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RELATIVE GRÖBNER–SHIRSHOV BASES FOR ALGEBRAS AND GROUPS

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Abstract. The notion of a relative Gröbner–Shirshov basis for algebras and groups is introduced. The relative composition lemma and relative (composition-)diamond lemma are established. In particular, it is shown that the relative normal forms of certain groups arising from Malcev's embedding problem are the irreducible normal forms of these groups with respect to their relative Gröbner–Shirshov bases. Other examples of such groups are given by showing that any group G in a Tits system (G, B, N, S) has a relative (B-)Gröbner–Shirshov basis such that the irreducible words are the Bruhat words of G.

§1. Introduction

In this paper we define the notion of a relative Gröbner–Shirshov basis and prove the relative Shirshov lemma and a relative composition-diamond lemma for Γ-algebras (Γ-semigroups, Γ-groups) presented by generators and defining relations. Here Γ is a fixed subgroup of the algebra (semigroup, group).

As illustrations, in subsequent sections we consider a number of examples of Γ-semigroups and Γ-groups for a nontrivial Γ, calculate relative Gröbner–Shirshov bases for them, and indicate some applications. Here are these examples.

1) Let Q_4 (see [4]) be the following semigroup:

$$Q_4 = \mathrm{sgr}\langle a_i, b_i, c_i, s_i, t_i, 1 \le i \le 4, v_0, v_1 \mid \\ a_i s_i = c_i v_0, b_i s_{i+1} = c_i v_1, b_i t_{i+1} = a_i t_i, 1 \le i \le 4\rangle,$$

where $s_5 = s_1$, $t_5 = t_1$.

Let R be the algebra of formal power series over Q_4 with coefficients in $GF(2)$. In other words, the algebra R is the completion of the semigroup algebra $GF(2)(Q_4)$ relative to the norm $2^{-\deg(f)}$, where $\deg(f)$ is the natural degree of an element $f \in GF(2)(Q_4)$. We fix the presentation of the multiplicative semigroup R^* of R that was found in [5]. Then R^* is a Γ-semigroup, where Γ is the group of units of R (Γ consists of series with nonzero constant terms).

By definition, to be a Γ-semigroup means that for any element q among the selected generators for R^* there are two isomorphic subgroups Γ_q and Γ'_q of Γ with

$$\gamma q = q\gamma',$$

where $\gamma \in \Gamma_q$ and $\gamma' \in \Gamma'_q$. In our case,

$$\Gamma_q = \{1 + qA \mid A \in R\}, \quad \Gamma'_q = \{1 + Aq \mid A \in R\}.$$

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It should be noted that the above Γ-semigroup structure of R^* was absolutely crucial in proving that R^* is embeddable into a group (see [6, 8, 9]), though R is not embeddable into any skew field (see §3). The relative composition-diamond lemma allows us to say that the main technical result of [6, 8, 9] consists in finding a relative Gröbner–Shirshov basis of the universal group $G(R^*)$ in an explicit form (see §3).

Recall that a universal group $G(P)$ of some semigroup presentation P is the group with the same generators and defining relations as P.

2) Let Q be any SNA-semigroup presentation in the sense of [8]. This is a semigroup with defining relations of the form

$$wh = uf,$$

where w, h, u, f are generators with some additional properties. One of the crucial properties of Q is that Q has no left (right) cycles of order not exceeding 3 in the sense of Adyan [1]; in particular, no relations of the form

$$wh = uf, \quad w_1h_1 = uf_1, \quad w_1h_2 = wf_2$$

may be fulfilled in Q simultaneously. This is not the case for the above semigroup Q_4. However, obviously, there are many examples of SNA-semigroups (cf. [8]).

Let k be a field, and let $\overline{kQ}$ be the algebra of formal power series over Q with coefficients in k. Again, the semigroup $\overline{kQ}^*$ is a Γ-semigroup, where Γ is the group of units of $\overline{kQ}$. The universal group $G(\overline{kQ}^*)$ is also a Γ-group. Again, the relative composition-diamond lemma makes it possible to state the main technical result of [8] about SNA-semigroups as follows: a relative Gröbner–Shirshov basis of the universal group $G(\overline{kQ}^*)$ can be found explicitly (see §3).

As a result, any semigroup $\overline{kQ}^*$ is embeddable into a group (see [8]).

3) Let (G, B, N, S) be a Tits system (see, e.g., [2]). Then G can be viewed as a Γ-group for $\Gamma = B$ (see §4). From this point of view, the Bruhat normal form for G is the irreducible normal form for G with respect to a relative Gröbner–Shirshov basis of G (see §5).

It should be noted that in the papers [4, 5, 6, 8, 9] cited above (see also [12]) the main goal was to solve the following Malcev embedding problem (cf. [23]) in the class of semigroup algebras:

Does there exist a semigroup algebra that is not embeddable into any skew field, but with the multiplicative semigroup embeddable into a group?

In §4 we discuss an analog of the Malcev problem for group algebras, as well as some other problems for them.

§2. Relative composition lemma

Let $X = \{x_i, i \in I\}$ be a well-ordered set (we transfer this order to I, that is, $x_i < x_j$ if $i < j$), and let Γ be a group, the elements of which will be denoted by small greek letters $\gamma, \delta, \ldots$ with indices. Consider the multiplication table

$$\gamma\delta = \mu \tag{1}$$

for Γ.

Suppose that for any $x \in X$ two isomorphic subgroups Γ_x and Γ'_x of Γ are fixed, together with an isomorphism

$$\partial_x : \Gamma_x \to \Gamma'_x. \tag{2}$$

Then we consider $k\langle X; \Gamma\rangle$, the Γ-free algebra over a field k and the set X (see [14]). In fact, this is an associative algebra with 1, generated by X and Γ, with the defining

relations (1) and

$$\gamma x = x\gamma', \tag{3}$$

where $x \in X$, $\gamma \in \Gamma_x$, $\gamma' = \partial_x(\gamma) \in \Gamma'_x$. If $\Gamma = 1$, then this algebra is simply the free algebra over k and X.

