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# $J_{p,m}$-INNER DILATIONS OF MATRIX-VALUED FUNCTIONS THAT BELONG TO THE CARATHÉODORY CLASS AND ADMIT PSEUDOCONTINUATION

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Abstract. The class $\ell^{p\times p}$ of matrix-valued functions $c(z)$ holomorphic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, having order $p$, and satisfying $\operatorname{Re} c(z) \geq 0$ in $D$ is considered, as well as its subclass $\ell^{p\times p}\Pi$ of matrix-valued functions $c(z) \in \ell^{p\times p}$ that have a meromorphic pseudocontinuation $c_-(z)$ to the complement $D_e = \{z \in \mathbb{C} : 1 < |z| \leq \infty\}$ of the unit disk with bounded Nevanlinna characteristic in $D_e$.

For matrix-valued functions $c(z)$ of class $\ell^{p\times p}\Pi$ a representation as a block of a certain $J_{p,m}$-inner matrix-valued function $\theta(z)$ is obtained. The latter function has a special structure and is called the $J_{p,m}$-inner dilation of $c(z)$. The description of all such representations is given.

In addition, the following special $J_{p,m}$-inner dilations are considered and described: minimal, optimal, $*$-optimal, minimal and optimal, minimal and $*$-optimal. Also, $J_{p,m}$-inner dilations with additional properties are treated: real, symmetric, rational, or any combination of them under the corresponding restrictions on the matrix-valued function $c(z)$. The results extend to the case where the open upper half-plane $\mathbb{C}_+$ is considered instead of the unit disk $D$. For entire matrix-valued functions $c(z)$ with $\operatorname{Re} c(z) \geq 0$ in $\mathbb{C}_+$ and with Nevanlinna characteristic in $\mathbb{C}_-$, the $J_{p,m}$-inner dilations in $\mathbb{C}_+$ that are entire matrix-valued functions are also described.

## §1. Introduction

The class $\ell^{p\times p}$ of matrix-valued functions $c(z)$ of order $p$ holomorphic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and such that $\operatorname{Re} c(z) \geq 0$ in $D$ is of interest for function theory, as well as for the theory of Hilbert space operators, the theory of passive linear dynamic systems, control theory, and stochastic processes theory (see [4, 9, 12, 16]).

Darlington's method is well known in the theory of passive linear circuits with lumped parameters. Development of this method required investigation of the subclass $\ell^{p\times p}\Pi$ of matrix-valued functions $c(z)$ that admit a meromorphic pseudocontinuation $c_-(z)$ to the complement of the unit disk $D_e = \{z \in \mathbb{C} : 1 < |z| \leq \infty\}$ with bounded Nevanlinna characteristic in $D_e$. The fact that $c_-(z)$ is a pseudocontinuation of $c(z)$ means that

$$c(\zeta) = \lim_{r\uparrow 1} c(r\zeta) = \lim_{r\downarrow 1} c_-(r\zeta) \quad \text{almost everywhere on } |\zeta| = 1.$$

In [3], the following Darlington representation of such matrix-valued functions was obtained:

$$c(z) = [a_{11}(z)\tau + a_{12}(z)][a_{21}(z)\tau + a_{22}(z)]^{-1}, \tag{1.1}$$

where $\tau$ is a constant matrix of order $p$ with $\operatorname{Re}\tau \geq 0$, and $A(z) = [a_{ij}(z)]_{i,j=1,2}$ is a $J_p$-inner matrix-valued function in $D$ with the signature matrix $J_p = \left[\begin{smallmatrix} 0 & -I_p \\ I_p & 0 \end{smallmatrix}\right]$; i.e., $A(z)$ is

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a matrix-valued function of order $2p$ meromorphic in $D$ and taking $J_p$-contractive values on $\Omega_A^+$ (the domain of holomorphy of $A(z)$ in $D$) and having $J_p$-unitary nontangential boundary values $A(\zeta)$ almost everywhere on the circle $T = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$:

$$A(z)^* J_p A(z) \le J_p, \quad z \in \Omega_A^+,$$
$$A(\zeta)^* J_p A(\zeta) = J_p \quad \text{for a.e. } \zeta \in T.$$

If a matrix-valued function $c(z)$ is of class $\ell^{p\times p}\Pi$, then its rank $m_c = \operatorname{rank} \operatorname{Re} c(\zeta)$ is constant for a.e. $\zeta \in T$, because $2 \operatorname{Re} c(\zeta)$ is a nontangential boundary value of the function $c(z) + c_-(\frac{1}{\bar{z}})$, which has bounded Nevanlinna characteristic in $D$. In (1.1), the matrix $\tau$ is such that $\operatorname{rank} \tau = m_c$, and there exists a representation of $c(z)$ in the form (1.1) with $\tau = \left[\begin{smallmatrix} I_{m_c} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for $1 \le m_c \le p$ and with $\tau = 0_{p\times p}$ for $m_c = 0$. In the Darlington method, the number $m_c$ is interpreted as the minimal number of the reduced scattering channels (see [1, 2]).

In this paper, we consider another representation of a matrix-valued function $c(z)$ of class $\ell^{p\times p}\Pi$; namely, we represent it as a block of a $J_{p,m}$-inner in $D$ matrix-valued function

$$\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & \delta(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}, \quad \delta(z) = c(z), \quad \text{for } m \ge m_c > 0 \tag{1.2}$$

and

$$\theta_0(z) = \begin{bmatrix} c(z) & I_p \\ I_p & 0 \end{bmatrix} \quad \text{for } m = m_c = 0, \tag{1.2*}$$

where

$$J_{p,m} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix} \quad \text{for } m > 0, \qquad J_{p,0} = J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix} \quad \text{for } m = 0. \tag{1.3}$$

The functions $\theta(z)$ of this type will be called *$J_{p,m}$-inner dilations* of $c(z)$. The following theorem is proved.

**Theorem 1.** *A matrix-valued function $c(z)$ belongs to the class $\ell^{p\times p}\Pi$ if and only if there exists a $J_{p,m}$-inner dilation $\theta$ of $c(z)$ of the form* (1.2). *Moreover, if $c \in \ell^{p\times p}\Pi$, then for the corresponding $J_{p,m}$-inner dilation $\theta$ of the form* (1.2) *we have $m \ge m_c \ge 0$, and there exists a $J_{p,m}$-inner dilation with $m = m_c$.*

The "if" part can easily be checked. To verify the "only if" part, we apply the method used in [1, 6] to obtain a representation of matrix-valued functions $s(z)$ of class $S^{p\times q}$ (see also [15]), i.e., a representation of $(p\times q)$-matrix-valued functions holomorphic and contractive in $D$ that admit a pseudocontinuation $s_-(z)$ in $D_e$ in the form of a block of an inner matrix-valued function $S(z)$ of order $n$, where

$$n = p + r_1 = q + r_2, \qquad r_1 = \operatorname{rank}(I_q - s(\zeta)s(\zeta)^*), \quad r_2 = \operatorname{rank}(I_p - s(\zeta)^* s(\zeta))$$

for a.e. $\zeta \in T$, and

$$S(z) = \begin{bmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{bmatrix}, \quad s_{12} = s(z). \tag{1.4}$$

In the same way (see [7]), a representation was obtained for an arbitrary $(p \times q)$-matrix-valued function $f(z)$ of bounded Nevanlinna characteristic in $D$ and having a pseudocontinuation $f_-(z)$ in $D_e$ of bounded Nevanlinna characteristic in $D_e$. Such a function can be represented as a block $w_{12}$ of some $j_{pq}$-inner in $D$ matrix-valued function $W(z)$:

$$W(z) = \begin{bmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{bmatrix}, \quad w_{12}(z) = f(z), \tag{1.5}$$

where

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.$$

A representation of $c(z) \in \ell^{p\times p}\Pi$ with $m_c > 0$ in the form (1.2) is not unique. In the present paper, all such representations with $m = m_c$ are described, much in the same way that the representations (1.4) of $s(z) \in S^{p\times q}\Pi$ were obtained earlier in [6]; see also the representations (1.5) in [7].

In §3, some special representations (1.2) are considered and described: minimal, optimal, $*$-optimal, minimal and optimal, minimal and $*$-optimal, which will be useful in constructing passive realizations of the impedance matrices $c(z)$. A separate paper will be devoted to this subject. In §4, the $J_{p,m}$-inner dilations with additional properties are described: real, symmetric, rational, and with various combinations of these properties, under the corresponding restrictions on $c(z)$. All these results are transferred to the case where the open half-plane $\mathbb{C}_+$ is considered instead of the unit disk $D$. In §5, for entire matrix-valued functions $c(z)$ with $\operatorname{Re} c(z) > 0$ in $\mathbb{C}_+$ and with bounded Nevanlinna characteristic in the lower half-plane $\mathbb{C}_-$, the $J_{p,m}$-inner dilations in $\mathbb{C}_+$ are described. They are still entire matrix-valued functions.

In subsequent papers we shall consider conservative and various passive (minimal, optimal, etc.) realizations of matrix-valued functions $c(z)$ of class $\ell^{p\times p}\Pi$ with $m_c > 0$ in the form of a resistance matrix of a dissipative system. Such realizations are constructed by considering the corresponding $J_{p,m}$-inner dilation $\theta(z)$ of $c(z)$ and a conservative transmission system with the transmission matrix $\theta(z)$.

We are also planning to consider the relationship between $J_{p,m}$-inner dilations and the theory of stochastic realization of discrete time stationary processes, developed by A. Lindquist, D. Picci, and their followers (see [18–20]). The analysis of precisely these papers brought the authors to the results presented here.

## Notation

$\mathbb{C}$ is the set of complex numbers;
$\mathbb{R}$ is the set of real numbers;
$\operatorname{Re} z = \frac{z+\overline{z}}{2}$ is the real part of $z \in \mathbb{C}$;
$\operatorname{Im} z = \frac{z-\overline{z}}{2}$ is the imaginary part of $z \in \mathbb{C}$;
$D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk;
$D_e = \{z \in \mathbb{C} : 1 < |z| \le \infty\}$ is the exterior of the open unit disk in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$;
$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the open upper half-plane;
$\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ is the open lower half-plane;
$T = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ is the unit circle;
$\operatorname{Re} A$ is the real part of the matrix $A$, i.e., $\operatorname{Re} A = \frac{A+A^*}{2}$;
$A^T$ is the transpose of the matrix $A$;
$\operatorname{rank} A$ is the rank of $A$;
$\operatorname{trace} A$ is the trace of $A$;
$\|A\|$ is the norm of $A$, which is the maximal singular number of $A$;
$I_m$ is the identity matrix of order $m$;

$\overline{\mathfrak{L}}$ is the closure of the set $\mathfrak{L}$ in the Hilbert space under consideration;
$f^{\sim}(z) = f(\overline{z})^*$;
$f^{\#}(z) = f(\frac{1}{\overline{z}})^*$.

