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CAN ONE SEE THE SIGNS OF STRUCTURE CONSTANTS?

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ABSTRACT. It is described how one can see the signs of action structure constants directly in the weight diagram of microweight and adjoint representations for groups of types E_6, E_7, and E_8. This generalizes the results of the preceding paper, "A third look at weight diagrams", where a similar algorithm was discussed for microweight representations of E_6 and E_7. The proofs are purely combinatorial and can be viewed as an elementary construction of Lie algebras and Chevalley groups of types E_l.

In the present paper, which is a sequel of [89, 66, 86], we prove most statements formulated without proofs in §3 of [66] and extend Theorems 1 and 2 of [86] to *all* microweight modules and to the adjoint modules of types A_l, D_l, and E_l.

We give an elementary construction of crystal bases in microweight representations and show how to sight-read the signs of the action structure constants of microweight representations (in a crystal base) and of adjoint representations (in a positive Chevalley base) directly from the weight diagram, alias, from the crystal graph. In particular, *this provides elementary purely combinatorial constructions of groups of types* E_6*,* E_7*, and* E_8 *as matrix groups*. These constructions could be stated in terms of the graphs depicted in Figures 1–5 in such a manner that any reference to Lie algebras and their representations could be avoided!

In [86], similar results and some of their consequences were established for microweight representations of types E_6 and E_7. There we used realizations of these modules in the unipotent radicals of maximal parabolic subgroups in Chevalley groups of types E_7 and E_8, respectively. In the present paper we propose straightforward proofs for groups of all types. These proofs are based exclusively upon

- identities for the structure constants for Lie algebras, and
- geometric and combinatorial properties of microweight representations.

Obviously, both for the microweight case and for the adjoint case *many* different methods are known for computing the signs; these methods are based on one of the following.

- ◦ Tits inductive algorithm [83, 23, 22, 33, 87].
- ◦ Frenkel–Kac cocycle [31, 74, 32, 39, 76, 87].
- ◦ Canonical bases of Lusztig–Kashiwara [40, 41], [49]–[51].
- ◦ Ringel's theory of Hall polynomials [71, 89].
- ◦ Frenkel–Lepowski cocycle.
- ◦ Littelmann path model [46]–[48], [56].

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○ Explicit constructions that involve representations of the classical algebras; see, for example, [10, 11, 13, 18, 21].

Thus, the present paper has no exclusive claim for novelty. On the other hand, a specific choice of signs of the structure constants is a *primary* technical difficulty in a vast majority of all usual calculations with Lie algebras and algebraic groups. While being inferior to many of the above approaches in terms of elegance, our present approach is Pareto optimal: it is much more *elementary* than most of the previous methods, and much more efficient for calculations *by hand* than all other elementary methods. Obviously, for computer calculations it is easier to use the habitual inductive algorithms, and it is more *efficient* to use a formula of Frenkel–Kac–Lepowski type; see [31, 32, 39, 76, 87].

Introduction

Let Φ be a reduced irreducible root system of rank l. Mostly, we are interested in the case when all roots of the root system Φ have the same length. We say that such systems are *simply-laced*, as opposed to the *multiply-laced* systems, which have roots of different lengths. Further, let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a fundamental root system of Φ; its elements are called *fundamental* (or, sometimes, *simple*) roots. Let Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively, and let $Q(\Phi) = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_l$ be the root lattice. We always employ the same numbering of the fundamental roots, as in [2]. As usual, $W = W(\Phi)$ denotes the Weyl group of the system Φ; w_α is the root reflection with respect to $\alpha \in \Phi$, whereas $s_i = w_{\alpha_i}$, $1 \le i \le l$, are the fundamental reflections. By $l(w)$ we denote the length of $w \in W$ relative to the Coxeter generators $s_1, \dots, s_l$.

Next, let $G = G(\Phi, R)$ be the *simply connected* Chevalley group of type Φ over a commutative ring R with 1. All necessary definitions can be found, for example, in [1, 14, 23, 55, 77, 78]; see also [4, 5], [84]–[86], and [89], where one can find many additional references. We fix a split maximal torus $T = T(\Phi, R)$ in G. As usual, $X = \{x_\alpha(\xi),\ \xi \in R\}$, $\alpha \in \Phi$, denotes the unipotent root subgroup in G, elementary with respect to T. For $\varepsilon \in R^*$, the element $w_\alpha(\varepsilon)$ is determined by the formula $w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_{-\alpha}(-\varepsilon^{-1})x_\alpha(\varepsilon)$. The *extended Weyl group* $\widetilde{W} = \widetilde{W}(\Phi)$, known also as the *Tits–Demazure group* [56], is generated by all $w_\alpha(1)$, $\alpha \in \Phi$. It is an extension of the usual Weyl group W by an elementary Abelian group of order 2^l.

As usual, $P(\Phi)$ is the lattice of integral weights of the root system Φ, while $P_{++}(\Phi)$ denotes the cone of *dominant* integral weights. Recall that every weight $\omega \in P_{++}(\Phi)$ is a nonnegative integral linear combination of the fundamental weights $\varpi_1, \dots, \varpi_l$. We fix a dominant weight $\omega \in P_{++}(\Phi)$, and denote by $V = V(\omega)$ the Weyl module of the group G with the highest weight ω. The corresponding representation $G \longrightarrow \mathrm{GL}(V)$ will be denoted by $\pi = \pi(\omega)$. By $\Lambda(\pi) = \Lambda(\omega)$ we denote the set of weights of the representation π *with multiplicities*. In V, an *admissible base* v^λ, $\lambda \in \Lambda(\omega)$, consisting of weight vectors can be chosen (i.e., v^λ is in fact a vector of weight λ, when λ is viewed as a weight in the usual sense, *without multiplicities*) such that the action of the root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, is described by matrices whose entries are polynomials in ξ with integer coefficients.

Weight diagrams were introduced by Evgenii Borisovich Dynkin in 1951 and were widely used by his students, in the first place by Ernest Borisovich Vinberg. However, they never appeared in their printed works, as Dynkin himself told me, mainly because it was technically difficult to incorporate pictures in a mathematical text in the pre-computer era! Roughly speaking, the *weight diagram* of a representation π is a marked — or, in Kashiwara's terminology, a *coloured* — oriented graph whose vertices correspond to the weights of π, usually with multiplicities, and in which two vertices λ and μ are joined by an arrow marked i (= of *colour* i), directed from μ to λ, if $\lambda - \mu = \alpha_i$ is the ith

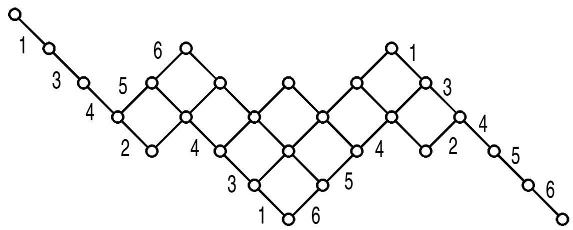

FIGURE 1. (E_6, ϖ_1).

fundamental root. The arrows are usually omitted in the notation, positive directions being right to left and bottom-up. Obviously, when the weights λ and/or μ are multiple, we must specify how exactly we interpret this identity.

In general, a solution of the multiplicity problem is *highly nontrivial* and was only obtained in the celebrated papers by George Lusztig and Masaki Kashiwara in the framework of quantum group theory [49]–[51], [40]–[42], where the correct analogs of the weight diagrams are the crystal graphs. Shortly thereafter, Peter Littelmann proposed a staggeringly beautiful elementary (but highly artful) approach towards the construction of these graphs, the *path model* [46]–[48]; see the brilliant expositions of this circle of ideas in [38, 53, 54]. However, for small representations, where almost all weights are extremal, it is not at all difficult to come up with a precise definition.

Weight diagrams are often used also as a shorthand notation of the corresponding *weight graphs*. As above, the vertices of the weight graph are all weights of the representation π, but now the edges correspond to all positive roots, and not merely to the fundamental roots, as in the case of weight diagrams. In other words, the weights λ and μ are joined by an arrow marked $\alpha \in \Phi^+$, directed from μ to λ, if $\lambda - \mu = \alpha$. Weight graphs often satisfy very strong regularity properties and are encountered in an enormous number of publications in combinatorics, finite geometries, and sphere packings; see [20] and the references there. For example, two of the most interesting graphs for us are the weight graphs of types (E_6, ϖ_1) and (E_7, ϖ_7). The corresponding weight diagrams are reproduced in Figures 1 and 2. They are commonly called the *Schläfli graph* and the *Gosset graph*, respectively. They first appeared in the theory of algebraic surfaces. Namely, consider the surface obtained from the projective plane $\mathbb{P}^2$ by blowing up l points in general position. Then the Schläfli graph described the configuration of the 27 lines arising on this surface for $l = 6$, whereas the Gosset graph describes the configuration of the 56 nonsingular rational curves with negative self-intersection for $l = 7$; see, for example, [16]. In the present paper we are mostly interested in the following two simplest special cases:

- *microweight* representations, when all weights are extremal;
- *adjoint* representations of simply-laced root systems, when the only nonextremal weight is the zero weight.

For the first of these cases, the multiplicity problem does not arise at all, since here all weights have multiplicity 1. Thus, for a microweight representation, the weight diagram (which, in this case, is isomorphic to the coset adjacency diagram of the Weyl group $W(\Phi)$ modulo the stabilizer of the highest weight) coincides with Kashiwara's *crystal graph*. This is explained by the fact that the microweight representations *do not depend on the temperature* or, in Kashiwara's language, *microweight representations do not melt*. The solution for adjoint representations, also in the multiply-laced case, proposed in [8, 84], can be described as follows. The set $\Lambda(\omega)$ is the union of Φ and the set consisting of the zero weights $\widehat{\alpha}_1, \dots, \widehat{\alpha}_l$, numbered by the simple roots. A vertex $\widehat{\alpha}_i$ corresponding to a

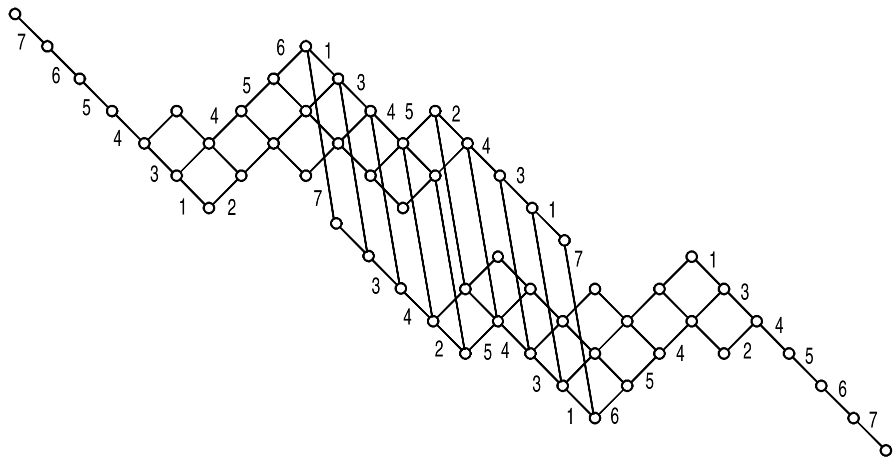

FIGURE 2. $(\mathrm{E}_7, \overline{\varpi}_7)$.