By definition, the Γ-words have the form

$$u = \gamma_0 x_1 \cdots x_k \gamma_k, \tag{4}$$

where $k \geq 0$, $\gamma_j \in \Gamma$, $x_{i_j} \in X$. We denote the projection of u to X by $[u]$, that is,

$$[u] = x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{5}$$

Also, we define

$$|u| = |[u]| = k;$$

this is the *length* or the *degree* of u.

It is easily seen that two Γ-words $u = \gamma_0 x_{i_1} \cdots x_{i_k} \gamma_k$ and $v = \delta_0 x_{j_1} \delta_1 \cdots x_{j_l} \delta_l$ are equal in the Γ-free algebra $k\langle X; \Gamma\rangle$ if and only if

$$k = l, \quad x_{i_s} = x_{j_s}, \quad 1 \leq s \leq k, \tag{6}$$

and

$$\gamma_0 = \delta_0 \gamma_{x_{i_1}}, \quad \gamma'_{x_{i_1}} \gamma_1 = \delta_1 \gamma_{x_{i_2}}, \ldots, \gamma'_{x_{i_k}} \gamma_k = \delta_k.$$

If $u = u_1 v u_2$ for some Γ-words, then v is called a Γ-subword of u. The relation $uv = wt$ implies immediately that $u = wa$ and $t = av$, or $w = ua$ and $v = at$, for some Γ-word a.

We fix a monomial order of X-words (5), such that $u < v \Rightarrow aub < avb$ for any X-words u, v, a, b, assuming that this order agrees with the order of X (clearly, this is possible). Then, it gives rise to a quasi-order of Γ-words:

$$u \preceq v \quad \text{if} \quad [u] \leq [v]. \tag{7}$$

Thus,

$$u \preceq v \Rightarrow aub \preceq avb$$

for any Γ-words u, v, a, b.

Let $f \in k\langle X; \Gamma\rangle$. We assume that f is presented in the form

$$f = \sum \alpha_i u_i,$$

where $\alpha_i \in k$ and the u_i are pairwise distinct Γ-words.

For the above polynomial f, we use $\bar{f}$ to denote a leading monomial of f. In general, there are several leading monomials of f. In other words, $\bar{f}$ is a maximal Γ-word occurring in f with a nonzero coefficient. If $\bar{f}$ is unique, then we call f a *strong* polynomial and $\bar{f}$ the *strong* leading monomial of f.

We say that f is *monic strong* if f is strong with leading coefficient 1 (i.e., the coefficient of $\bar{f}$ is 1).

Let $S \subset k\langle X; \Gamma\rangle$. Then we call S a *monic strong set* if any $s \in S$ is a monic strong polynomial. From now on, we use S to denote some set of monic strong polynomials.

The *composition* of two monic strong polynomials f, g can be defined, in essence, in the same way as in the case where $\Gamma = 1$ (see, e.g., [15]). Namely, we define

$$(f, g)_w = \begin{cases} fv - ug & \text{if} \quad w = \bar{f}v = u\bar{g}, |\bar{f}| > |u|, \\ f - ugv & \text{if} \quad w = \bar{f} = u\bar{g}v, \end{cases} \tag{8}$$

where w, v, u are Γ-words. In the second case of (8), the transformation

$$f \mapsto f - ugv \tag{9}$$

is called elimination of the leading word (briefly, ELW) of g in f. In both cases of (8), for any leading word $\overline{(f,g)_w}$ we have

$$\overline{(f,g)_w} < w, \tag{10}$$

though $(f,g)_w$ may fail to be a strong polynomial.

Let f be a polynomial (not necessarily strong) such that a leading monomial $\bar{f}$ of f is of the form

$$\bar{f} = u\bar{g}v,$$

where g is a monic strong polynomial. Suppose that f is $\bar{f}$-monic, that is, $\bar{f}$ occurs in f with coefficient 1. Then the transformation (9) (elimination of the leading word of g in f) is applicable. In this case, for the polynomial $f - ugv$ the number of occurrences of leading words that are quasi-equal to $\bar{f}$ is smaller than that for f. By using this crucial observation, we obtain the following important property: any sequence of ELW's of strong monic polynomials is finite; i.e., any sequence

$$f_1 \to f_2 \to \cdots \to f_n \to \cdots$$

is finite, where the leading word of a strong monic polynomial is eliminated at each step.

A composition of the form (8) is said to be *trivial relative to* S (more precisely, relative to S and w), and we write $(f,g)_w \equiv \mod (S,w)$ if

$$(f,g)_w = \sum \alpha_i u_i s_i v_i, \tag{11}$$

where $\alpha_i \in k$, $s \in S$, u_i, v_i are Γ-words, and

$$u_i \bar{s}_i v_i \prec w. \tag{12}$$

In particular, if $(f,g)_w$ goes to zero under the action of ELW's of S, then $(f,g)_w$ is trivial relative to S, w (see the proof of the "if" part in the next theorem).

A monic strong set S is called a *relative* Γ-Gröbner–Shirshov basis (in $k\langle X;\Gamma\rangle$) if any composition $(f,g)_w$ of elements f,g of S is trivial $\mathrm{mod}(S,w)$.

Remark 2.1. For the first time, condition (11) (for $\Gamma = 1$) was stated in [10] for the case of Lie algebras; then it was used in [11] for associative algebras, and also in [22] for commutative algebras. In fact, condition (11) is easier to check, as compared to a stronger condition involving ELW's, though the latter condition is algorithmic (for a recursive S). For the definition of a (relative) Gröbner–Shirshov basis, any of these two conditions can be used (see below).