## §2. Preliminary information about matrix-valued functions belonging to the Nevanlinna class or to its subclasses

**2.1. Basic classes of matrix-valued functions.** A measurable $(p \times q)$-matrix-valued function $f(\zeta)$ on the unit circle $T$ belongs to the space $L_r^{p\times q}$ with $1 \le r < \infty$ if

$$\|f\|_r^r = \frac{1}{2\pi} \int_{|\zeta|=1} \operatorname{trace}\{f(\zeta)^* f(\zeta)\}^{\frac{r}{2}} \, |d\zeta| < \infty,$$

and to the space $L_\infty^{p\times q}$ if

$$\operatorname{ess\,sup}\{\|f(\zeta)\| : \zeta \in T\} < \infty.$$

A $(p \times q)$-matrix-valued function $f(z)$ holomorphic in $D$ belongs to the Hardy class $H_r^{p\times q}$ with $1 \le r < \infty$ if

$$\|f\|_r^r = \sup_{\rho<1} \int_{|\zeta|=1} \operatorname{trace}\{f(\rho\zeta)^* f(\rho\zeta)\}^{\frac{r}{2}} \, |d\zeta| < \infty,$$

and to the class $H_\infty^{p\times q}$ if

$$\|f\|_\infty = \sup\{\|f(z)\| : z \in D\} < \infty.$$

A $(p \times q)$-matrix-valued function $s(z)$ holomorphic in $D$ belongs to the Schur class $S^{p\times q}$ if $s(z)^* s(z) \le I_q$ for all $z \in D$.

A $(p \times p)$-matrix-valued function $c(z)$ holomorphic in $D$ belongs to the Carathéodory class $\ell^{p\times p}$ if

$$\operatorname{Re} c(z) = \frac{c(z) + c(z)^*}{2} \ge 0 \quad \text{for all} \quad z \in D.$$

A $(p \times q)$-matrix-valued function $f(z)$ meromorphic in $D$ belongs to the Nevanlinna class $N^{p\times q}$ of matrix-valued functions of bounded characteristic if it can be represented in the form

$$f = h^{-1} g,$$

where $g \in H_\infty^{p\times q}$ and $h \in H_\infty$ $(= H_\infty^{1\times 1})$.

It should be noted that the class $N^{p\times q}$ contains the classes $H_r^{p\times q}$, $1 \le r \le \infty$, $S^{p\times q}$, and $\ell^{p\times p}$ (for $p = q$). An arbitrary function $f(z) \in N^{p\times q}$ has nontangential boundary values $f(\zeta)$ almost everywhere on the circle $T$. Therefore, in particular, the limit $f(\zeta) = \lim_{\rho\uparrow 1} f(\rho\zeta)$ exists for a.e. $\zeta \in T$, and $f(z)$ is uniquely determined by the boundary values $f(\zeta)$ on a set of positive Lebesgue measure of $T$.

Observe that $\operatorname{Re} c(\zeta) \in L_1^{p\times p}$ for an arbitrary matrix-valued function $c(z) \in \ell^{p\times p}$.

For any class $\mathfrak{X}^{p\times q}$ of matrix-valued functions, we shall write $\mathfrak{X}$ instead of $\mathfrak{X}^{1\times 1}$ and $\mathfrak{X}^p$ instead of $\mathfrak{X}^{p\times 1}$.

A matrix-valued function $f(z) \in S^{p\times q}$ is *inner* ($*$-*inner*) if $f(\zeta)^* f(\zeta) = I_q$ (respectively, $f(\zeta) f(\zeta)^* = I_p$) for a.e. $\zeta \in T$. The class of inner matrix-valued functions is denoted by $S_{\text{in}}^{p\times q}$ and the class of $*$-inner functions by $S_{*\text{in}}^{p\times q}$. Note that the classes $S_{\text{in}}^{p\times q}$ and $S_{*\text{in}}^{p\times q}$ are not empty for $p \ge q$ and for $p \le q$, respectively, and for $p = q$ we have $S_{\text{in}}^{p\times p} = S_{*\text{in}}^{p\times p}$.

If $f$ and $h$ are matrix-valued functions, $f \in H_\infty^{p\times q}$ and $h \in H_2^q$, then $fh \in H_2^p$ and

$$\|fh\|_2 \le \|f\|_\infty \|h\|_2;$$

therefore, the operator $M_f$ of multiplication by a matrix-valued function $f$,

$$\big(M_f h\big)(z) = f(z) h(z),$$

is a well-defined bounded operator from $H_2^q$ to $H_2^p$.

A matrix-valued function $f(z) \in H_\infty^{p\times q}$ is *outer* (*$*$-outer*) if $\overline{M_f H_2^q} = H_2^p$ (respectively, $\overline{M_{f^\sim} H_2^p} = H_2^q$, where $f^\sim(z) = f(\overline{z})^*$).

The next lemma is important for what follows.

**Lemma 1.** *A matrix-valued function $f(z) \in H_\infty^{p\times q}$ is outer if and only if*

(1) rank $f(z) = p$ *for at least one point $z \in D$, and*

(2) *every matrix-valued function $g \in H_\infty^{r\times q}$ such that*

$$g(\zeta)^* g(\zeta) \le f(\zeta)^* f(\zeta) \quad \text{for almost all } \zeta \in T$$

*satisfies the inequality*

$$g(z)^* g(z) \le f(z)^* f(z) \quad \text{for all } z \in D.$$

*Moreover, in that case* rank $f(z) = p$ *for all $z \in D$*, rank $f(\zeta) = p$ *for almost all $\zeta \in T$, and $g(z) = b(z)f(z)$ for some $b \in S^{r\times p}$.*

The proof can be found in [11, p. 214, Proposition 2.4; p. 223, Proposition 4.1].

Every nonzero matrix-valued function $f(z) \in H_\infty^{p\times q}$ admits an inner-outer factorization of the form

$$f(z) = b(z)\varphi(z),$$

where $b(z) \in S_{\text{in}}^{p\times r}$ and $\varphi(z)$ is an outer matrix-valued function of class $H_\infty^{r\times q}$ for some $r \le \min\{p,q\}$. This representation is essentially unique; i.e., it is unique up to the replacement of $b(z)$ by $b(z)u$ and $\varphi(z)$ by $u^*\varphi(z)$ for some constant unitary matrix $u$ of order $r$. Moreover, $r = \operatorname{rank} f(z)$ for all $z \in D$ except possibly a finite or countable set of points where $\operatorname{rank} f(z) < r$. This set has no accumulation points in $D$.

Every nonzero matrix-valued function $f(z) \in H_\infty^{p\times q}$ admits also an essentially unique $*$-outer-$*$-inner factorization

$$f(z) = \varphi(z)b(z).$$

An arbitrary square matrix-valued function $f(z) \in H_\infty^{p\times p}$ with $\det f(z) \not\equiv 0$ in $D$ admits inner-outer and outer-inner factorizations

$$f = b_1\varphi_1 = \varphi_2 b_2,$$

where $b_i \in S_{\text{in}}^{p\times p}$ and the $\varphi_i$ are outer matrix-valued functions of class $H_\infty^{p\times p}$ for $i = 1, 2$.

**2.2. The Smirnov class.** A $(p \times q)$-matrix-valued function $f(z)$ holomorphic in $D$ belongs to the Smirnov class $N_+^{p\times q}$ if it has a representation of the form $f = h^{-1}g$, where $g \in H_\infty^{p\times q}$ and $h$ is a scalar outer function in $H_\infty$. A matrix-valued function $f(z)$ is an outer function of class $N_+^{p\times q}$ if in the above representation $g$ is an outer function belonging to $H_\infty^{p\times q}$. In this case we shall write $f(z) \in N_{\text{out}}^{p\times q}$. It is clear that $N_{\text{out}}^{p\times q} \subset N_+^{p\times q} \subset N^{p\times q}$.

A matrix-valued function $f$ of the Smirnov class $N_+^{p\times p}$ belongs to $N_{\text{out}}^{p\times p}$ if and only if $\det f(z) \ne 0$ for all $z \in D$ and $f^{-1} \in N_+^{p\times p}$; therefore,

$$f \in N_{\text{out}}^{p\times p} \iff f^{-1} \in N_{\text{out}}^{p\times p}.$$

The maximum principle is true in the Smirnov class.

**Lemma 2.** *If $f \in N_+^{p\times q}$, then for $1 \le r < \infty$ we have*

$$\sup_{\rho<1} \int_{|\zeta|=1} \Big(\operatorname{trace}\{f(\rho\zeta)^* f(\rho\zeta)\}\Big)^{\frac{r}{2}} |d\zeta| = \int_{|\zeta|=1} \Big(\operatorname{trace}\{f(\zeta)^* f(\zeta)\}\Big)^{\frac{r}{2}} |d\zeta| \le \infty,$$

$$\sup_{z\in D} \|f(z)\| = \operatorname*{ess\,sup}_{\zeta\in T}\{\|f(\zeta)\|\} \le \infty.$$

For scalar functions $f \in N_+$ this assertion was proved by Smirnov (see [13]), and for matrix-valued functions it was proved by Ginzburg [14, 17].

Let $f(z) \in N^{p\times q}$. Then an ordered pair of matrix-valued functions $\{b_1, b_2\}$, where $b_1(z) \in S_{\text{in}}^{p\times p}$ and $b_2(z) \in S_{\text{in}}^{q\times q}$ are such that $b_1(z)f(z)b_2(z) \in N_+^{p\times q}$, is called a *denominator* of $f(z)$. A denominator of $f(z)$ of the form $\{u, b\}$, where $u$ is a unitary matrix of order $p$, is said to be *right*, and a denominator of the form $\{b, v\}$, where $v$ is a unitary matrix of order $q$, is said to be *left*. We denote by

$$\operatorname{Den}(f) = \{\{b_1, b_2\} : b_1 \in S_{\text{in}}^{p\times p}, b_2 \in S_{\text{in}}^{q\times q}, b_1 f b_2 \in N_+^{p\times q}\}$$

the set of all denominators of $f(z)$, and by

$$\operatorname{Den}^r(f) = \{\{u, b\} \in \operatorname{Den}(f), u = \text{const}\},$$

$$\operatorname{Den}^l(f) = \{\{b, v\} \in \operatorname{Den}(f), v = \text{const}\}$$

the sets of all right and left denominators of $f(z)$, respectively. Note that for an arbitrary matrix-valued function $f \in N^{p\times q}$ the sets $\operatorname{Den}^r(f)$ and $\operatorname{Den}^l(f)$ are not empty (see [2, 6]).

