$ and any test function $F=F(x,t)\in C_0^\infty(\mathbb{R}^n\times(0,T))$. Without loss of generality we may assume that $\operatorname{supp}F\subseteq B_{R_0}\times(0,T)$ for some $R_0>0$ ($T>0$ is fixed). Consider the solution $\psi\equiv\psi_R$ of the backward problem

$$\begin{cases}\rho\,\partial_t\psi+\Delta\psi=-F & \text{in } Q_{R,T},\\ \psi=0 & \text{in } \partial B_R\times(0,T),\\ \psi=0 & \text{in } B_R\times\{T\},\end{cases} \tag{2.23}$$

where $R>R_0$ (the index R is omitted in what follows). Then

$$\int_0^T\int_{B_R}w\,F\,dx\,dt=-\int_0^T\int_{\partial B_R}w\,\partial_\nu\psi\,d\sigma\,dt; \tag{2.24}$$

we have used the fact that $w=0$ in $B_R\times\{0\}$ and $\psi=0$ in $(B_R\times\{T\})\cup(\partial B_R\times(0,T))$. Identity (2.24) shows that the conclusion will follow if we check the relation

$$\lim_{R\to\infty}\int_0^T\int_{\partial B_R}w\,\partial_\nu\psi\,d\sigma\,dt=0. \tag{2.25}$$

Integrating by parts over $(0, T)$ yields

$$\int_0^T \int_{\partial B_R} w\, \partial_\nu \psi \, d\sigma\, dt = \int_{\partial B_R} \int_0^T \partial_t W \, \partial_\nu \psi \, dt\, d\sigma = -\int_{\partial B_R} \int_0^T W\, \partial_t(\partial_\nu \psi)\, dt\, d\sigma;$$

we have used the fact that $W = 0$ in $\partial B_R \times \{0\}$ and $\partial_\nu \psi = 0$ in $\partial B_R \times \{T\}$. The regularity of ρ (see assumption (H_0)–(i)) implies $\psi \in C^2(\overline{Q}_{R,T})$. Consequently,

$$\left| \int_0^T \int_{\partial B_R} w\, \partial_\nu \psi \, d\sigma\, dt \right| \le \max_{\partial B_R \times [0,T]} |\partial_\nu(\partial_t \psi)| \int_0^T \int_{\partial B_R} |W| \, d\sigma\, dt. \tag{2.26}$$

Set $\varphi \equiv \varphi_R := \partial_t \psi$; from (2.23) we see that φ satisfies the backward problem

$$\begin{cases} \rho\, \partial_t \varphi + \Delta \varphi = -\partial_t F & \text{in } Q_{R,T}, \\ \varphi = 0 & \text{in } \partial B_R \times (0,T), \\ \varphi = 0 & \text{in } B_R \times \{T\} \end{cases}$$

for any $R > R_0$.

Observe that, by the maximum principle, φ is bounded in $Q_{R,T}$ uniformly with respect to R. We put

$$M := \max_{\overline{Q}_{R,T}} |\varphi|$$

and consider the problem

$$\begin{cases} \rho\, \partial_t \chi + \Delta \chi = 0 & \text{in } (B_R \setminus \bar{B}_{R_0}) \times (0,T) =: K_{R,T}, \\ \chi = M & \text{in } \partial B_{R_0} \times (0,T), \\ \chi = 0 & \text{in } \partial B_R \times (0,T), \\ \chi = z & \text{in } (B_R \setminus \bar{B}_{R_0}) \times \{T\}, \end{cases} \tag{2.27}$$

where

$$z(x) := M \, \frac{|x|^{2-n} - R^{2-n}}{R_0^{2-n} - R^{2-n}} \quad (x \in B_R \setminus \bar{B}_{R_0}). \tag{2.28}$$

It is easily seen that φ is a subsolution of (2.27) (recall that $\operatorname{supp} F \subseteq B_{R_0} \times (0,T)$, so that $\partial_t F \equiv 0$ in $K_{R,T}$). On the other hand, the function z defined in (2.28) is a solution of the same problem; hence, by the maximum principle, $\varphi \le z$ in $K_{R,T}$. Since $\varphi = z$ in $\partial B_R \times (0,T)$, we also have

$$\partial_\nu \varphi > -\frac{(n-2)M}{R_0^{2-n} - R^{2-n}} R^{1-n} \quad \text{in } \partial B_R \times (0,T). \tag{2.29}$$

Similarly, $\varphi \ge -z$ in $K_{R,T}$, and

$$\partial_\nu \varphi < \frac{(n-2)M}{R_0^{2-n} - R^{2-n}} R^{1-n} \quad \text{in } \partial B_R \times (0,T). \tag{2.30}$$