With the above concepts at hand, now we can formulate the relative composition lemma. This is an analog (even a generalization) of Shirshov's composition lemma [25] (see, e.g., [15]). In what follows, $\mathrm{id}(S)$ stands for the two-sided ideal of the algebra $k\langle X;\Gamma\rangle$ generated by S.

Theorem 2.1 (Relative composition lemma)**.** *Let $S \subset k\langle X;\Gamma\rangle$ be a strong monic set of the Γ-free algebra $k\langle X;\Gamma\rangle$. Then S is a relative Gröbner–Shirshov basis if and only if for any $f \in \mathrm{id}(S)$ we have $\bar{f} = a\bar{s}b$ for some leading monomial $\bar{f}$ of f and $s \in S$.*

Proof. The "only if" part. Let S be a relative Gröbner–Shirshov basis, and let $f \in \mathrm{id}(S)$. Then

$$f = \sum_{i=1}^{n} \alpha_i \alpha_i s_i b_i, \tag{13}$$

where $\alpha_i \in k$, the α_i, b_i are Γ-words, and $s_i \in S$. It is easily seen that each $a_i s_i b_i$ is a monic strong polynomial with the leading word $a_i \bar{s}_i b_i$. Now, we arrange these leading

words in nonincreasing order by

$$w = a_1\bar{s}_1b_1 = a_2\bar{s}_2b_2 = \cdots = a_k\bar{s}_k \succeq a_{k+1}\bar{s}_{k+1}b_{k+1} \succeq \cdots \succeq a_n\bar{s}_nb_n$$

and $w \neq a_{k+i}s_{k+i}b_{k+i}$, $i \geq 1$. If $k = 1$, then on the right-hand side of (13) there are no Γ-words equal to $a_1\bar{s}_1b_1$. This shows that $\bar{f} = a_1\bar{s}_1b_1$ is a leading word of f. Let $k \geq 2$, and let

$$a_1\bar{s}_1b_1 = a_2\bar{s}_2b_2, \quad [a_1][\bar{s}_1][b_1] = [a_2][\bar{s}_2][b_2]. \tag{14}$$

The following three cases arise.

1) *The subwords* $[\bar{s}_1]$ *and* $[\bar{s}_2]$ *do not intersect.* For example, let

$$[a_2] = [a_1][\bar{s}_1][d], \quad [b_1] = [d][\bar{s}_2][b_2],$$

where $[d]$ is an X-word. Then (14) implies that

$$a_2 = a_1\bar{s}_1d, \quad b_1 = d\bar{s}_2b_2$$

for some Γ-word d.

Consequently,

$$\begin{aligned}\alpha_1a_1\bar{s}_1b_1 + \alpha_2a_2s_2b_2 &= (\alpha_1+\alpha_2)a_1s_1b_1 - \alpha_2(a_1s_1b_1 - a_2s_2b_2)\\ &= (\alpha_1+\alpha_2)a_1s_1b_1 - \alpha_2(a_1s_1d\bar{s}_2b_2 - a_1\bar{s}_1ds_2b_2)\\ &= (\alpha_1+\alpha_2)a_1s_1b_1 - \alpha_2(a_1s_1d(\bar{s}_2 - s_2)b_2 + a_1(s_1 - \bar{s}_1))ds_2b_2),\end{aligned}$$

where

$$a_1s_1d(\bar{s}_2 - s_2)b_2 = \sum \gamma_jc_js_jd_j, \quad c_j\bar{s}_jd_j \prec w,$$

and

$$a_1(s_1 - \bar{s}_1)ds_2b_2 = \sum \delta_jc'_js_jd'_j, \quad c'_j\bar{s}_jd'_j \prec w.$$

The above identities allow us to rewrite (13) with a smaller k.

2) *The subwords* $[\bar{s}_1]$ *and* $[\bar{s}_2]$ *in* (14) *have nonempty intersection, but none of them includes the other.* For example, let

$$[\bar{s}_1] = [a][c], \quad [\bar{s}_2] = [c][b], \quad [c] \neq 1.$$

Then (14) implies that

$$a_2 = a_1a, \quad b_1 = bb_2, \quad \bar{s}_1b = a\bar{s}_2 = w_1, \quad |\bar{s}_1| > |a|,$$

where a, b are some Γ-words. Indeed,

$$\bar{s}_1 = a'c', \quad \bar{s}_2 = c'b', \quad a_1a'c'b_1 = a_2c'b'b_2$$

for some Γ-words a', c', b'. Then

$$a_2 = a_1a'\gamma, \quad \gamma c' = c'\delta, \quad b_1 = \delta b'b_2$$

for some γ, δ in Γ. It remains to put $a = a'\gamma$, $b = \delta b'$.

Now we can deduce the formula

$$\begin{aligned}\alpha_1a_1s_1b_1 + \alpha_2a_2s_2b_2 &= (\alpha_1+\alpha_2)a_1s_1b_1 - \alpha_2(a_1s_1b_1 - a_2s_2b_2)\\ &= (\alpha_1+\alpha_2)a_1s_1b_1 - \alpha_2(a_1s_1bb_2 - a_1as_2b_2)\\ &= (\alpha+\alpha_2)a_1s_1b_1 - \alpha_2a_1(s_1b - as_2)b_2\\ &= (\alpha_1+\alpha_2)a_1s_1b_1 - \alpha_2a_1(s_1,s_2)_{w_1}b_2.\end{aligned}$$

Since S is a Gröbner–Shirshov basis, the composition $(s_1,s_2)w_1$ is trivial $\mathrm{mod}(S,w_1)$, and we obtain

$$a_1(s_1,s_2)_{w_1}b_2 = \sum \gamma_ja_1c_js_jd_jb_2,$$

where $s_j \in S$ and

$$c_j\bar{s}_jd_j \prec w_1.$$

Then

$$a_1 c_j \bar{s}_j d_j b_2 \prec a_1 w_1 b_2 = w.$$

By using the above relations, we can rewrite (13) with a smaller k.

