Алгебра и анализ
Том 19 (2007), № 4

St. Petersburg Math. J.
Vol. 19 (2008), No. 4, Pages 545–560
S 1061-0022(08)01009-1
Article electronically published on May 9, 2008

A NEW MEASURE OF GROWTH FOR GROUPS AND ALGEBRAS

WALDEMAR HOŁUBOWSKI

Abstract. The notion of a bandwidth growth is introduced, which generalizes the growth of groups and the bandwidth dimension, first discussed by J. Hannah and K. C. O'Meara for countable-dimensional algebras. The new measure of growth is based on certain infinite matrix representations and on the notion of growth of nondecreasing functions on the set of natural numbers. Two natural operations are defined on the set $\Omega^\star$ of growths. With respect to these operations, $\Omega^\star$ forms a lattice with many interesting algebraic properties; for example, $\Omega^\star$ is distributive and dense and has uncountable antichains.

This new notion of growth is applied in order to define bandwidth growth for subgroups and subalgebras of infinite matrices and to study its properties.

§1. Introduction

The notion of growth plays a very important role in group theory. Let S be a generating set of a group G (S is assumed to be closed under inversion and not containing the identity). The length of a group element g with respect to S is defined as the distance between g and the identity in the Cayley graph of G with respect to S. Let $f(n)$ be the number of elements of G in the ball of radius n centered at the identity. The function f is monotone nondecreasing and can be extended to the nonnegative real numbers. One can define a certain equivalence relation on the set of such monotone nondecreasing functions. The equivalence class of f is called the growth of G. It does not depend on the choice of S; thus, it is an invariant of the group G itself.

For any finitely generated infinite group G, the following trichotomy holds: either G is of polynomial growth, or G is of intermediate growth, or else G is of exponential growth. By Gromov's theorem [Gr], the class of groups of polynomial growth is precisely the class of all virtually nilpotent groups. The class of groups of exponential growth contains, for example, all nonelementary hyperbolic groups [GH]. Tits' alternative [Tits] asserts that among linear groups there are no groups of intermediate growth. However, in [Gri1, Gri2], R. Grigorchuk discovered examples of groups of intermediate growth, thus solving a famous Milnor problem [Mil]. Presently, there is a rich theory of such groups.

An analog of group growth exists also for algebras, and its construction is similar. By an *algebra* we always mean an associative algebra with identity over a field. Let A be a finitely generated algebra over a field $\mathbb{F}$ with a generating set $a_1, \dots, a_m$. We set $V^0 = \mathbb{F}$ and denote by V^n the subspace spanned by all monomials in $a_1, \dots, a_m$ of length n, for all $n \geq 1$. Then $A = \bigcup_{n=0}^\infty A_n$, where $A_n := \mathbb{F} + V + V^2 + \dots + V^n$. Denote by Φ the set of all functions $f : \mathbb{N} \to \mathbb{R}$ that are positive valued and monotone increasing. The function $d_V(n) = \dim_{\mathbb{F}}(A_n)$ depends on the space V generated by $a_1, \dots, a_m$, rather than on these elements themselves, and can be viewed as an element of Φ. For $f, g \in \Phi$ we set $f \leq^\star g$ if and only if there exist $c, m \in \mathbb{N}$ such that $f(n) \leq c \cdot g(nm)$ for almost

2000 *Mathematics Subject Classification.* Primary 15A30, 16P90, 20E07; Secondary 06D99, 16S50, 17B60, 20E15.

Key words and phrases. Growth of groups, growth of algebras, bandwidth growth, string.

all $n \in \mathbb{N}$, and $f \sim g$ if and only if $f \leq^\star g$ and $g \leq^\star f$. For $f \in \Phi$ the equivalence class $[f] \in \Phi/\sim$ is called the *growth* of f. The order $\leq^\star$ induces a partial order $\leq$ on $\Phi/\sim$. The growth $[d_V(n)]$ is an invariant of the algebra A.

Usually, it is hard to find the growth of an algebra explicitly. Moreover, the growth itself is not a suitable tool for considering subalgebras, homomorphic images, and Ore extensions. It is more interesting to study the asymptotic behavior of monotone increasing functions. The notion of dimension most useful for practical purposes is that of the Gelfand–Kirillov dimension, also called the GK-dimension. It is defined as follows [KL]:

$$GK\text{-}\dim(A) = \sup_V \overline{\lim} \log_n d_V(n),$$

where the supremum is taken over all finite-dimensional subspaces V of the algebra A. The GK-dimension can be 0, 1, any real number belonging to the interval $[2, \infty)$ or ∞.

In [HO1, HO2, O], Hannah and O'Meara introduced and studied a new notion of growth for algebras. This notion does not involve the growth of an algebra in terms of generators. Instead, it is based on certain infinite matrix representations. Finite matrices play a prominent role in the representation theory of groups and associative algebras. On the other hand, infinite matrices play quite a minor role. The reason is that in the infinite case there are no obvious analogs of such tools as determinant, trace, or rank. A result by Goodearl, Menal, and Moncasi [GMM] shows that every countable-dimensional algebra A over a field F can be embedded in the algebra $A_{rc}(\infty, F)$ of row- and column-finite infinite matrices over F. In the sequel we fix such an embedding and identify A with its image under this embedding. All nonzero entries of the elements of A are fairly close to the principal diagonal. Now one can ask how far from the principal diagonal such nonzero entries can get. Hannah and O'Meara introduced a quantitative measure for this, the growth curve, which bounds the bandwidth of an algebra element.

We say that $f : \mathbb{N} \to \mathbb{R}^+$, $f(n) = n + h(n)$, is a growth curve for $a = (a_{ij}) \in A$ $(i, j \in \mathbb{N})$ if $a_{nk} = 0 = a_{kn}$ for all $k > f(n)$. An element a has growth at most $h(n)$, or, for short, has $O(h(n))$ growth, provided there exists $c > 0$ such that $c \cdot (n + h(n))$ is the growth curve for a. An algebra A has growth $O(h(n))$ if each element $a \in A$ has growth $O(h(n))$. The growth of order $O(n)$ is said to be *linear*. Now, the main result of [HO2] can be stated as follows: every countable-dimensional algebra A over a field F can be embedded in $A_{rc}(\infty, F)$ as a subalgebra of linear growth. This allows us to define the bandwidth dimension of a countable-dimensional algebra as

$$\inf\{r \in \mathbb{R}, r \geq 0 \mid A \text{ embeds in } A_{rc}(\infty, F) \text{ with } O(n^r) \text{ growth}\}.$$

In [O] it was proved that the bandwidth dimensions of finitely generated algebras over any field F fill the entire unit interval $[0, 1]$. This shows that bandwidth dimension is smoother than GK-dimension. The bandwidth dimension of a free algebra on two generators is 0, whereas its GK-dimension is ∞. Thus, the bandwidth dimension of this algebra takes the smallest possible value, while its GK-dimension is the largest possible. This shows that bandwidth dimension may lead to a new approach, especially to algebras of GK-dimension ∞. Certain bandwidth dimensions express interesting purely ring-theoretic properties of algebras (see [HO2] for details).

Unfortunately, in general this notion of bandwidth dimension seems to be not very useful, because the set

$$B(r) = \{a \in A_{rc}(\infty, F) \mid a \text{ has growth } O(n^r)\}$$

is a subalgebra of $A_{rc}(\infty, F)$ only if $r \in [0, 1]$. A careful search through the *Mathematical Reviews* has revealed no progress on this topic (see also the comments in [KL]). Hannah and O'Meara raised the problem as to whether this notion admits an appropriate generalization to the case of uncountable-dimensional algebras.

In the present paper we solve this problem. Namely, we generalize the notion of bandwidth dimension and growth of groups. This generalization is based on the idea similar to that used to study permutation groups in the paper [Kov] by Koval′chuk. We consider monotone nondecreasing functions from $\mathbb{N} \cup \{\infty\}$ to $\mathbb{N} \cup \{\infty\}$. The classes of these functions under a certain equivalence relation are called growths. We define two natural operations on the set $\Omega^\star$ of growths. With respect to these operations, $\Omega^\star$ forms a lattice with many interesting algebraic properties. In particular, $\Omega^\star$ is dense and distributive and has uncountable antichains.

With every infinite matrix a we associate the lower and the upper bound functions $f(n)$ and $g(n)$. In some sense, the functions f and g give bounds on the width of the *band* along the principal diagonal that contains all nonzero entries of the matrix a. In other words, all nonzero entries are confined between two curves defined by the functions f and g (see Figure 1).

