ZERO SETS OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE WITH ADDITIONAL CONDITIONS ON THE REAL AXIS

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Abstract. A complete description of the zero sets is obtained for some well-known classes of entire functions of exponential growth.

We denote by $B$ the class of entire functions of exponential type bounded on the real axis; by $C$, the class of Cartwright’s entire functions, i.e., entire functions $f$ of exponential type with the property

$$(1) \int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2} \, dx < \infty;$$

and by $D$, the class of entire functions $f$ of exponential type with the following property: for each sequence $h_m \in \mathbb{R}$ there exists a subsequence $h'_m$ and a function $\tilde{f} \not\equiv 0$ such that the sequence $f(z + h'_m)$ converges uniformly to a function $\tilde{f}$ on every compact subset of $\mathbb{C}$.

It is easily seen that $D \subset B \subset C$. The classes $B$ and $C$ are well known and play an important role in various topics of complex and functional analysis (see [1, 2, 3]); the class $D$ is an extension of the well-known class of sine-type functions (see [2]), and it inherits some properties of sine-type functions.

Here we consider sequences $\{a_k\}$ of complex numbers without finite limit points; note that each $\{a_k\}$ may appear in the sequence with a finite multiplicity. We say that a sequence $\{a_k\}$ is the zero set for an entire function $f$ if $f(a) \neq 0$ for $a \not\in \{a_k\}$ and $f(a) = f'(a) = \cdots = f^{(p-1)}(a) = 0$, $f^{(p)}(a) \neq 0$, where $p$ is the multiplicity of $a$ in $\{a_k\}$.

The zero set of any function $f \in C$ satisfies the following conditions:

$$(2) \sum_{a_k \neq 0} |\text{Im } a_k^{-1}| < \infty;$$

$$(3) \text{the limit } \lim_{R \to \infty} \sum_{0 < |a_k| < R} a_k^{-1} \text{ exists;}$$

and

$$(4) \lim_{R \to \infty} R^{-1} \text{card}\{a_k : 0 < |a_k| < R, |\arg a_k| \leq \alpha\} = d/2\pi,$$

$$(5) \lim_{R \to \infty} R^{-1} \text{card}\{a_k : 0 < |a_k| < R, |\arg a_k - \pi| \leq \alpha\} = d/2\pi,$$

for every $0 < \alpha \leq \pi/2$, where $d$ is the width of the indicator diagram of $f$ (see [2] p. 127)).

Thus, every function $f \in C$ has the form

$$(6) f(z) = Az^s e^{i\lambda z} \lim_{R \to \infty} \prod_{0 < |a_k| < R} (1 - z/a_k)$$

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with \( A \in \mathbb{C}, \lambda \in \mathbb{R} \) (see [2] p. 130). However, there are zero sets satisfying (2)–(5) and such that no entire function with this zero set belongs to \( C \).

In [1] Appendix VI, B. Levin investigated zero sets \( \{a_k\}, k \in \mathbb{Z} \), for functions of class \( B \) with the additional property \( \sup_{k} |a_k - h| < \infty \) for some \( h \in \mathbb{R} \). In the present paper we obtain a complete description of the zero sets for the classes \( B, C, \) and \( D \).

In what follows, we put \( n(c, t) = \text{card}\{a_k : |a_k - c| \leq t\} \) for \( c \in \mathbb{C} \). Assume also that

\[
(7) \quad n(0, t) = O(t), \quad t \to \infty,
\]

and

\[
(8) \quad n(0, t + 1) - n(0, t) = o(t), \quad t \to \infty.
\]

It is clear that (7) and (8) are preserved if we replace \( n(0, t) \) with \( n(c, t) \).

**Theorem 1.** A sequence \( \{a_k\} \subset \mathbb{C} \) is the zero set of some function \( f \in C \) if and only if it satisfies conditions (3), (7), and (8) and

\[
(9) \quad \int_{-\infty}^{\infty} \left[ \int_{0}^{\infty} |n(b, t) - n(x, t)|t^{-1}dt \right] dx < \infty
\]

for some point \( b \in \mathbb{R} \setminus \{a_k\} \).

**Theorem 2.** A sequence \( \{a_k\} \subset \mathbb{C} \) is the zero set of some function \( f \in B \) if and only if it satisfies conditions (3), (7), and (8), and

\[
(10) \quad \sup_{x \in \mathbb{R}} \int_{0}^{\infty} |n(b, t) - n(x, t)|t^{-1}dt < \infty
\]

for some point \( b \in \mathbb{R} \setminus \{a_k\} \).

