Алгебра и анализ
Том 20 (2008), № 3

St. Petersburg Math. J.
Vol. 20 (2009), No. 3, Pages 447–463
S 1061-0022(09)01055-3
Article electronically published on April 7, 2009

ON SOME NONUNIFORM CASES OF THE WEIGHTED SOBOLEV AND POINCARÉ INEQUALITIES

F. I. MAMEDOV AND R. A. AMANOV

Abstract. Weighted inequalities $\|f\|_{q,\nu,B_0} \le C\sum_{j=1}^{n}\|f_{xj}\|_{p,\omega_j,B_0}$ of Sobolev type ($\operatorname{supp} f \subset B_0$) and of Poincaré type ($\bar{f}_{\nu,B_0}=0$) are studied, with different weight functions for each partial derivative f_{x_j}, for parallelepipeds $B_0 \subset E_n, n \ge 1$. Also, weighted inequalities $\|f\|_{q,\nu} \le C\|Xf\|_{p,\omega}$ of the same type are considered for vector fields $X=\{X_j\}$, $j=1,\dots,m$, with infinitely differentiable coefficients satisfying the Hörmander condition.

§1. Introduction

This paper is devoted to weighted versions of the classical inequalities

$$\|f\|_{q,B_0} \le C\|\nabla f\|_{p,B_0} \quad \Big(q=\frac{pn}{n-p},\ \ n>p\ge 1\Big) \tag{1.1}$$

of Sobolev type if $\operatorname{supp} f \subset B_0$ and of Poincaré type if $\bar{f}_{B_0}=0$ (see [1]). In terms of a certain system of parallelepipeds (balls of the quasimetric $|\cdot|_{\bar\sigma}$), we study the inequalities

$$\|f\|_{q,\nu,B_0} \le C\sum_{j=1}^{n}\|f_{x_j}\|_{p,\omega_j,B_0} \quad (q\ge p\ge 1), \quad f\in \operatorname{Lip}(B_0), \tag{1.2}$$

of the above types, with different weight functions for each partial derivative f_{x_j} (Theorems 2.1 and 2.2). For $p=2$, such inequalities may turn out to be useful when applying the general method (see [2, 3, 4]) for the study of regularity, the Harnack inequality, and the Wiener criterion for weak solutions of degenerate elliptic equations of the form

$$\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)=0, \tag{1.3}$$

where $A=\|a_{ij}(x)\|$ is a real symmetric matrix such that there exists $\mu\in(0,1]$ with

$$\mu\sum_{j=1}^{n}\omega_j(x)\xi_j^2 \le A\xi\cdot\xi \le \mu^{-1}\sum_{j=1}^{n}\omega_j(x)\xi_j^2 \tag{1.4}$$

for all $\xi\in E_n$.

This case has been studied relatively poorly, in comparison with the uniform case ($\omega_j(x)\equiv\omega(x)$, $j=1, 2, \dots, n$); a summary of the results pertaining to the uniform case was given, e.g., in [5, Theorem 5].

2000 *Mathematics Subject Classification.* Primary 46E35.

Key words and phrases. Sobolev and Poincaré inequalities, Carnot-Carathéodory metric, Besicovitch property.

The work of the first author was supported in part by INTAS (grant no. 8792).

A modern approach to the study of these issues is related to applying the Carnot–Carathéodory metric in the case of the degenerate vector field

$$X_1 = \omega_1(x)\frac{\partial}{\partial x_1}, \quad \dots, \quad X_n = \omega_n(x)\frac{\partial}{\partial x_n}, \tag{1.5}$$

which, in its turn, is given in terms of some family of curves ("horizontal curves"). Since the coefficients in (1.5) may fail to be smooth, in [3, 8, 9] the Hörmander condition was replaced with appropriate conditions of a geometric nature.

We also consider the weighted inequalities

$$\|f\|_{q,\nu,B_0} \le C\|Xf\|_{p,\omega,B_0} \quad (q \ge p \ge 1), \quad u \in \mathrm{Lip}(B_0), \tag{1.6}$$

for vector fields $X = \{X_j\}$, $j = 1, \dots, m$, with C^∞-coefficients that satisfy the Hörmander condition [10] of finite rank

$$\operatorname{rank}\operatorname{Lie}[X_1, \dots, X_m] = n, \tag{1.7}$$

where B_0 is a metric ball of a special Carnot–Carathéodory metric determined by the vector field.