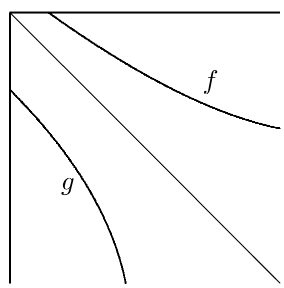

FIGURE 1

These bounds are in natural agreement with the addition and multiplication of matrices. This allows us to state our results in the language of universal algebra (see [BS]), which simultaneously comprises semigroups, groups, rings, algebras, and Lie algebras.

In terms of growths related to lower and upper bounds, we define four subsets of the universal algebra $X_c(\infty, R)$ of column-finite infinite matrices. These subsets are universal subalgebras.

Next, we define two lattices of subalgebras isomorphic to the lattice $\Omega^\star$. We prove that for any subset Y of $X_c(\infty, R)$ there exist smallest growths ω_1, ω_2 such that Y is contained in the subalgebra $X(\omega_1, \omega_2)$ determined by these growths. This observation allows us to give a general definition of a bandwidth growth. In the rest of the paper we study properties of the bandwidth growth.

The paper is organized as follows. In §2 we introduce the set Ω_∞ of monotone nondecreasing functions, the set $\Omega^\star$ of growths, and two binary operations on $\Omega^\star$ under which $\Omega^\star$ becomes a lattice, and formulate some basic properties of this lattice. In §3 we define staircase functions and prove results stated in §2. In the next section we introduce the notion of a bandwidth growth and study its properties. §§5 and 6 contain results and examples which show that our language of growths unifies many previous definitions and results and gives a new insight into the structure of groups and algebras of infinite matrices.

I am very grateful to Prof. V. Sushchanskiĭ for discussion and bringing the paper [Kov] to my attention.

§2. GROWTH OF FUNCTIONS

Let $\mathbb{N}$ be the set of positive integers with the natural order. We extend this order to the set $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ by assuming that $n < \infty$ for all $n \in \mathbb{N}$. The set $P(\mathbb{N})$ of all functions

$f : \mathbb{N}_\infty \to \mathbb{N}_\infty$ forms a semigroup with respect to the composition of functions. By Ω_∞ we denote the subsemigroup of all functions $f \in P(\mathbb{N})$ such that $f(n+1) \geq f(n) > n$ for all $n \in \mathbb{N}$ and $f(\infty) = \infty$. Clearly, the semigroup Ω_∞ has no identity element.

We define an order on Ω_∞ as follows:

$$f \prec g \quad \text{if and only if} \quad \begin{array}{c}\text{there exists } n_0 \text{ such that} \\ f(n) < g(n) \text{ for all } n > n_0.\end{array}$$

As usual, $f < g$ means that $f(n) < g(n)$ for all natural n. Let f^k denote the kth power of f with respect to composition. We write $f \ll g$ if $f^k \prec g$ for all k. Let $M(f) = \{h \in \Omega_\infty : f \ll h\}$.

We define an equivalence relation on the set Ω_∞:

$$f \sim g \quad \text{if and only if} \quad M(f) = M(g).$$

This equivalence relation can be characterized in terms of $\prec$.

Proposition 1. *$f \sim g$ if and only if there exists $k \in \mathbb{N}$ such that $f \prec g^k$ and $g \prec f^k$.*

Proof. If $f \prec g^k$ for some k, then $M(g) \subseteq M(f)$. Conversely, assume that $M(g) \subseteq M(f)$ but $g \not\prec f^k$ for all natural k. Fixing an s, we denote by $\{n_{s,i}\}_{i\in\mathbb{N}}$ the monotone increasing sequence of natural numbers m such that $g(m) > f^s(m)$. Set $n_i = n_{i,i}$. Since $\{n_{s+1,i}\}_{i\in\mathbb{N}} \subset \{n_{s,i}\}_{i\in\mathbb{N}}$, we have $n_i < n_{i+1}$. Now we can define a new function h by setting $h(n) = g(n_{i+1})$ for all $n \in (n_i, n_{i+1}]$. Then $h(n+1) \geq h(n) = g(n_{i+1}) > n_{i+1} \geq n$, so that $h \in \Omega$. Furthermore, h is not in $M(g)$ since h and g coincide on an infinite set. On the other hand, for fixed k and $t \geq k$, $n \in (n_t, n_{t+1}]$, we have $h(n) = g(n_{t+1}) > f^{t+1}(n_{t+1}) > f^k(n)$. This shows that $f \ll h$ and $h \in M(f) \subseteq M(g)$, a contradiction. $\square$

The equivalence classes of $\sim$ will be called *growths.* The set of all growths will be denoted by $\Omega^\star$.

Proposition 2. *Each growth ω is a subsemigroup of Ω_∞.*

Proof. It suffices to prove that ω is closed under composition of functions. If $f \sim g$, then $g \prec f^k$ for some k, whence $f \circ g \prec f^{k+1}$. On the other hand, $f \prec f \circ g \prec (f \circ g)^{k+1}$ and so $f \circ g \sim f$. $\square$

We define two important functions in Ω_∞:

$$f_\infty(n) = \infty \quad \text{and} \quad f_0(n) = n + 1,$$

and the corresponding growths $\omega_\infty = [f_\infty]$ and $\omega_0 = [f_0]$. Obviously, $f \in \omega_\infty$ if and only if there exists $n \in \mathbb{N}$ such that $f(n) = \infty$, and $g \in \omega_0$ if and only if there exists $d \in \mathbb{N}$ such that $g(n) \leq n + d$ for all $n \in \mathbb{N}$.

We define a partial order in $\Omega^\star$ as follows:

$$\omega_1 \leq \omega_2 \quad \text{if} \quad M(f) \supseteq M(g) \quad \text{for some} \quad f \in \omega_1, g \in \omega_2.$$

Clearly, $\omega_0 \leq \omega \leq \omega_\infty$ for all $\omega \in \Omega^\star$. By Proposition 1, this order can be characterized as follows: $[f] < [g]$ if and only if there exists k_0 such that $f \prec g^k$ and $g \not\prec f^k$ for all $k \geq k_0$. Moreover, $[f]$ and $[g]$ are incomparable if $f \not\prec g^k$ and $g \not\prec f^k$ for all k. For example, it is easily seen that the following growths form an increasing sequence: $[n+1] < [2n] < [n^2] < [e^n]$. We say that f is exponentially greater than g if $[e^g] < [f]$.

For $\omega_1 = [f]$ and $\omega_2 = [g]$ we can define two new growths:

$$\omega_1 \vee \omega_2 = [\max\{f, g\}], \quad \omega_1 \wedge \omega_2 = [\min\{f, g\}].$$

Obviously, $\omega_1 \vee \omega_2 \geq \omega_1, \omega_2$ and $\omega_1 \wedge \omega_2 \leq \omega_1, \omega_2$. In fact, the following lemma implies that $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ are well-defined lattice operations.

Lemma 1. *If $\omega_3 > \omega_1 \wedge \omega_2$, then $\omega_3 > \omega_1$ or $\omega_3 > \omega_2$. If $\omega_4 < \omega_1 \vee \omega_2$, then $\omega_4 < \omega_1$ or $\omega_4 < \omega_2$.*

This lemma and the following theorem, which summarizes the main properties of growths, will be proved in §3.

Theorem 1. *With respect to the operations $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$, the set $\Omega^\star$ of growths forms a lattice having the following properties*:

a) *$\Omega^\star$ has the smallest element ω_0 and the largest element ω_∞.*

b) *For any ω such that $\omega < \omega_\infty$ there exists a strictly monotone increasing sequence of growths $\omega = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_n < \cdots$ such that each ω_{i+1} is exponentially greater than ω_i, $i = 0, 1, \ldots$.*

c) *$\Omega^\star$ is dense, i.e., for any $\omega_1 < \omega_2$ $(\omega_1, \omega_2 \in \Omega^\star)$, there exists $\omega_3 \in \Omega^\star$ such that $\omega_1 < \omega_3 < \omega_2$.*

d) *$\Omega^\star$ has neither atoms nor coatoms.*

e) *For any ω, $\omega_0 < \omega < \omega_\infty$, there exists an uncountable family of pairwise incomparable growths that are not comparable with ω (an uncountable antichain).*

f) *$\Omega^\star$ is a distributive (and thus modular) lattice.*

g) *$\Omega^\star$ is a complete lattice.*

Observe that $\Omega^\star$ is not a Boolean algebra. Analogs of statements b), c), and e) of Theorem 1 were stated without proof in [Kov].