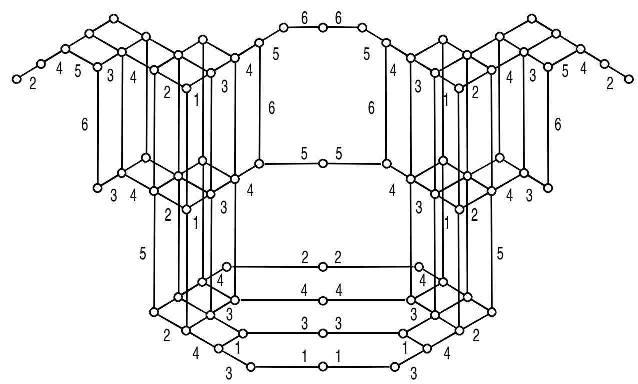

FIGURE 3. $(\mathrm{E}_6, \overline{\varpi}_2)$.

zero weight is only joined to α_i and to $-\alpha_i$. As was shown by Robert Marsh [52], this construction *precisely* leads to the crystal graph. For the classical groups, this statement also follows from the explicit construction of the crystal graphs in the paper [42] by Kashiwara and Nakashima, or from the paper [46] by Littelmann.

In Figures 3–5 we reproduce the weight diagrams of the adjoint representations of types E_l, $l = 6, 7, 8$. Diagrams of *all* microweight and adjoint representations, as well as the short-root representations, are collected in [59]. These diagrams arise in various areas of mathematics, starting with representation theory, the theory of algebraic groups, invariant theory, up to algebraic K-theory, algebraic geometry, differential geometry, combinatorics, finite geometry, nonassociative algebras, mathematical physics, etc. A thorough discussion of weight diagrams and their use, as well as many further related references can be found in [19, 26, 28, 35, 36, 57, 58], [64]–[68], [73, 79, 89, 90, 84, 86, 89]. Lately, weight diagrams have again and again amply demonstrated their utmost usefulness in the structure theory of algebraic groups itself [3]–[7], [88], and also in algebraic geometry [12, 24, 30, 37, 59, 60, 63, 82] and combinatorics [29, 69, 70, 80], [90]–[92]. To convince oneself of this, it suffices to compare the brilliant paper [59] by Sergei Nikolenko and Nikita Semenov with those of the preceding papers on Schubert calculus on exceptional varieties, where weight diagrams had not been used!

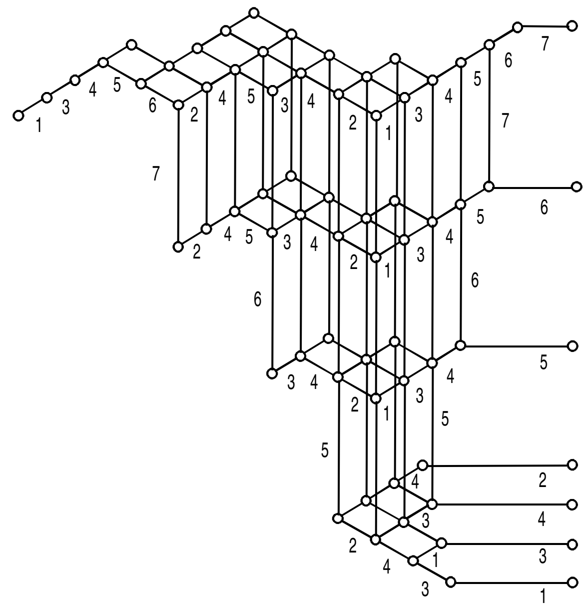

FIGURE 4. (E_7, ϖ_1).

Initially weight diagrams emerged as a purely combinatorial object describing the Bruhat order or the Duflo order on the cosets of the Weyl group $W = W(\Phi)$ modulo a parabolic subgroup $W_J = \langle s_j,\ j \in J\rangle$ for some $J \subseteq \Pi$ = a *first look*. Obviously, the vertices of such a diagram correspond to the Weyl orbit of the highest weight, i.e., to the *extremal* weights alone, so that the multiplicity problem as such does not arise. For example, the vertices of the diagram (E_7, ϖ_7), reproduced in Figure 2, can be interpreted as cosets $W/W_J = W(\mathrm{E}_7)/W(\mathrm{E}_6)$. Two cosets w_1W and w_2W are joined by a bond marked i if $w_1W = s_iw_2W$. The positive direction is designated by the increase of weight, $\lambda(w_2) = \lambda(w_1) + 1$. The results by Robert Proctor [67, 68] show that for a microweight representation this diagram is precisely the Hasse diagram of the order opposite to the induced Bruhat order on W/W_J; see, for example, [66] or [84].

Soon thereafter it became clear that weight diagrams are extraordinarily convenient for describing the action of a Lie algebra/algebraic group *up to signs* = a *second look*; see [55, 78]. Namely, the vectors from $V = V(\omega)$ can be conceived as follows. Fix an admissible base v^λ, $\lambda \in \Lambda(\omega)$ of $V(\omega)$. Then a vector $a = \sum a_\lambda v^\lambda \in V$, $a_\lambda \in R$, $\lambda \in \Lambda(\omega)$, which is written simply as $a = (a_\lambda)$ in the sequel, is depicted by a marked graph obtained from the weight diagram (Φ, ω) by placing a_λ to the vertex that corresponds to the weight λ. The root unipotent $x_{\alpha_i}(\xi)$, $\alpha_i \in \pm\Pi$, $\xi \in R$, corresponding to a positive/negative fundamental root acts along the edges of the diagram marked i in the positive/negative direction. For an arbitrary root $\alpha \in \Phi$, the action of $x_\alpha(\xi)$ is described by directed paths with marks determined by the coefficients in the expansion of α with respect to the fundamental roots, in any order. For example, for a microweight representation π we have $x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha}$, where the $c_{\lambda\alpha}$ are called the structure constants of the representation π. See [55, 78, 66, 84, 86, 89] for further details.

Yet, in the papers by Hideia Matsumoto [55] and Michael Stein [78], as well as in subsequent publications, all calculations were performed *up to sign*. Let us quote the corresponding passage from [78]: "It should be noted that in describing the elementary transformations no attempt to fix signs has been made." For many applications this approach is perfectly adequate, but some subtler calculations with exceptional groups

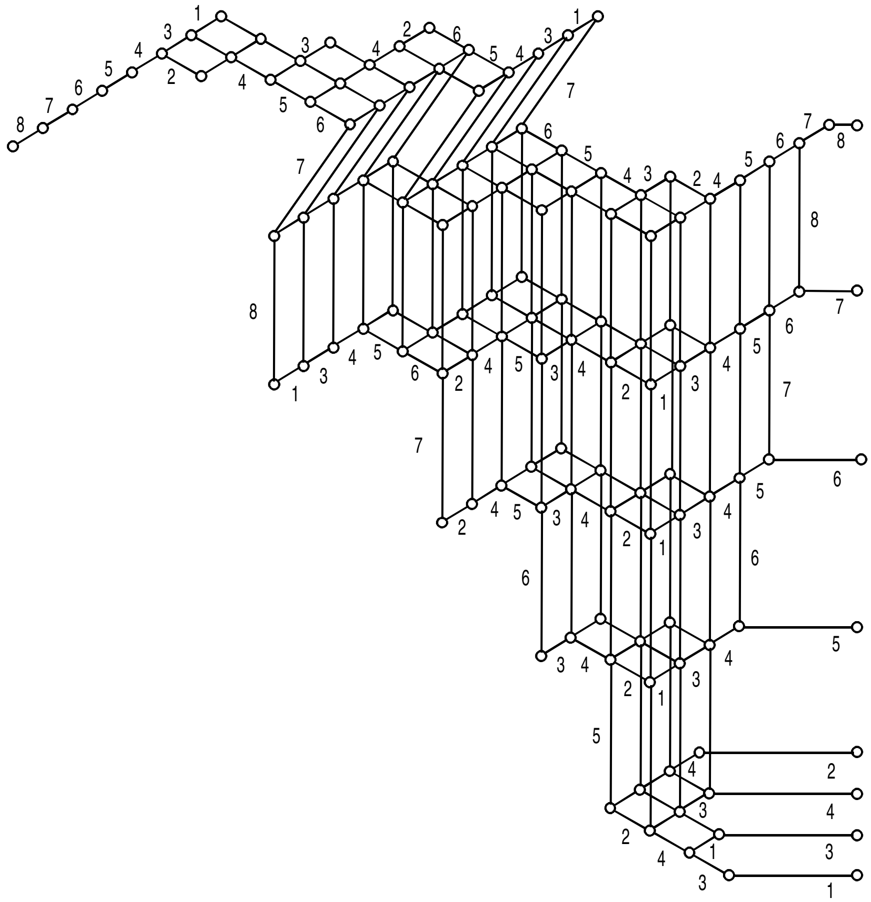

FIGURE 5. $(\mathrm{E}_8, \overline{\varpi}_8)$.

require a specific knowledge of signs. In such cases they were usually calculated by means of elegant, but not very practical, inductive procedures, similar to the famous algorithm by Jacques Tits [83, 23]. This algorithm gives a correct answer in a positive Chevalley base straight away. However, to calculate the signs for exceptional groups by hand, using such algorithms is next to impossible. That is why calculations were performed on a computer, and tabulated; see, for example, [22, 33, 34, 87].[1]

For the adjoint representation, there is an extremely simple and computationally efficient algorithm that determines the signs, proposed by Igor Frenkel and Victor Kac [31] and Graham Segal [74] in the simply-laced case, and by Claus Ringel [71] in general; see also the description of this algorithm in [32, 39, 76, 89, 87]. However, a *natural* choice of the bilinear form and the sign function does not lead to the positive Chevalley base. Only in the papers by A. Cohen, R. Griess, and I. Lisser [25, 27] and of the present author [87] has it been shown how the structure constants can be obtained in a positive Chevalley base by this algorithm. Namely, [27] modified the bilinear form, whereas [87]

[1] Here I quote *some* papers I know where these calculations were performed in the context of algebraic groups. There is no doubt that similar calculations were carried out — and probably published — *dozens of times* by experts in Lie algebras, Lie groups, differential geometry, algebraic geometry, representation theory and mathematical physics. Here I make no attempt to adduce a complete bibliography in this direction. First, compiling a similar bibliography is well beyond human abilities; second, a paper in a research journal could not accommodate it anyway.

modified the sign function. Frenkel and Lepowsky generalized this algorithm to other types of representations, including the microweight ones.