A denominator $\{\tilde{b}_1, \tilde{b}_2\} \in \operatorname{Den}(f)$ is called a *divisor* of a denominator $\{b_1, b_2\} \in \operatorname{Den}(f)$ if $b_1(z) = u(z)\tilde{b}_1(z)$ and $b_2(z) = \tilde{b}_2(z)v(z)$, where $u(z) \in S_{\text{in}}^{p\times p}$ and $v(z) \in S_{\text{in}}^{q\times q}$. Such a divisor is said to be *trivial* if $u(z) = \text{const}$ and $v(z) = \text{const}$.

A denominator of a matrix-valued function $f \in N^{p\times q}$ is *minimal* if it has no nontrivial divisors in $\operatorname{Den}(f)$.

**Lemma 3.** *For an arbitrary matrix-valued function $f(z) \in N^{p\times q}$ there exists a minimal right* (*left*) *denominator $\{u, b_2\} \in \operatorname{Den}^r(f)$* ($\{b_1, v\} \in \operatorname{Den}^l(f)$). *It is unique up to a right* (*left*) *unitary factor of $b_2$* (*respectively, $b_1$*) *and up to a unitary matrix $u$* (*respectively, $v$*).

**Lemma 4.** (a) *Suppose $f(z) \in N^{p\times q}$ and $\{b_1, b_2\} \in \operatorname{Den}(f)$. Then there exists a minimal denominator $\{\hat{b}_1, \hat{b}_2\}$ of $f$ that is a divisor of $\{b_1, b_2\}$.*

(b) *Let $f(z)$ be a rational $(p \times q)$-matrix-valued function. Then $f(z) \in N^{p\times q}$, and the inner matrix-valued functions $\hat{b}_1$ and $\hat{b}_2$ in its arbitrary minimal denominator $\{\hat{b}_1, \hat{b}_2\}$ are rational.*

The proofs of Lemmas 3 and 4 can be found in [2, 6].

An arbitrary nonzero matrix-valued function $f(z) \in N_+^{p\times q}$ has an essentially unique inner-outer factorization of the form

$$f(z) = b_1(z)\varphi_1(z), \quad \text{where } b_1 \in S_{\text{in}}^{p\times r} \quad \text{and} \quad \varphi_1 \in N_{\text{out}}^{r\times q},$$

and an essentially unique $*$-outer-$*$-inner factorization of the form

$$f(z) = \varphi_2(z)b_2(z), \quad \text{where } \varphi_2^{\sim} \in N_{\text{out}}^{p\times r} \quad \text{and} \quad b_2 \in S_{*\text{in}}^{r\times q}.$$

In these representations we have $r = \operatorname{rank} f(z)$ for all $z \in D$ except possibly an at most countable set of points where $\operatorname{rank} f(z) < r$. This set has no accumulation points in $D$.

The following inclusions are true:

$$S^{p\times q} \subset H_2^{p\times q} \subset N_+^{p\times q} \quad \text{and} \quad \ell^{p\times p} \subset N_+^{p\times p}.$$

**2.3. The class $\Pi^{p\times q}$.** A $(p \times q)$-matrix-valued function $f_-$ meromorphic in $D_e = \{z \in \mathbb{C} : 1 < |z| \le \infty\}$ is called a *pseudocontinuation* of $f$, $f \in N^{p\times q}$, if $f_-^{\#} \in N^{q\times p}$ and

$$f(\zeta) := \lim_{\rho\uparrow 0} f(\rho\zeta) = \lim_{\rho\downarrow 0} f_-(\rho\zeta) \quad \text{for a.e. } \zeta \in T.$$

The subclass of all $f \in N^{p\times q}$ that have a pseudocontinuation $f_-$ to $D_e$ will be denoted by $\Pi^{p\times q}$. The intersection of $\mathfrak{X}^{p\times q}$ and $\Pi^{p\times q}$ will be denoted by $\mathfrak{X}^{p\times q}\Pi$.

For a given $f \in \Pi^{p\times q}$, a pseudocontinuation $f_-$ is unique, because the function $f_-^{\#}$ of class $N^{q\times p}$ in $D$ is uniquely determined by its boundary values $f_-(\zeta)^* = f(\zeta)^*$.

If for a matrix-valued function $f \in \Pi^{p\times q}$ we consider its boundary values $f(\zeta)$ and its pseudocontinuation $f_-$, then as a result we get a matrix-valued function defined everywhere on the complex plane except possibly some set of Lebesgue measure zero on the circle $T$ and isolated singularities, namely, the poles of $f$ and $f_-$. This matrix-valued function will be denoted in the same way as the initial one, i.e., $f(z)$. The set where this function is holomorphic will be denoted by $\Omega_f$, and $\Omega_f^+ := \Omega_f \cap D$, $\Omega_f^- := \Omega_f \cap D_e$.

We have

$$S_{\mathrm{in}}^{p\times p} \subset \Pi^{p\times p},$$

and moreover, the pseudocontinuation $s_-$ of $s \in S_{\mathrm{in}}^{p\times p}$ can be obtained by the symmetry principle,

$$s_-(z) = [s_-^{\#}(z)]^{-1}, \quad z \in D, \quad \det s\Big(\frac{1}{\overline{z}}\Big) \neq 0,$$

from the identity

$$s(\zeta)s(\zeta)^* = s(\zeta)^*s(\zeta) = I_p \quad \text{for a.e. } \zeta \in T.$$

The following fact, implied by the results by Douglas, Shapiro, and Shields [10], is very important.

**Lemma 5.** *Let $f \in H_2^{p\times q}$. Then $f \in H_2^{p\times q}\Pi$ if and only if there exists a matrix-valued function $b \in S_{\mathrm{in}}^{p\times p}$ such that $b(\zeta)^* f(\zeta) = g(\zeta)^*$ for a.e. $\zeta \in T$, where $g(\zeta)$ stands for the nontangential boundary values of some matrix-valued function $g(z) \in H_2^{q\times p}$. Moreover, $b(z)$ can be taken in the form $b(z) = \eta(z) I_p$, where $\eta \in S_{\mathrm{in}}$.*

*Proof.* For $p = q = 1$ the result is contained in [10].

Let $p \neq 1$ or $q \neq 1$. The matrix-valued function $f \in H_2^{p\times q}$ has a pseudocontinuation to the exterior of the unit disk $D_e$ if and only if each of its entries is of class $H_2\Pi$ and satisfies the conclusion of the lemma for $p = q = 1$. Therefore, the required $b \in S_{\mathrm{in}}^{p\times p}$ of the form $b(z) = \eta(z)I_p$, $\eta \in S_{\mathrm{in}}$, exists. The function $\eta(z)$ can be taken as the product of all functions of class $S_{\mathrm{in}}$ given by the scalar version of the lemma for each entry of $f$. Lemma 5 is proved. □

Let $f \in \Pi^{p\times q}$, and let

$$r_f = \max\{\operatorname{rank} f(z) : z \in \Omega_f\};$$

then $\operatorname{rank} f(z) = r_f$ for all $z \in \Omega_f$ except possibly a set of isolated points, and moreover, $\operatorname{rank} f(\zeta) = r_f$ for a.e. $\zeta \in T$.

Note that all rational $(p \times q)$-matrix-valued functions belong to $\Pi^{p\times q}$.

**Rosenblum–Rovnyak Theorem.** *Suppose $f \in \Pi^{p\times p}$, $r = r_f$, and $f(\zeta) \geq 0$ for a.e. $\zeta \in T$. Then:*

1) *the factorization problem*

$$g(\zeta)^* g(\zeta) = f(\zeta) \quad \textit{for a.e. } \zeta \in T$$

*has a solution $g = \varphi \in N_{\mathrm{out}}^{r\times p}$ unique up to a constant left unitary factor of order $r$; every solution $\varphi \in N_{\mathrm{out}}^{r\times p}$ belongs to $\Pi^{r\times p}$;*

2) *the dual factorization problem*

$$\omega(\zeta)\omega(\zeta)^* = f(\zeta) \quad \textit{for a.e. } \zeta \in T$$

*has a solution $\omega = \psi$ such that $\psi^\sim \in N_{\mathrm{out}}^{r\times p}$, unique up to a constant right unitary factor of order $r$, and $\psi \in \Pi^{p\times r}$;*

3) *a matrix-valued function $f$ belongs to $L_1^{p\times p}$ if and only if $\varphi \in H_2^{r\times p}$ $(\psi \in H_2^{p\times r})$;*

4) *if* $f \in L_1^{p\times p}$, *then the set of solutions* $g \in H_2^{r\times p}$ *of the direct factorization problem can be described by the formula* $g = b_1\varphi$, *where* $b_1 \in S_{\mathrm{in}}^{r\times r}$, *and the set of solutions* $\omega \in H_2^{p\times r}$ *of the dual factorization problem can be described by the formula* $\omega = \psi b_2$, *where* $b_2 \in S_{\mathrm{in}}^{r\times r}$;
5) *a* $(p\times p)$*-matrix-valued function* $f$ *is rational if and only if the solutions* $\varphi$ *and* $\psi$ *of the direct and dual factorization problems are rational matrix-valued functions of size* $r \times p$ *and* $p \times r$, *respectively*;
6) *for a rational* $(p \times p)$*-matrix-valued function* $f$, *the set of rational solutions* $g \in H_2^{r\times p}$ *of the direct factorization problem can be described by the formula* $g = b_1\varphi$, *where* $b_1$ *is a rational inner matrix-valued function of order* $r$, *and the set of rational solutions* $\omega \in H_2^{p\times r}$ *of the dual factorization problem can be described by the formula* $\omega = \psi b_2$, *where* $b_2$ *is a rational inner matrix-valued function of order* $r$;
7) *if* $g \in N^{r\times p}$ *is a solution of the direct factorization problem, then* $g \in \Pi^{r\times p}$ *and*

$$g^{\#}(z)g(z) = f(z) \quad \text{for all } z \in \Omega_g \cap \Omega_{g^{\#}},$$

*and if* $\omega \in N^{p\times r}$ *is a solution of the dual factorization problem, then* $\omega \in \Pi^{p\times r}$ *and*

$$\omega(z)\omega^{\#}(z) = f(z) \quad \text{for all } z \in \Omega_\omega \cap \Omega_{\omega^{\#}}.$$

*Proof.* The results stated in the theorem are contained, e.g., in the book [12]. □

## §3. $J_{p,m}$-INNER DILATIONS

**3.1. Necessary information about matrix-valued functions of classes $P(J)$ and $U(J)$.** We let $J$ denote a signature matrix, i.e., a matrix of order $m$ such that

$$J^* = J, \quad J^2 = I_m.$$

A matrix $\theta$ of order $m$ said to be $J$-*contractive* if

$$\theta^* J\theta \le J,$$

and it is $J$-*unitary* if

$$\theta^* J\theta = J.$$

Equivalent conditions are $\theta J\theta^* \le J$ and $\theta J\theta^* = J$, respectively.