§3. Staircase functions

In general, functions of a given growth can have a very complicated structure, and are hard to handle. However, in each growth we can find a lot of very simple functions, the so-called staircase functions.

Definition 1. Let $N = \{n_i\}_{i \in \mathbb{N}}$ be an ascending sequence of natural numbers such that $1 = n_0 < n_1 < n_2 < \cdots$. A *staircase function* corresponding to the sequence N is defined by $S_N(n) = n_{i+1}$ for all $n \in [n_i, n_{i+1})$.

Every growth can be represented by a staircase function.

Proposition 3. *Every growth ω is the equivalence class of a staircase function.*

Proof. Let $f \in \omega$. We define a sequence $N = \{n_i\}$ as follows: $n_0 = 1$, $n_{i+1} = f(n_i) = f^{i+1}(1)$. For all $n \in [n_i, n_{i+1})$ we have $f(n) \le f(n_{i+1}) = n_{i+2} = S_N^2(n)$. On the other hand, $S_N(n) = n_{i+1} = f(n_i) \le f(n)$, which implies $f \sim S_N$. $\square$

Of course, different functions can be equivalent to the same staircase function, and every growth contains many different staircase functions. The class of staircase functions is well behaved under composition. If $n \in [n_i, n_{i+1})$, then $S_N^k(n) = n_{i+k}$.

Lemma 2. *For two sequences $N = \{n_i\}_{i \in \mathbb{N}}$ and $M = \{m_j\}_{j \in \mathbb{N}}$, we have $S_N(n) \prec S_M^k(n)$ if and only if all but finitely many intervals $[m_j, m_{j+1})$ contain at most $k-1$ terms of the sequence N.*

Proof. Indeed, if $S_M \prec S_N^k$ and $n_s, \ldots, n_{s+k} \in [m_i, m_{i+1})$ for some i, then $S_M(n_s) = m_{i+1} > n_{s+k} = S_N^k(n_s)$. On the other hand, if $n \in [n_i, n_{i+1}) \cap [m_j, m_{j+1})$, then $S_M(n) = m_{j+1}$, $S_N^k(n) = n_{i+k}$. If, moreover, $[m_j, m_{j+1})$ contains at most $k-1$ terms of N, then $n_{i+k} > m_{j+1}$ and $S_M(n) \prec S_N^k(n)$. $\square$

Now we characterize the order on growths in terms of the corresponding staircase functions. The following proposition, which follows easily from Lemma 2, plays a fundamental role in the sequel.

Proposition 4. *For two arbitrary sequences N and M we have*:

a) $[S_N] = [S_M]$ *if and only if there are bounds for the number of terms of N in the intervals $[m_i, m_{i+1})$ and for the number of terms of M in the intervals $[n_j, n_{j+1})$;*

b) $[S_N] < [S_M]$ *if and only if there is a bound for the number of terms of N in the intervals $[m_i, m_{i+1})$ and there is no such bound for the number of terms of M in the intervals $[n_j, n_{j+1})$;*

c) $[S_N]$ *and* $[S_M]$ *are incomparable if and only if there are no bounds for the number of terms of N and M in the intervals $[m_i, m_{i+1})$ and $[n_j, n_{j+1})$, respectively.*

Lemma 3. *If $N \subset M$, then $[S_M] \le [S_N]$. Conversely, if $\omega_1 \le \omega_2$, then there exist sequences N and M such that $\omega_1 = [S_M]$, $\omega_2 = [S_N]$, and $N \subset M$.*

Proof. By Proposition 3, $\omega_1 = [S_N]$ and $\omega_2 = [S_M]$ for some N, M. Since $\omega_1 \le \omega_2$, the intervals $[m_i, m_{i+1})$ contain at most k terms of N for some fixed k. Let $[m_{i_1}, m_{i_1+1})$, $\dots, [m_{i_s}, m_{i_s+1}), \dots$ be a sequence of all intervals containing at least one term of N. We set $n'_s = m_{i_s}$ and $N' = \{n'_s\}$. Then $N' \subset M$. The intervals $[n'_s, n'_{s+1})$ contain at most k terms of N. On the other hand, the intervals $[n_j, n_{j+1})$ contain at most one term of N', and it follows that $S_N \sim S_{N'}$. □

Now, we characterize the operations $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ in terms of staircase functions. For two sequences N and M, we have $\min(S_N, S_M) = S_{N \cup M}$, where $N \cup M$ is the usual join of two sets.

We introduce a new operation on sequences. For two sequences N and M, we define a new sequence $L = N \vee M$ as follows. If $n_1 \le m_1$, we set $l_1 = n_1$. Suppose that $l_1, \dots, l_{2i-1}$ have already been constructed. Then l_{2i} is the smallest term of M greater than l_{2i-1}, and l_{2i+1} is the smallest term of N greater than l_{2i}. Obviously, $\max(S_N, S_M) \sim S_{N \vee M}$, and we have the following result.

Lemma 4. *If $N \sim N'$ and $M \sim M'$, then $N \cup M \sim N' \cup M'$ and $N \vee M \sim N' \vee M'$.*

Proof. Let $N = \{n_i\}$, $N' = \{n'_i\}$, $M = \{m_i\}$, $M' = \{m'_i\}$, and $N \cup M = K = \{k_i\}$, $N' \cup M' = K' = \{k'_i\}$. There exists a natural number n such that the intervals $[n_i, n_{i+1})$ and $[m_j, m_{j+1})$ contain at most n terms of N' and M', respectively, and the intervals $[n'_i, n'_{i+1})$ and $[m'_j, m'_{j+1})$ contain at most n terms of N and M, respectively. In this case the intervals $[k_i, k_{i+1})$ and $[k'_i, k'_{i+1})$ contain at most $2n$ terms of K' and K, respectively. This shows that $K \sim K'$.

Now, let $L = N \vee M$, $L' = N' \vee M'$. An interval $[l_{2i-1}, l_{2i})$ is contained in the interval $[m_j, m_{j+1})$, where $m_{j+1} = l_{2i}$, and thus contains at most n terms of M' for some n. Since $[l_{2i-1}, l_{2i})$ contains at most n terms of M', it contains at most $2n$ terms of L'. Moreover, $[l_{2i}, l_{2i+1})$ contains at most n terms of N', and thus it contains at most $2n$ terms of L'. Now, the conclusion follows by interchanging L and L' in this argument. □

Proof of Lemma 1. Let $\omega_1 = [S_N]$, $\omega_2 = [S_M]$, $K = N \cup M$, $\omega_3 = [S_K]$. Since $K \supset N, M$, by Lemma 2 we have $[S_N] \ge [S_K]$, $[S_M] \ge [S_K]$. Now, suppose that $\omega \not< \omega_3$ for $\omega = [S_{K'}]$, $K' = \{k'_i\}$. Then there is no bound for the number of terms of K in the intervals $[k'_i, k'_{i+1})$. Suppose that $\omega \le \omega_1$ and $\omega \le \omega_2$. Then there exists n such that $[k'_i, k'_{i+1})$ contains at most n terms of N and M, and thus at most $2n$ terms of K, a contradiction.