3) *One of the subwords* $[\bar{s}_1]$, $[\bar{s}_2]$ *is a subword of the other.*

For example, let

$$[\bar{s}_1] = [a][\bar{s}_2][b].$$

Arguing as in the preceding case, we deduce from (14) that

$$a_2 = a_1 a'\gamma, \quad \gamma\bar{s}_2 = \bar{s}_2\delta, \quad \delta b_2 = b' b_1$$

for some Γ-words a', b' and some elements $\gamma, \delta \in \Gamma$. Putting $a = a'\gamma$, $b = \delta^{-1}b'$, we see that

$$w_1 = \bar{s}_1 = a\bar{s}_2 b, \quad a_2 = a_1 a, \quad b_2 = b b_1$$

for some Γ-words a, b. Now, we can deduce the formula

$$\begin{aligned}
\alpha_1 a_1 s_1 b_1 + \alpha_2 a_2 s_2 b_2 &= (\alpha_1 + \alpha_2) a_1 s_1 b_1 - \alpha_2 (a_1 s_1 b_1 - a_2 s_2 b_2) \\
&= (\alpha_1 + \alpha_2) a_1 s_1 b_1 - \alpha_2 (a_1 s_1 b_1 - a_1 a s_2 b b_1) \\
&= (\alpha_1 + \alpha_2) a_1 s_1 b_1 - \alpha_2 a_1 (s_1 - a s_2 b) b_1 \\
&= (\alpha_1 + \alpha_2) a_1 s_1 b_1 - \alpha_2 a_2 (s_1, s_2)_{w_1} b_1.
\end{aligned}$$

As before,

$$a_1 (s_1, s_2)_{w_1} b_1 = \sum \gamma_j a_1 c_j s_j d_j b_1,$$

where $s_j \in S$ and

$$c_j \bar{s}_j d_j \succeq w_1, \quad a_1 c_j \bar{s}_j d_j b_1 \prec a_1 w_1 b_1 = w.$$

After this, we can use the above relations to rewrite (13) with a smaller k.

Thus, by induction on k, we can easily complete the proof of the "only if" part.

The "if" part. First, we let $(f, g)_w$ be a composition of elements of S relative to w. Then

$$(f, g)_w = \sum \beta_j u_j, \quad u_j \prec w.$$

By using ELW's of S in $(f, g)_w$, we can present $(f, g)_w$ in a form h such that any leading monomial $\bar{h}$ of h contains no subwords $\bar{s}$, $s \in S$. Clearly, we have $h = 0$, because $h \in \mathrm{id}(S)$. By the construction of h,

$$(f, g)_w = \sum \alpha_i a_i s_i b_i, \quad s_i \in S, \quad a_i \bar{s}_i b_i \prec w,$$

i.e., $(f, g)_w = 0 \,(\mathrm{mod}(S, w))$, and precisely this is required for S to be a relative Gröbner–Shirshov basis. To clarify the last formula, let u_j be a leading monomial in the initial presentation of $(f, g)_w$, $u_j \prec w$. We know that $u_j < w$. Suppose that $u_j = a\bar{s}b$ for some $s \in S$. Then

$$(f, g)_w - \beta_j asb$$

has a smaller number of leading monomials than $(f, g)_w$, and $a\bar{s}b \prec w$. After a finite number of such steps, we can obtain the zero polynomial and, with it, the required presentation of $(f, g)_w$. □

Theorem 2.2 implies the following useful proposition. This is an analog (again, a generalization) of the composition-diamond lemma (see, e.g., [15]). We say that a Γ-word u is *irreducible* relative to some S (Irr-word for short) if $u \neq a\bar{s}b$, $s \in S$. The set of all Irr-words relative to S is denoted by $\mathrm{Irr}(S)$ (this set was denoted by $PBW(S)$ in [15]).

Theorem 2.2 (Relative composition-diamond lemma)**.** *Let $S \subset k\langle X;\Gamma\rangle$ be a monic strong set. Then S forms a relative Gröbner–Shirshov basis in $k\langle X;\Gamma\rangle$ if and only if $\mathrm{Irr}(S)$ is a basis over k of the algebra $k\langle X;\Gamma|S\rangle$, that is, the Γ-algebra with generators X and defining relations S.*

Proof. The "only if" part. Using ELW's of S, we can easily check that every $f \in k\langle X;\Gamma\rangle$ can be presented modulo $\mathrm{id}(S)$ as a linear combination of Irr-words. On the other hand, all the Irr-words are linearly independent $\mathrm{mod}\,\mathrm{id}(S)$.

The "if" part. Suppose that a composition $(f,g)_w$, where $f,\ g \in S$, is nontrivial. Then we can use ELW's of S to present $(f,g)_w$ as a nontrivial linear combination of Irr-words plus some element of $\mathrm{id}(S)$. This contradicts the linear independence of the Irr-words. □

The following corollary also characterizes the relative Gröbner–Shirshov bases in $k\langle X;\Gamma\rangle$. This statement is well known for the case where $\Gamma = 1$ (it dates back to [25] and [10]).

Corollary 2.3. *Let $S \subset k\langle X;\Gamma\rangle$ be a monic strong subset of $k\langle X;\Gamma\rangle$. Then S is a relative Gröbner–Shirshov basis in $k\langle X;\Gamma\rangle$ if and only if any composition of elements of S can be reduced to zero by* ELW*'s of S.*

Proof. The "only if" part. By using ELW's of S, we can represent a composition $(f,g)_w$, $f,\ g \in S$, as a linear combination of Irr-words. By the relative composition lemma, this linear combination must be trivial.