In the present paper we propose to *see* the action structure constants directly in the weight diagram. Namely, relatively recently it has been discovered that, as a matter of fact, weight diagrams *completely* describe the action *including signs* = a *third look*. Indeed, the crystal bases possess remarkable positivity properties. Using these properties, it is possible to completely recover the action by the diagram. For the cases of (E_6, ϖ_1) and (E_7, ϖ_7), such an algorithm and an elementary proof, by using only a realization of these modules in the unipotent radicals of parabolic subgroups, were produced in [86]. In the present paper we continue this line and propose the corresponding algorithms and direct *elementary* proofs, which do not involve quantum deformations, for all microweight and some adjoint modules; see Theorems 1–3. We discuss also some further closely related results. The (much more technically demanding) case of two root lengths is considered in a joint paper by the present author and Eugene Plotkin.

We illustrate the basic idea of our method by the example of the weight diagram (E_6, ϖ_1), depicted in Figure 1. First of all, we fix a *positive* Chevalley base in the Lie algebra of type E_6 and a *crystal* base of the module $V = V(\varpi_1)$. In this case all nonzero action structure constants for fundamental and negative fundamental roots are equal to $+1$. Assume we wish to precisely determine the matrix of a unipotent root element $x_\alpha(\xi)$ in this base, for a root α that is not necessarily fundamental. Consider, for instance, the root $\alpha = \alpha_2 + \alpha_4$. The action of the element $x_\alpha(\xi)$ is described by all possible paths with marks $\{2, 4\}$, in any order. In the diagram we see that these marks occur three times in the order $(2, 4)$ and three times in the order $(4, 2)$. It turns out that this precisely means that three of the nonzero structure constants $c_{\lambda\alpha}$ are equal to $+1$, while the other three are equal to -1. In §§5–7 we give a precise description of a procedure that allows us to determine the sign of a structure constant by the sequence of marks in a diagram.

We believe that the results of the present paper may be of independent interest, since they give an *entirely elementary approach towards calculations in exceptional groups, accessible to an undergraduate student*. However, the main applications we have in mind are the *geometry of exceptional groups* and an *a priori* proof of the theorem on *decomposition of unipotents* for the corresponding representations. Namely, in the paper [9], we proposed a new approach to the proof of the main structure theorems for Chevalley groups over commutative rings, based on the geometry of their minimal modules. Roughly, the idea of this approach can be described as follows. All calculations necessary for the proof of the main structure theorems can be organized in such a way that only the calculations of the following two types are used: first, *elementary* calculations, based exclusively on Steinberg relations; second, calculations touching only one column or one row of a matrix (for the reasons that we do not discuss here, they were mentioned in [9] as *stable* calculations). A specific method to reduce general calculations to the elementary and the stable ones, proposed in [9], was called there *decomposition of unipotents*.

For classical groups *in vector representations*, the realization of the program outlined in [9] does not present significant technical complications, even in much more general situations; see, in particular, [81, 84] and the references therein. At the same time, it turned out that to verify all the details for exceptional groups was not nearly as easy as we initially believed. To convince oneself that the signs of the action and equations agree, we had to make essential use of extensive computer calculations. The sketch of the proof we outlined in [84] is incomplete exactly in this respect.[2] In the paper [86] we gave an *a priori* proof of the main theorem on decomposition of unipotents for *microweight*

[2]As a curiosity, we mention that in the papers [4, 5, 7], as a further development of the method of decomposition of unipotents, we proposed a new geometric approach to the proof of structure theorems

representations of the groups of types E_6 and E_7. The results of the present paper form part of the proof of a similar theorem for other microweight and adjoint representations. Such a proof is the goal of a series of joint papers by the present author and Plotkin, [89, 3], *Fortsetzung folgt*. A description of the entire project can be found in the papers [84, 89, 81] by the author, Plotkin, and Stepanov. There, as well as in [5, 85, 86, 66], one can find many additional references pertaining to Chevalley groups over rings, weight diagrams, structure constants, etc.

The present paper is organized as follows. In §§1 and 2 we prove some elementary facts on structure constants of Lie algebras and their representations. In §§3 and 4 we produce two elementary proofs of the existence of a crystal base in a microweight representation, Theorem 1. In §§5–7 we prove Theorems 2 and 3, which address the problem of how to read off the signs directly from the weight diagrams, both for microweight representations and for adjoint representations of types E_6, E_7, and E_8. Finally, in §8 we list some obvious corollaries to our results.

§1. Structure constants of Lie algebras

In the present section we recall some standard facts related to Lie algebras.

1. Identities for structure constants. We fix a Chevalley base $\{e_\alpha,\ \alpha \in \Phi,\ h_i,\ 1 \leq i \leq l\}$ in the simple complex Lie algebra L of type Φ, where the e_α are root nilpotents, whereas the $h_i = h_{\alpha_i}$ are the corresponding fundamental coroots (recall that $h_\alpha = [e_\alpha, e_{-\alpha}]$). The set $\{e_\alpha,\ \alpha \in \Phi\}$ is often called a *Chevalley system*. The structure constants $N_{\alpha\beta}$, $\alpha, \beta \in \Phi$, are determined by the relation $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$. By the definition of a Chevalley base, all $N_{\alpha\beta}$ are integers. It will be convenient for us to require additionally that $N_{\alpha\beta} > 0$ for all extra-special pairs; see [83, 23, 33, 87]. For simply-laced systems, this last requirement means exactly that $N_{\alpha_i\beta} = 1$ whenever $\alpha_i + \beta \in \Phi^+$ has the following property: if $\alpha_j + \gamma = \alpha_i + \beta$ for a fundamental root α_j and a positive root γ, then $j > i$. A Chevalley base with this property is said to be *positive*. We always assume that our Chevalley base is positive.

We summarize the properties of the structure constants to be used permanently in the sequel; see, in particular, [83, 23], or the further references contained in [87, 89]. First,

$$N_{\alpha\beta} = N_{-\beta,-\alpha} = -N_{-\alpha,-\beta} = -N_{\beta\alpha}. \tag{C1}$$

The following two properties are slightly less obvious:

$$\frac{N_{\alpha\beta}}{(\gamma,\gamma)} = \frac{N_{\beta\gamma}}{(\alpha,\alpha)} = \frac{N_{\gamma\alpha}}{(\beta,\beta)} \tag{C2}$$

if $\alpha + \beta + \gamma = 0$, and, finally,

$$\frac{N_{\alpha\beta}N_{\gamma\delta}}{(\alpha+\beta,\alpha+\beta)} + \frac{N_{\beta\gamma}N_{\alpha\delta}}{(\beta+\gamma,\beta+\gamma)} + \frac{N_{\gamma\alpha}N_{\beta\delta}}{(\gamma+\alpha,\gamma+\alpha)} = 0 \tag{C3}$$

if $\alpha + \beta + \gamma + \delta = 0$. For a simply-laced system, these formulas simplify:

$$N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha} \quad \text{if} \quad \alpha + \beta + \gamma = 0, \tag{C4}$$

$$N_{\alpha\beta}N_{\gamma\delta} + N_{\beta\gamma}N_{\alpha\delta} + N_{\gamma\alpha}N_{\beta\delta} = 0 \quad \text{if} \quad \alpha + \beta + \gamma + \delta = 0. \tag{C5}$$

As a matter of fact, it is easy to observe that in the last formula at most two summands can be nonzero, so that it is equivalent to the following piece of a 2-cocycle:

$$N_{\beta\gamma}N_{\alpha,\beta+\gamma} = N_{\alpha+\beta,\gamma}N_{\alpha\beta}. \tag{C6}$$

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for Chevalley groups over commutative rings, *the proof from the Book*, which makes no reference whatsoever to the signs of structure constants or equations! At the same time, a further development of this method in [88], necessary to accommodate the case of Steinberg groups, again assumes ability to control at least some signs.

In the sequel we always consider the set Φ^+ of positive roots in a regular order that is lexicographic on roots of a given height, the *height lexicographic order*. This means that $\alpha < \beta$ when $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ or when $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$ and $m_1m_2\dots m_l > n_1n_2\dots n_l$, where $\alpha = \sum m_i\alpha_i$, $\beta = \sum n_i\alpha_i$. Since all m_i, n_i are *digits*, the last inequality can be interpreted as an inequality for integers; see [87] for the details.

2. Positive Chevalley base. Now we pass to the adjoint representation. In this case ω is simply the maximal root of Φ with respect to the fixed order (with the only exception of $\Phi = \mathrm{A}_l$, it is in fact a fundamental weight, for instance, $\omega = \varpi_2$ for $\Phi = \mathrm{E}_6$, $\omega = \varpi_1$ for $\Phi = \mathrm{E}_7$, and $\omega = \varpi_8$ for $\Phi = \mathrm{E}_8$); $V = L_{\mathbb{Z}} \otimes R$, where $L_{\mathbb{Z}}$ is the Chevalley algebra of type Φ over $\mathbb{Z}$ (i.e., the integral linear span of the Chevalley base e_α, h_i in L). By the very definition, the Chevalley base $e_\alpha = e_\alpha \otimes 1$, $h_i = h_i \otimes 1$ is admissible. In the sequel we usually write simply e_α instead of v^α and, respectively, h_i instead of $v^{\widehat{\alpha}_i}$. The action of the root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, in this base is described by the following well-known formulas; see, for instance, [23].

(A1) On the $\mathfrak{sl}_2$-subalgebra $\langle e_\alpha, h_\alpha, e_{-\alpha}\rangle$, where $h_\alpha = [e_\alpha, e_{-\alpha}]$ is the coroot corresponding to α, the root element $x_\alpha(\xi)$ acts as follows:

$$x_\alpha(\xi)e_{-\alpha} = e_{-\alpha} + \xi h_\alpha - \xi^2 e_\alpha, \quad x_\alpha(\xi)h_\alpha = h_\alpha - 2\xi e_\alpha, \quad x_\alpha(\xi)e_\alpha = e_\alpha.$$

Observe that the element h_α is an integral linear combination of h_i's.