Let $\omega_4 = [L]$, $L = N \vee M$, $L = \{l_i\}$, and let $L_1 = \{l_{2i-1}\}$, $L_2 = \{l_{2i}\}$. Then $L_1 \sim L_2 \sim L$. Since $L_1 \subset N$, $L_2 \subset M$, we have $[S_L] \ge [S_N], [S_M]$. Now, suppose that $\omega \not< \omega_4$, $\omega = [S_{L'}]$. There is no bound for the number of terms of M in the intervals $[l_i, l_{i+1})$, and thus the same must be true for the intervals $[l_{2i-1}, l_{2i})$, or for the intervals $[l_{2i}, l_{2i+1})$. But $[l_{2i-1}, l_{2i})$ is contained in some $[m_s, m_{s+1})$, and $[l_{2i}, l_{2i+1})$ is contained in some $[n_t, n_{t+1})$, which completes the proof. □

Proof of Theorem 1. From Lemma 3 it follows that $\Omega^\star$ is a lattice. Part a) is obvious. For b), let $\omega = [f]$. We set $g_1(n) = f^n(n)$. Then $\omega_1 = [g_1] > \omega$. Now, let $\omega_k = [g_k]$, where $g_k(n) = g_{k-1}^n(n)$. We have $\omega < \omega_1 < \omega_2 < \cdots < \omega_k < \cdots$.

c) Assume that $\omega_1 = [S_N]$, $\omega_2 = [S_M]$, $N = \{n_i\}$, $M = \{m_j\}$, $\omega_1 < \omega_2$. By Proposition 4 b), the intervals $[n_i, n_{i+1})$ contain a bounded number of terms of M, but there is no bound for the number of terms of N in the intervals $[m_j, m_{j+1})$. Let $\{[m_{j_s}, m_{j_{s+1}}]\}$, $s = 1, 2, 3, \dots$, be a sequence of intervals such that $[m_{j_s}, m_{j_s+1})$ contains more than s^2 terms of N. We denote these terms by $n_{i_s+1}, \dots, n_{i_s+s^2}$ and put $K' = \bigcup_{s=1}^{\infty}\{n_{i_s}, n_{i_s+s}, \dots, n_{i_s+s(s-1)}\}$ and $K = M \cup K'$, $K = \{k_i\}$. Let $\omega_3 = [S_K]$. Since $M \subset K$, we have $\omega_3 \leq \omega_2$ and the inequality is strict, because every interval $[m_{j_s}, m_{j_s+1})$ contains s terms of K. On the other hand, every interval $[n_i, n_{i+1})$ contains at most k_0 terms of M and at most 1 term of K'. It follows that $\omega_1 \leq \omega_3$. But every interval $[k_i, k_{i+1})$ coincides with some interval $[n_{i_s}, n_{i_s+s})$ and contains s terms of N, so that $\omega_1 < \omega_3 < \omega_2$.

d) follows easily from c).

e) If $\omega_0 < \omega < \omega_\infty$, then the density implies that we can find ω_1, ω_2 such that $\omega_0 < \omega_1 < \omega < \omega_2 < \omega_\infty$. Let $\omega = [S_N]$, $\omega_1 = [S_M]$, $\omega_2 = [S_K]$. Taking some subsequence $N' \subseteq N$, we may assume that each interval $[n'_i, n'_{i+1})$ contains three consecutive m_j and three consecutive k_l. Let A be an infinite subset of $\mathbb{N}$. We represent $A = A_1 \cup A_2$ as a disjoint sum of odd and even terms $A_1 = \{a_{2s+1}\}$, $A_2 = \{a_{2s}\}$. Clearly, a new function F_A can be defined as follows. On the intervals $[n'_i, n'_{i+1})$ the function F_A takes the value S_N if $i \in \mathbb{N} \setminus A$; the value S_K if $i \in A_1$; and the value S_M if $i \in A_2$. After that, F_A can be interpolated in such a way that $F_A \in \Omega_\infty$. The growths $[S_N]$ and $[F_A]$ are incomparable.

For each real number r we choose an infinite sequence $\{b_n\}$ of distinct rational numbers that converge to r, and define Γ_r to be the set $\{b_n : n = 1, 2, \dots\}$. Recall that Γ is called a moiety if $|\Gamma| = |\mathbb{Q} \setminus \Gamma|$. The set $\{\Gamma_r : r \in \mathbb{R}\}$ is an uncountable family of moieties, and $\Gamma \cap \Gamma'$ is finite for any two distinct members Γ, Γ' of this family. This is a classical result by W. Sierpiński, proved in 1928 in [S]. Let $\phi : \mathbb{Q} \to \mathbb{N}$ be any bijection and let $\Delta_r = \phi(\Gamma_r)$.

Then the family of growths $\{[F_{\Delta_r}]\}_{r \in \mathbb{R}}$ is an uncountable family of pairwise incomparable growths, as desired.

f) Suppose that $\omega_1 = [S_N]$, $\omega_2 = [S_M]$, $\omega_3 = [S_K]$ are pairwise incomparable. We look at $S_{N \cup M} = \min(S_N, S_M)$ and $S_{N \cup K} = \min(S_N, S_K)$. Let $S_{N \cup M} \sim S_{N \cup K}$, $L = N \cup M$, $L' = N \cup K$. Then there exists k such that each interval $[l_i, l_{i+1})$ contains at most k terms of L'. However, either $[l_i, l_{i+1}) = [m_s, m_{s+1})$, or $[l_i, l_{i+2}) = [m_s, m_{s+1})$, so that $[m_s, m_{s+1})$ contains at most $2k$ terms of N. By symmetry, we obtain $S_N \sim S_M$, a contradiction.

Now, suppose that, for $\omega_1 = [N]$, $\omega_2 = [M]$, $\omega_3 = [K]$, the growths ω_1, ω_3 and ω_2, ω_3 are pairwise incomparable and $\omega_1 < \omega_2$. By Lemma 2, we may assume that $M \subset N$. If we assume additionally that $\omega_1 \wedge \omega_3 = \omega_2 \wedge \omega_3$, by the same argument as above we obtain a contradiction with $\omega_1 < \omega_2$. This shows that $\Omega^\star$ contains no diamond M_5 and no pentagon N_5 as sublattices; thus it is distributive [BS]. Obviously, distributivity implies modularity for lattices.

g) Suppose that we have an arbitrary family $\{\omega_\alpha\}_{\alpha \in I}$ of growths. We choose any function f_α in ω_α. We construct two functions f and g by setting $f(n) = \max_\alpha\{f_\alpha(n)\}$ and $g(n) = \min_\alpha\{g_\alpha(n)\}$. Clearly, $f, g \in \Omega_\infty$. The construction implies that $[f]$ and $[g]$ are an upper and a lower bound for the family $\{\omega_\alpha\}_{\alpha \in I}$. $\square$

§4. Bandwidth growth

Let R be an associative ring with 1. By $M(\infty, R)$ we denote the set of all $(\mathbb{N}\times\mathbb{N})$-matrices. Addition of matrices in $M(\infty, R)$ is well defined, but multiplication is not. Moreover, multiplication is not necessarily associative in $M(\infty, R)$, even when it is defined, as the following example shows [DDR]. Let

$$x = \begin{pmatrix} 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0\dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

and

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

then both $(xy)z$ and $x(yz)$ are defined, but $(xy)z \neq x(yz)$.

We cite two important results from [DDR]. If $R = F$ is a field, then:

1) any nonassociative countable-dimensional F-algebra can be embedded in $M(\infty, F)$;

2) any finite or countable groupoid can be embedded in $M(\infty, F)$.

In this paper we are mostly interested in associative structures, so that we need to impose some restrictions to ensure that multiplication is well behaved. A matrix is said to be column-finite if each of its columns contains only finitely many nonzero entries; the row-finite matrices are defined similarly. We denote by $CFM(\infty, R)$ the set of all column-finite matrices and by $RCFM(\infty, R)$ the set of all row- and column-finite matrices.

Definition 2. For a given matrix $a = (a_{ij}) \in M(\infty, R)$, we say that a function $f \in P(\mathbb{N})$ is a *row bound* for a if $a_{n,m} = 0$ for all $m > f(n)$. A function $g \in P(\mathbb{N})$ is a *column bound* for a if $a_{n,m} = 0$ for all $n > g(m)$. We assume that the boundedness conditions are trivially satisfied if $f(n) = \infty$ or $g(n) = \infty$.

In some sense, the functions f and g give bounds on the width of the *band* along the principal diagonal that contains all nonzero entries of the matrix a.

Proposition 5. *For every $a \in M(\infty, R)$, there exists the smallest row bound of a in Ω_∞, denoted by B_r^a. For every $a \in M(\infty, R)$, there exists the smallest column bound of a in Ω_∞, denoted by B_c^a.*

Proof. We put $B_r^a(1) = m$, where $m = \max(2, \max(n : a_{1n} \neq 0))$, or $B_r^a(1) = \infty$ if there is no such m. Let $B_r^a(n+1) = \max(n+2, B_r^a(n), m)$, where $m = \max(k : a_{nk} \neq 0)$, or $B_r^a(n+1) = \infty$ if there is no such m. Similarly, we define $B_c^a(n)$. From this definition it follows that $B_r^a(n)$ and $B_c^a(n)$ have the required properties. □

We call B_r^a the *upper bound* for a and B_c^a the *lower bound* for a. The lower and upper bounds are well behaved under multiplication of matrices. If $B_r^a, B_r^b \leq f$, then $B_r^{ab} \leq f^2$ (see Figure 2). Similarly, if $B_c^a, B_c^b \leq g$, then $B_c^{ab} \leq g^2$.