Using (2.29) and (2.30), from (2.26) we deduce the inequality

$$\begin{aligned} \left| \int_0^T \int_{\partial B_R} w\, \partial_\nu \psi \, d\sigma\, dt \right| &\le \frac{(n-2)M}{R_0^{2-n} - R^{2-n}} R^{1-n} \int_0^T \int_{\partial B_R} |W| \, d\sigma\, dt \\ &= C_R \int_0^T \fint_{\partial B_R} |W| \, d\sigma\, dt \end{aligned}$$

with

$$C_R := \frac{(n-2)M|\partial B_1|}{R_0^{2-n} - R^{2-n}}.$$

Since, as $R \to \infty$,

$$\fint_{\partial B_R} |W(x,t)|\, d\sigma \to 0$$

uniformly with respect to $t \in [0,T]$ (see (2.22)), the above inequality shows that (2.25) is true. This completes the proof. □

Remark 2.2. The above uniqueness result implies that the entire family $\{u_R\}$ of solutions of problem (2.9) converges as $R \to \infty$ to a bounded solution of (1.1) satisfying (1.5).

§3. Neumann conditions: Proofs

First, we prove Theorem 1.4.

Proof of Theorem 1.4. Let $T > 0$ be fixed arbitrarily. For any $R > 0$, we set $u_{0,R} := \zeta_R\, u_0$, where $\zeta_R \in C_0^\infty(B_R)$, $0 \le \zeta_R \le 1$, $\zeta_R = 1$ in $B_{R/2}$. Consider the auxiliary problem

$$\begin{cases} \rho\, \partial_t u = \Delta u & \text{in } Q_{R,T}, \\ \partial_\nu u = 0 & \text{in } \partial B_R \times (0,T), \\ u = u_{0,R} & \text{in } B_R \times \{0\}. \end{cases} \tag{3.1}$$

The existence and uniqueness of solutions, as well as comparison results for problem (3.1), are proved by the usual methods (see, e.g., [12]).

Let u_R denote a unique solution of problem (3.1); by comparison, we have

$$|u_R| \le \|u_0\|_\infty \quad \text{in } Q_{R,T}. \tag{3.2}$$

Let $R \to \infty$. By compactness, there exists a sequence $\{u_{R_k}\}$ that converges uniformly on any compact subset of $\mathbb{R}^n \times (0,T)$ to a bounded solution $\hat{u}$ of problem (1.1); in fact, we have

$$|\hat{u}| \le \|u_0\|_\infty \quad \text{in } \mathbb{R}^n \times (0,T) \tag{3.3}$$

for any $T > 0$. Moreover, (3.1) implies that

$$\int_{B_{R_k}} \rho\, u_{R_k}(x,t)\, dx = \int_{B_{R_k}} \rho\, u_{0,R_k}(x)\, dx$$

for any R_k. Since $\rho \in L^1(\mathbb{R}^n)$ by assumption (see (H_2)), inequality (3.2) allows us to pass to the limit as $R_k \to \infty$ in the above identity; thus, $\hat{u} \in \mathcal{N}$.

The remaining claims concerning $\hat{u}$ follow easily from the energy estimate:

$$\frac{1}{2}\int_{B_{R_k}} \rho\, u_{R_k}^2(x,t)\, dx + \int_0^t \int_{B_{R_k}} \left|\nabla u_{R_k}(x,\tau)\right|^2 dx\, d\tau = \frac{1}{2}\int_{B_{R_k}} \rho\, u_{0,R_k}^2\, dx \quad (t > 0)$$

as $R_k \to \infty$. This completes the proof. □

Proof of Theorem 1.5. Let u_1, u_2 be two bounded solutions of class $\mathcal{N}$ of problem (1.1), and let $w := u_1 - u_2$. Observe that

$$\int_{\mathbb{R}^n} \rho\, w\,(x,t)\, dx = 0 \quad \text{for any } t \ge 0.$$

Set

$$W(x,t) := \int_0^t w(x,\tau)\, d\tau \quad ((x,t) \in S).$$

By Theorem 1.1, there exists $A \in \mathrm{Lip}(\overline{\mathbb{R}}_+)$ such that

$$\lim_{R \to \infty} \fint_{\partial B_R} |W(x,t) - A(t)|\, d\sigma = 0$$

uniformly with respect to $t \in [0,T]$. Let $a \equiv A' \in L^\infty(\mathbb{R}_+)$, and let a_R denote a smooth approximation of a such that $a_R(0)=0$ and $\|a_R\|_\infty \le \|a\|_\infty$ for any $R>0$, and $a_R \to a$ in $L^1(0,T)$ for any $T>0$ as $R\to\infty$. For any fixed $T>0$, we consider the problem