**Theorem 3.** A sequence \( \{a_k\} \subset \mathbb{C} \) is the zero set of some function \( f \in D \) if and only if it satisfies conditions (3), (7), and (8), and

\[
(11) \quad \sup_{x \in \mathbb{R}} \left| \int_{0}^{\infty} |n(0, t) - n(x, t)|t^{-1}dt \right| < \infty.
\]

The proofs of Theorems 1–3 are based on the following lemma.

**Lemma 1.** Suppose a sequence \( \{a_k\} \subset \mathbb{C} \setminus \{0\} \) satisfies (3), (7), and (8). Then

\[
(12) \quad g(z) = \lim_{R \to \infty} \prod_{|a_k| < R} (1 - z/a_k)
\]

is a well-defined entire function of a finite exponential type, and

\[
(13) \quad \log |g(z)| = \int_{0}^{\infty} |n(0, t) - n(z, t)|t^{-1}dt
\]

for all \( z \in \mathbb{C} \). Moreover, if \( z_0 \) is a point of the sequence \( \{a_k\} \) with multiplicity \( l = n(z_0, 0) > 0 \), then

\[
(14) \quad \log \left| \frac{g^{(l)}(z_0)}{l!} \right| = \int_{0}^{\infty} \frac{n(0, t) - n(z_0, t)}{t} dt + \int_{0}^{1} \frac{n(0, t) - n(z_0, t) + l}{t} dt.
\]

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1. For example, take \( a_1 = -e^{2} \) and consider reals \( a_k < a_1, \ k = 2, \ldots \), such that the number \( n(r) = \text{card}\{a_k : a_k \geq -r\} \) is the integral part of \( r/\log^2 r \) for each \( r \geq e^{2} \). Then (2)–(4) are fulfilled and the function (3) with \( s = 0 \) does not satisfy (4) because for \( x > e^{2} \) we have

\[
\log |f(x)| = \int_{1}^{\infty} \log(1 + x + t) dt \quad (x > e^{2})
\]

(this idea belongs to Professor A. Grishin).

2. Theorems 1 and 2 were announced (with my permission) in the survey [4] p. 45.
Proof of Lemma 1. Relation (17) implies that the integral \( \int_0^\infty n(0,t)t^{-3}dt \) is finite; hence,

\[
\sum |a_k|^2 = \int_0^\infty t^{-2}dn(0,t) < \infty.
\]

Let \( K \) be a fixed disk in the complex plane. Then

\[
\sum_{R<|a_k|<R'} \log \left(1 - \frac{z}{a_k}\right) - \sum_{R<|a_k|<R'} \frac{z}{a_k} \to 0
\]
as \( R, R' \to \infty \) uniformly in \( z \in K \). Hence, uniformly in \( z \in K \) we have

\[
\lim_{R \to \infty} \sum_{|a_k|>R} \log \left(1 - \frac{z}{a_k}\right) = 0,
\]

and the function \( g \) is well defined. Now, conditions 3 and 7 show that the function \( 12 \) is of a finite exponential type.

Note that 13 is trivial for \( z = a_k \). Fix \( z \notin \{a_k\} \). Suppose \( \varepsilon > 0 \) is such that \( n(0,\varepsilon) = n(z,\varepsilon) = 0 \). Then for any \( R < \infty \) we have

\[
\sum_{|a_k| \leq R} (\log |z - a_k| - \log |a_k|) = \sum_{\varepsilon \leq |a_k - z| \leq R} \log |z - a_k| - \sum_{\varepsilon \leq |a_k| \leq R} \log |a_k|
\]

\[
+ \sum_{|a_k| \leq R, |z - a_k| > R} \log |z - a_k| - \sum_{|z - a_k| \leq R, |a_k| > R} \log |z - a_k|.
\]

Furthermore,

\[
\sum_{\varepsilon \leq |a_k - z| \leq R} \log |z - a_k| = n(z, R) \log R - \int_0^R n(z, t)t^{-1}dt
\]

and

\[
(\log R)[n(z, R) - n(0, R)] + \sum_{|a_k| \leq R, |z - a_k| > R} \log |z - a_k|
\]

\[
- \sum_{|z - a_k| \leq R, |a_k| > R} \log |z - a_k|
\]

\[
= \sum_{|a_k| \leq R, |z - a_k| > R} \log \frac{|z - a_k|}{R} + \sum_{|z - a_k| \leq R, |a_k| > R} \log \frac{R}{|z - a_k|}.
\]

Observe that if \( |z - a_k| > R \) and \( |a_k| \leq R \), then \( |a_k| > R - |z| \) and \( |z - a_k| \leq |z| + R \).