Remark. By Proposition 6, for every $a \in M(\infty, R)$ there exist growths ω_1 and ω_2 such that $\omega_1 = [B_r^a]$, $\omega_2 = [B_c^a]$. Moreover, if $\omega_3 \not\succ \omega_1$, then none of $f \in \omega_3$ is a row bound for a, and if $\omega_4 \not\succ \omega_2$, then none of $g \in \omega_4$ is a column bound for a.

On the other hand, for every ω we can find $a \in M(\infty, R)$ such that $\omega = [B_r^a]$. Let $\omega = [S_N]$ for some staircase function S_N. It suffices to take an upper unitriangular matrix

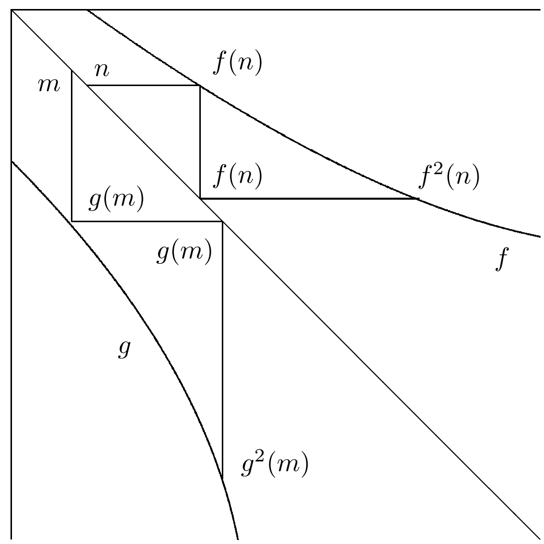

FIGURE 2

with consecutive finite blocks on the principal diagonal of sizes $n_1,\ n_2 - n_1, n_3 - n_2, \dots$ and with all entries above the diagonal equal to 1. A similar result is valid for column bounds.

Obviously, $CFM(\infty, R)$ forms a ring under the addition and multiplication of matrices; sometimes this ring is denoted by $M_c(\infty, R)$. Its subring of row- and column-finite matrices is denoted by $M_{rc}(\infty, R)$. The group of invertible column-finite infinite matrices is denoted by $\mathrm{GL}_c(\infty, R)$, and its subgroup of row and column-finite matrices is denoted by $\mathrm{GL}_{rc}(\infty, R)$. The corresponding semigroups are denoted by $S_c(\infty, R)$ and $S_{rc}(\infty, R)$.

For a commutative ring R, we can define an associative R-algebra $A_c(\infty, R)$ of column-finite matrices and also its subalgebra $A_{rc}(\infty, R)$. With respect to the Lie bracket $[a, b] = ab - ba$, they form the Lie algebra $L_c(\infty, R)$ and its Lie subalgebra $L_{rc}(\infty, R)$.

Since our results are valid for groups, semigroups, rings, algebras, and Lie algebras, and can be proved similarly in all these cases, we state them in the language of universal algebra.

In the text below, X_c means one of

$$M_c(\infty, R), \quad \mathrm{GL}_c(\infty, R), \quad S_c(\infty, R), \quad A_c(\infty, R), \quad L_c(\infty, R)$$

and X_{rc} means the corresponding row- and column-finite analog. With any two growths ω_1 and ω_2 we associate the following four subsets of X_c:

$$\begin{aligned}
X(\omega_1, \omega_2) &= \{a \in X_c : B_c^a \leq f,\ B_r^a \leq g \text{ for some } f \in \omega_1,\ g \in \omega_2\},\\
X(\omega_1, \widehat{\omega}_2) &= \{a \in X_c : B_c^a \leq f,\ B_r^a \ll g \text{ for some } f \in \omega_1,\ g \in \omega_2\},\\
X(\widehat{\omega}_1, \omega_2) &= \{a \in X_c : B_c^a \ll f,\ B_r^a \leq g \text{ for some } f \in \omega_1,\ g \in \omega_2\},\\
X(\widehat{\omega}_1, \widehat{\omega}_2) &= \{a \in X_c : B_c^a \ll f,\ B_r^a \ll g \text{ for some } f \in \omega_1,\ g \in \omega_2\}.
\end{aligned}$$

In the case where $X_c = \mathrm{GL}_c(\infty, R)$ we modify the above definitions by additionally imposing the same restrictions on a^{-1}. For example,

$$X(\omega_1, \omega_2) = \{a \in X_c : B_c^a, B_c^{a^{-1}} \leq f,\ B_r^a, B_r^{a^{-1}} \leq g \text{ for some } f \in \omega_1,\ g \in \omega_2\}.$$

To cover some important cases we make a rather natural assumption that $X(\widehat{\omega}_0,-)$ (respectively, $X(-,\widehat{\omega}_0)$) means that in all matrices only a finite number of entries below (respectively, above) the principal diagonal are nonzero.

$$\begin{tikzcd}[column sep=small] & X(\omega_1,\omega_2) \arrow[dl, dash] \arrow[dr, dash] & \\ X(\widehat{\omega}_1,\omega_2) \arrow[dr, dash] & & X(\omega_1,\widehat{\omega}_2) \arrow[dl, dash] \\ & X(\widehat{\omega}_1,\widehat{\omega}_2) & \end{tikzcd}$$

FIGURE 3

From the remark it follows that $X(\omega_1,\omega_2) = X(\omega_3,\omega_4)$ if and only if $\omega_1 = \omega_3$ and $\omega_2 = \omega_4$. In general, we have the Hasse diagram of inclusions shown in Figure 3, with the only exceptions being

$$X(\widehat{\omega}_\infty,\omega_\infty) = X(\omega_\infty,\omega_\infty) = X_c \text{ and } X(\widehat{\omega}_\infty,\widehat{\omega}_\infty) = X(\omega_\infty,\widehat{\omega}_\infty) = X_{rc}.$$

Theorem 2. *For any two growths ω_1 and ω_2, the sets*

$$X(\widehat{\omega}_1,\widehat{\omega}_2), \quad X(\omega_1,\widehat{\omega}_2), \quad X(\widehat{\omega}_1,\omega_2), \quad X(\omega_1,\omega_2)$$

are subalgebras of the algebra X_c, in the sense of universal algebra. If $\omega_1,\omega_2 < \omega_\infty$, then $X(\omega_1,\omega_2)$ is a subalgebra of X_{rc}.

When $\omega_1 = \omega_2$, we simply write $X(\omega) = X(\omega,\omega)$ and $X(\widehat{\omega}) = X(\widehat{\omega},\widehat{\omega})$. The remark above immediately implies the following result.

Theorem 3. *The lattices $\{X(\omega)\}$ and $\{X(\widehat{\omega})\}$ are isomorphic to the lattice $\Omega^\star$ of growths.*

The following result is a key for applications of growths to the study of universal subalgebras.

Theorem 4. *If Y is a subset of X_c, then there exist smallest ω_1 and ω_2 in $\Omega^\star$ such that $Y \subset X(\omega_1,\omega_2)$.*

Proof. If $Y = \{y_\alpha\}_{\alpha\in I}$, then, as in the remark above, we can construct two families $\{f_\alpha\}_{\alpha\in I}$ and $\{g_\alpha\}_{\alpha\in I}$ of upper and lower bounds. We construct two functions f and g as follows: $f(n) = \max_\alpha\{f_\alpha(n)\}$ and $g(n) = \max_\alpha\{g_\alpha(n)\}$; then, clearly, $f,g \in \Omega_\infty$. This construction implies that $[f]$ and $[g]$ are upper and lower bounds for the family $\{y_\alpha\}_{\alpha\in I}$, so that $Y \subset X([g],[f])$.

In the case of groups we must modify the definitions of f and g by considering the upper and lower bounds not only of the elements of Y themselves, but also of their inverses. When Y is a subgroup, this is not necessary. $\square$

Definition 3. We say that the subset Y of $X_c(\infty,R)$ in Theorem 4 has bandwidth growth (ω_1,ω_2). When $\omega_1 = \omega_2$, we simply say that Y has bandwidth growth ω.

Now we state two problems concerning growths.

Problem 1. Which algebraic properties of a universal algebra X can be deduced from the fact that X can be embedded in $X(\omega_1,\omega_2)$?

Problem 2. Describe properties of $X(\omega_1,\omega_2)$ in terms of ω_1,ω_2.