Put

$$P = (I_m + J)/2, \quad Q = (I_m - J)/2. \tag{3.1}$$

For a $J$-contractive matrix $\theta$, the matrix

$$S = (Q + P\theta)(P + Q\theta)^{-1} \tag{3.2}$$

is well defined. It is called the Potapov–Ginzburg transform of $\theta$. Since

$$I_m - S^*S = (P + Q\theta)^{*-1}(J - \theta^* J\theta)(P + Q\theta)^{-1},$$

we have the following statement (see [2, 14]).

**Lemma 6.** *A matrix* $\theta$ *is* $J$*-contractive if and only if the matrix* $S$ *defined by* (3.2) *is contractive* ($\|S\| \le 1$).

The matrix $\theta$ can be expressed in terms of $S$ by the formula

$$\theta = (SQ - P)^{-1}(Q - SP).$$

Consider the Potapov class $P(J)$ of matrix-valued functions $\theta(z)$ meromorphic in the unit disk $D$ and having $J$-contractive values at every point of $D$ where it is holomorphic, i.e.,

$$\theta(z)^* J\theta(z) \leq J, \quad z \in \Omega_\theta^+. \tag{3.3}$$

Such matrix-valued functions will also be called $J$-contractive.

By Lemma 6, $\theta(z)$ is $J$-contractive if and only if the matrix-valued function

$$S(z) = (Q + P\theta(z))(P + Q\theta(z))^{-1}, \quad z \in \Omega_\theta^+,$$

extends continuously to the entire $D$ so that $S(z) \in S^{m\times m}$. Here $P$ and $Q$ are defined by (3.1). Since

$$\theta(z) = (S(z)Q - P)^{-1}(Q - S(z)P),$$

any $J$-contractive matrix-valued function $\theta(z)$ can be represented as a ratio of bounded matrix-valued functions holomorphic in $D$. Therefore, the following is true.

**Lemma 7.** *Any $J$-contractive matrix-valued function has bounded Nevanlinna characteristic.*

In other words, $P(J) \subset N^{m\times m}$. This implies that any $J$-contractive matrix-valued function $\theta(z)$ has radial limit values almost everywhere on $T$,

$$\theta(\zeta) = \lim_{\rho\uparrow 1} \theta(\rho\zeta),$$

and these limit values on a subset of $T$ of positive Lebesgue measure determine the function $\theta(z)$ uniquely. Passing to the limit in (3.3), we get

$$\theta(\zeta)^* J\theta(\zeta) \leq J \quad \text{for a.e. } \zeta \in T. \tag{3.4}$$

We shall be interested in $J$-contractive matrix-valued functions $\theta(z)$ with $J$-unitary boundary values, i.e., the functions such that

$$\theta(\zeta)^* J\theta(\zeta) = J \quad \text{for a.e. } \zeta \in T. \tag{3.5}$$

Such matrix-valued functions are said to be *$J$-inner*. The class of $J$-inner matrix-valued functions will be denoted by $U(J)$. Clearly, $U(I_p) = S_{\text{in}}^{p\times p}$.

*Remark* 1. Condition (3.5) implies that $\det\theta(\zeta) \neq 0$ a.e. on $T$ for $\theta \in U(J)$; therefore, $\det\theta(z) \neq 0$ for $z \in \Omega_\theta^+$ except probably some subset of $\Omega_\theta^+$ without accumulation points in $D$. It follows that $\theta(z)^{-1} \in N^{m\times m}$, and the pseudocontinuation $\theta_-$ of $\theta$, defined by the "symmetry principle"

$$\theta_-(z) = J[\theta^\#(z)]^{-1}J,$$

has bounded Nevanlinna characteristic in $D_e$. The boundary values

$$\theta_-(\zeta) = \lim_{\rho\downarrow 1} \theta_-(\rho\zeta) \quad (\text{for a.e. } \zeta \in T)$$

coincide almost everywhere with the boundary values $\theta(\zeta)$ of $\theta(z)$. Thus, $U(J) \subset \Pi^{m\times m}$.

*Remark* 2. Let $J_p = \left[\begin{smallmatrix} 0 & -I_p \\ -I_p & 0 \end{smallmatrix}\right]$. For a matrix-valued function $c(z)$ of order $p$ we define $\theta_0(z)$ by (1.2$*$). It is easy to check that

1) $c \in \ell^{p\times p} \iff \theta_0 \in P(J_p)$;

2) $c \in \ell^{p\times p}$ and $\operatorname{Re} c(z) = 0$ for a.e. $\zeta \in T \iff \theta_0 \in U(J_p)$.

**3.2. Proof of Theorem 1.** Let $m \geq 0$. Consider the signature matrix $J_{p,m}$ defined by (1.3).

A matrix-valued function $\theta(z) \in U(J_{p,m})$ is called a $J_{p,m}$-unitary dilation of $c(z) \in \ell^{p\times p}$ if it has the block structure of type (1.2) for $m > 0$ and of type (1.2*) for $m = 0$.

Now, we start proving Theorem 1 (see the Introduction).

**Proof of the "only if" part.** Let $c \in \ell^{p\times p}\Pi$. If $m_c = 0$, then the matrix-valued function $\theta_0$ defined in (1.2*) is a unique $J_{p,0}$-inner dilation of $c(z)$.

Now, let $m_c > 0$. The matrix-valued function $\operatorname{Re} c(\zeta)$, which is nonnegative for a.e. $\zeta \in T$, is the boundary value of the function $c(z) + c^{\#}(z)$, belonging to the Nevanlinna class $N^{p\times p}$. Since $\operatorname{Re} c(\zeta) \in L_1^{p\times p}$ for $c \in \ell^{p\times p}$, the Rosenblum–Rovnyak theorem shows that the factorization problem

$$2\operatorname{Re} c(\zeta) = g(\zeta)^* g(\zeta) \quad \text{for a.e. } \zeta \in T \tag{3.6}$$

is solvable in $H_2^{m\times p}$, and its solution satisfies the condition $g \in H_2^{m\times p}\Pi$ for $m = m_c$. Therefore, by Lemma 5, there exists a matrix-valued function $b \in S_{in}^{m\times m}$ such that $b(\zeta)^* g(\zeta) = \omega(\zeta)^*$, where $\omega(\zeta)$ is the boundary value of $\omega \in H_2^{p\times m}$. Put

$$\alpha = b, \quad \beta = g, \quad \gamma = \omega, \quad \delta = c, \quad \theta = \begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \delta & I_p \\ 0 & I_p & 0 \end{bmatrix}.$$

Then the following identity is true:

$$\theta(\zeta)^* J_{p,m}\theta(\zeta) = J_{p,m} \quad \text{for a.e. } \zeta \in T. \tag{3.7}$$

Indeed, for a.e. $\zeta \in T$ we have

$$\begin{aligned}
\theta(\zeta)^* J_{p,m}\theta(\zeta) &= \begin{bmatrix} \alpha(\zeta)^* & \gamma(\zeta)^* & 0 \\ \beta(\zeta)^* & \delta(\zeta)^* & I_p \\ 0 & I_p & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix} \begin{bmatrix} \alpha(\zeta) & \beta(\zeta) & 0 \\ \gamma(\zeta) & \delta(\zeta) & I_p \\ 0 & I_p & 0 \end{bmatrix} \\
&= \begin{bmatrix} \alpha(\zeta)^*\alpha(\zeta) & \alpha(\zeta)^*\beta(\zeta) - \gamma(\zeta)^* & 0 \\ \beta(\zeta)^*\alpha(\zeta) - \gamma(\zeta) & \beta(\zeta)^*\beta(\zeta) - 2\operatorname{Re}\delta(\zeta) & -I_p \\ 0 & -I_p & 0 \end{bmatrix} \\
&= \begin{bmatrix} b(\zeta)^* b(\zeta) & b(\zeta)^* g(\zeta) - \omega(\zeta)^* & 0 \\ g(\zeta)^* b(\zeta) - \omega(\zeta) & g(\zeta)^* g(\zeta) - 2\operatorname{Re} c(\zeta) & -I_p \\ 0 & -I_p & 0 \end{bmatrix} \\
&= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix} \\
&= J_{p,m}.
\end{aligned}$$

Therefore, $\theta$ is $J_{p,m}$-unitary almost everywhere on $T$. It remains to show that $\theta$ is $J_{p,m}$-contractive in $D$. For this, we use the Potapov–Ginzburg transform of $\theta(z)$:

$$P = (I_{2p+m} + J_{p,m})/2, \quad Q = (I_{2p+m} - J_{p,m})/2, \quad \tilde{S}(z) = (Q + P\theta(z))(P + Q\theta(z))^{-1}.$$

We have

$$\tilde{S}(z) = \begin{bmatrix} \alpha(z) - \frac{1}{2}\beta(z)\bigl(I_p + \frac{1}{2}\delta(z)\bigr)^{-1}\gamma(z) & \beta(z)\bigl(I_p + \frac{1}{2}\delta(z)\bigr)^{-1} & 0 \\ 0 & 0 & I_p \\ -\bigl(I_p + \frac{1}{2}\delta(z)\bigr)^{-1}\gamma(z) & \bigl(I_p - \frac{1}{2}\delta(z)\bigr)\bigl(I_p + \frac{1}{2}\delta(z)\bigr)^{-1} & 0 \end{bmatrix}.$$

Clearly, the matrix-valued function $s = (I_p - \frac{1}{2}\delta)(I_p + \frac{1}{2}\delta)^{-1}$ belongs to the Schur class $S^{p\times p}$, because $\frac{1}{2}\delta \in \ell^{p\times p}$. The function $(I_p + \frac{1}{2}\delta(z))^{-1}$ $(= \frac{1}{2}(I_p + s))$ is of class $H_\infty^{p\times p}$. This implies that all blocks of $\tilde{S}$ belong to the Smirnov classes of appropriate sizes. Therefore, $\tilde{S} \in N_+^{(2p+m)\times(2p+m)}$, and (3.7) is equivalent to the identity

$$\tilde{S}(\zeta)^*\tilde{S}(\zeta) = I_{2p+m} \quad \text{for a.e. } \zeta \in T.$$

By the maximum principle for the Smirnov class, $\tilde{S}$ is contractive in $D$:

$$\tilde{S}(z)^*\tilde{S}(z) \le I_{2p+m}, \quad z \in D.$$

Now, we can use Lemma 6 to conclude that the resulting matrix-valued function $\theta$ is $J_{p,m}$-contractive in $D$:

$$\theta(z)^* J_{p,m}\theta(z) \le J_{p,m}, \quad z \in \Omega_\theta^+. \tag{3.8}$$

Thus, the function $\theta$, meromorphic in $D$, belongs to $U(J_{p,m})$. Moreover, it has a block structure as in (1.2), so that $\theta$ is a $J_{p,m}$-inner dilation of $c(z)$ with $m = m_c$.