Also, if \( |a_k| > R \) and \( |z - a_k| \leq R \), then \( |a_k| \leq R + |z| \) and \( |z - a_k| > R - |z| \). Hence, as \( R \to \infty \), the right-hand side of (18) does not exceed the quantity

\[
O(|z/R|n(0, R) - n(0, R - |z|)) + O(|z/R|n(0, R + |z|) - n(0, R)).
\]

Recalling (3), we see that (19) tends to zero for fixed \( z \) as \( R \to \infty \). Therefore, (13) follows from (17) and (16). In order to get (14), it suffices to apply (13) to the function \( g(z)(z - z_0)^{-\ell} \). Lemma 1 is proved.

Proof of Theorems 1–3. Relation (15) shows that the limits as \( R \to \infty \) of the sums \( \sum_{|a_k - c| < R}(a_k - c)^{-1} \) and \( \sum_{|a_k - c| < R}(a_k)^{-1} \) exist simultaneously; the latter sum differs from \( \sum_{|a_k - c| < R}(a_k)^{-1} \) in no more than \( n(0, R + |c|) - n(0, R - |c|) = o(R) \) terms, and the modulus of each of these terms does not exceed \( (R - |c|)^{-1} \). Therefore, (3) is invariant with respect to the shift of the origin. Applying Lemma 1 to the function
lim_{R \to \infty} \prod_{|a_k-c|<R} (1-z/(a_k-c))$, we see that \( \int_0^\infty [n(c,t) - n(x,t)]t^{-1}dt^+ = O(|x|) \) as \( |x| \to \infty \). Therefore, \( \Box \) is also invariant with respect to the shift of the real parameter \( b \). Hence, there is no loss of generality in assuming that \( 0 \not\in \{a_k\} \) and \( b = 0 \) in \( \Box \) and \( \Box \).

Lemma 1 immediately shows that, under the hypotheses of Theorems 1 or 2, the function \( \Box \) belongs to \( C \) or to \( B \), respectively. Let \( \{a_k\} \) satisfy the hypothesis of Theorem 3. Using \( \Box \), \( \Box \), and the Jensen formula

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |g(z + e^{i\theta})|d\theta = \log \left| g^{(i)}(z) \right| + \int_0^1 \frac{n(z,t) - n(z,0)}{t} dt
\]

with \( l = n(z,0) \), we get

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |g(z + e^{i\theta})|d\theta = \int_1^\infty \frac{n(0,t) - n(z,t)}{t} dt + \int_0^1 \frac{n(0,t)}{t} dt.
\]

Hence, \( \Box \) and the inequality

\[
|g(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |g(z + e^{i\theta})|d\theta
\]

imply that \( g \) is bounded on \( \mathbb{R} \). The Phragmén–Lindelöf theorem yields the boundedness of \( g(z) \) on every horizontal strip of finite width. Therefore, for each sequence \( t_m \in \mathbb{R} \) there exists a subsequence \( t'_m \) such that the \( f(z + t_m') \) converge uniformly to a function \( h(z) \) on every compact subset of \( \mathbb{C} \). If \( h(z) \equiv 0 \) on \( \mathbb{C} \), then \( \int_0^{2\pi} \log |g(t_m' + e^{i\theta})|d\theta \to \infty \). By \( \Box \), this contradicts \( \Box \); hence \( g \in D \).

To prove the “only if” part, note that, by \( \Box \), every \( f \in C \) with \( f(0) \neq 0 \) has the form

\[
f(z) = A e^{i\lambda z} g(z)
\]

with \( g \) as in \( \Box \). Conditions \( \Box \) and \( \Box \) follow immediately from \( \Box \) and \( \Box \). Using Lemma 1, we see that the zero set of \( f \) satisfies condition \( \Box \) and, when \( f \in B \), condition \( \Box \). If \( f \in D \), then \( f \) and \( g \) are bounded on \( \mathbb{R} \) and, by the Phragmén–Lindelöf theorem, on every horizontal strip of finite width. Hence, if \( \Box \) is false, then, by \( \Box \),

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |g(t_i + e^{i\theta})|d\theta \to \infty
\]

for some sequence \( t_j \) of reals. By the properties of averages of subharmonic functions, we have

\[
\int_{|u+iv-t_j|<1} \log |g(u+iv)|du dv \to -\infty.
\]

Since \( \log |g| \) is bounded from above on the strip \( |\text{Im} z| < 2 \), we see that

\[
\int_{|u+iv-t_j-s|<2} \log |g(u+iv)|du dv \to -\infty
\]

uniformly in \( s \in [-1, 1] \). Using the properties of averages of subharmonic functions once again, we obtain

\[
\sup_{-1 \leq s \leq 1} \log |g(t_j + s)| \to -\infty.
\]