Inequalities (1.6) find applications to the study of regularity, the Harnack inequality, etc., for solutions of subelliptic equations

$$\sum_{j=1}^{m} X_j^*(A(x, u, Xu)) = 0 \tag{1.8}$$

(see, e.g., [11, 12, 13]), where X_j^* denotes the formal conjugate to X_j, and $A : D \times E_1 \times E_n \to E_n$ is a Carathéodory function satisfying the coercivity condition (see [13]).

§2. Definitions, notation, and formulation of results

To adequately formulate our results, we list some definitions and notation. The n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$, $n \ge 1$, is denoted by E_n. Suppose some metric (or quasimetric) $\rho(x, y)$ is given in E_n. We put $\rho(x) = \rho(x, 0)$. In particular, given a system $\bar\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ of positive numbers, we can consider the quasimetric $\rho(x, y) = \max_{1\le i\le n}\{|x_i - y_i|^{1/\sigma_i}\}$; by analogy with the Euclidean metric, we denote this $\rho(x, y)$ by $|x-y|_{\bar\sigma}$. $B(x, r) = \{y \in E_n : \rho(x, y) < r\}$ is the metric ball centered at $x \in E_n$ of radius $r > 0$; $d(\Omega) = \sup\{\rho(x, y) : x, y \in \Omega\}$ is the metric diameter of a domain Ω. We write $r(B)$ for the radius of a metric ball B. We put $e_j(B) = \sup\{|x_j - y_j| : x, y \in B\}$; this is the length of the jth edge of B. For a measurable set E, its Lebesgue measure is denoted by $|E|$. For an integrable nonnegative function f and a measurable set E, we denote $f(E) = \int_E f(x)\,dx$.

$\mathrm{Lip}(D)$ is the space of functions Lip-continuous in D. $\mathrm{Lip}_0(D)$ will denote the subspace in $\mathrm{Lip}(D)$ formed by the functions that vanish on the boundary ∂D of the domain $D \subset E_n$.

We denote by $L_{p,\nu}(D)$ the space of measurable functions in D for which the norm

$$\|f\|_{p,\nu,D} = \left(\int_D |f(x)|^p \nu(x)\,dx\right)^{1/p}, \quad p \ge 1,$$

is finite. If this leads to no confusion, we shall simply write $\|f\|_{p,\nu}$ in place of $\|f\|_{p,\nu,D}$. The quantities

$$\bar f_{\nu,D} = \frac{1}{\nu(D)}\int_D f(x)\nu(x)\,dx, \quad \bar f_D = \bar f_{1,D},$$

are the mean values of f in D.

Let ω be a nonnegative locally integrable function, $\Omega \Subset E_n$ a compact set, and ρ a metric (or quasimetric) in E_n. We say that ω satisfies the condition A_p $(= A_p(\Omega, \rho, dx))$, $1 \le p < \infty$, if

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}(x)\,dx\right)^{p-1} \le C_p < \infty \quad \text{for} \quad 1 < p < \infty, \tag{2.1}$$

$$\frac{1}{|B|}\int_B \omega(x)\,dx \le C_1 \operatorname*{ess\,inf}_{x\in B} \omega(x) \quad \text{for} \quad p = 1, \tag{2.2}$$

for all metric balls $B = B(x,t)$ with $x \in \Omega$, $0 < t \le 5d(\Omega)$. We say that a function v satisfies the condition A_∞ $(= A_\infty(\Omega, \rho, dx))$ if there exist constants C, $\delta > 0$ such that

$$v(E)/v(B) \le C(|E|/|B|)^\delta \tag{2.3}$$

for any metric ball $B = B(x,t)$ with $x \in \Omega$, $0 < t \le 5d(\Omega)$, and any subset $E \subset B$.

Suppose we are given a pair of nonnegative functions v, ω such that $v, \omega^{-1/(p-1)} \in L_{1,\mathrm{loc}}$ for $1 < p < \infty$ and $\omega^{-1} \in L_{\infty,\mathrm{loc}}$ for $p = 1$ (these conditions will be assumed throughout).

We say that the functions v, ω satisfy the *local balance condition* in Ω if

$$\frac{r(B)}{r(B_1)}\frac{|B_1|}{|B|}\left(\frac{v(B)}{v(B_1)}\right)^{1/q}\left(\frac{\omega^{-1/(p-1)}(B)}{\omega^{-1/(p-1)}(B_1)}\right)^{(p-1)/p} \le C_{pq} < \infty \tag{2.4}$$

for all metric balls $B = B(x,t)$, $B_1 = B(x, 2d(\Omega))$, where $x \in \Omega$, $0 < t \le 2d(\Omega)$.