The "if" part. This part is clear: see the text after (12) and the proof of the "if" part of Theorem 2.3. □

Recall that

$$k\langle X;\Gamma|S\rangle = k\langle X;\Gamma\rangle/\,\mathrm{id}(S)$$

is the Γ-algebra generated by X with the set of defining relations S. We define the Γ-semigroups and Γ-groups in a similar way.

Let $X = \{x_i, i \in I\}$, Γ, Γ_x, Γ'_x, $x \in X$, be as above. By a Γ-free semigroup $\mathrm{sgr}\langle X;\Gamma\rangle$ we mean a semigroup with 1 that is generated by X and Γ, with the defining relations (1) and (3). The elements of $\mathrm{sgr}\langle X;\Gamma\rangle$ are the Γ-words (4) with the equality relation (6).

Let $S = \{(u_j, v_j), j \in J\}$ be a set of pairs of Γ-words. As usual, we introduce the Γ-semigroup with defining relations S by

$$P = \mathrm{sgr}\langle X;\Gamma \mid u_j = v_j,\ j \in J\rangle = \mathrm{sgr}\langle X;\Gamma \mid S\rangle \tag{15}$$

(we identify the relation $u = v$ with the pair (u,v)). Then the semigroup algebra $k(P)$ is the Γ-algebra

$$k(P) = \langle X;\Gamma \mid u_i - v_j = 0,\ j \in J\rangle = \langle X;\Gamma \mid S\rangle \tag{16}$$

(again we identify the relation $u-v = 0$ with the pair (u,v)). Clearly, a relative Gröbner–Shirshov basis of the semigroup algebra (16) does not depend on k and is simply a set of semigroup relations, say S^{comp}. This set S^{comp} is called a relative Gröbner–Shirshov basis of the semigroup (15).

The Γ-free group $\mathrm{gr}\langle X;\Gamma\rangle$ is defined as the group generated by X and Γ with the defining relations (1) and (3). In fact, this is the free Γ-semigroup generated by $X \cup X^{-1}$, with defining relations

$$x_i x_i^{-1} = 1 \quad \text{and} \quad x_i^{-1} x_i = 1, \quad \text{where} \quad x_i \in X.$$

Now, we can easily define the Γ-group

$$G = \mathrm{gr}\langle X;\Gamma \mid S\rangle$$

generated by X with defining relations S. The group algebra $k(G)$ has the following presentation:

$$k(G) = \langle X \cup X^{-1}; \Gamma \mid x_i x_i^{-1} = 1, x_i^{-1} x_i = 1, S \rangle. \tag{17}$$

A relative Gröbner–Shirshov basis of the algebra (17) will be called a relative Gröbner–Shirshov basis of G with respect to the set of generators X and the order of the group of X-words.

§3. Groups like $G(\overline{GF(2)(Q_0)})$, $Q_0 = \mathrm{sgr}\langle w, h, u, f \mid wh = uf \rangle$

We denote by $GF(2)(Q_0)$ the semigroup algebra of Q_0 over $GF(2)$. Then $\overline{GF(2)(Q_0)}$ is the completion of $GF(2)(Q_0)$ up to the algebra of formal infinite power series. It is a domain (see [23]). Let $\overline{GF(2)(Q_0)}^*$ denote the multiplicative semigroup of nonzero elements. In [5], a certain representation of this semigroup was found in terms of generators and relations; we fix this presentation of $\overline{GF(2)(Q_0)}^*$. We use $G_0 = G(\overline{GF(2)(Q_0)}^*)$ to denote the universal group of fractions of the semigroup in question (recall that this is the group with the same generators and defining relations as $\overline{GF(2)(Q_0)}^*$. An abstract definition of groups "like" G_0 was given in [14]. Explicit examples of such groups can be found in [8]. These are groups of the form $G(\overline{GF(2)(Q)}^*)$, where Q is the so-called SNA-semigroup with defining relations $w_i h_i = u_i f_i$, where w_i, h_i, u_i, $f_i \in X$, under some conditions. Here are some properties of these groups.

1) They are Γ-groups, where Γ is the group of invertible series (units) of $\overline{GF(2)(Q)}$. For any p in the fixed system of generators of the semigroup $\overline{GF(2)(Q)}^*$, we have

$$\Gamma_p = \{1 + pA\}, \quad \Gamma'_p = \{1 + Ap\},$$

where $A \in \overline{GF(2)(Q)}$.

2) In [8], for any $G = G(\overline{GF(2)(Q)}^*)$, where Q is an SNA-semigroup, we constructed a rewriting system (semi-Thue-system)

$$\sqcap = [\Sigma; \Gamma \mid A_j \to B_j, j \in J]$$

such that the set of canonical words (that contain no A_j, $j \in J$) is a Γ-basis of G (i.e., any Γ-word is equal to a unique canonical Γ-word). Actually, for any $j \in J$ we have

$$A_j > B_j,$$

in the "tower" order of the $\sum$-words.

Now we briefly describe the tower order of words. First, we let $X = Y \cup Z$ be a well-ordered set such that $z > y$ for any $z \in Z$, $y \in Y$. Suppose that the set Y^* of Y-words can be well ordered in agreement with the order of all Y and with concatenation of words. Any X-word has the form

$$u = u_0 z_1 u_1 \cdots u_k z_k u_{k+1},$$

where $k \geq 0$, $u_i \in Y^*$, $s_i \in Z$. We define

$$wt(u) = (k, u_0, z_1, u_1, \ldots, u_k, z_k, u_{k+1})$$

and order all the wt's lexicographically. Then the tower order of the X-words is defined as follows:

$$u > v \Leftrightarrow wt(u) > wt(v).$$

Clearly, this is a well-ordering compatible with concatenation of words.

Property 2) above and Theorem 2.3 (the relative composition-diamond lemma) imply the following statement.