$$\begin{cases} \rho\,\partial_t w = \Delta w & \text{in } Q_{R,T}, \\ w = a_R(\cdot) & \text{in } \partial B_R \times (0,T), \\ w = 0 & \text{in } B_R \times \{0\} \end{cases} \tag{3.4}$$

and let w_R be its solution. Arguing as in the proof of Theorem 1.2, and using Remark 2.2, we see that

$$w = \lim_{R\to\infty} w_R \tag{3.5}$$

uniformly on compact subsets of $\mathbb{R}^n \times (0,T)$.

Now we observe that any solution w_R of problem (3.4) satisfies the identity

$$\int_0^t \int_{B_R} \{\rho\, w_R\, \partial_t\phi - \nabla w_R \nabla\phi\}\, dx\, d\tau = \int_{B_R} \rho\, w_R(x,t)\phi(x,t)\, dx \tag{3.6}$$

for any $t\in(0,T]$ and any $\phi \in \mathrm{Lip}([0,T]; C^1(\overline{B}_R))$ such that $\phi=0$ on $\partial B_R \times [0,T]$. As in [13], we define

$$\varphi_R(x,t) := -\int_t^T w_R(x,\tau)\, d\tau + \int_t^T a_R(\tau)\, d\tau \quad ((x,t)\in \overline{Q}_{R,T}).$$

It is easily seen that

- $\varphi_R = 0$ in $(B_R\times\{T\}) \cup (\partial B_R \times (0,T))$;
- we have

$$\nabla\varphi_R(x,t) = -\int_t^T \nabla w_R(x,\tau)\, d\tau,$$
$$\partial_t\varphi_R(x,t) = w_R(x,t) - a_R(t) \quad \text{in } Q_{R,T},$$

whence $\varphi_R \in \mathrm{Lip}([0,T]; C^1(\overline{B}_R))$. We set $t=T$ and $\phi=\varphi_R$ in (3.6); using the above properties of φ_R, we obtain

$$\begin{aligned} \int_0^T \int_{B_R} \rho\, w_R^2(x,t)\, dx\, dt + \int_0^T \int_{B_R} \nabla w_R(x,t) \Big\{ \int_t^T \nabla w_R(x,\tau)\, d\tau \Big\}\, dx\, dt \\ = \int_0^T a_R(t) \int_{B_R} \rho w_R(x,t)\, dx\, dt. \end{aligned} \tag{3.7}$$

Since

$$\nabla w_R(x,t) \Big\{ \int_t^T \nabla w_R(x,\tau)\, d\tau \Big\} = -\frac{1}{2}\partial_t \Big| \int_t^T \nabla w_R(x,\tau)\, d\tau \Big|^2,$$

(3.7) implies

$$\begin{aligned} \int_0^T \int_{B_R} \rho\, w_R^2(x,t)\, dx\, dt + \frac{1}{2}\int_{B_R} \Big| \int_0^T \nabla w_R(x,t)\, dt \Big|^2 dx \\ = \int_0^T a_R(t) \int_{B_R} \rho w_R(x,t)\, dx\, dt. \end{aligned} \tag{3.8}$$

Thus

$$\int_0^T \int_{B_R} \rho\, w_R^2(x,t)\, dx\, dt \le \int_0^T a_R(t) \int_{B_R} \rho w_R(x,t)\, dx\, dt \tag{3.9}$$

for any $R > 0$. Since

$$\lim_{R\to\infty} \int_{B_R} \rho w_R(x,t)\,dx = \int_{\mathbb{R}^n} \rho\, w\,(x,t)\,dx = 0 \tag{3.10}$$

for any $t \in [0,T]$, inequality (3.9) allows us to conclude that

$$\lim_{R\to\infty} \int_0^T \int_{B_R} \rho\, w_R^2(x,t)\,dx\,dt = \int_0^T \int_{\mathbb{R}^n} \rho\, w^2(x,t)\,dx\,dt = 0.$$

Consequently, $w \equiv 0$, so that $u_1 = u_2$ in $\mathbb{R}^n \times (0,T)$ for any $T > 0$; the result follows. □

Remark 3.1. It is worth pointing out an alternative proof of Theorem 1.5. Let w, W, w_R be defined as in the proof above, and let

$$W_R(x,t) := \int_0^t w_R(x,\tau)\,d\tau \quad ((x,t) \in Q_{R,T}).$$

Observe that, by relation (3.5) (which is fulfilled uniformly on compact subsets), $W_R \to W$ in $\mathbb{R}^n \times [0,T]$ as $R \to \infty$.