In the next section we establish some results on both problems for groups and algebras of infinite matrices.

§5. Bandwidth growths for groups and algebras

As we observed in the Introduction, every countable-dimensional algebra over a field F can be embedded in $A_{rc}(\infty, F)$ [GMM]. In fact, one can be much more specific here: by Theorem 4 such an algebra can be embedded in $A(\omega_1, \omega_2)$ for some ω_1, ω_2.

Now, we can formulate the main results of [O, HO2] in terms of bandwidth growth.

Theorem 5. *For every finitely generated associative algebra A over a field F, there exists a minimal real $\alpha \in [0, 1]$ such that A can be embedded in $A([n+n^\alpha])$. For every $\alpha \in [0, 1]$ there exists a finitely generated algebra A such that A can be embedded in $A([n + n^\alpha])$ and cannot be embedded in $A([n + n^\beta])$ for $0 \le \beta < \alpha$.*

Theorem 6. *Every countable-dimensional associative algebra over a field can be embedded in $A([n + n])$.*

Generally speaking, the proofs for algebras cannot be applied to groups directly. Thus, it seems that the following result may be of interest.

Corollary 1. *The groups of units of a countable-dimensional associative algebra over a field can be embedded in $G([n+n])$. Let A be a finitely generated associative algebra over a field F. Then there exists the smallest $\alpha \in [0, 1]$ such that the group of units $U(A)$ can be embedded in $G([n + n^\alpha])$ and cannot be embedded in $G([n + n^\beta])$ for $\beta < \alpha$.*

A block-diagonal matrix $a = \operatorname{diag}(a_1, a_2, a_3, \dots) \in \mathrm{GL}_c(\infty, R)$ with finite blocks along the principal diagonal is called a string. Its inverse equals $a^{-1} = \operatorname{diag}(a_1^{-1}, a_2^{-1}, a_3^{-1}, \dots)$, and is also a string. However, the product of two strings may fail to be a string. The set of all finite products of strings forms a subgroup of $\mathrm{GL}_c(\infty, R)$, and we refer to [HV] for a detailed study of its properties. A string $b = \operatorname{diag}(1, \dots, 1, b_n, 1, \dots)$ with only one nontrivial block is called a *bead*.

We identify the group of all permutations of $\mathbb{N}$ with its regular representation $\mathrm{Sym}(\mathbb{N})$ in $\mathrm{GL}_c(\infty, R)$. Let $\mathrm{Sym}(\omega) = \mathrm{Sym}(\mathbb{N}) \cap G(\omega)$ and $\mathrm{Sym}(\widehat{\omega}) = \mathrm{Sym}(\mathbb{N}) \cap G(\widehat{\omega})$.

A subgroup $\mathrm{Sym}(\widehat{\omega}_0)$ is called the finitary subgroup; it consists of all permutations that move only finitely many elements. Since for a permutation a we have $a^{-1} = a^t$, every subgroup of $\mathrm{Sym}(\mathbb{N})$ is contained in some $G(\omega) \cap S(\mathbb{N})$. E. A. Koval′chuk described normal subgroups of $\mathrm{Sym}(\omega_0)$ [Kov].

Now we are in a position to prove the following result.

Theorem 7. *The group $\mathrm{Sym}(\omega)$ is generated by strings. In fact, every element of $\mathrm{Sym}(\omega)$ is a string or a product of two strings from $\mathrm{Sym}(\omega)$.*

Proof. Suppose that $a \in \mathrm{Sym}(\omega)$ and a is not a string. We construct two strings b, c such that $c \cdot a \cdot b$ is the identity matrix.

Choose any $n_1 > 1$ and let m_1 be the minimal integer such that all nonzero entries in the first n_1 columns are in the first m_1 rows, and all nonzero entries in the first n_1 rows are in the first m_1 columns. We put $n_2 = \max\{n_1 + 1, m_1\}$. Now, let m_2 be the minimal integer such that all nonzero entries in the first n_2 columns are in the first m_2 rows, and all nonzero entries in the first n_2 rows are in the first m_2 columns. We set $n_3 = \max\{n_2 + 1, m_2\}$. Repeating this procedure we obtain an infinite sequence $n_1 < n_2 < n_3 < \cdots$.

Now, we can represent a in the following block-tridiagonal form:

$$a = \begin{pmatrix} u_1 & w_1 & & & \\ v_1 & u_2 & w_2 & & \\ & v_2 & u_3 & w_3 & \\ & & v_3 & u_4 & w_4 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The blocks on the principal diagonal are square matrices $u_1, u_2, u_3, \ldots$ of sizes $n_1 \times n_1$, $(n_2 - n_1) \times (n_2 - n_1)$, $(n_3 - n_2) \times (n_3 - n_2), \ldots$. The matrices $v_1, v_2, \ldots$ have sizes $(n_2 - n_1) \times n_1, (n_3 - n_2) \times (n_2 - n_1), \ldots$, whereas the matrices $w_1, w_2, \ldots$ have the corresponding sizes $n_1 \times (n_2 - n_1), (n_2 - n_1) \times (n_3 - n_2), \ldots$. Moreover, all nonzero entries of a occur in blocks $u_i, v_i, w_i, i \geq 1$.

We set $b_0 = c_0 = 0$ and let $b_k = n_{2k}$, $c_k = n_{2k-1}$. This gives us two partitions of natural numbers, $B_k = \{b_{k-1} + 1, \ldots, b_k\}$ and $C_k = \{c_{k-1} + 1, \ldots, c_k\}$.

Each of the first b_1 columns of a contains exactly one unit element, and we rearrange them in such a way that the column that has a unit in the kth row, $k \in C_1$, becomes the kth column. Such an arrangement of columns is not unique, since the remaining columns with units below the c_1th row can be placed arbitrarily. After this rearrangement, the block u_1 becomes the identity matrix of size $n_1 \times n_1$, and the blocks w_1, v_1 are both changed to zero matrices. This rearrangement of columns is equivalent to multiplication of a by a bead $\mathrm{diag}(b_1, 1, 1, \ldots)$ on the right.

Similarly, we can rearrange the columns with indices in B_2 in such a way that u_3 becomes the identity matrix and w_3, v_3 are both changed to zero matrices. This rearrangement of columns is equivalent to multiplication of $a \cdot \mathrm{diag}(b_1, 1, 1, \ldots)$ by the bead $\mathrm{diag}(1, \ldots, 1, b_2, 1, 1, \ldots)$ on the right.

Proceeding like that, we can simultaneously rearrange the columns with indices in B_1, B_2, $B_3, \ldots$. Such a rearrangement corresponds to multiplication of a by the string $b = \mathrm{diag}(b_1, b_2, b_3, \ldots)$ on the right.

Similarly, we can rearrange the rows with indices in C_2 in such a way that u_2 becomes the identity matrix and w_2, v_2 are both changed to zero matrices. This rearrangement of rows is equivalent to multiplication of a by the bead $\mathrm{diag}(1, \ldots, 1, c_2, 1, 1, \ldots)$ on the left.

Now, it is evident that similar operations can be applied simultaneously to the rows with indices in C_2, C_3, $C_4, \ldots$. This rearrangement is equivalent to multiplication by the string $c = \mathrm{diag}(1, \ldots, 1, c_2, c_3, c_4, \ldots)$ on the left. The resulting product $c \cdot a \cdot b$ is the identity matrix. Since both c and b belong to $\mathrm{Sym}(\omega)$, this proves Theorem 7. □

Theorem 8. *The group* $\mathrm{Sym}(\widehat{\omega}_0)$ *of finitary permutations is a normal subgroup of* $\mathrm{Sym}(\mathbb{N})$.

Proof. It suffices to show that the conjugate of a finitary permutation by a string is a finitary permutation. Our claim then follows from Theorem 7. Let $b = \mathrm{diag}(b_1, 1, 1, 1, \ldots)$ be a finitary permutation, and let $a = \mathrm{diag}(a_1, a_2, a_3, \ldots)$ be a string. We choose s in such a way that the size of b_1 is not greater than the sum t of the sizes of $a_1, \ldots, a_s$. Then the matrix $a^{-1} \cdot b \cdot a = \mathrm{diag}(b_1', 1, 1, 1, \ldots)$, where the size of b_1' equals t, is finitary. □

An involution $a \in \mathrm{Sym}(\mathbb{N})$ is said to be *elementary* if all 2-cycles in a are of the form $(i, i+1)$.