(A2) If the roots α and β are linearly independent, we still have

$$x_\alpha(\xi)h_\beta = h_\beta - 2(\alpha,\beta)/(\beta,\beta)e_\alpha.$$

The formula expressing $x_\alpha(\xi)e_\beta$ is slightly more complicated in general, and we reproduce it only in the case of simply-laced systems we are interested in. Here

$$x_\alpha(\xi)e_\beta = e_\beta + N_{\alpha\beta}\xi e_{\alpha+\beta}.$$

The positivity property of a Chevalley base and anticommutativity can be expressed as follows:

(B1) $N_{\alpha_i\beta} = 1$ if α_i is a simple root with the smallest index such that there is an edge labeled with i and entering $\alpha_i + \beta$;

(B2) $N_{\alpha_i\alpha_j} = -1$ if $i > j$.

It turns out that these rules, together with the fact that structure constants form a piece of a 2-cocycle, already suffice to restore the action of the elements $x_\alpha(\xi)$ in the case where α is a fundamental or a negative fundamental root. First of all, observe that without loss of generality we may assume that β is positive. Indeed, $N_{\alpha\beta} = -N_{-\alpha,-\beta}$, so that in the sequel we always assume that $\beta > 0$. Moreover, by (C2), we have $N_{\alpha\beta} = N_{-\alpha,\beta+\alpha}$, so that in what follows we may assume that $\alpha = \alpha_i$ is a fundamental root. The following result is a minor variation on the theme of Tits [83]; see also [23]. Nevertheless, for the reader's convenience we reproduce its proof, since it is very short and contains an *algorithm* on how to determine the structure constants $N_{\pm\alpha_i\beta}$ by the diagram.

Lemma 1. *The structure constants $N_{\pm\alpha_i\beta}$ are completely determined by the properties* (B1)*,* (B2)*, and* (C6)*.*

Proof. As we already observed above, without loss of generality we may assume that $\beta \in \Phi^+$. The structure constants $N_{\alpha_i\beta} = N_{-\alpha_i,\beta+\alpha_i}$ are easily calculated by induction. If the height $\mathrm{ht}(\beta)$ of a root β equals 1, these constants are completely determined by the properties (B1) and (B2). In turn, if $\mathrm{ht}(\beta) \geq 2$, then the sign of $N_{\alpha_i\beta}$ can be calculated as follows. Assume that for all roots γ of smaller height the structure constants $N_{\alpha_j\gamma}$ are already known for all j. If there is a unique edge that enters the vertex corresponding to $\alpha_i + \beta$ in the positive direction, or if i is the smallest index j such that there is an edge labeled with j and entering $\alpha_i + \beta$, then $N_{\alpha_i\beta} = 1$ by (B1). If there are several edges entering $\alpha_i+\beta$ and j is the smallest index such that there is an edge labeled with j that enters $\alpha_i+\beta$ in the positive direction, then we already know that $N_{\alpha_j,\beta+\alpha_i-\alpha_j} = 1$. Consider the root $\beta+\alpha_i-\alpha_j$. By assumption, $(\beta+\alpha_i, \alpha_i) = (\beta+\alpha_i, \alpha_j) = 1$; thus, should $\alpha_i + \alpha_j$ be a root, the equality $(\beta + \alpha_i, \alpha_i + \alpha_j) = 2$ would imply $\beta + \alpha_i = \alpha_i + \alpha_j$, and finally, $\beta = \alpha_j$, which is impossible because $\mathrm{ht}(\beta) \geq 2$. Now, it is clear that $(\alpha_i, \alpha_j) = 0$ and thus $(\beta + \alpha_i - \alpha_j, \alpha_i) = 1$, so that $\gamma = \beta - \alpha_j$ is also a root. At this point, the constants $N_{\alpha_j\gamma}$ and $N_{\alpha_j\gamma}$ are already known by the inductive hypothesis. Together with (C6), this determines $N_{\alpha_i\beta}$. This completely determines the signs of the structure constants for fundamental and negative fundamental roots. □

§2. Action structure constants

Here we collect some elementary facts on minimal representations of Lie algebras, which will be used repeatedly in the sequel.

1. Basic representations. We keep the notation from the introduction. Recall that a representation π is *basic* if the Weyl group $W = W(\Phi)$ has a unique orbit on the set $\Lambda^*(\omega)$ of nonzero weights. Thus, either all weights are extremal (= the *microweight case*), or a unique nonextremal weight is the zero weight 0. It is easy to see, compare [55], that this definition is equivalent to the following: if the difference $\alpha = \lambda - \mu$ of two nonzero weights λ and μ of the representation π is a (fundamental) root, then $\mu = w_\alpha\lambda$. This immediately implies the following result; see [86, Lemma 2].

Lemma 2. *Let π be a basic representation, and let $\alpha, \beta, \alpha + \beta \in \Phi$ be such that $\lambda, \lambda + \alpha, \lambda + \beta, \lambda + \alpha + \beta \in \Lambda(\pi)$ for some weight λ. Then at least one of the following is true*:

- *one of the weights $\lambda, \lambda + \alpha, \lambda + \beta, \lambda + \alpha + \beta$ is zero*;
- *$(\alpha, \beta) = 0$.*

We need the following restatement of this lemma.

Lemma 3. *Let $\alpha, \beta \in \Phi$, and let $\lambda \in \Lambda(\pi)$ be such that λ, $\lambda + \alpha$, $\lambda + \beta$, $\lambda + \alpha + \beta$ are nonzero weights of a basic representation π. Assume that at least one of the following is true*:

- *at least one of the roots α, β is long*;
- *all three roots α, β, $\alpha + \beta$ are short*;
- *both α and β are fundamental.*

Then $\alpha + \beta$ is not a root.

Proof. If $\alpha + \beta \in \Phi$, then $(\alpha, \beta) = 0$ by the previous lemma. This is impossible if all three roots $\alpha, \beta, \alpha + \beta$ have the same length. If exactly one of the roots α and β, say α, is long, then $|\alpha + \beta| > |\alpha|$, which is impossible. Thus, we can assume that both roots α and β are short and their sum $\alpha + \beta$ is a long root. However, a unique irreducible root system where orthogonal short fundamental roots can exist is the system of type C_l. But in this case any two such roots generate a subsystem of type $\mathrm{A}_1 + \mathrm{A}_1$, and not of type B_2. This shows that $\alpha + \beta$ cannot be a root. □

The statement of the lemma fails without the assumption that all occurring weights are nonzero. Indeed, $\lambda = 0$ and any two nonorthogonal fundamental roots α, β provide a counterexample. It also fails for nonzero weights without the assumption that both short roots α and β are fundamental. Indeed, consider the representation (C_l, ϖ_1), $l \geq 2$, and take $\lambda = -e_1$, $\alpha = e_1 - e_2$, $\beta = e_1 + e_2$. Then all four weights λ, $\lambda + \alpha = -e_2$, $\lambda + \beta = e_2$, $\lambda + \alpha + \beta = e_1$ are nonzero, but nevertheless, $\alpha + \beta = 2e_1 \in \Phi$. The same examples show that the analog of Lemma 3 in [86] fails either without the assumption that the representation π is microweight, or when Φ is multiply laced. However, for the cases we are interested in, the following analog of [86, Lemma 3] is true, which suffices for our purposes.

Lemma 4. *Let π be a basic representation, and let $\alpha, \beta, \alpha+\beta \in \Phi$, where either one of the roots α, β is long, or all three roots α, β, and $\alpha+\beta$ are short.*

- *If for a given weight $\lambda \in \Lambda(\pi)$, $\lambda \neq 0$, we have $\lambda+\alpha+\beta \in \Lambda(\pi)$ and $\lambda+\alpha+\beta \neq 0$, then either $\lambda+\alpha$ or $\lambda+\beta$ is in $\Lambda(\pi)$, but not both.*
- *If π is a microweight representation, the roots α, β are short, $\alpha+\beta$ is long, and $\lambda+\alpha+\beta \in \Lambda(\pi)$, then $\lambda+\alpha, \lambda+\beta \in \Lambda(\pi)$.*

Proof. For microweight representations this is almost exactly Lemma 3 in [86]. In fact, if one of the roots α, β is long and their sum is a root, or all three roots $\alpha, \beta, \alpha+\beta$ are short, then α and β cannot be orthogonal. Thus, by Lemma 2, the sums $\lambda+\alpha$ and $\lambda+\beta$ cannot be weights of the representation π simultaneously. On the other hand, if one of them is not a weight, then $w_\alpha(\lambda) = \lambda, \lambda-\alpha$ and $w_\alpha(\lambda) = \lambda, \lambda-\beta$, or, what is the same, $(\lambda, \alpha) \geq 0$ and $(\lambda, \beta) \geq 0$. But then also $(\lambda, \alpha+\beta) \geq 0$, which contradicts our assumption $w_{\alpha+\beta}(\lambda) = \lambda+\alpha+\beta$.

Now, let π be an adjoint representation. Since all roots Φ have the same length and $\alpha+\beta \in \Phi$, we conclude that α and β cannot be orthogonal. Thus, in the case where all four weights $\lambda, \lambda+\alpha, \lambda+\beta, \lambda+\alpha+\beta$ are distinct from 0, we can argue as above, with the use of Lemma 2. This means that it only remains to consider the case where one of the weights $\lambda+\alpha$ or $\lambda+\beta$ is zero (they cannot be both zero because $\alpha \neq \beta$). Let, for example, $\lambda+\alpha = 0$ or, what is the same, $\lambda = -\alpha$. Then $\lambda+\alpha+\beta = \beta$, and thus, $\lambda+\beta = \beta-\alpha$ cannot be a weight of the representation π because $\lambda(\pi)$ cannot contain α-chains $\beta+\alpha, \beta, \beta-\alpha$ with $\beta \neq 0$.