Obviously, (3.4) implies that W_R satisfies the problem

$$\begin{cases} \rho\,\partial_t W = \Delta W & \text{in } Q_{R,T}, \\ W = A_R(\cdot) & \text{in } \partial B_R \times (0,T), \\ W = 0 & \text{in } B_R \times \{0\} \end{cases} \tag{3.11}$$

for any $R > 0$, with A_R defined as in (2.12). Multiplying the first equation in (3.11) by $W_R - A_R(\cdot)$ and integrating in time, we easily get

$$\begin{aligned} &\frac{1}{2}\int_{B_R} \rho\, W_R^2(x,t)\,dx + \int_0^t \int_{B_R} \left|\nabla W_R\,(x,\tau)\right|^2 dx\,d\tau \\ &\qquad = \int_0^t A_R(\tau) \int_{B_R} \rho\, w_R(x,\tau)\,dx\,d\tau \quad (t \in [0,T]), \end{aligned} \tag{3.12}$$

whence

$$\frac{1}{2}\int_{B_R} \rho\, W_R^2(x,t)\,dx \le \int_0^t A_R(\tau) \int_{B_R} \rho\, w_R(x,\tau)\,dx\,d\tau \tag{3.13}$$

for any $R > 0$, $t \in [0,T]$. Using (3.10) and (3.13), we obtain

$$\lim_{R\to\infty} \int_{B_R} \rho\, W_R^2(x,t)\,dx = \int_{\mathbb{R}^n} \rho\, W^2(x,t)\,dx = 0$$

for any $t \in [0,T]$. Again, this shows that $w \equiv 0$ in $\mathbb{R}^n \times (0,T)$ for any $T > 0$, and the conclusion follows.

To prove Theorem 1.7, we need the following statement.

Lemma 3.2. *Let u be any bounded solution of class $\mathcal{N}$ of problem* (1.1)*. Then*

$$\lim_{t\to\infty} \int_{B_R} |\nabla\, u\,(x,t)|^2\,dx = 0 \tag{3.14}$$

for any $R > 0$.

The proof of Lemma 3.2 is a standard adaptation of the proof given in [6] for $n = 1, 2$; we omit this.

In what follows, we set

$$\fint_{B_R} v\,dx := \frac{1}{|B_R|}\int_{B_R} v\,dx.$$

Proof of Theorem 1.7. Fixing $R > 0$, we define

$$D(x,t) := u(x,t) - \fint_{B_R} u(x,t)\,dx \quad ((x,t) \in \mathbb{R}^n \times [0,\infty)).$$

The Poincaré inequality and Lemma 3.2 yield

$$\lim_{t\to\infty} \int_{B_R} |D(x,t)|^2\,dx \le C'_R \lim_{t\to\infty} \int_{B_R} |\nabla\, u(x,t)|^2\,dx = 0 \tag{3.15}$$

with some constant $C'_R > 0$.

By classical results, on the compact subsets of $\mathbb{R}^n$ the solution $u(\cdot,t)$ satisfies Hölder estimates uniformly with respect to time in $[\tau,\infty)\,(\tau > 0)$. Then, by the Ascoli–Arzelà theorem, there exists a sequence $\{t_k\}$, $t_k \to \infty$, and a function $\tilde{u} \in C(\mathbb{R}^n)$ such that

$$\lim_{t_k\to\infty} u(\cdot,t_k) = \tilde{u} \tag{3.16}$$

uniformly on the compact subsets of $\mathbb{R}^n$. It follows that

$$\lim_{t_k\to\infty} \fint_{B_R} u(x,t_k)\,dx = \fint_{B_R} \tilde{u}\,dx. \tag{3.17}$$

From (3.15)–(3.17) we deduce that

$$\Big\| \tilde{u} - \fint_{B_R} \tilde{u}\,dx \Big\|_{L^2(B_R)} = \lim_{t_k\to\infty} \|D(\cdot,t_k)\|_{L^2(B_R)} = 0,$$

whence $\tilde{u}$ = constant. Using the conservation law (1.6), we conclude that

$$\lim_{t_k\to\infty} u(\cdot,t_k) = \frac{\int_{\mathbb{R}^n} \rho\, u_0\,dx}{\int_{\mathbb{R}^n} \rho\,dx}.$$

It is easily seen that the same limit is attained along any diverging sequence $\{t_k\}$; the conclusion follows. □

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