Theorem 9. *The group* $\mathrm{Sym}(\omega_0)$ *is generated by elementary involutions.*

Proof. For any string $a = \mathrm{diag}(a_1, a_2, a_3, \ldots)$ in $\mathrm{Sym}(\omega_0)$ the sizes of the blocks $a_1, a_2, \ldots$ are bounded by a natural k. Thus, each block can be viewed as a permutation in $\mathrm{Sym}(k)$. Every permutation in $\mathrm{Sym}(k)$ is a product of at most $f(k)$ elementary involutions, and it follows that the same is true for the string a. Thus, when a is the product of two strings, each of these strings is a product of elementary involutions. □

Similar results for subgroups of the group $UT_c(\infty, R)$ of unitriangular matrices were proved in [Hol4]. Moreover, if $R = F$ is a finite field, then $UT_c(\infty, R) \cap G(\omega_0)$ contains free subgroups [Hol, OS]. In [Hol1], it was proved that almost all k-generated subgroups in $UT_c(\infty, R)$ are free of rank k in the profinite topology. Similar results for semigroups of infinite triangular matrices were proved in [Hol2].

Another interesting related observation is that every finitely generated free algebra can be embedded in $A(\omega_0)$; see [HO2].

§6. Examples and remarks

Examples. In Figure 4, stars designate the area that contains all nonzero entries for matrices in the corresponding subalgebras of X_c. Of course, for the group case there are further nonzero entries at the principal diagonal.

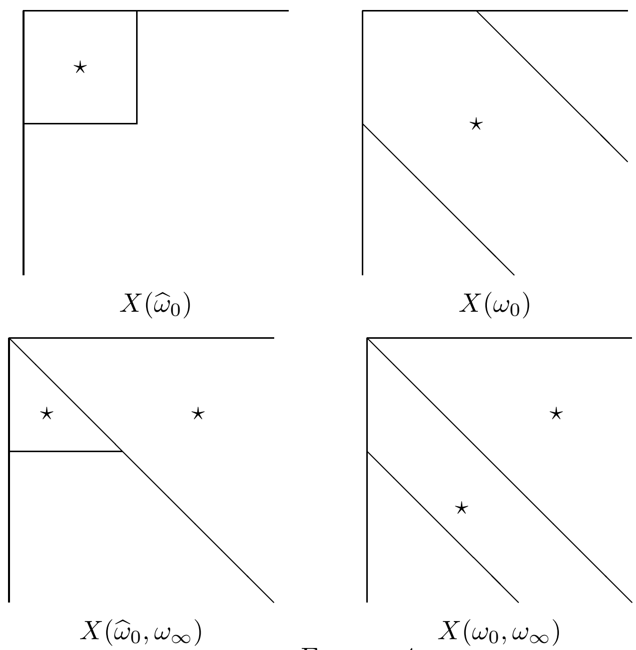

Figure 4

a) $X(\widehat{\omega}_0, \widehat{\omega}_0) = X(\widehat{\omega}_0)$. In the case of rings (algebras, Lie algebras) we get the ring $M(\widehat{\omega}_0)$ (or, respectively, the algebra $A(\widehat{\omega}_0)$, the Lie algebra $L(\widehat{\omega}_0)$) of all infinite matrices possessing only finitely many nonzero entries. Every intermediate ring M, $M(\widehat{\omega}_0) \leq M \leq M_c(\infty, R)$, is called an infinite matrix ring. See [AF, AS, CCS] for a survey of a rich theory of such rings and for references. In the case of groups, $G(\widehat{\omega}_0)$ is known as the stable general linear group $\mathrm{GL}(R)$, one of the main objects of algebraic K-theory (see [M], [HO]).

The Lie algebra $L(\widehat{\omega}_0)$ contains an infinite rank affine Lie algebra of traceless matrices, which is the Kac–Moody algebra corresponding to the infinite affine matrix A_∞^+ [Kac].

b) $X(\omega_0, \omega_0) = X(\omega_0)$. The ring $M(\omega_0)$ is a ring where every one-sided ideal is generated by idempotents, but which is not von Neumann regular; see [T]. In the case of Lie algebras, $L(\omega_0)$ is the Lie algebra of all matrices having only a finite number of nonzero diagonals. It is related to central extensions of affine Kac–Moody Lie algebras (or, in the case of groups, to Kac–Moody groups); see [Kac]. The group $G(\omega_0)$ is the smallest irreducible subgroup of $\mathrm{GL}_c(\infty, R)$. Goodearl [KL] raised the question as to whether an algebra of finite Gelfand–Kirillov dimension can always be embedded in $A(\omega_0)$. An affirmative answer would provide a useful tool to distinguish between various algebras of infinite Gelfand–Kirillov dimension.

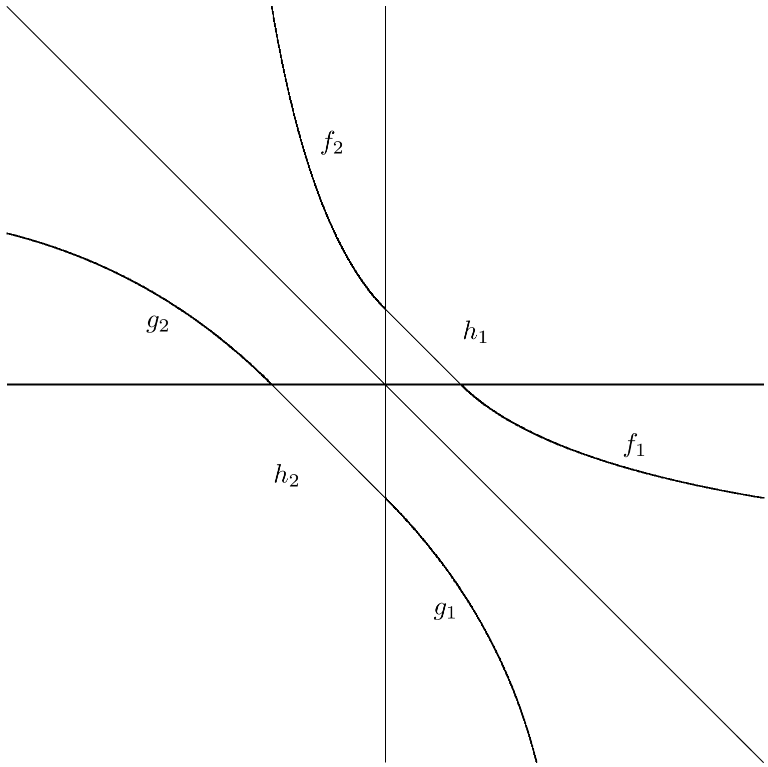

FIGURE 5

c) $X(\widehat{\omega}_0, \omega_\infty)$. The group $G(\widehat{\omega}_0, \omega_\infty)$ is known as the Vershik–Kerov group. It was first considered in an appendix by Vershik and Kerov to the Russian translation of [J]. See [V] for an extended version of that appendix. Arbarello, De Concini, and Kac [ACK] studied the properties of some of its central extensions. The paper [VK] was devoted to the study of its structure, characters, and unitary representations, [Hol3] contains a description of its parabolic subgroups and establishes a version of the Bruhat decomposition for $G(\widehat{\omega}_0, \omega_\infty)$. In [ACK], some results concerning the algebra $A(\widehat{\omega}_0, \omega_\infty)$ can be found.

d) $X(\omega_0, \omega_\infty)$. The ring $M(\omega_0, \omega_\infty)$ contains the endomorphism ring L of Laurent polynomials. Every endomorphism in L corresponds to a matrix whose entries are constant along the diagonals. In other words, for each k we have $a_{i,i+k} = a_{j,j+k}$ for all i, j.

Possible generalizations. a) A definition of bandwidth growth for row- and column-finite matrices over $\mathbb{Z}$ uses the same ideas as in the case of $\mathbb{N}$, but is more complicated. Now, in terms of four growths we can define 16 subalgebras in a similar way. As in Figure 5, we have two possibilities for each of the two upper and the two lower bounds f_1, f_2, g_1, g_2, with the same Hasse diagram as for the Boolean algebra of subsets of a 4-element set. For technical reasons, we take linear functions $h_i(n) = -n + d_i$ in the first and the third quadrants.