**Proof of the "if" part.** Suppose a matrix-valued function $c(z)$ has a $J_{p,m}$-inner dilation $\theta$ of type (1.2) or (1.2∗) with $m \ge 0$. In accordance with Remark 1, the condition $\theta \in U(J_{p,m})$ implies $\theta \in \Pi^{(2p+m)\times(2p+m)}$. Therefore, the matrix-valued function $c(z)$ is in $\Pi^{p\times p}$, being a block of its dilation $\theta(z)$. The property $c \in \ell^{p\times p}$ follows from Remark 2 if $m = 0$. If $m > 0$, then the same property follows from the inequality $\operatorname{Re} c(z) \ge \beta(z)^*\beta(z)$ for $z \in \Omega_\theta^+$. The latter inequality is a consequence of (3.8). If a matrix-valued function $c(z)$ of order $p$ is meromorphic in $D$ and $\operatorname{Re} c(z) \ge 0$ for $z \in \Omega_c^+$, then $c(z)$ may have only removable singularities, so that we can view it as holomorphic in $D$. Therefore, $c \in \ell^{p\times p}\Pi$.

Now, we prove that $m \ge m_c$. If $m > 0$ for $\theta$, then

$$2\operatorname{Re} c(\zeta) = \beta(\zeta)^*\beta(\zeta) \quad \text{for a.e. } \zeta \in T,$$

and $m_c = \operatorname{rank} \beta(\zeta)$ on $T$, whence $m \ge m_c$. If $m = 0$, then $\theta = \theta_0$, and therefore, by Remark 2, $c(z)$ is of class $\ell^{p\times p}$ and $\operatorname{Re} c(\zeta) = 0$ a.e. on $T$. In this case, $c(z)$ has a pseudocontinuation $c_-(z) = -c^\#(z)$, whence $c \in \ell^{p\times p}\Pi$ and $m = m_c\,(= 0)$. The theorem is proved. □

In what follows, we shall consider only $J_{p,m}$-inner dilations of $c \in \ell^{p\times p}\Pi$ with the minimal possible $m$, i.e., with $m = m_c$.

**3.3. Description of the set of $J_{p,m}$-inner dilations with $m = m_c$.** A $J_{p,m}$-inner dilation $\theta$ of a matrix-valued function $c \in \ell^{p\times p}\Pi$ with $m = m_c > 0$ is not unique. Indeed, if $b_1 \in S_{\text{in}}^{m\times m}$ and $b_2 \in S_{\text{in}}^{m\times m}$, then the function

$$\theta_1 = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \theta \begin{bmatrix} b_2 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}$$

is also a $J_{p,m}$-inner dilation of $c(z)$.

Let

$$\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & \delta(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}$$

be an arbitrary $J_{p,m}$-inner dilation of $c \in \ell^{p\times p}\Pi$, and let $m = m_c > 0$. Since $\theta(\zeta)$ is $J_{p,m}$-unitary, (3.7) is fulfilled almost everywhere on $T$. This is equivalent to the following family of scalar identities for a.e. $\zeta \in T$:

$$\alpha(\zeta)^*\alpha(\zeta) = I_m, \quad \alpha(\zeta)^*\beta(\zeta) = \gamma(\zeta)^*, \tag{3.9}$$

$$2\operatorname{Re} c(\zeta) = \beta(\zeta)^*\beta(\zeta). \tag{3.10}$$

By (3.8), we have

$$\begin{bmatrix} I_m - \alpha(z)^*\alpha(z) & \gamma(z)^* - \alpha(z)^*\beta(z) & 0 \\ \gamma(z) - \beta(z)^*\alpha(z) & 2\operatorname{Re} c(z) - \beta(z)^*\beta(z) & 0 \\ 0 & 0 & 0 \end{bmatrix} \ge 0 \tag{3.11}$$

for $z \in \Omega_\theta^+$, whence

$$\alpha(z)^*\alpha(z) \le I_m, \quad \beta(z)^*\beta(z) \le 2\operatorname{Re} c(z), \quad z \in \Omega_\theta^+. \tag{3.12}$$

Furthermore, (3.12) implies that the matrix-valued functions $\alpha$ and $\beta$, meromorphic in $D$, may have only removable singularities. Therefore, we may assume that $\alpha$ and $\beta$ are holomorphic in $D$. In particular, for $\beta$ we have

$$\frac{1}{2\pi}\int_{|\zeta|=1}\|\beta(\rho\zeta)\xi\|^2\,|d\zeta| \le \frac{1}{2\pi}\int_{|\zeta|=1} 2(\operatorname{Re} c(\rho\zeta)\xi,\xi)\,|d\xi| = 2(\operatorname{Re} c(0)\xi,\xi) < \infty,$$

where $\xi \in \mathbb{C}^p$. Hence, $\beta$ is an $H_2^{m\times p}$-solution of the factorization problem (3.10), and $\alpha$ belongs to $S_{\mathrm{in}}^{m\times m}$ by (3.9) and (3.12).

Relation (3.7) is equivalent to

$$\theta(\zeta)J_{p,m}\theta(\zeta)^* = J_{p,m} \quad \text{for a.e. } \zeta \in T. \tag{3.13}$$

Therefore,

$$\alpha(\zeta)\alpha(\zeta)^* = I_m, \quad \alpha(\zeta)\gamma(\zeta)^* = \beta(\zeta), \tag{3.14}$$

$$2\operatorname{Re} c(\zeta) = \gamma(\zeta)\gamma(\zeta)^* \quad \text{for a.e. } \zeta \in T. \tag{3.15}$$

Since (3.8) is equivalent to

$$\theta(z)J_{p,m}\theta(z)^* \le J_{p,m}, \quad z \in \Omega_\theta^+, \tag{3.16}$$

in a similar way we show that $\gamma$ is a solution of the factorization problem (3.15), and it belongs to $H_2^{p\times m}$.

The matrix-valued functions $\beta$ and $\gamma$ can be represented in the form

$$\beta(z) = b_1(z)\varphi(z), \quad \gamma(z) = \psi(z)b_2(z), \tag{3.17}$$

where $\varphi$ and $\psi$ are an outer and a $*$-outer solution of the factorization problems (3.10) and (3.15). They have rank $m$ and belong to the classes $H_2^{m\times p}$ and $H_2^{p\times m}$, respectively; $b_1$ and $b_2$ are of class $S_{\mathrm{in}}^{m\times m}$. The Rosenblum–Rovnyak theorem ensures that such "maximal" solutions $\varphi$ and $\psi$ of problems (3.10) and (3.15) exist. The functions $\varphi(z)$ and $\psi(z)$ are uniquely determined by $c(z)$, up to a unitary left or right constant matrix, respectively.

Since $c \in \ell^{p\times p}\Pi$ and $m = m_c > 0$, the matrix-valued function $\operatorname{Re} c(\zeta)$ (nonnegative for a.e. $\zeta \in T$) is the boundary value of a matrix-valued function of class $N^{p\times p}$ in the disk $D$. Therefore, since $\operatorname{rank}\operatorname{Re} c(\zeta) = m$ a.e. on $T$, the function $\operatorname{Re} c(\zeta)$ has a principal minor of order $m$ different from zero a.e. on $T$, whereas any principal minor of order exceeding $m$ is identically zero. Without loss of generality we may assume that such a principal minor of order $m$ is at the upper left corner of the matrix $\operatorname{Re} c(\zeta)$. To arrive at this case, we can always make a permutation of the rows of $\operatorname{Re} c(\zeta)$, together with the same permutation of the columns. As a result, we get a matrix-valued function $h(\zeta) = K\operatorname{Re} c(\zeta)K^*$ that is determined now by $\tilde{c} = KcK^*$; hence $h(\zeta) = \operatorname{Re}\tilde{c}(\zeta)$, where $K$ is a constant orthogonal matrix. A dilation $\theta$ of $c$ can be obtained from the dilation $\tilde{\theta}$ of $\tilde{c}$ by multiplying $\tilde{\theta}$ from the left and from the right by $J_{p,m}$-unitary matrices $\begin{bmatrix} I_m & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}$ and $\begin{bmatrix} I_m & 0 & 0 \\ 0 & K^* & 0 \\ 0 & 0 & {}^*K \end{bmatrix}$, respectively. Under our assumption, an outer and a $*$-outer solution of the factorization problems (3.10) and (3.15) have the following form:

$$\begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix}, \quad \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

where $\varphi_1$ and $\psi_1$ are matrix functions of order $m$ with $\det\varphi_1 \not\equiv 0$ and $\det\psi_1 \not\equiv 0$. They could be determined uniquely by imposing the normalization conditions $\varphi_1(0) > 0$ and $\psi_1(0) > 0$. These solutions will be called the normalized outer and $*$-outer solutions of (3.10) and (3.15), respectively. They will be denoted by $\varphi_N$ and $\psi_N$. In the general case, we assume that $\varphi_N = \left[\begin{smallmatrix}\varphi_1 & \varphi_2\end{smallmatrix}\right] K$ and $\psi_N = K^* \left[\begin{smallmatrix}\psi_1 \\ \psi_2\end{smallmatrix}\right]$, where $K$ is the orthogonal matrix considered above. The normalization conditions are $\varphi_1(0) > 0$ and $\psi_1(0) > 0$.

In what follows, a description of all $J_{p,m}$-inner dilations of the form (1.2) with $m = m_c$ will be given for $c \in \ell^{p\times p}\Pi$ with $m_c > 0$. This description will involve the normalized outer ($\varphi_N$) and $*$-outer ($\psi_N$) solutions of the factorization problems (3.10) and (3.15), as well as the matrix-valued function $s_c(\zeta)$ defined by the relation

$$s_c(\zeta)\psi_N(\zeta)^* = \varphi_N(\zeta) \quad \text{for a.e. } \zeta \in T. \tag{3.18}$$

This function was introduced in [8] in a more general setting. There it was called a scattering suboperator, because precisely this suboperator arises when we consider interior scattering channels in conservative resistance systems with resistance matrix equal to $c(z)$. In the same paper, it was shown that the matrix-valued function $s_c(\zeta)$ plays a role in investigating minimal passive resistance systems with the impedance matrix $c(z)$. Note that if $s_c(\zeta)$ is defined by (3.18), then it has unitary values, i.e.,

$$s_c(\zeta)^* s_c(\zeta) = I_m \quad \text{for a.e. } \zeta \in T. \tag{3.19}$$

Moreover, if $c \in \ell^{p\times p}\Pi$, then $s_c(\zeta)$ is the nontangential boundary value of a matrix-valued function $s_c(z)$ of class $N^{p\times p}$. The latter function is defined by

$$s_c(z)\psi_N^{\#}(z) = \varphi_N(z), \quad z \in \Omega_{\varphi_N} \cap \Omega_{\psi_N^{\#}} \cap \Omega_{s_c}. \tag{3.20}$$

Relation (3.20) is equivalent to the formula

$$s_c(z) = \varphi_1(z)\psi_1^{\#}(z)^{-1}, \quad z \in \Omega_{\varphi_1} \cap \Omega_{\psi_1^{\#}}, \tag{3.21}$$

which in turn is equivalent to the identity

$$s_c(z) = \varphi_1^{\#}(z)^{-1}\psi_1(z), \quad z \in \Omega_{\varphi_1^{\#} \cap \Omega_{\psi_1}}. \tag{3.22}$$

We show that (3.20) and (3.21) are indeed equivalent. Let (3.20) be fulfilled. Then

$$s_c(z)\begin{bmatrix}\psi_1^{\#}(z) & \psi_2^{\#}(z)\end{bmatrix} = \begin{bmatrix}\varphi_1(z) & \varphi_2(z)\end{bmatrix},$$

whence

$$s_c(z)\psi_1^{\#} = \varphi_1(z), \quad s_c(z)\psi_2^{\#}(z) = \varphi_2(z).$$

Therefore, (3.21) is true.