Finally, let α, β be short roots such that their sum is a long root. We scale the inner product in such a way that $(\gamma, \gamma) = 2$ for every long root γ (for C_l this differs from the usual scaling by $\sqrt{2}$). Then, since π is microweight, we have $(\lambda, \alpha+\beta) = \pm 1$ and, since $\lambda+\alpha+\beta$ is a weight, we have $(\lambda, \alpha+\beta) = -1$. This is possible only if $(\lambda, \alpha) = (\lambda, \beta) = -1/2$, i.e., if $\lambda+\alpha, \lambda+\beta \in \Lambda(\pi)$. ☐

2. Action structure constants. For the representations in question, it is especially easy to explicitly describe the action of the root unipotents $x_\alpha(\xi)$. In particular, if λ is a nonzero weight distinct from $-\alpha$ for the adjoint case, then

$$x_\alpha(\xi) v^\lambda = \begin{cases} v^\lambda + c_{\lambda\alpha} \xi v^{\lambda+\alpha} & \text{if } \lambda+\alpha \in \Lambda(\pi), \\ v^\lambda & \text{otherwise.} \end{cases} \tag{A1}$$

The constants $c_{\lambda\alpha} = \pm 1$ are called *action structure constants*; recall that in the adjoint case all roots have the same length. One of our main goals in the present paper is the construction of an algorithm that makes it possible to read off the signs of the action structure constants from the diagram.

We start with some obvious properties of these constants. Denote by $\Lambda^*(\pi) = \Lambda^*(\omega)$ the set of all nonzero weights of the representation π (for microweights we have $\Lambda^*(\omega) = \Lambda(\omega)$). In the case where V is realized as an internal Chevalley module, it is natural to select the elements $v^\gamma = x_\gamma(1) U_r(2)$ as a base of V. In this base we have $c_{\gamma\alpha} = N_{\alpha\gamma}$, so that (C1), (C2), and $|\gamma| = |\alpha+\gamma|$ imply that $c_{\gamma\alpha} = N_{\alpha\gamma} = N_{-a-\gamma,\alpha} = N_{-\alpha,\alpha+\gamma} = c_{\gamma+\alpha,-\alpha}$. The same is true in general.

Lemma 5. *For all $\alpha \in \Phi$ and all $\lambda \in \Lambda^*(\pi)$ such that $\lambda+\alpha \in \Lambda^*(\pi)$, we have*

$$c_{\lambda\alpha} = c_{\lambda+\alpha,-\alpha}.$$

Proof. For an adjoint representation this follows from the above formula. Therefore, we let π be a microweight representation. The action of the extended Weyl group W on V is monomial relative to any admissible base. In particular, this means that $w(1)v^{\lambda} = \pm v^{\lambda+\alpha}$. On the other hand, calculating $w(1)v^{\lambda}$ by means of the expression $w_{\alpha}(1) = x_{\alpha}(1)x_{-\alpha}(-1)x_{\alpha}(1)$ and applying (A1), we get $w(1)v^{\lambda} = (1 - c_{\lambda+\alpha,-\alpha}c_{\lambda\alpha})v^{\lambda} + c_{\lambda\alpha}v^{\lambda+\alpha}$. Comparing these two expressions, we see that $c_{\lambda+\alpha,-\alpha}c_{\lambda\alpha} = 1$. □

Lemma 5 was a restatement of (C2); we shall need also an appropriate analog of (C3). Let us look at the *small squares* in the weight diagram of the representation π. They correspond to the situation where $\lambda, \lambda+\alpha, \lambda+\beta, \lambda+\alpha+\beta \in \Lambda(\pi)$ for some weight λ and some *fundamental* roots α and β. When we speak of an adjoint representation or of internal Chevalley modules, (C6) can be rewritten in the form $c_{\lambda\alpha}c_{\lambda+\alpha,\beta} = c_{\lambda\beta}c_{\lambda+\beta,\alpha}$, or in other words, the product of the structure constants corresponding to the four sides of a small square, equals 1. This remains valid for all basic modules.

Lemma 6. *Let π be a basic representation, and let $\lambda, \lambda+\alpha, \lambda+\beta, \lambda+\alpha+\beta \in \Lambda(\pi)$ for some weight λ and for some fundamental roots α and β. Then*

$$c_{\lambda\alpha}c_{\lambda+\alpha,\beta} = c_{\lambda\beta}c_{\lambda+\beta,\alpha}.$$

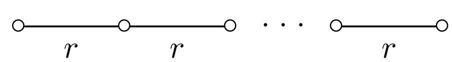
Proof. Indeed, Lemma 3 asserts that $\alpha+\beta$ is not a root, so that $x_{\alpha}(1)$ and $x_{\beta}(1)$ commute. Now, calculating $x_{\alpha}(1)x_{\beta}(1)v^{\lambda} = x_{\beta}(1)x_{\alpha}(1)v^{\lambda}$ in two ways, we get the required identity. □

§3. Crystal base of a microweight representation

Here we give an elementary construction of a crystal base of a microweight representation. For such a base, all signs of the action structure constants for fundamental and negative fundamental roots are equal to $+1$. The author has no doubt whatsoever that such a construction should be well known to experts, but has never seen a published proof.

1. Restriction of a microweight representation. Our proof critically depends on the following two observations, due to Chrys Parker and Gerhard Röhrle [61, 62].

Lemma 7. *Let π be a microweight representation, α a fundamental root of Φ, and $K = \Phi \setminus \{\alpha\}$. Then the diagram obtained from the weight diagram (Φ, π) by contracting each K-connected component to a point is an α_r-chain*:



This lemma rephrases Corollary 3.7 in [61]. In fact, in [62], one can find a much more general result, asserting that for any (not necessarily maximal!) subset $K \subseteq \Pi$ the poset of K-connected components of the weight diagram is ranked by the K-level. In [61, 62] this claim was stated in a geometric language, but actually this is simply a property of double cosets $W_K \setminus W/W_J$, where W_J denotes the stabilizer of the highest weight ω of the representation π in the Weyl group. The vertices of the weight diagram correspond to the cosets W/W_J, and the K-connected components correspond to the double cosets $W_K \setminus \mathrm{W}/W_J$; compare [66].

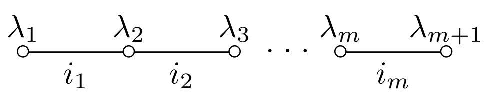
Lemma 8. *Let π be a microweight representation, $K \subseteq \Pi$ a set of fundamental roots, and Φ_K a subsystem in Φ generated by K. Then the restriction of π to $G(\Phi_K, R) \leq G(\Phi, R)$ decomposes into a direct sum of microweight modules of type Φ_K (possibly, some of them are trivial).*

This lemma is a special case of Theorem 2.4 in [61], where much more detailed information can be found. In fact, these irreducible components correspond to the K-connected components of the weight diagram or, what is the same, to the double cosets $W_K \backslash \mathrm{W} / W_J$, and in [61, 62] one can find recipes on how to calculate their highest weights, dimensions, etc.

2. Crystal base of a microweight representation. We approach our first principal result. Certainly, the statement below is well known (this is a *very* special case of results by Lusztig and Kashiwara), but all the usual proofs of this fact are *much* less elementary. The base we construct in the proof of this theorem is called a *crystal base*. An elementary proof for the microweight modules that can be realized as internal Chevalley modules was proposed in [86], and in the next section we reproduce the basic construction that works in that case. The proof reproduced below relies exclusively on Lemmas 5–8.

Theorem 1. *Let (V, π) be a microweight representation. Then there exists an admissible base v^λ, $\lambda \in \Lambda(\pi)$, such that $c_{\lambda\alpha} = +1$ for all fundamental and negative fundamental roots α.*

Proof. We prove this theorem by induction on $l = \mathrm{rk}(\Phi)$. As a base of induction we take the case where the weight diagram is a chain (in fact, it would even suffice to assume that $\Phi = A_1$, but it is more convenient to treat a slightly more general case, so that we could later repeat the same argument word for word, for the induction step). By Lemma 6, it suffices to verify that $c_{\lambda\alpha} = +1$ for fundamental roots α. Now, let the weight diagram of the representation π be a chain, where $\lambda_1, \dots, \lambda_{m+1}$ are the weights



of the representation π (some of the indices $i_1, i_2, \dots, i_m$ may be equal, but this does not influence our arguments). Next, let $v^1, v^2, \dots, v^{m+1}$ be an admissible base of V. Assume that it is already proved that $c_{\lambda\alpha} = +1$ for all weights $\lambda = \lambda_j$, $h < j \leq m+1$, but $c_{\lambda\alpha} = -1$ for $\lambda = \lambda_h$ and $\alpha = \alpha_{i_{h-1}}$. In this case we modify the base v^i as follows: we leave all vectors v^j, $h \leq j \leq m+1$, unchanged, and change the signs of all vectors v^j, $1 \leq j \leq h-1$. Then all action structure constants do not change, except for a unique constant $c_{\lambda\alpha}$ with $\lambda = \lambda_h$ and $\alpha = \alpha_{i_{h-1}}$, which now equals $+1$. Proceeding like that, we can eventually make all action structure constants for fundamental roots to be $+1$.

Now, assume that the theorem has already been proved for all microweight representations of groups of rank less than l. Let Φ_K be the subsystem of Φ generated by the set $K = \Pi \setminus \{\alpha_l\}$ of fundamental roots distinct from α_l. Consider the restriction of π to the subgroup $G(\Phi_K, R) \leq G(\Phi, R)$ corresponding to the embedding of root systems $\Phi_K \leq \Phi$. By Lemma 8, all irreducible components of this restriction are themselves microweight modules corresponding to a root system of smaller rank, so that by the inductive hypothesis we may assume that $c_{\lambda\alpha} = +1$ for all fundamental roots except, possibly, $\alpha = \alpha_l$.