If ω satisfies A_p, then (2.4) turns into the condition

$$\frac{r(B)}{r(B_1)} \cdot \left(\frac{v(B)}{v(B_1)}\right)^{1/q} \le C'_{pq}\left(\frac{\omega(B)}{\omega(B_1)}\right)^{1/p}. \tag{2.5}$$

In Theorem 2.2, for the quasimetric $|\cdot|_{\bar\sigma}$ and a fixed ball $\Omega = B_0$, instead of A_∞ we assume the following condition $A'_\infty (= A_\infty(\Omega, |\cdot|_{\bar\sigma}, \chi_\Omega dx))$: there exist $C, \delta > 0$ such that

$$\frac{v(E\cap\Omega)}{v(B\cap\Omega)} \le C\left(\frac{|E\cap\Omega|}{|B\cap\Omega|}\right)^\delta \tag{2.6}$$

for any metric ball $B = B(x,t)$, $x \in \Omega$, $0 < t \le 5d(\Omega)$ and any subset $E \subset B$.

The Sobolev inequality to be proved in the case of the quasimetric $|\cdot|_{\bar\sigma}$ looks like the following.

Theorem 2.1. *Let $1 \le p \le q < \infty$, and let B_0 be a fixed ball of the quasimetric $|\cdot|_{\bar\sigma}$ in E_n. Suppose that $v \in A_\infty(B_0, |\cdot|_{\bar\sigma}, dx)$. If the condition*

$$|B|^{-1} e_j(B)(v(B))^{1/q}(\omega_j^{-1/(p-1)}(B))^{(p-1)/p} \le A_{pq} < \infty, \quad j = 1, \dots, n, \tag{2.7}$$

is fulfilled for any metric ball $B = B(x,t)$, $x \in B_0$, $0 \le t \le 2d(B_0)$, then there exists a positive number C_0, depending on n, q and the constants C and δ occurring in (2.3), *such that for any $f \in \mathrm{Lip}_0(B_0)$ we have*

$$\left(\int_{B_0} |f(x)|^q v(x)\,dx\right)^{1/q} \le C_0 A_{pq} \sum_{j=1}^n \left(\int_{B_0} |f_x|^p \omega_j(x)\,dx\right)^{1/p}. \tag{2.8}$$

Remark 2.1. The proof shows that, instead of the quasimetric $|\cdot|_{\bar\sigma}$, in Theorem 2.1 (as well as in Theorem 2.2) we could choose an arbitrary quasimetric with the Besicovitch property, provided that the metric balls are convex (we need this for obtaining estimate (3.7)).

The next result concerns the Poincaré inequality for the quasimetric $|\cdot|_{\bar\sigma}$.

Theorem 2.2. *Let $1 \le p \le q < \infty$, and let B_0 be a fixed ball of the quasimetric $|\cdot|_{\bar\sigma}$ in E_n. Suppose that v is a function satisfying* (2.6)*. If the condition*

$$e_j(B)|B|^{-1}(v(B\cap B_0))^{1/q}(\omega_j^{-1/(p-1)}(B\cap B_0))^{(p-1)/p} \le A_{pq} < \infty, \quad j = 1,\dots,n, \tag{2.9}$$

is fulfilled for any quasimetric ball $B = B(x,t)$, $x \in B_0$, $0 < t \le 2d(B_0)$, then there exists a positive number C_0, depending on n, q and the constants C and δ occurring in (2.6)*, such that for any $f \in \operatorname{Lip}(B_0)$ we have*

$$\left(\int_{B_0} |f - \bar f_{v,B_0}|^q v(x)\,dx\right)^{1/q} \le C_0 A_{pq} \sum_{j=1}^n \left(\int_{B_0} |f_{x_j}|^p \omega_j(x)\,dx\right)^{1/p}. \tag{2.10}$$

As applications of Theorems 2.1 and 2.2, we present the following two examples, which pertain to the case where $p = 2$.

Example 2.1. Let $\beta = (\beta_1,\dots,\beta_n)$ be a fixed vector with nonnegative components, and let $\eta \ge 0$ be such that $\max_{1\le j\le n}\beta_j \le (\sum_{k=1}^n \beta_k + n\eta)/2$. Next, let the numbers σ_i for the quasimetric $|\cdot|_{\bar\sigma}$ be defind by $\sigma_i = \frac{\eta+\beta_i}{2}$, $i = 1,\dots,n$. Suppose that

$$\frac{1}{q} - \frac{1}{2} + \frac{\eta}{n\eta + \sum_{k=1}^n \beta_k} = 0. \tag{2.11}$$