Theorem 3.1. *Let*

$$\sqcap = [\Sigma; \Gamma \mid A_i \to B_j, j \in J]$$

be the rewriting system for $G = G(\overline{GF(2)(Q)}^*)$, *where* Q *is an* SNA*-semigroup* [8]. *Then the set*

$$S = \{A_j - B_j, j \in J\} \subset GF(2)\langle \Sigma; \Gamma \rangle$$

is the relative Γ*-Gröbner–Shirshov basis of* G *with respect to the tower order of words.*

Corollary 3.2 ([8])**.** *Any semigroup* $\overline{GF(2)(Q)}^*$, *where* Q *is an* SNA*-semigroup, is embeddable in a group.*

Now, we return to SNA-semigroups Q [8]. A crucial property of Q is that Q has no left (right) cycles of order not exceeding 3 in the sense of Adyan [1]; in particular, Q admits no relations of the form

$$wh = uf, \quad w_1h_1 = uf_1, \quad w_1h_2 = wf_2. \tag{18}$$

On the contrary, the semigroup Q_4 (see [8, 9]) is defined by the same relations as in the Introduction:

$$a_is_i = c_iv_0, \quad b_is_{i+1} = c_iv_1, \quad b_it_{i+1} = a_it_i, \tag{19}$$

where $1 \le i \le 4$, $s_5 = s_1$, $t_5 = t_1$. This system of relations is similar to (18). Consequently, Q_4 is not an SNA-semigroup.

Here we note that the algebra $GF(2)(Q_4)$ is not embeddable into a division algebra; it is not even invertible. (A domain D is said to be invertible if $D \subset D_1$, where D_1 is a domain such that any nonzero element of D is invertible in D_1). Indeed, suppose that $D = GF(2)(Q_4)$ is contained in D_1 with the above property. Then in D_1 we have

$$v_0^{-1}v_1 = s_i^{-1}a_i^{-1}b_is_{i+1} = s_i^{-1}t_it_{i+1}^{-1}s_{i+1}, \quad 1 \le i \le 4,$$

whence

$$(v_0^{-1}v_1)^4 = 1.$$

This means that

$$(v_0^{-1}v_1 - 1)^4 = 0$$

in D_1, and $v_0^{-1}v_1 = 1$, so that $v_0 = v_1$ in D, a contradiction.

Remark 3.3. The above argument shows that the algebra $GF(2)(Q_4)$ cannot be embedded in any domain D_1 with $Q_4 \subset U(D_1)$, where $U(D_1)$ is the group of units of D_1.

Let $\overline{GF(2)(Q_4)}$ be the completion of $GF(2)(Q_4)$. Then the main results of [6, 8, 9] take the following form.

Main Theorem. *The semigroup* $\overline{GF(2)(Q_4)}^*$ *is embeddable in a group.*

For the proof of this theorem, the concept of an "almost" Γ-normal form was constructed for the group $G_4 = G(\overline{GF(2)(Q_4)}^*)$. Here, as before, Γ is the group of units of $\overline{GF(2)(Q_4)}$, and

$$\Gamma_q = \{1 + qA\}, \quad \Gamma'_q = \{1 + A_q\},$$

where q is among the fixed generators of $\overline{GF(2)(Q_4)}^*$, and $A \in \overline{GF(2)(Q_4)}$. The only problem ("almost") is that we changed the generator v_1 of G to

$$p = v_0^{-1}v_1.$$

But all arguments of [8] and [9] are still valid for v_1 in place of p. This can be checked directly. We only need to change the presentation of the first Γ-group G_1 in [9]:

$$G_1 = \langle v_0; v_1; \Gamma \mid (v_0^{-1}v_1)^4 = 1 \rangle.$$

Thus, the relative Gröbner–Shirshov basis of this new Γ-group G_1 consists only of trivial relations and the relations

$$\begin{aligned} v_0^{-1}v_1v_0^{-1}v_1 &= v_1^{-1}v_0v_1^{-1}v_0, \\ v_0v_1^{-1}v_0v_1^{-1} &= v_1v_0^{-1}v_1v_0^{-1}, \end{aligned} \tag{20}$$

where the left-hand sides are maximal words in the tower order relative to

$$\{v_0^{-1}, v_0\} \dot\cup \{v_1^{-1}, v_1\},$$

and $v_0^{-1} < v_0 < v_1^{-1} < v_1$.

So, the results of [8] and [9] lead to the following statement.

Theorem 3.4. *Let*

$$\sqcap_4 = [\Sigma; \Gamma \mid A_j \to B_j,\ j \in J]$$

be the rewriting (semi-Thue) system constructed for $G(\overline{GF(2)(Q_4)}^*)$ *in* [9], *with* (20) *in place of the transformations* $p^4 \to 1$, $p^{-1} \to p^3$. *Then the relations*

$$S_4 = \{A_j - B_j, j \in J\} \tag{21}$$

constitute a relative (Γ)*-Gröbner–Shirshov basis for the group* $G(\overline{GF(2)(Q_4)}^*)$ *with respect to the tower order of group words.*

The main theorem mentioned above follows immediately from Theorem 3.4. On the other hand, we could use the relative composition-diamond lemma (Theorem 2.3) to prove Theorem 3.4 by checking that all possible compositions of the polynomials (21) are trivial. In principle, this approach would be easier than the rather complicated arguments used in [8] and [9], based on the notion of a "group with a relative standard basis" [7].

§4. Digression to some open problems for semigroup and group algebras

The following are some classes of domains (i.e., associative rings without divisors of zero, see [13]):

$\mathcal{D}_0$ — all domains;

$\mathcal{D}_1$ — all domains with multiplicative semigroups embeddable into groups;

$\mathcal{D}_2$ — all invertible domains (see §3);

$\mathcal{D}_3$ — all domains embeddable into fields; then

$$\mathcal{D}_0 \supseteq \mathcal{D}_1 \supseteq \mathcal{D}_2 \supseteq \mathcal{D}_3. \tag{22}$$

Malcev's famous example [23] shows that $\mathcal{D}_0 \neq \mathcal{D}_1$. In particular, it follows that $\mathcal{D}_0 \neq \mathcal{D}_3$, which solves the problem of Van der Waerden (see [26]).