It only remains to understand why all structure constants $c_{\lambda\alpha}$ for $\alpha = \alpha_l$ can be made equal to $+1$ as well. Indeed, let us look at how two K-components can be joined. Take two weights λ, μ such that $\lambda + \alpha, \mu + \alpha$ are also weights. Suppose that λ, μ lie in the same connected K-component X or, what is the same, that $\lambda - \mu \in Q(\Phi)$. Then $(\lambda + \alpha) - (\mu + \alpha) = \lambda - \mu \in Q(\Phi)$ also lie in the same connected K-component, say Y. We show that $c_{\lambda\alpha} = c_{\mu\alpha}$. In fact, if $\beta = \lambda - \mu \in K$ is a fundamental root, then this is precisely Lemma 7 (by the inductive hypothesis, $c_{\mu\beta} = c_{\mu+\alpha,\beta} = +1$). In general, there exists a chain $\lambda = \lambda_1, \lambda_2, \dots, \lambda_m = \mu$, such that all consecutive differences $\lambda_i - \lambda_{i-1}$ belong to K and, thus, $c_{\lambda\alpha} = c_{\lambda_1\alpha} = c_{\lambda_2\alpha} = \cdots = c_{\lambda_m\alpha} = c_{\mu\alpha}$. This means that we can speak of the sign of $c_{X\alpha}$. Now, take the graph obtained from the weight diagram π by contracting every K-connected component to a point. Lemma 7 asserts that this graph is in fact a chain, so that we can repeat the argument used to prove the base of induction word for word. The only difference is that now we must simultaneously change the signs of all base vectors corresponding to the weights in the same K-component. But this does not change the signs of the action structure constants within the component, which finishes the proof of the theorem. □

§4. Internal Chevalley modules

As usual, we denote by $U = U(\Phi, R)$ the subgroup generated by all root subgroups X_α corresponding to the positive roots $\alpha \in \Phi^+$, and by $B = TU$ the standard Borel subgroup. For any set $J \subseteq \Pi$ of fundamental roots we denote by Φ_J the subsystem of Φ generated by J, while $P_J \geq B$ denotes the corresponding standard parabolic subgroup in G, L_J stands for its Levi factor (which is a reductive group of type Φ_J), and $U_J = \langle X_\alpha,\ \alpha \in \Phi^+ \setminus \Phi_J \rangle$ stands for its unipotent radical. Recall that P_J is the semidirect product $P_J = L_J U_J$, where L_J acts on the normal subgroup U_J by conjugation. We consider also the opposite parabolic subgroup P_J^- with the same Levi subgroup L_J and with the opposite unipotent radical $U_J^- = \langle X_{-\alpha},\ \alpha \in \Phi^+ \setminus \Phi_J \rangle$. In the case where $R = K$ is a field, $L_J = T\langle X_\alpha,\ \alpha \in \Phi_J \rangle$.

In fact, we are mostly interested in the case of maximal parabolic subgroups. We fix r, $1 \leq r \leq l$, and set $J = J_r = \Pi \setminus \{\alpha_r\}$. The corresponding maximal parabolic subgroup, its Levi subgroup, and its unipotent radical will be denoted by P_r, L_r, and U_r, respectively. Setting $\Sigma_r = \Phi^+ \setminus \Phi_J$, we denote by $\Sigma_r(h)$ the set of all roots $\alpha \in \Sigma_r$ of α_r-level h. In other words, $\Sigma_r(h) = \{\alpha = \sum m_i \alpha_i,\ m_r = h\}$. Clearly, $\Sigma_r(h) = \varnothing$ for $h > h_r$, where h_r denotes the coefficient with which α_r enters into the expansion of the maximal root. The union of all $\Sigma_r(h)$, $h \geq k$, will be denoted by $\widetilde{\Sigma}_r(k)$; we have $\widetilde{\Sigma}_r(1) = \Sigma_r$. Set $U_r(h) = \prod X_\alpha$, where the product is taken over all roots $\alpha \in \widetilde{\Sigma}_r(h)$, in arbitrary order. The Chevalley commutator formula implies that $U_r(h)$ is indeed a subgroup in U_r, and $U_r(1) = U_r$. Actually, $U_r(h)$ almost always coincides with the hth term of the lower central series of U_r, but we have defined $U_r(h)$ as the product of all root subgroups corresponding to the roots of level at least h precisely with the purpose of not having to discuss the possible small exceptions.

The structure of the consecutive factors $U_r(h)/U_r(h+1)$ as L_r-modules is thoroughly studied; see, for instance, [17, 72].

Lemma 9. *For any r, $1 \leq r \leq l$, and h, $1 \leq h_r$, the quotient group*

$$U_r(h)/U_r(h+1) \cong \bigoplus X_\alpha, \qquad \alpha \in \Sigma_r(h),$$

is the Weyl module of L_r with the highest weight ω, where ω is the element of $\Sigma_r(h)$ of highest height.

Here we talk of the direct sum of Abelian groups, whereas in the definition of $U_r(h)$ we used the product of subgroups of a given group. The above statement is a trivial special case of the results of [17], where we replaced irreducible modules by Weyl modules, to avoid any discussion of the very bad primes.

Since we are interested not in the ambient group G itself, but rather in the commutator subgroup[3] $G_r = [L_r, L_r]$ of a Levi factor L_r, we shall slightly modify our notation. In

[3] We emphasize that, when speaking of the normalizers, commutator subgroups, central series, etc. of the group G, we mean the commutator subgroup of L_r *in the sense of algebraic groups*, and not the commutator subgroup of its group of points. This distinction becomes crucial for groups over rings, where G_r always differs from the abstract commutator subgroup by a K_1-functor.

the sequel Φ denotes what was denoted by Φ_J, whereas the root system of the ambient group will be denoted by Δ. When r is fixed, we write simply Σ, $\Sigma(h)$ instead of Σ_r, $\Sigma_r(h)$. Thus, $\Delta = \Phi \cup \Sigma \cup (-\Sigma)$. The largest (and in most cases most interesting) is the factor $\Sigma/\Sigma(2)$. The following two cases are of particular importance:

- U_r is Abelian, i.e., $U_r(2) = 1$,
- U_r is extra-special, i.e., $U_r(2) = X_\delta$, where δ is the maximal root of Φ.

As a most interesting example of an Abelian unipotent radical, we can mention the 27-dimensional module for E_6, which can be realized as U_7 in the group of type E_7, whereas a most interesting instance of a module realized within an extra-special radical is the 56-dimensional module for E_7, isomorphic to $U_8/U_8(2)$ in the group of type E_8.

Now we are in a position to rethink the proof of Theorem 1. For the modules that can be realized as internal Chevalley modules, *usually* we can take $v^\lambda = x_\lambda(1)$ or $v^\lambda = x_\lambda(1)U_r(2)$ as a crystal base, where a weight λ is interpreted as a root of the ambient root system. In some cases, the sign should be alternated by height, $v^\lambda = x_\lambda((-1)^\lambda)$; see [86] for details. For polyvector representations of a group of type A_l, the usual base of decomposable polyvectors $e^{i_1} \wedge \cdots \wedge e^{i_m}$, $i_1 < \cdots < i_m$ is a crystal base. At the same time, for other classical groups, the usual choice of signs in the vector representation consists in setting half of the signs for fundamental roots α equal to -1. This leads to a much more transparent rule for the signs of other roots than a crystal base. But for groups of types E_6 and E_7 we do not see any other *natural* choice of signs.

§5. The sign rule: Inductive step

In the present section we establish an analog of Theorem 2 of [86]. We wish to find a sign rule that would enable us to find the structure constant directly from the diagram.

1. The sign rule: inductive step. Our algorithm is based upon the following inductive step, which rephrases Lemma 5 of [86]. Obviously, now we can assume neither that all structure constants $c_{\lambda\beta}$ for fundamental roots β are positive, nor the validity of Lemma 4, and this leads to a somewhat more complicated formula. Consider the following situation: $\alpha \in \Phi \subseteq \Pi$, $\lambda \in \Lambda(\pi)$, and β is the first fundamental root such that $\alpha - \beta \in \Phi$. With the exception of the case where both roots β and $\alpha - \beta$ are short, while the root α is long, Lemma 4 implies that if $\lambda + \beta \notin \Phi$, then $\lambda + \alpha - \beta \in \Phi$. On the other hand, if $\lambda + \beta \in \Phi$, then $\lambda + \alpha - \beta \notin \Phi$. In all cases except for $\Phi = B_l$ and $\Phi = C_l$, only this situation can possibly occur, and we single it out as a separate proposition.

Proposition 1. *Let $\alpha \in \Phi \subseteq \Pi$, $\lambda \in \Lambda(\pi)$, and let β be the first fundamental root such that $\alpha - \beta \in \Phi$. If either at least one of the roots $\beta, \alpha - \beta$ is long, or all three roots $\alpha, \beta, \alpha - \beta$ are short, then*

$$c_{\lambda\alpha} = \begin{cases} c_{\lambda+\alpha-\beta,\beta} c_{\lambda,\alpha-\beta} & \textit{if } \ \lambda + \alpha - \beta \in \Lambda(\pi), \quad \lambda + \beta \notin \Lambda(\pi), \\ c_{\lambda\beta} c_{\lambda+\beta,\alpha-\beta} & \textit{if } \ \lambda + \alpha - \beta \notin \Lambda(\pi), \quad \lambda + \beta \in \Lambda(\pi). \end{cases}$$

Proof. We invoke the Chevalley commutator formula to express $x_\alpha(1)$ via $x_\beta(1)$ and $x_{\alpha-\beta}(1)$. If all three roots $\alpha, \beta, \alpha - \beta$ have the same length, then, by our choice of structure constants, $x_\alpha(1) = [x_\beta(1), x_{\alpha-\beta}(1)]$. This means that

$$x_\alpha(1)v^\lambda = x_\beta(1)x_{\alpha-\beta}(1)x_\beta(-1)x_{\alpha-\beta}(-1)v^\lambda.$$

Calculating the right-hand side of this expression, we immediately get the desired conclusion.

Now, suppose that exactly one of the roots β and $\alpha - \beta$ is long. Again, by the very choice of structure constants, the Chevalley commutator formula implies that

$$x_\alpha(1) = x_\gamma(\pm 1)[x_\beta(1), x_{\alpha-\beta}(1)],$$

where $\gamma = \alpha + \beta$ if β is short, and $\gamma = 2\alpha - \beta$ if $\alpha - \beta$ is short. The sign of $x_\gamma(\pm 1)$ can be made precise, but it does not play any role here, because it does not influence the subsequent calculation of $c_{\lambda\alpha}$. Now, exactly the same calculation as in the case of roots of equal length leads to one of the first two identities in the statement of the proposition. Since $\gamma > \alpha$, the presence of the factor $x_\gamma(\pm 1)$ cannot influence the value of the constant $c_{\lambda\alpha}$. This finishes the proof of the proposition. □

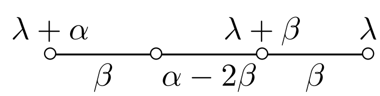
2. The sign rule: the case of short roots. It remains to consider the case where both roots β and $\alpha - \beta$ are short, whereas α is long. For the representations we consider, this case only occurs for $\Phi = \mathrm{B}_l$ and $\Phi = \mathrm{C}_l$. In this case we cannot employ the Chevalley commutator formula for $x_\beta(1)$ and $x_{\alpha-\beta}(1)$ because of the coefficient 2, which pops up there. The Chevalley commutator formula for $x_\beta(1)$ and $x_{\alpha-2\beta}(1)$ has the form

$$[x_\beta(1), x_{\alpha-2\beta}(1)] = x_{\alpha-\beta}(N_{\beta,\alpha-2\beta})x_\alpha(N_{\beta,a-2\beta}N_{\beta,\alpha-\beta}/2);$$

see, for instance, [14, 23, 76]. In the C_l case, for any long root $\alpha \neq \alpha_l$ there exists a unique fundamental root β such that $\alpha - \beta$ is a root, and this root β is short. Moreover, β is also the first fundamental root γ such that $(\alpha - \beta) - \gamma$ is a root. Thus, in this case, both signs on the right-hand side are equal to $+1$. In the case of B_l, there is a unique situation where α is long and β is short provided β is the first fundamental root such that $\alpha - \beta$ is a root. Specifically, this appears if and only if $\alpha = \alpha_{l-1} + 2\alpha_l = e_{l-1} + e_l$ and $\beta = \alpha_l$. But in this case, β is no longer the first fundamental root γ such that $(\alpha - \beta) - \gamma$ is a root, so that $\gamma = \alpha_{l-1}$. Thus, in this case, both signs on the right-hand side are equal to -1, and a straightforward calculation shows that $c_{\lambda\alpha} = +1$. In the case of $\Phi = \mathrm{C}_l$, the Chevalley commutator formula can be rewritten in the form $x_\alpha(1) = x_{\alpha-\beta}(-1)[x_\beta(1), x_{\alpha-2\beta}(1)]$. In this case the sign of $c_{\lambda\alpha}$ can be determined as follows.