Conversely, assume (3.21). The factorization problems (3.10) and (3.15) ensure that $\varphi_N^{\#}\varphi_N = \psi_N\psi_N^{\#}$, i.e.,

$$\begin{bmatrix}\varphi_1^{\#}\varphi_1 & \varphi_1^{\#}\varphi_2 \\ \varphi_2^{\#}\varphi_1 & \varphi_2^{\#}\varphi_2\end{bmatrix} = \begin{bmatrix}\psi_1\psi_1^{\#} & \psi_1\psi_2^{\#} \\ \psi_2\psi_1^{\#} & \psi_2\psi_2^{\#}\end{bmatrix},$$

whence $\varphi_1^{\#}\varphi_2 = \psi_1\psi_2^{\#}$, and $\varphi_2 = \varphi_1^{\#-1}\psi_1\psi_2^{\#}$. Since $\varphi_1^{\#}\varphi_1 = \psi_1\psi_1^{\#}$, we have $\varphi_1 = \varphi_1^{\#-1}\psi_1\psi_1^{\#} = s_c\psi_1^{\#}$. Therefore, (3.20) is true.

Now, we state and prove a theorem that yields a complete description of the set of all $J_{p,m}$-inner dilations of $c(z)$.

**Theorem 2.** *Let $c \in \ell^{p\times p}\Pi$, and let $m = m_c > 0$. Consider the matrix-valued functions $\varphi_N \in H_2^{m\times p}$ and $\psi_N \in H_2^{p\times m}$ that are the normalized outer and $*$-outer solutions of the factorization problems*

$$2\operatorname{Re} c(\zeta) = \varphi(\zeta)^*\varphi(\zeta) \quad \text{and} \quad 2\operatorname{Re} c(\zeta) = \psi(\zeta)\psi(\zeta)^* \quad \text{for a.e. } \zeta \in T,$$

*respectively. Let $s_c$ be defined by* (3.20) *and $\vartheta$ by the formula*

$$\vartheta(z) = \begin{bmatrix} s_c(z) & \varphi_N(z) & 0 \\ \psi_N(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}. \tag{3.23}$$

*Let $\{b_1, b_2\}$ be a denominator of the matrix-valued function $s_c \in N^{p\times p}$. Put*

$$\begin{aligned} \alpha(z) &= b_1(z)s_c(z)b_2(z), \quad \beta(z) = b_1(z)\varphi_N(z),\\ \gamma(z) &= \psi_N(z)b_2(z), \quad \delta(z) = c(z), \end{aligned} \tag{3.24}$$

*and*

$$\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & \delta(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}. \tag{3.25}$$

*Then $\theta(z)$ is a $J_{p,m}$-unitary dilation of $c(z)$, and it has a unique representation of the form*

$$\theta(z) = \begin{bmatrix} b_1(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \vartheta(z) \begin{bmatrix} b_2(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}, \tag{3.26}$$

*where $\{b_1, b_2\} \in \mathrm{Den}(s_c)$. All $J_{p,m}$-inner dilations of $c \in \ell^{p\times p}\Pi$ can be obtained in this way.*

*Proof.* Let $\theta(z)$ be a $J_{p,m}$-inner dilation of $c \in \ell^{p\times p}\Pi$ of the form (1.2). Then, as has been shown above, its blocks $\beta$ and $\gamma$ are solutions of the factorization problems (3.10) and (3.15). They belong to $H_2^{m\times p}$ and $H_2^{p\times m}$, respectively, and $\alpha$ belongs to $S_{\mathrm{in}}^{m\times m}$. The functions $\beta$ and $\gamma$ can be represented as in (3.17), where $\varphi = \varphi_N$ and $\psi = \psi_N$ are the normalized outer and $*$-outer solutions of the factorization problems (3.10) and (3.15), belonging to $H_2^{m\times p}$ and $H_2^{p\times m}$, respectively, and $b_1 \in S_{\mathrm{in}}^{m\times m}$, $b_2 \in S_{\mathrm{in}}^{m\times m}$. The functions $\varphi_N$ and $\psi_N$ determine the function $s_c(z)$ uniquely by one of the formulas (3.20)–(3.22). The function $s_c(z)$ satisfies (3.19).

Next, since $\gamma(\zeta)^* = \alpha(\zeta)^*\beta(\zeta)$ a.e. on $T$, we have $b_2(\zeta)^*\psi_N(\zeta)^* = \alpha(\zeta)^*b_1(\zeta)\varphi_N(\zeta)$, and with the help of (3.18) we get $\alpha = b_1 s_c b_2$.

This leads to the following parametrization for the blocks of the dilation $\theta$:

$$\alpha = b_1 s_c b_2, \quad \beta = b_1\varphi_N, \quad \gamma = \psi_N b_2.$$

Therefore,

$$\theta(z) = \begin{bmatrix} b_1(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \vartheta(z) \begin{bmatrix} b_2(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix},$$

where $\vartheta$ is uniquely determined by $c$, by formula (3.23).

Thus, the freedom in the choice of the dilation reduces to the inner matrix-valued functions $b_1$ and $b_2$. Nevertheless, these functions are not arbitrary, because the following condition must be fulfilled:

$$b_1 s_c b_2 \in S_{\mathrm{in}}^{m\times m}, \tag{3.27}$$

where $s_c \in N^{m\times m}$ is uniquely determined by $c$ in accordance with one of the formulas (3.20)–(3.22). The boundary condition (3.19) yields the equivalence (3.27) and the relation

$$b_1 s_c b_2 \in N_+^{m\times m}. \tag{3.28}$$

Therefore, the dilation $\theta \in U(J_{p,m})$ of $c \in \ell^{p\times p}\Pi$ has the form (3.26), where $\vartheta$ is uniquely determined by $c(z)$, and $\{b_1, b_2\}$ is a denominator of $s_c$, i.e., $\{b_1, b_2\} \in \mathrm{Den}(s_c)$.

Conversely, if an arbitrary matrix-valued function $c \in \ell^{p\times p}\Pi$ with $m_c > 0$ is given, then we can construct functions $s_c$ and $\vartheta$ by (3.20)–(3.22) and (3.23). Furthermore, we take an arbitrary denominator $\{b_1, b_2\}$ of $s_c$ and consider the function

$$\theta(z) = \begin{bmatrix} b_1(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \vartheta(z) \begin{bmatrix} b_2(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}.$$

A direct calculation shows that (3.24) and (3.25) are fulfilled and that $\theta$ has $J_{p,m}$-unitary boundary values a.e. on $T$.

The matrix-valued function $\vartheta(z)$ defined by (3.23) has $J_{p,m}$-unitary boundary values almost everywhere on the unit circle $T$, but in general it may fail to be $J_{p,m}$-unitary. It is such if and only if $s_c \in N_+^{m\times m}$. However, by the choice of $b_1$ and $b_2$, the Potapov–Ginzburg transform of $\theta$ is contractive in $D$, which can be checked in the same way as was done in the proof of (3.8) in Theorem 1. Using Lemma 6, we conclude that $\theta$ is $J_{p,m}$-contractive in the disk $D$. Therefore, $\theta \in U(J_{p,m})$. The theorem is proved. □

**3.4. Minimal and optimal $J_{p,m}$-inner dilations.** The result obtained allows us to describe the set of all minimal (in a sense) $J_{p,m}$-inner dilations of the form (1.2) with $m = m_c$.

If two $J_{p,m}$-inner matrix-valued functions $\theta$ and $\theta_1$ satisfy $\theta = \theta_1\theta_2$, where $\theta_2$ is also a $J_{p,m}$-inner matrix-valued function, then $\theta_1$ is called a left divisor of $\theta$. We say that such a divisor is *trivial* if $\theta = \theta_1 U$, where $U$ is a $J_{p,m}$-unitary matrix. A right divisor and a right trivial divisor are defined in a similar way.

A dilation $\theta \in U(J_{p,m})$ is said to be *minimal* if it admits no nontrivial $J_{p,m}$-inner left and right divisors of the form $\begin{bmatrix} w(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}$, i.e., if $\theta$ cannot be represented in the form

$$\theta(z) = \begin{bmatrix} u(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \tilde{\theta}(z) \begin{bmatrix} v(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}, \tag{3.29}$$

where $\tilde{\theta} \in U(J_{p,m})$, $u \in S_{\text{in}}^{m\times m}$, $v \in S_{\text{in}}^{m\times m}$, and at least one of the functions $u(z)$ and $v(z)$ is not constant.

We are going to find out for what denominators $\{b_1, b_2\}$ of $s_c$ formula (3.26) produces minimal $J_{p,m}$-inner dilations.

**Theorem 3.** *A $J_{p,m}$-inner dilation $\theta$ of a matrix-valued function $c \in \ell^{p\times p}\Pi$ with $m = m_c > 0$ is minimal if and only if the corresponding denominator $\{b_1, b_2\}$ of $s_c$, occurring in* (3.26)*, is minimal.*

*Proof.* Let $\theta$ be a $J_{p,m}$-inner dilation of $c \in \ell^{p\times p}\Pi$, and let $\{b_1, b_2\}$ be the denominator of $s_c$ that corresponds to $\theta$ by formula (3.26). Suppose that this dilation is minimal, i.e., (3.29) is fulfilled, where $\tilde{\theta} \in U(J_{p,m})$, $u \in S_{\text{in}}^{m\times m}$, $v \in S_{\text{in}}^{m\times m}$, and either $u(z)$ or $v(z)$ is not constant. Then $\tilde{\theta}(z)$ is a $J_{p,m}$-inner dilation of $c(z)$ with some denominator$\{\tilde{b_1}, \tilde{b_2}\}$ of $s_c$. Since the representation of $\theta(z)$ in the form (3.26) is unique, we have $b_1(z) = u(z)\tilde{b_1}(z)$ and $b_2(z) = \tilde{b_2}(z)v(z)$; i.e., $\{\tilde{b_1}, \tilde{b_2}\}$ is a nontrivial divisor of the denominator $\{b_1, b_2\}$. Conversely, if we have such a divisor, we get relation (3.29) in which at least one of the inner functions, either $u(z)$ or $v(z)$, is not constant. The theorem is proved. □

A dilation $\theta \in U(J_{p,m})$ of $c \in \ell^{p\times p}\Pi$ of the form (1.2) with $m = m_c$ is said to be *optimal* if $\beta$ in (1.2) is an outer matrix-valued function; it is $*$-*optimal* if $\gamma$ in (1.2) is a $*$-outer matrix-valued function.