Then there exists a positive constant $C = C(n,\eta,\beta)$ such that for any $f \in \operatorname{Lip}_0(B_R^a)$ we have

$$\left(\frac{1}{|B_R^a|}\int_{B_R^a} |f(x)|^q\,dx\right)^{1/q} \le C_0 R^{\eta/2} \sum_{j=1}^n \left(\frac{1}{|B_R^a|}\int_{B_R^a} |x|_{\bar\sigma}^{\beta_j} |f_{x_j}|^2\,dx\right)^{1/2}. \tag{2.12}$$

Example 2.2. Let vectors $\beta = (\beta_1,\dots,\beta_n)$, $\bar\sigma = (\sigma_1,\dots,\sigma_n)$ and numbers q, η be as in Example 2.1. Then there exists a constant $C = C(n,\eta,\beta)$ such that for any $f \in \operatorname{Lip}(B_R^a)$ we have inequality (2.12), where f is replaced with $f - \bar f_{B_R^a}$.

Remark 2.2. It is of interest to consider the approach where the parallelepipeds of the metric $|\cdot|_{\bar\sigma}$ are replaced by an arbitrary system of parallelepipeds $\{P\}$, without any reference to a metric or quasimetric; however, $\{P\}$ and the system of weights $\{v,\omega_1,\dots,\omega_n\}$ must be coordinated so as to allow the verification of (2.7) (or (2.9)). Also, $\{P\}$ should possess the following properties.

1) The Besicovitch property: let $A \subset E_n$ be an arbitrary bounded set (in the Euclidean metric), and let $G \subset \{P\}$ be a subsystem that covers A, i. e., there exists $\{P(x)\} \subset G$ such that $A \subset \bigcup_{x\in A} P(x)$. Then the covering G admits a finite or countable subcovering $\{P_i\}_{i\in N}$ of finite multiplicity $C_n > 0$, i.e.,

$$\exists \{P_i\}_{i\in N} \subset G, \quad A \subset \bigcup_{i\in N} P_i, \quad \sum_{i\in N} \chi_{p_i}(x) \le C_n. \tag{2.13}$$

2) For every $\gamma \in (0, d_0)$, where $d_0 \in (0,1)$ is some number, every compact set $K \subset E_n$, and almost every $x \in \operatorname{int} K$, there exists a parallelepiped $P_\gamma(x) \in G$ such that

$$|P_\gamma(x) \setminus K| = \gamma |P_\gamma(x)|. \tag{2.14}$$

Note that a certain criterion for a system of parallelepipeds in E_n to possess the Besicovitch property was proved in [14].

The next results of the paper pertain to weighted inequalities of Sobolev type ($\operatorname{supp} f \subset B_0$) and of Poincaré type ($\bar{f}_{\nu,B_0} = 0$):

$$\left(\frac{1}{v(B_0)}\int_{B_0}|f|^q v(x)\,dx\right)^{1/q} \le Cr(B_0)\left(\frac{1}{\omega(B_0)}\int_{B_0}|Xf|^p\omega(x)\,dx\right)^{1/p}, \qquad q \ge p \ge 1, \quad f \in \operatorname{Lip}(B_0), \tag{2.15}$$

for vector fields $X = \{X_j\}$, $j = 1, \ldots, m$, where $X_j = \sum_{k=1}^n b_{jk}(x)\frac{\partial}{\partial x_k}$ with C^∞-coefficients; we assume the Hörmander condition (1.7). For $f \in \operatorname{Lip}(E_n)$, we define $X_j f = \sum_{k=1}^n b_{jk}(x)\frac{\partial f}{\partial x_k}$, $j = 1, 2, \ldots, m$; then Xf is the vector with the components $X_j f$ and with length $|Xf|^2 = \sum_{i=1}^m |X_i f|^2$.

Following the conventional definitions (see, e.g., [15, 16]), we say that an absolutely continuous curve $\gamma : [a,b] \to E_n$ is admissible ("horizontal") if there exist functions $c_j(t)$, $a \le t \le b$, satisfying

$$\sum_{j=1}^n c_j^2(t) \le 1, \quad \dot{\gamma} = \sum_{j=1}^n c_j(t)X_j(\gamma(t)). \tag{2.16}$$