Proposition 2. *Suppose* $\Phi = \mathrm{C}_l$, $\alpha \in \Phi \setminus \Pi$, $\lambda \in \Lambda(\pi)$, *and let* β *be the first fundamental root such that* $\alpha - \beta \in \Phi$. *If both roots* β *and* $\alpha - \beta$ *are short, while* α *is long, then* $c_{\lambda\alpha} = -c_{\lambda+\beta,\alpha-2\beta}$.

Proof. We obtain the necessary formula by carrying through the same calculation as above, for the chain of additions. Recall that $c_{\lambda\beta} = c_{\lambda+\alpha-\beta,\beta} = +1$ for a microweight



representation, and at the moment when $x_{\alpha-\beta}(-\varepsilon)$ enters the game the summand $v^{\lambda+\beta}$ is already cancelled, so that $x_{\alpha-\beta}(-\varepsilon)$ cannot influence the coefficient of $v^{\lambda+\alpha}$. This finishes the proof of the proposition. □

§6. The sign rule: Microweight representations

> Nevsky prospect has the following striking property: it consists of a space for public circulation; it is surrounded by numbered houses; the numbering agrees with the order of houses — and the search of the requisite house is immensely simplified.
>
> Andrei Belyi. *Petersburg*

To pass from fundamental roots to the general case, we need the notion of the *canonical string* associated with a root $\alpha \in \Phi$. The canonical string $\mathrm{cn}(\alpha)$ of a fundamental root $\alpha = \alpha_i$ equals i. If $\mathrm{ht}(\alpha) \geq 2$, then the canonical string $\mathrm{cn}(\alpha)$ of a root α is defined as follows. Let i be the smallest index such that $\alpha - \alpha_i$ is a root. Then $\mathrm{cn}(\alpha)$ is obtained by preassigning i to $\mathrm{cn}(\alpha-\alpha_i)$. For example, $\mathrm{cn}(\alpha_1+\alpha_3)=13$, while the canonical strings of the maximal roots in E_6, E_7, and E_8 are reproduced below:

$$\mathrm{cn}\binom{12321}{2} = 24315423456,$$

$$\mathrm{cn}\binom{234321}{2} = 13425431654234567,$$

$$\mathrm{cn}\binom{2465432}{3} = 87654231435426543176542345678.$$

Now, we are all set to pass to the second main theme of the present paper. First, we expound a slightly easier algorithm, which allows us to calculate signs for microweight representations. In [86], this algorithm was stated slightly differently, in terms of nasty indices, as we do in the next section. Here, we describe it in terms of the following notion.

Let $\lambda\in\Lambda(\pi)$, $\alpha\in\Phi$. Then the *phantom sum* of a weight λ and a positive root α is the weight $\lambda\boxplus\alpha$ defined inductively as follows.

• For a fundamental root α, we set $\lambda\boxplus\alpha=\lambda+\alpha$ if $\lambda+\alpha\in\Lambda(\pi)$, and $\lambda\boxplus\alpha=\lambda$ otherwise.

• If α is not fundamental and β is the first fundamental root such that $\alpha-\beta$ is a root, then $\lambda\boxplus\alpha=(\lambda\boxplus\beta)\boxplus(\alpha-\beta)$.

The *phantom difference* $\lambda\boxminus\alpha$ is defined similarly. Namely, if α is a fundamental root, then $\lambda\boxminus\alpha=\lambda-\alpha$ if $\lambda-\alpha\in\Lambda(\pi)$, and $\lambda\boxminus\alpha=\lambda$ otherwise. If α is not a fundamental root, $\lambda\boxminus\alpha=(\lambda\boxminus\beta)\boxminus(\alpha-\beta)$.

If Φ is a simply-laced system and $\lambda\boxplus\alpha\in\Lambda(\pi)$, then Lemma 4 exactly means that $(\lambda+\alpha)\boxminus\beta-\lambda\boxplus\beta=\alpha-\beta$. If Φ is multiply laced, a root α is long, and a root β is short, then $(\lambda+\alpha)\boxminus\beta-\lambda\boxplus\beta=\alpha-2\beta$.

In the sequel we use two distinct notions of the distance between two weights $\lambda,\mu\in\Lambda(\pi)$. One of them is the usual distance $h(\lambda,\mu)$ between two weights in the weight diagram. In other words, if λ and μ are two comparable weights and $\lambda\geq\mu$, then $h(\lambda,\mu)=\mathrm{ht}(\lambda-\mu)$. In general, $h(\lambda,\mu)=h(\lambda,\nu)+h(\nu,\mu)$, where ν is the supremum of λ and μ. For microweight representations, $h(\lambda,\mu)$ can be described as the length of a shortest element w of the Weyl group such that $w\lambda=\mu$. Also, we use the distance $d(\lambda,\mu)$ between λ and μ in the weight graph. This distance equals 0 if $\lambda=\mu$; it equals 1 if $\lambda-\mu$ is a root; it equals 2 if $\lambda-\mu$ is a sum of two roots, etc.

For simplicity sake, we state the following result only for simply-laced root systems. Observe that only this case is of practical interest: for each of $\Phi=B_l$ and $\Phi=C_l$ there exists exactly one microweight representation. For C_l this is the natural vector representation, where the choice of signs is obvious, whereas for B_l the signs are easily determined by the signs for D_l. Moreover, the following theorem can easily be extended to these cases, by using Proposition 2 alongside with Proposition 1.

Theorem 2. *Let Φ be a simply-laced root system, and v^λ a crystal base of a microweight representation π. Then for any $\alpha\in\Phi$ and any $\lambda\in\Lambda(\pi)$ such that $\lambda+\alpha\in\Lambda(\pi)$ we have*

$$c_{\lambda\alpha}=(-1)^{h(\lambda\boxplus\alpha,\lambda)-1}.$$

Proof. We prove the theorem by induction on α. If the root α is fundamental, then the formula is true (indeed, $c_{\lambda\alpha}=+1$, while $h(\lambda\boxplus\alpha,\lambda)=1$). If α is not fundamental and β is the first fundamental root such that $\alpha-\beta$ is a root, then by Proposition 1 we have $c_{\lambda\alpha}=c_{\lambda,\alpha-\beta}$ if $\lambda+\beta$ is not a weight; in other words, $\lambda\boxplus\beta=\lambda$ and $c_{\lambda\alpha}=-c_{\lambda+\beta,\alpha-\beta}$ if $\lambda+\beta$ is a weight, i.e., $\lambda\boxplus\beta=\lambda+\beta$. By the inductive hypothesis, in the first case we have

$$c_{\lambda\alpha}=c_{\lambda,\alpha-\beta}=(-1)^{h(\lambda\boxplus(\alpha-\beta),\lambda)-1},$$

but

$$\lambda \boxplus \alpha = (\lambda \boxplus \beta) \boxplus (\alpha - \beta) = \lambda \boxplus (\alpha - \beta),$$

and this proves the required formula.

In the second case, by the inductive hypothesis we have

$$c_{\lambda\alpha} = -c_{\lambda+\beta,\alpha-\beta} - (-1)^{h(\lambda \boxplus (\alpha-\beta),\lambda)-1}.$$

But here

$$\lambda \boxplus \alpha = (\lambda \boxplus \beta) \boxplus (\alpha - \beta) = (\lambda + \beta) \boxplus (\alpha - \beta),$$

whence

$$\begin{aligned} h(\lambda \boxplus \alpha, \lambda) &= h((\lambda + \beta) \boxplus (\alpha - \beta), \lambda) \\ &= p((\Lambda + \beta) \boxplus (\alpha - \beta), \Lambda + \beta) - h(\Lambda + \beta, \Lambda) \\ &= h((\lambda + \beta) \boxplus (\alpha - \beta), \lambda + \beta) - 1, \end{aligned}$$

and this proves the required formula in this case as well. □

Norman Wildberger [91, 92] independently proposed a construction of Chevalley groups of types E_6 and E_7 in microweight representations, similar to that discussed in [86], but somewhat more complicated. Namely, he considered not the weight diagram, but the weight graph, and thus was forced to simultaneously specify the signs of the action constants for all root elements, rather than only for the fundamental ones, as we did; see [92], where this was actually accomplished for G_2.

§7. The sign rule: Adjoint representation

The host, overall, was a nice person. He used to refer to the month of May as Adarmapagon, to June as Hardat, to July as Terma, to August as Mederme, and to his own house as Eleusis. He would sometimes sign his name by digits, as follows: 15, 18, 4, 10, 5, 12, 19, 10, 5, 8, 7, 7, and conclude it with a flourish.

Constantin Vaginov. *The works and days of Svistonov*

For adjoint representations we continue to assume that Φ is a simply-laced system. Even in this case the sign rule is somewhat more complicated than for microweight representations; see [89] for the general case.

We could prove an analog of Theorem 1, but then we would be obliged to consider two distinct Chevalley bases, because the root elements $x_\alpha(\xi)$ and the weight vectors should be regarded with respect to *distinct* bases. For one base it is impossible to satisfy the condition that all signs here be equal to $+1$ for the fundamental roots; this immediately follows from anticommutativity. Indeed, if the two roots α and β are fundamental, then, obviously, $N_{\alpha\beta} = -N_{\beta\alpha}$, so that the corresponding action constants cannot be both positive. We believe that simultaneous consideration of two distinct bases would introduce unnecessary technical complications. Therefore, we renounce the requirement that all action structure constants for fundamental roots be equal to $+1$.