The example given by Bokut in [6, 8, 9] shows that $\mathcal{D}_1 \neq \mathcal{D}_2$. The examples given by Bowtell [3] and Klein [20], together with the result of Gerasimov in [18], show that $\mathcal{D}_2 \neq \mathcal{D}_3$ (see also [17]). In fact, these three examples give a complete solution to the Malcev problem.

Remark 4.1. A. I. Malcev formulated his problem ($\mathcal{D}_1 \neq \mathcal{D}_3$) around 1937 (see [24, p. 5]). At the Moscow ICM, 1966, the solution of the Malcev problem was announced by L. A. Bokut, by P. M. Cohn on behalf of his student A. Bowtell, and by S. Amitsur on behalf of his student A. A. Klein. Next year, three papers on this subject were published (see [6, 3, 20]). The paper [6] was submitted by A. I. Malcev to the Russian Academy of Sciences Doklady. A full proof of Bokut's example appeared in [8, 9], as has already been mentioned.

In fact, the Malcev and Bokut examples are semigroup algebras. Note that, up to now, the Bokut example is the only known example of a semigroup algebra in $\mathcal{D}_1 \setminus \mathcal{D}_2$.

To present the situation for semigroup algebras, we let $\mathcal{S}_i$ be the intersection of $\mathcal{D}_i$ with the class of semigroup algebras, $0 \le i \le 3$. Then

$$\mathcal{S}_0 \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supseteq \mathcal{S}_3. \tag{23}$$

However, so far there are no examples of semigroup algebras in $\mathcal{D}_2 \setminus \mathcal{D}_3$.

Very interesting open problems arise if the domains in (23) are group algebras over a field. Consider the following classes of algebras:

$\mathcal{K}$ — all group algebras of torsion free groups;

$\mathcal{K}_0$ — all group algebras without zero divisors (i.e., domains);

$\mathcal{K}_1$ — all group algebras that are domains with multiplicative semigroups embeddable into groups;

$\mathcal{K}_2$ — all invertible group algebras;

$\mathcal{K}_3$ — all group algebras embeddable into fields.

Then

$$\mathcal{K} \supseteq \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \mathcal{K}_3. \tag{24}$$

The question as to whether $\mathcal{K} \ne \mathcal{K}_0$ becomes the famous Kaplansky zero-divisor problem [19].

The question as to whether $\mathcal{K}_0 \ne \mathcal{K}_3$ is a problem similar to the Van der Waerden problem in the class of group algebras.

The question as to whether $\mathcal{K}_1 \ne \mathcal{K}_3$ is a problem similar to the Malcev problem in the class of group algebras.

So far, all the proper embedding questions in (24) remain open and difficult problems.

§5. Tits systems

In this section, we shall show that the well-known Bruhat decompositions of elements of a group G involved in a Tits system (G, B, N, S) are Irr-words with respect to a relative (B)-Gröbner–Shirshov basis of G (see, e.g., [2]). The definition of a Tits system (G, B, N, S) is as follows. First, let G be a group, and let B and N be subgroups of G. Suppose $T = B \cap N$ is a normal subgroup of N and $S \subset W = N/B \cap N$.

Then, we present B, N, and T by the following multiplication tables:

$$\begin{aligned} B &= \langle \{b\} \mid bb' = b'' \rangle, \\ N &= \langle \{n\} \mid nn' = n'' \rangle, \\ T &= \langle \{t\} = B \cap N \mid tt' = t'' \rangle. \end{aligned} \tag{25}$$

A system (G, B, N, S) is called a Tits system if the following axioms are fulfilled (see [2]).

(T1) The set $B \cup N$ generates G, i.e.,

$$(\forall g \in G) \quad g = b_0 n_1 \cdots b_{k-1} n_k b_k, \tag{26}$$

where $k \le 0$, $b_i \in B$, $n_i \in N$.

(T2) S generates W and all elements of S are of order 2.

(T3) $(\forall s \in \{s\})$, $sBw \subset BwB \cup BswB$.

(T4) $(\forall s \in \{s\}) \quad sBs \not\subseteq B$.

Since $\{w\}$ is a system of representatives of the cosets nT, $n \in N$, we have

$$\begin{aligned} &(\forall n \in N) && n = wt, \\ &(\forall w, w' \in \{w\}) && ww' = w''t, \\ &(\forall t \in T, w \in \{w\}) && tw = wt', \end{aligned} \tag{27}$$

where all w's belong to $\{w\}$ and all t's belong to T. The third identity in (27) shows that, for a fixed w, the map

$$t \to t'$$

is an automorphism of T.

By (26) and (27), we have

$$(\forall g \in G) \quad g = b_0 w_1 b_1 \cdots w_k b_k, \tag{28}$$

where $k \geq 0$, $b_i \in B$, $w_i \in \{w\}$. Thus, G is generated by B and $\{w\}$.

Next, axiom (T2) implies that

$$(\forall w \in \{w\}) \quad w = s_1 \cdots s_n t, \quad s_i \in S, \quad t \in T, \tag{29}$$

$$(\forall s \in \{s\}) \quad s^2 = t, t \in T. \tag{30}$$

Indeed,

$$\begin{aligned} w &= s_1 t_1 s_2 t_2 s_3 t_3 \cdots s_n t_n \\ &= s_1 s_2 t_1' t_2 s_3 t_3 \cdots s_n t_n \\ &= s_1 s_2 s_3 t_1'' t_2' t_3 \cdots s_n t_n \\ &= \cdots \\ &= s_1 \cdots s_n t. \end{aligned}$$

Axiom (T3) shows that

$$(\forall s \in \{s\}, b \in B, w \in \{w\}) \quad sbw = b_1 s^\delta w b_2, \tag{31}$$

where $\delta = 0, 1$; b_1, $b_2 \in B$.