On the other hand, \( f(z + t'_j) \to h(z) \neq 0 \) for some subsequence \( t'_j \) uniformly on the compact subsets of \( \mathbb{C} \), and so \( \Box \) is impossible. Theorems 1–3 are proved completely. \( \Box \)

**Remark 1.** It can be shown that all these theorems are valid with the function \( \tilde{n}(c,t) = \text{card}\{a_k : |\text{Re}(a_k - c)| \leq t, |\text{Im}(a_k - c)| \leq t\} \) instead of \( n(c,t) \) as well.
Remark 2. The additional condition
\[
\limsup_{y \to \pm \infty} \int_0^\infty |n(b, t) - n(iy, t)|(|y||t|)^{-1} dt \leq \sigma
\]
remains valid in Theorems 1–3 as long as the condition \(\sigma \leq 1\) holds.

In Theorems 1–3 gives a complete description of the zero sets for the corresponding functions.

Example. Let \(\alpha(t)\) be a strictly monotone increasing concave function on \(\mathbb{R}^+\) with the properties \(\alpha(0) = 0, 1 \leq \alpha'(t) \leq 1 + O(t^{-1})\) as \(t \to \infty\), and \(\alpha(t) \geq t + 1 + \varepsilon\) for large \(t\) with some \(\varepsilon > 0\). Consider a sequence of reals \(\{a_k\}, k \in \mathbb{Z} \setminus \{0\}\), such that \(\alpha(a_k) = k\) and \(a_{-k} = -a_k, k \in \mathbb{N}\). Note that
\[
n(0, t) - n(x, t) = 2E[\alpha(t)] + E[\alpha(x - t)] - E[\alpha(x + t)]
\]
for \(x > t\), and
\[
n(0, t) - n(x, t) = 2E[\alpha(t)] - E[\alpha(x - t)] - E[\alpha(x + t)]
\]
for \(0 < x < t\); here \(E[x]\) is the integral part of \(x\). Since \(a_{n+1} - a_n = (\alpha'(^1t))^{-1}\) with some \(^1t \in (a_n, a_{n+1})\), we see that
\[
0 \leq 1 - (a_{n+1} - a_n) \leq (a_{n+1} - a_n)^{-1} - 1 \leq \alpha'(a_n) - 1 < C/a_n < 1/2
\]
for \(n \geq n_0\), and
\[
\left| \int_{a_n}^{a_{n+1}} (\alpha(t) - E[\alpha(t)] - 1/2) dt \right| \leq \int_{a_n}^{a_{n+1}} \left( \int_{a_n}^{t} \alpha'(u) du - 1/2 \right) dt \leq C'/a_n.
\]
Therefore for any \(N < \infty\) we have
\[
\left| \int_{a_n}^{a_{n+N}} (\alpha(t) - E[\alpha(t)] - 1/2) dt \right| \leq C' \log a_N.
\]
Now it easily follows that all the integrals
\[
\int_1^x \frac{\alpha(t) - E[\alpha(t)] - 1/2}{t} dt, \quad \int_1^x \frac{\alpha(x \pm t) - E[\alpha(x \pm t)] - 1/2}{t} dt,
\]
\[
\int_x^\infty \frac{\alpha(t \pm x) - E[\alpha(t \pm x)] - 1/2}{t} dt,
\]
are bounded uniformly in \(x \to \infty\). Hence, up to a uniformly bounded term, the integral
\[
\int_0^\infty |n(0, t) - n(x, t)|t^{-1} dt
\]
equals
\[
\int_1^x \frac{2E[\alpha(t)] - 2t}{t} dt + \int_1^{x-1} \frac{\alpha(x - t) - \alpha(x + t) + 2t}{t} dt
\]
\[
+ \int_x^\infty \frac{2\alpha(t) - \alpha(t - x) - \alpha(x + t)}{t} dt.
\]
Since \(\alpha(t)\) is concave, the third integral in (22) is positive. Furthermore, for \(1 < t < x - 1\) we have
\[
\alpha(x - t) - \alpha(x + t) + 2t = \int_t^x 1 - \alpha'(x + u) du \geq \frac{-C \log \frac{x + t}{x - t}}{x - t}.
\]
Now it is easy to check that the second integral in (22) is uniformly bounded from below. Obviously, the first integral in (22) is unbounded from above as \(x \to \infty\). Thus, the sequence \(\{a_k\}\) is not the zero set of any function \(f \in B\).
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