The next assertion follows from Theorem 3 and Lemma 3.

**Theorem 4.** *All optimal $J_{p,m}$-inner dilations of a matrix-valued function $c \in \ell^{p\times p}\Pi$ are described by the formula*

$$\theta(z) = \begin{bmatrix} u & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \vartheta(z) \begin{bmatrix} b(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix},$$

*where $\{u, b\} \in \mathrm{Den}^r(s_c)$ and $u$ is a unitary matrix of order $m$. Moreover, an optimal dilation $\theta$ is minimal if and only if the corresponding right denominator $\{u, b\}$ of $s_c$ is minimal. Such a denominator exists and is essentially unique.*

*All $*$-optimal $J_{p,m}$-inner dilations of a matrix-valued function $c \in \ell^{p\times p}\Pi$ are described by the formula*

$$\theta(z) = \begin{bmatrix} b(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \vartheta(z) \begin{bmatrix} v & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix},$$

*where $\{b, v\} \in \mathrm{Den}^l(s_c)$ and $v$ is a unitary matrix of order $m$. Moreover, a $*$-optimal dilation $\theta$ is minimal if and only if the corresponding left denominator $\{b, v\}$ of $s_c$ is minimal. Such a denominator exists and is essentially unique.*

## §4. Real, symmetric, and rational $J_{p,m}$-inner dilations

**4.1. Real $J_{p,m}$-inner dilations.** Matrix-valued functions $f(z)$ real in the disk $D$, i.e., satisfying $\overline{f(\overline{z})} = f(z)$, appear very often in applications. For this reason, we consider real matrix-valued functions $c(z) \in \ell^{p\times p}\Pi$ and their $J_{p,m}$-inner dilations.

A denominator $\{b_1, b_2\}$ of $f \in N^{p\times q}$ is said to be *real* if $b_1(z)$ and $b_2(z)$ are real matrix-valued functions.

**Lemma 8.** *For a real matrix-valued function $f(z) \in N^{p\times q}$, its minimal right and minimal left denominators can be chosen to be real.*

*Proof.* See Theorem 6.1 in [2]. □

**Theorem 5.** *A real matrix-valued function $c(z) \in \ell^{p\times p}\Pi$ with $m_c > 0$ has a real $J_{p,m}$-inner dilation $\theta(z)$ of the form (1.2) with $m = m_c$. All such $J_{p,m}$-inner dilations are described by formula (3.26) with real denominators of $s_c(z)$. In particular, minimal optimal as well as minimal $*$-optimal $J_{p,m}$-inner dilations can be chosen to be real.*

*Proof.* Since $c(z) \in \ell^{p\times p}\Pi$ is real, $2\,\mathrm{Re}\, c(\zeta)$ is also real. Therefore, the functions $\overline{\varphi_N(\overline{z})}$ and $\overline{\psi_N(\overline{z})}$ are solutions of the factorization problems (3.10) and (3.15), together with $\varphi_N(z)$ and $\psi_N(z)$. Since they are outer and $*$-outer matrix-valued functions, respectively, and they satisfy the normalization conditions $\overline{\varphi_1(0)} > 0$, $\overline{\psi_1(0)} > 0$, we have $\varphi_N(z) = \overline{\varphi_N(\overline{z})}$ and $\psi_N(z) = \overline{\psi_N(\overline{z})}$ because such solutions are unique. Therefore, $\varphi_N(z)$ and $\psi_N(z)$ are real, and hence, $s_c(z)$ and $\vartheta(z)$ are also real. Moreover, the $J_{p,m}$-inner dilation $\theta(z)$ is real if and only if the denominator $\{b_1(z), b_2(z)\}$ of $s_c(z)$ has real inner matrix-valued functions $b_1$ and $b_2$. Combined with Theorem 3 and Lemma 8, this fact shows that a minimal optimal and a minimal $*$-optimal $J_{p,m}$-inner dilation of the real matrix-valued function $c(z)$ can be chosen to be real. □

**4.2. Symmetric $J_{p,m}$-inner dilations.** In applications, the case of symmetric matrix-valued functions $f(z)$, i.e., those for which $f(z)^T = f(z)$ for all $z \in \Omega_f$, is also important. In this subsection we consider symmetric $J_{p,m}$-inner dilations of symmetric matrix-valued functions $c(z) \in \ell^{p\times p}\Pi$.

The denominators of the form $\{b(z), b(z)^T\}$ for a function $f(z)$ belonging to the Nevanlinna class are said to be *symmetric*. Such denominators always exist, because for a scalar function $\eta \in S_{in}$ such that $\eta f \in N_+^{p\times q}$ the pair $\{\eta I_p, \eta I_p\}$ is a symmetric denominator of $f$.

**Theorem 6.** *Any symmetric matrix-valued function $c(z) \in \ell^{p\times p}\Pi$ with $m_c > 0$ has a symmetric $J_{p,m}$-inner dilation $\theta$. Moreover, all symmetric $J_{p,m}$-inner dilations are described by formula* (3.26)*, where $\{b_1, b_2\}$ is a symmetric denominator of $s_c$, i.e., $b_2(z) = b_1(z)^T$.*

*Proof.* Since $c(z) \in \ell^{p\times p}\Pi$ is symmetric, the outer function $\psi_N(z)^T$ and the $*$-outer function $\varphi_N(z)^T$ are, respectively, solutions of the factorization problems (3.10) and (3.15), together with the normalized solutions $\varphi_N(z)$ and $\psi_N(z)$. Moreover, they satisfy the same normalization conditions. Therefore, $\psi_N(z) = \varphi_N(z)^T$, which implies that the matrix-valued functions $s_c(z)$ and $\vartheta(z)$ are symmetric. Moreover, the dilation $\theta(z)$ given by (3.26) is symmetric if and only if the denominator $\{b_1, b_2\}$ of $s_c(z)$ satisfies the condition $b_2(z) = b_1(z)^T$. □

We say that a symmetric denominator $\{b(z), b(z)^T\} \in \mathrm{Den}(f)$ is *minimal symmetric* if it has no nontrivial symmetric divisors. Such denominators of $s_c$ correspond to minimal symmetric $J_{p,m}$-inner dilations of $c(z)$, i.e., to those admitting no representation of the form (3.29) with $\tilde\theta(z)^T = \tilde\theta(z)$ and with nonconstant $u(z)$ and $v(z)\,(= u(z)^T)$. The existence of such dilations follows from the next statement.

**Lemma 9.** *For every symmetric denominator $\{b(z), b(z)^T\}$ of $f \in N^{m\times m}$, there exists a minimal symmetric denominator $\{\hat b(z), \hat b(z)^T\}$ that is a divisor of the initial one. In the scalar case* ($m = 1$) *the function $f(z)$ has a unique minimal symmetric denominator* (*up to a constant factor $\kappa$ with $|\kappa| = 1$*).

Note that the minimal symmetric denominator $\{\hat b(z), \hat b(z)^T\}$ of $f \in N^{p\times p}$ may fail to be its minimal denominator. For the proof of Lemma 9, see [2] (Lemma 6.1 and the Remark after it).

Now, we consider the case where $c(z) \in \ell^{p\times p}\Pi$ is both real and symmetric. The following results are consequences of Theorems 6.3 and 6.4 in [2].

**Theorem 7.** *For a real symmetric matrix-valued function $c(z) \in \ell^{p\times p}\Pi$ with $m_c > 0$, there exists a real symmetric $J_{p,m}$-inner dilation $\theta(z)$ of the form* (1.2)*. All such dilations are described by formula* (3.26) *with real symmetric denominators $\{b_1, b_2\}$ of $s_c(z)$.*

*Any real symmetric $J_{p,m}$-inner dilation $\theta(z)$ can be represented as in* (3.29)*, where $\tilde\theta(z)$ is a real minimal symmetric $J_{p,m}$-inner dilation of $c(z)$. Moreover, the functions $u(z) \in S_{\mathrm{in}}^{m\times m}$ and $v(z) \in S_{\mathrm{in}}^{m\times m}$ satisfy $\overline{u(\overline{z})} = u(z)$ and $v(z) = u(z)^T$.*

Note that, for a given real symmetric matrix-valued function $c(z) \in \ell^{p\times p}\Pi$, its real minimal symmetric $J_{p,m}$-inner dilation is not unique. However, for a scalar $c(z) \in \ell\Pi$ its real minimal symmetric $J_{p,m}$-inner dilation is unique, because we have an essentially unique minimal symmetric denominator of $s_c(z)$.

**4.3. Rational $J_{p,m}$-inner dilations.** In this subsection we consider the case where $c(z) \in \ell^{p\times p}\Pi$ is a rational matrix-valued function.

We say that a denominator $\{b_1, b_2\}$ of $f \in N^{p\times q}$ is *rational* if $b_1(z)$ and $b_2(z)$ are rational.

If $c \in \ell^{p\times p}\Pi$ is rational, then so is $c(z) + c^{\#}(z)$. Therefore, the normalized solutions $\varphi_N(z)$ and $\psi_N(z)$ of the factorization problems (3.10) and (3.15) are rational matrix-valued functions of size $m \times p$ and $p \times m$, respectively (see assertions 5 and 6 of the Rosenblum–Rovnyak theorem). This implies that the functions $s_c(z)$ and $\vartheta(z)$ defined by (3.21) and (3.23) are also rational. The $J_{p,m}$-inner dilation $\theta$ is rational if and only if so is the denominator $\{b_1(z), b_2(z)\}$ of $s_c(z)$. Assertion (b) of Lemma 4 and Theorem 3 imply that the minimal $J_{p,m}$-inner dilations of a rational $c \in \ell^{p\times p}\Pi$ are rational.

Thus, the following theorem is true.