Observe that for Lie algebras of infinite complex matrices over $\mathbb{Z}$ we can distinguish two important cases. If $\omega_1 = \omega_2 = \omega_3 = \omega_4 = \widehat{\omega}_0$, we get the Lie algebra of finite width matrices (indexed by $\mathbb{Z}$), containing an affine Kac–Moody algebra as a subalgebra $\mathfrak{sl}(\infty)$ of traceless matrices. If $\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega_0$, we get the Lie algebra $\mathfrak{gl}_J$ of generalized Jacobi matrices with finitely many nonzero diagonals. Its central extension $\mathfrak{gl}_J^\star(\infty)$ plays an important role in representation theory. Many classical constructions in the theory of finite-dimensional Lie algebras carry over to this algebra. Some important infinite-dimensional Lie algebras can be embedded in $\mathfrak{gl}_J^\star(\infty)$; thus, the representations of $\mathfrak{gl}_J^\star(\infty)$ furnish representations of these algebras. For instance, this works for all affine Kac–Moody algebras and for the Virasoro algebra; see [Kac]. The author plans to return to a more detailed study of bandwidth growths over $\mathbb{Z}$ in a separate paper.

b) The notion of bandwidth growth can be used to study subgroup lattices of other infinite-dimensional algebraic objects, such as the automorphism group of a free group of countable rank [GH1], or the automorphism group of a homogeneous tree of countable valency [GH2].

References

[AS] G. Abrams and J. J. Simón, *Isomorphisms between infinite matrix rings: A survey*, Algebra and its Applications (Athens, OH, 1999), Contemp. Math., vol. 259, Amer. Math. Soc., Providence, RI, 2000, pp. 1–12. MR1778492 (2001i:16047)

[ACK] E. Arbarello, C. De Concini, and V. G. Kac, *The infinite wedge representation and the reciprocity law for algebraic curves*, Theta Functions — Bowdoin, 1987, Part 1 (Brunswick, ME, 1987), Proc. Sympos. Pure Math., vol. 49, Part 1, Amer. Math. Soc., Providence, RI, 1989, pp. 171–190. MR1013132 (90i:22034)

[AF] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, 2nd ed., Grad. Texts in Math., vol. 13, Springer-Verlag, New York, 1992. MR1245487 (94i:16001)

[BS] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Grad. Texts in Math., vol. 78, Springer-Verlag, New York–Berlin, 1981. MR0648287 (83k:08001)

[CCS] V. Camillo, F. J. Costa-Cano, and J. J. Simón, *Relating properties of a ring and its ring of row and column finite matrices*, J. Algebra **244** (2001), 435–449. MR1859035 (2002i:16039)

[C] R. G. Cooke, *Infinite matrices and sequence spaces*, Macmillan & Co., Ltd., London, 1950. MR0040451 (12:694d)

[DDR] R. O. Davies, M. P. Drazin, and M. L. Roberts, *Universal properties of infinite matrices*, J. Algebra **180** (1996), 402–411. MR1378536 (97a:16059)

[Dixon] J. D. Dixon, *Most finitely generated permutation groups are free*, Bull. London Math. Soc. **22** (1990), 222–226. MR1041134 (91c:20005)

[GH] É. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progr. Math., vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1990. MR1086648 (92f:53050)

[GMM] K. R. Goodearl, P. Menal, and J. Moncasi, *Free and residually Artinian regular rings*, J. Algebra **156** (1993), no. 2, 407–432. MR1216477 (94f:16025)

[Gri1] R. I. Grigorchuk, *On the Milnor problem of group growth*, Dokl. Akad. Nauk SSSR **271** (1983), no. 1, 30–33; English transl., Soviet Math. Dokl. **28** (1983), no. 1, 23–26. MR0712546 (85g:20042)

[Gri2] ———, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985; English transl., Math. USSR-Izv. **25** (1985), no. 2, 259–300. MR0764305 (86h:20041)

[Gr] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. No. 53 (1981), 53–73. MR0623534 (83b:53041)

[GH1] C. K. Gupta and W. Hołubowski, *Automorphisms of a free group of infinite rank*, Algebra i Analiz **19** (2007), no. 2, 74–85; English transl., St. Petersburg Math. J. **19** (2008), 215–223. MR2333897

[GH2] ———, *Automorphisms of rooted tree of countable valency* (in preparation).

[HO] A. J. Hahn and O. T. O'Meara, *The classical groups and K-theory*, Grundlehren Math. Wiss., Bd. 291, Springer-Verlag, Berlin, 1989. MR1007302 (90i:20002)

[HO1] J. Hannah and K. C. O'Meara, *A new measure of growth for countable-dimensional algebras*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 2, 223–227. MR1215312 (94c:16035)

[HO2] ———, *A new measure of growth for countable-dimensional algebras*. I, Trans. Amer. Math. Soc. **347** (1995), no. 1, 111–136. MR1282887 (95h:16027)

[Hol] W. Hołubowski, *Free subgroups of the group of infinite unitriangular matrices*, Internat. J. Algebra Comput. **13** (2003), 81–86. MR1970869 (2004c:20090)

[Hol1] ———, *Most finitely generated subgroups of infinite unitriangular matrices are free*, Bull. Austral. Math. Soc. **66** (2002), 419–423. MR1939204 (2003i:20092)

[Hol2] ———, *The ubiquity of free subsemigroups of infinite triangular matrices*, Semigroup Forum **66** (2003), 231–235. MR1953501 (2003m:20076)

[Hol3] ———, *Parabolic subgroups of Vershik–Kerov's group*, Proc. Amer. Math. Soc. **130** (2002), 2579–2582. MR1900864 (2003b:20074)

[Hol4] ———, *Subgroups of unitriangular group of infinite matrices* (in preparation).

[HV] W. Hołubowski and N. A. Vavilov, *Infinite general linear group* (in preparation).

[J] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Math., vol. 682, Springer, Berlin, 1978. MR0513828 (80g:20019)

[Kac] V. G. Kac, *Infinite-dimensional Lie algebras*, Cambridge Univ. Press, Cambridge, 1990. MR1104219 (92k:17038)

[Kov] E. A. Koval′chuk, *Groups of growth-bounded permutations*, Problems in Group Theory and Homological Algebra, Yaroslav. Gos. Univ., Yaroslavl′, 1988, pp. 18–28. (Russian) MR1174988

[KL] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand–Kirillov dimension*, Grad. Stud. Math., vol. 22, Amer. Math. Soc., Providence, RI, 2000. MR1721834 (2000j:16035)

[L] Yu. G. Leonov, *Representation of finitely approximative* 2*-groups by infinite unitriangular matrices over a two element field* (to appear).

[M] B. A. Magurn, *An algebraic introduction to K-theory*, Encyclopedia Math. Appl., vol. 87, Cambridge Univ. Press, Cambridge, 2002. MR1906572 (2003g:19001)

[Mil] J. Milnor, *Problem* 5603, Amer. Math. Monthly **75** (1968), 685–686. MR1534960

[OS] A. Oliĭnyk and V. I. Sushchanskiĭ, *A free group of infinite unitriangular matrices*, Mat. Zametki **67** (2000), no. 3, 382–386; English transl., Math. Notes **67** (2000), no. 3–4, 320–324. MR1779471 (2001f:20047)

[OS1] ———, *Representations of free products by infinite unitriangular matrices over finite fields*, Internat. J. Algebra Comput. **14** (2004), 741–749. MR2104778 (2005i:20034)

[O] K. C. O'Meara, *A new measure of growth for countable-dimensional algebras*. II, J. Algebra **172** (1995), no. 1, 214–240. MR1320631 (96b:16023)

[S] W. Sierpiński, *Sur une décomposition d'ensembles*, Monatsh. Math. Phys. **35** (1928), 239–242. MR1549531

[Tits] J. Tits, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270. MR0286898 (44:4105)

[T] D. V. Tjukavkin, *Rings all of whose one-sided ideals are generated by idempotents*, Comm. Algebra **17** (1989), no. 5, 1193–1198. MR0993398 (90h:16024)

[V] A. M. Vershik, *Asymptotic aspects of the representation theory of symmetric groups*, Selecta Math. Soviet. **11** (1992), no. 2, 159–179. MR1166626

[VK] A. M. Vershik and S. V. Kerov, *On an infinite-dimensional group over a finite field*, Funktsional. Anal. i Prilozhen. **32** (1998), no. 3, 3–10; English transl., Funct. Anal. Appl. **32** (1998), no. 3, 147–152 (1999). MR1659651 (2000b:22006)

Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland

E-mail address: w.holubowski@polsl.pl

Received 15/AUG/2006

Translated by N. A. VAVILOV