First, we observe that without loss of generality we may assume that both roots α and β are positive. Indeed, we already know that we may assume that one of them is positive. Let, for example, $\alpha > 0$, whereas $\beta < 0$. We can suppose that $\alpha + \beta > 0$ (if this is not the case, use the relation $N_{\alpha\beta} = N_{-\beta,-\alpha}$). Now, $N_{\alpha\beta} = N_{\beta,-\alpha-\beta} = N_{\alpha+\beta,-\beta}$, where both roots $\alpha + \beta$ and $-\beta$ are positive.

Now, to calculate $N_{\alpha\beta}$ for all $\alpha, \beta > 0$, we proceed as follows. Let $i_1 \dots i_h$ be the canonical string of the root α, where $h = \mathrm{ht}(\alpha)$. We are looking for a path in the *negative* direction that starts at $\lambda_1 = \beta + \alpha$, finishes at $\mu_1 = \beta$, and has marks $i_1, \dots, i_h$ *in the same order*. Such a path may fail to exist. If there exists an edge labeled with

i_1 and emanating from λ_1 in the negative direction, we set $\lambda_2 = \lambda_1 - \alpha_{i_1}$, $\mu_2 = \mu_1$, and say that $i = i_1$ is *nasty* for β if $N_{\alpha_i,\lambda_2} = -1$. Otherwise, there exists an edge labeled with i_1 and departing from μ_1 in the positive direction. In this case we set $\lambda_2 = \lambda_1$, $\mu_2 = \mu_1 + \alpha_{i_1}$, and say that $i = i_1$ is *nasty* for β if $N_{\alpha_i,\mu_1} = 1$. Next, if there exists an edge labeled with i_2 and emanating from λ_2 in the negative direction, we set $\lambda_3 = \lambda_3 - \alpha_{i_2}$, $\mu_3 = \mu_2$; otherwise we set $\lambda_3 = \lambda_2$, $\mu_3 = \mu_2 + \alpha_{i_2}$. In this case, we say that $i = i_2$ is nasty provided the same formulas as above are satisfied with λ_2 replaced by λ_3 and μ_1 replaced by μ_2. We proceed like that until we arrive at the end of the canonical string. Finally, let $n(\alpha, \beta)$ be the number of labels in the canonical string of α that are nasty for β. Then Proposition 1 immediately implies the following result.

Theorem 3. *Let e_α, $\alpha \in \Phi$, be a positive Chevalley system. Then for any two positive roots α, β we have*

$$N_{\alpha\beta} = (-1)^{n(\alpha,\beta)}.$$

Talking of the structure constants of the adjoint representation, it is impossible not to mention an extraordinary elegant method proposed (in the case of simply-laced systems) by Frenkel and Kac [31]. Under this approach, $N_{\alpha\beta}$ is calculated in the form $N_{\alpha\beta} = \varepsilon(\alpha)\varepsilon(\beta)\varepsilon(\alpha+\beta)(-1)^{f(\alpha,\beta)}$, where $f(\alpha, \beta)$ is a certain bilinear form on the weight lattice $Q(\Phi)$, and $\varepsilon(\alpha)$ is a sign function, equal to 0 if α is not a root and to ± 1 if $\alpha \in \Phi$ is a root. Unfortunately, the obvious choices of the bilinear form and the sign function do not lead to a positive Chevalley base. To get a positive base, one should modify either the form f, see [25, 27], or the sign function ε; see [87]. Ringel [71] suggested a remarkable generalization of this construction to multiply-laced systems, in terms of Hall polynomials. An algorithm resulting from Ringel's construction was reproduced in [89]. The action structure constants for some representations of exceptional groups were tabulated in various papers; see the references in [6].

§8. Some corollaries

In this section we state some corollaries and complements to our main results.

1. Exceptional symmetries. In Proposition 1 of [86] we observed that for the representations (E_6, ϖ_1) and (E_7, ϖ_7) the constants $c_{\lambda\alpha}$ possess the following remarkable properties.[4]

Theorem 4. *Let π be one of the representations (E_6, ϖ_1) or (E_7, ϖ_7).*

- *If δ is the maximal root, then $c_{\lambda\delta} = +1$ for all $\lambda, \lambda + \alpha \in \Lambda(\pi)$.*
- *For any root α, the parity of the number of negative structure constants $c_{\lambda\alpha}$ is opposite to the parity of $\mathrm{ht}(\alpha)$.*
- *For any $\alpha \in \Phi$ the last nontrivial action constant $c_{\lambda\alpha}$ equals $+1$.*

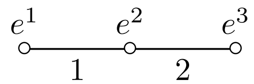
These properties do not generalize to other microweight representations. An *a priori* explanation of the additional symmetry in the cases of (E_6, ϖ_1) and (E_7, ϖ_7) can be stated as follows.

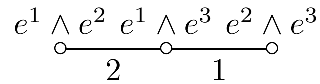
Lemma 10. *Let μ be the lowest weight of the representation (E_6, ϖ_1) or (E_7, ϖ_7), and let $i_1 \dots i_h$ be the canonical string of the maximal root $\delta \in \Phi$. Then for all $j = 1, \dots, h$ the partial sum $\mu + \alpha_{i_j} + \dots + \alpha_{i_h}$ is a weight of the representation π.*

Proof. This is a straightforward verification using Figures 1 and 2. □

[4]Initially, these properties were observed during computer experiments with Mathematica 2.0 for DEC RISC.

However, in this form this is completely wrong already for (E_6, ϖ_6) and (D_l, ϖ_1), $l \geq 3$. In fact, already $\Phi = \mathrm{A}_2$ is a counterexample. In the Lie algebra $\mathfrak{sl}(3, R)$, consider the Chevalley system consisting of the standard matrix units e_{ij}, $1 \leq i \neq j \leq 3$. Then the standard base e^1, e^2, e^3 is a crystal base of the vector representation $V(\varpi_1)$, and $e^1 \wedge e^2, e^1 \wedge e^3, e^2 \wedge e^3$ is a crystal base of the dual bivector representation $V(\varpi_2)$. Denote





by μ the minimal weight in each of these cases. Then $c_{\mu\delta} = +1$ in the first case, but $c_{\mu\delta} = -1$ in the second case. Indeed,

$$e_{13}(e^2 \wedge e^3) = e^2 \wedge e^1 = -e^1 \wedge e^2.$$

Thus, the first of these representations is a counterexample to the second item of the theorem, whereas the second one is a counterexample to the first and third items!

2. Action of the maximal root. We have seen that Theorem 4 as stated does not generalize to other microweight representations. However, part of its first item survives in the following form.

Proposition 3. *All nontrivial structure constants $c_{\lambda\delta}$ are equal.*

The proof is based on the following two easy facts. Here, as before, $K = \Pi \setminus \{\alpha_k\}$, where α_k is the fundamental root adjacent to the maximal root in the affine Dynkin diagram.

Lemma 11. *The set of weights $\lambda \in \Lambda(\pi)$ such that $\lambda - \delta \in \Lambda(\pi)$ coincides with the highest K-connected component $X \subseteq \Lambda(\pi)$.*

Proof. First, observe that since ω is a microweight, we have $(\omega, \delta) = +1$. Thus, every weight $\lambda \in X$ in the K-connected component containing ω has the form $\lambda = \omega - x$ for some $x \in Q(\Phi)$. However, $(\lambda, \delta) = +1$ because $(x, \delta) = 0$. This means that $\lambda - \delta = w_\delta \lambda$ belongs to $\Lambda(\pi)$. This shows that $\lambda - \delta \in \Lambda(\pi)$ for all $\lambda \in X$. Conversely, if λ does not belong to X, then λ has the form $\lambda = \omega - x - n\alpha_k$ for some $x \in Q(\Phi)$ and some $n \in \mathbb{N}$, and thus $(\lambda, \delta) = 1 - n \leq 0$, so that $\lambda - \delta$ cannot be a weight. $\square$

The following version of Lemma 6 is proved similarly.

Lemma 12. *Let π be a microweight representation, and let $\Lambda, \lambda+\alpha, \lambda+\beta, \lambda+\alpha+\beta \in \Lambda(\pi)$ for some weight λ and some roots α and β, where the root α is long. Then $c_{\lambda\alpha} c_{\lambda+\alpha,\beta} = c_{\lambda\beta} c_{\lambda+\beta,\alpha}$.*

Proposition 3 immediately follows from the preceding two lemmas and the following two facts: the maximal root is long, and all constants $c_{\lambda\beta}$ for fundamental roots β are equal to $+1$.

3. Action of the extended Weyl group. Since the elements $w_\alpha(\varepsilon)$ are defined in terms of $x_\alpha(\xi)$, it is clear that the results of Subsection 1 completely describe also the action of the extended Weyl group $\widetilde{W}$, and not only of the usual Weyl group W. This action accounts for the signs in many other problems related to G, including, in particular, the description of invariants of G, and now we describe it explicitly. For example, for microweight representations we immediately get the following result.

Lemma 13. *Assume that $\alpha\in\Phi$, $\lambda\in\Lambda(\pi)$, and $\varepsilon\in R^*$. Then*

$$w_\alpha(\varepsilon)v^\lambda=\begin{cases}v^\lambda & \text{if}\quad \lambda\pm\alpha\notin\Lambda(\pi),\\ c_{\lambda\alpha}\varepsilon v^{\lambda+\alpha} & \text{if}\quad \lambda+\alpha\in\Lambda(\pi),\\ -c_{\lambda,-\alpha}\varepsilon^{-1}v^{\lambda-\alpha} & \text{if}\quad \lambda-\alpha\in\Lambda(\pi).\end{cases}$$

Now the action of the extended Weyl group can be determined completely by Theorem 2. However, in many cases it is more convenient to calculate it directly from the diagram, by using Proposition 2. Namely, if α is a fundamental root, then, by Theorem 1, the formula in the preceding lemma simplifies to $w_\alpha(\varepsilon)v^\lambda=\varepsilon v^{\lambda+\alpha}$ if $\lambda+\alpha\in\Lambda(\pi)$, and to $w_\alpha(\varepsilon)v^\lambda=-\varepsilon^{-1}v^{\lambda-\alpha}$ if $\lambda-\alpha\in\Lambda(\pi)$. In other words, the sign does not change when we move along an edge in the positive direction and is inverted when we move in the negative direction. Now, to find wv^λ for an arbitrary $w\in\widetilde{W}$, it suffices to express w as a product of fundamental generators $w_\alpha(1)$, $\alpha\in\Pi$, and apply this rule to every factor.

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