Moreover,

$$(\forall w_1, w_2 \in \{w\}, b \in B) \quad w_1 b w_2 = b_1 w b_2, \tag{32}$$

for some b_1, $b_2 \in B$, $w \in \{w\}$. Indeed, by (29), w_1 is representable in the form $w_1 = s_1, \ldots, s_n t$; next, $b_0 = tb \in B$, and (31) implies

$$w_1 b w_2 = s_1 \cdots s_n t b w_2 = s_1 \cdots s_n b_0 w_2 = b_1 s_1^{\delta_1} \cdots s_n^{\delta_n} w_2 b_2' = b_1 w t' b_2' = b_1 w b_2,$$

where $\delta_i = 0, 1$.

Finally, (T4) means that

$$(\forall s \in \{s\} \ \exists b \in B) \quad sbs \notin B. \tag{33}$$

The above axioms (more precisely, T1–T3) imply that G is generated by B and $\{w\}$, with the defining relations (25), (27), and (31). For any $w \in \{w\}$, we define

$$\begin{aligned} \Gamma_w &= \{\gamma \in B \mid (\exists \gamma' \in B)\ \gamma w = w \gamma'\}, \\ \Gamma_w' &= \{\gamma' \in B \mid (\exists \gamma \in B)\ w \gamma' = \gamma w\}. \end{aligned} \tag{34}$$

Then Γ_w, Γ_w' are two isomorphic subgroups of B that contain T. Also, the isomorphism $\Gamma_w \to \Gamma_w'$, $\gamma \mapsto \gamma'$, extends the automorphism $T \to T$, $t \mapsto t'$.

Now we see that G is indeed a B-group with the following defining B-group relations:

$$\begin{aligned} w_1 b w_2 &= b_1 w b_2, \\ w_1 w_2 &= w t. \end{aligned} \tag{35}$$

By using (35), we can show that any element g of G can be presented in the form $g = b_1 w b_2$.

Clearly, this is the Bruhat decomposition of g. It is well known that a Bruhat decomposition is unique:

$$b_1 w b_2 = b_1' w' b_2' \Rightarrow w = w' \tag{36}$$

(see, e.g., [2]). Then we have

$$\gamma = b_1^{'-1} b_1 \in \Gamma_w, \quad \gamma' = b_2' b_2^{-1} \in \Gamma_w'.$$

This shows the uniqueness of the Bruhat decomposition (36) in the sense of B-words.

Now, using Theorem 2.3 (the relative composition-diamond lemma) and assuming that $\{w\}$ is well ordered and the set of B-words on $\{w\}$ is ordered in the deg-lex order, we obtain the following result.

Theorem 5.1. *Let (G, B, N, S) be a Tits system. The relations*

$$w_1 b w_2 = b_1 w b_2,$$
$$w_1 w_2 = wt,$$

where w, $w_i \in \{w\}$, b, $b_i \in B$, $t \in T$, form a relative (B)-Gröbner–Shirshov basis of G as a B-group in the generators $\{w\}$. Next, the Irr-words of G with respect to the above relative Gröbner–Shirshov basis are the Bruhat decompositions (words) of G.

We illustrate Theorem 5.1 by $G = SL_2(k)$. We follow the notation of [21].

Suppose k is a field, $b \in k$, $a \in k \setminus \{0\}$. Write

$$u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \text{and} \quad w = s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $T = \{t(a), a \in k \setminus \{0\}\}$ is a maximal torus subgroup of G, $U = \{u(b), b \in k\}$ is a unipotent subgroup of G,

$$B = UT = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in k \setminus \{0\}, b \in k \right\}$$

is a Borel subgroup of G, and

$$N = T \cup wT = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & a^{-1} \\ a & 0 \end{pmatrix}, a \in k \setminus \{0\} \right\}.$$

Clearly, N is a subgroup of G and $N \cap B = T \lhd N$ is a normal subgroup of N. Then $(G, B, N, S = \{wT\})$ is a Tits system and the following relations are valid:

$$\begin{aligned} wu(a)w &= u(a)wt(-a)u(a), \\ w^2 &= t(-1), \\ u(b_1)u(b_2) &= u(b_1 + b_2), \\ t(a_1)t(a_2) &= t(a_1 a_2), \\ t(a)u(b) &= u(ba^2)t(a), \\ t(a)w &= wt(a). \end{aligned} \tag{37}$$

Moreover,

$$sBs = \left\{ \begin{pmatrix} a^{-1} & 0 \\ b & a \end{pmatrix} \right\} \not\subset B.$$

In this case $\Gamma_w = \Gamma_w' = \Gamma$. System (37) is a B-relative Gröbner–Shirshov basis of G in the deg order of B-words in $\{w\}$. Moreover, in this case, (37) may be regarded as an (absolute) Gröbner–Shirshov basis if U is well ordered, T is ordered arbitrarily, and then the words in $U \cup T$ are ordered by using deg-lex order, assuming $u(b) > t(a)$. Then, we order the words in $(U \cup T) \dot\cup \{w\}$ in the tower order. Now, by direct computation of compositions, it is easy to show that (37) is complete under compositions not only as a B-set, but in the absolute sense.

Using the usual composition-diamond lemma ($\Gamma = 1$), we obtain the following statement.

Theorem 5.2. *Any element of $SL_2(k)$ has a unique presentation in the form*

$$u(b)t(a) \quad or \quad u(b)wu(c)t(a).$$

Results such as Theorems 5.1 and 5.2 can be checked not only for $SL_2(k)$, but also for any Chevalley group (see [16, Theorem 8.4.3]), by applying the composition-diamond lemma with $\Gamma = 1$.

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