**Theorem 8.** *For a rational matrix-valued function $c(z) \in \ell^{p\times p}\Pi$ with $m_c > 0$, there exists a rational $J_{p,m}$-inner dilation $\theta(z)$ of the form* (1.2). *All rational $J_{p,m}$-inner dilations are described by formula* (3.26) *with rational denominators of $s_c(z)$. All minimal $J_{p,m}$-inner dilations of $c(z)$ are rational. Moreover,*

1) *for a rational real matrix-valued function $c(z) \in \ell^{p\times p}\Pi$ with $m_c > 0$, there exists a rational real $J_{p,m}$-inner dilation $\theta(z)$ of the form* (1.2). *All rational $J_{p,m}$-inner dilations are described by formula* (3.26) *with rational real denominators of $s_c(z)$;*
2) *an arbitrary rational and symmetric matrix-valued function $c(z) \in \ell^{p\times p}\Pi$ with $m_c > 0$ admits a rational symmetric $J_{p,m}$-inner dilation $\theta(z)$, and all rational and symmetric $J_{p,m}$-inner dilations are described by formula* (3.26) *with rational and symmetric denominators $\{b, b^T\} \in \mathrm{Den}(s_c)$. In the class of all rational and symmetric $J_{p,m}$-inner dilations there exist rational, minimal, and symmetric $J_{p,m}$-inner dilations of $c(z)$;*
3) *for a rational, real, and symmetric matrix-valued function $c(z) \in \ell^{p\times p}\Pi$ with $m_c > 0$, there exists a rational, real, and symmetric $J_{p,m}$-inner dilation $\theta(z)$ of the form* (1.2). *All such dilations are described by formula* (3.26) *with rational, real, and symmetric denominators $\{b_1, b_2\}$ of $s_c(z)$. An arbitrary rational, real, and symmetric $J_{p,m}$-inner dilation $\theta(z)$ of $c(z)$ can be represented in the form* (3.29), *where $\tilde{\theta}(z)$ is a rational, real, minimal, and symmetric $J_{p,m}$-inner dilation of $c(z)$; moreover, $u(z) \in S_{\mathrm{in}}^{m\times m}$ and $v(z) \in S_{\mathrm{in}}^{m\times m}$ are rational, $\overline{u(\overline{z})} = u(z)$, and $v(z) = u(z)^T$.*

## §5. Entire matrix-valued functions of class $\ell^{p\times p}\Pi(\mathbb{C}_+)$

Instead of the unit disk $D$, we can consider the open upper half-plane

$$\mathbb{C}_+ = \{z : \mathrm{Im}\ > 0\}.$$

Then a class $\mathfrak{X}$ of matrix-valued functions on $D$ turns into the corresponding class $\mathfrak{X}(\mathbb{C}_+)$ of functions on $\mathbb{C}_+$, with the exception of the Hardy space $H_2^{p\times q}$, which will be discussed below.

The class $\ell^{p\times p}(\mathbb{C}_+)$ consists of all matrix-valued functions $c(z)$ of order $p$ holomorphic in $\mathbb{C}_+$ and such that $\mathrm{Re}\, c(z) \geq 0$ in $\mathbb{C}_+$. Its subclass $\ell^{p\times p}\Pi(\mathbb{C}_+)$ consists of all $c(z)$ belonging to $\ell^{p\times p}(\mathbb{C}_+)$ and having a pseudocontinuation $c_-(z)$ of bounded Nevanlinna characteristic in the lower half-plane $\mathbb{C}_- = \{z : \mathrm{Im\, z} < 0\}$:

$$c(\mu) = \lim_{\nu\downarrow 0} c(\mu + i\nu) = \lim_{\nu\downarrow 0} c_-(\mu - i\nu) \quad \text{for a.e. } \mu \in \mathbb{R},$$

where $c_-(z) = h_2(z)^{-1}h_1(z)$, and $h_1$ is a matrix-valued function of order $p$ holomorphic and bounded in $\mathbb{C}_-$, and $h_2$ is a scalar bounded function holomorphic in $\mathbb{C}_-$. For $c(z) \in \ell^{p\times p}(\mathbb{C}_+)$, the condition $(1 + \mu^2)^{-1}\,\mathrm{Re}\, c(\mu) \in L_1^{p\times p}(\mathbb{C}_+)$ is fulfilled, and the rank $m_c$ of $c(z) \in \ell^{p\times p}\Pi(\mathbb{C}_+)$, i.e., $\mathrm{rank}\,\mathrm{Re}\, c(\mu)$, is constant a.e. on $\mathbb{R}$. If $m_c > 0$, then the factorization problems on $\mathbb{R}$ of the types (3.10) and (3.15) are solvable in the class of all $g$ and $\omega$ such that $(z+i)^{-1}g(z) \in H_2^{m\times p}(\mathbb{C}_+)$ and $(z+i)^{-1}\omega(z) \in H_2^{p\times m}(\mathbb{C}_+)$, where $H_2^{k\times l}(\mathbb{C}_+)$ is the Hardy class of $(k\times l)$-matrix-valued functions $h(z)$ that are holomorphic in $\mathbb{C}_+$ and satisfy

$$\sup_{\nu>0}\int_{-\infty}^{\infty} \mathrm{trace}\{h(\mu+i\nu)^* h(\mu+i\nu)\}d\mu < \infty.$$

A solution $g = \varphi$ of the factorization problem of the form (3.10) is outer if and only if the closed linear span of the vector-valued functions $e^{izt}(z+i)^{-1}\varphi(z)\xi$ with $t > 0$ and $\xi \in \mathbb{C}^p$ is the entire space $H_2^m(\mathbb{C}_+)$.

A matrix-valued function $\psi(z)$ is $*$-outer in $\mathbb{C}_+$ if $\psi(z)^T$ is outer in $\mathbb{C}_+$.

All results of the preceding sections have their analogs for matrix-valued functions on the upper half-plane. They can easily be deduced from the corresponding results in the unit disk by the change of variables $\lambda = i\frac{1-z}{1+z}$, which transfers $D$ onto $\mathbb{C}_+$.

Instead of real matrix-valued functions, we consider functions $f(z)$ defined on $\mathbb{C}_+$ and possessing the property

$$\overline{f(-\overline{z})} = f(z).$$

These functions will be called *I-real*, because this is the class of function invariant under the operator $I$ mapping $f(z)$ to $\overline{f(-\overline{z})}$.

We restrict ourselves to the case where $c \in \ell^{p\times p}\Pi(\mathbb{C}_+)$ is an entire function.

**Lemma 10.** *If $f \in N^{p\times q}(\mathbb{C}_+)$ is an entire matrix-valued function, then its minimal denominators $\{b_1, b_2\}$ are pairs of entire functions.*

See [3].

By theorems of M. G. Kreĭn and of Rosenblum–Rovnyak (see [12]), if $f(\mu) \geq 0$ and $f(z)$ is an entire $(p\times p)$-matrix-valued function of bounded Nevanlinna characteristic in $\mathbb{C}_+$ and $\mathbb{C}_-$, i.e., $f \in \Pi^{p\times p}(\mathbb{C}_+)$, then an outer solution $g = \varphi$ of the factorization problem of the type (3.10) is an entire matrix-valued function of class $\Pi^{m\times p}(\mathbb{C}_+)$. Therefore, $\varphi^{\sim}(z)$ is an entire matrix-valued function of class $\Pi^{p\times m}(\mathbb{C}_+)$. Hence, by Lemma 10, if $\{I_m, b\}$ is its right minimal denominator, then the function $b \in S_{\text{in}}^{m\times m}(\mathbb{C}_+)$ is entire, whence

$$b(\mu)^*\varphi(\mu) = \omega(\mu)^*, \tag{5.1}$$

where $\omega(\mu)$ is a boundary value of an entire matrix-valued function $\omega(z) \in N_+^{p\times m}(\mathbb{C}_+)$.

Put

$$\alpha(z) = b(z), \quad \beta(z) = \varphi(z), \quad \gamma(z) = \omega(z), \quad \delta(z) = c(z),$$
$$\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & \delta(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}.$$

Clearly, the matrix-valued function $\theta(z)$ constructed in this way is an entire, optimal, and $J_{p,m}$-inner dilation of $c(z)$. Now, we show that it is minimal.

Relation (5.1) implies that

$$\gamma(z) = \varphi_N^{\sim}(z)b(z) = \psi_N(z)d(z) \tag{5.2}$$

for some $d \in S_{\text{in}}^{m\times m}(\mathbb{C}_+)$. Consequently, $\varphi_1^{\sim}(z)b(z) = \psi_1(z)d(z)$, whence $\alpha(z) = b(z) = s_c(z)d(z)$, i.e., $\{I_m, d\} \in \text{Den}^r(s_c)$. We prove that this right denominator of $s_c$ is minimal. Let $d(z) = w_1(z)w_2(z)$, where $w_1, w_2 \in S_{\text{in}}^{m\times m}(\mathbb{C}_+)$ and $\{I_m, w_1(z)\} \in \text{Den}^r(s_c)$. Then $b(z) = [s_c(z)w_1(z)]w_2(z)$, so that $w_2(z)$ is a right divisor of $b(z)$. Being divisors of the entire matrix-valued function $b$, the matrix-valued functions $s_cw_1$ and $w_2$ are entire. We have $\varphi^{\sim}(z)b(z) = \psi(z)w_1(z)w_2(z)$, whence $\varphi^{\sim}(z)[s_c(z)w_1(z)] = \psi(z)w_1(z)$. This implies that $\{I_m, s_cw_1\}$ is a right denominator of $\varphi^{\sim}(z)$. Since $\{I_m, b\}$ is the minimal right denominator of $\varphi^{\sim}(z)$, we have $w_2(z) = \text{const}$; i.e., $\{I_m, d\}$ is the minimal right denominator of $s_c(z)$. Therefore, by Theorem 3, the optimal $J_{p,m}$-inner dilation $\theta(z)$ of $c(z)$ is minimal.

Similar arguments apply to a $*$-outer solution of the factorization problem of type (3.15) on $\mathbb{R}$. Thus, the following theorem is true.

**Theorem 9.** *Let $c(z) \in \ell^{p\times p}\Pi(\mathbb{C}_+)$ be an entire matrix-valued function with $m = m_c$. Then any optimal minimal $J_{p,m}$-inner dilation of $c(z)$ in $\mathbb{C}_+$ is entire, and any $*$-optimal minimal $J_{p,m}$-inner dilation of $c(z)$ is entire. Moreover,*

1) *any entire $I$-real matrix-valued function $c(z)$ admits an entire, $I$-real, minimal, and optimal $J_{p,m}$-inner dilation, as well as an entire, $I$-real, minimal, and $*$-optimal $J_{p,m}$-inner dilation*;
2) *if $c(z) \in \ell^{p\times p}\Pi(\mathbb{C}_+)$ is an entire symmetric matrix-valued function, then it admits an entire, minimal, and symmetric $J_{p,m}$-inner dilation*;
3) *any entire, $I$-real, and symmetric matrix-valued function $c(z)$ admits an entire, $I$-real, minimal, and symmetric $J_{p,m}$-inner dilation.*

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