Let a distance $\rho(x,y)$ between $x, y \in E_n$ be defined as the infimum of all $T > 0$ for which there exists an admissible curve $\gamma : [0,T] \to E_n$ with $\gamma(0) = x$, $y(T) = y$. The resulting space (E_n, ρ) is metric; in the literature it is called the Carnot–Carathéodory space (for a modern presentation, see [7]). In [17], many important properties of the above metric can be found; in particular, we mention the doubling property for the Lebesgue measure of metric balls: for any compact set $\Omega \subset E_n$ there exists a number $C_1 \ge 1$ such that

$$|B(x_0, 2r)| \le C_1|B(x_0, r)|, \quad x_0 \in \Omega, \tag{2.17}$$

$0 < r \le 5d(\Omega)$. The quantity $Q = \log_2 C_1$ is called the *size of homogeneity* of the field relative to the compact set K. This quantity plays the role of a dimension (n) for vector fields with property (1.7) (see, e.g., [18, 19]). Inequality (2.17) implies that

$$|B(x_0,\rho)|/|B(x_0,r)| \ge C(\rho/r)^Q, \quad 0 \le \rho < r, \tag{2.18}$$

where $0 < r \le 5d(\Omega)$.

The space (E_n, ρ) described above is locally compact and equipped with a curve length. In [18, Lemma 3.7], with the help of the Arzelà lemma, it was proved that there exists a continuous curve $\gamma_{xy}(t)$ that connects two given points $x, y \in E_n$ and is such that for any $z \in \{\gamma_{xy}(t)\}$ we have

$$\rho(x,y) = \rho(x,z) + \rho(z,y). \tag{2.19}$$

The Poincaré inequality for vector fields was obtained in [20] for $q = p > 1$. This result was refined in [21] to cover the case where $1 < p < Q$, $q = pQ/(Q-p)$. The important case where $p = 1$, $q = Q/(Q-1)$ was proved in [22]; this case merges with (more precisely, is equivalent to) the isoperimetric inequality

$$\min(|E \cap B|, |B \setminus E|)^{(Q-1)/Q} \le C\left(\frac{r(B)}{|B|^{1/Q}}\right)P_X(E;B), \tag{2.20}$$

where $C > 0$ depends on n, Q, the compact set Ω, and the field $\{X_j\}$. In (2.20) it is assumed that $B = B(x_0, r)$, $x_0 \in \Omega$, and $0 < r < 2d(\Omega)$; $P_X(E;B)$ denotes the relative perimeter of a set $E \subset B$. We follow [18, p. 1083] to recall the definition of $P_X(E,B)$.

Let

$$F(D) = \left\{\Phi = (\Phi_1, \ldots, \Phi_n) \in C_0^1(D, E_n) : \|\Phi\|_\infty = \sup_{x \in D}\left(\sum_{j=1}^n |\Phi_j(x)|^2\right)^{1/2} \le 1\right\}.$$

We put

$$\operatorname{Var}_X(f;D) = \sup_{\Phi \in F(D)} \int_D f(x) X^* \Phi dx, \tag{2.21}$$

$$P_X(E;D) = \operatorname{Var}_X(\chi_E; D), \tag{2.22}$$

where χ_E is a characteristic function of the set E and $X_j^* = \sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{ij}(x))$, $j = 1, \dots, m$.

Sobolev-type inequalities for vector fields were studied, e.g., in [11, 12, 23, 24].

As to weighted results for vector fields, we mention the papers [21, 22], where the case of $q > p \geq 1$ was treated and a balance condition was imposed on the left and right weights in (2.15). In those papers it was assumed that the left weight belongs to the doubling class, while the right weight is of A_p-class in the metric of the field $\{X_j\}$.

Our result pertaining to the Sobolev inequality for vector fields $\{X_j\}$ reads as follows.

Theorem 2.3. *Suppose* $1 \leq p \leq q < \infty$, $\Omega \Subset E_n$, *and* $B_0 = B(x_0, r_0)$, *where* $x_0 \in \Omega$, $0 < r_0 < 2d(\Omega)$; *suppose* B_0 *is a fixed ball in the metric* (2.16) *corresponding to a vector field* $X = \{X_j\}, j = 1, \dots, m$, *and satisfying condition* (1.7). *Assume that* $v \in A_\infty(B_0, \rho, dx)$ *with respect to the same metric. Next, assume that the balance condition* (2.4) *is fulfilled in* B_0 (*i.e.,* $\Omega = B_0$).

Then in B_0 *we have*

$$\begin{aligned}\left(\frac{1}{v(B_0)} \int_{B_0} |f(x)|^q v(x) dx\right)^{\frac{1}{q}} &\leq C r(B_0) \left(\frac{1}{|B_0|} \int_{B_0} \omega(x) dx\right)^{\frac{1}{p}} \\ &\times \left(\frac{1}{|B_0|} \int_{B_0} \omega^{-1/(p-1)}(x)\, dx\right)^{\frac{p-1}{p}} \left(\frac{1}{\omega(B_0)} \int_{B_0} |Xf|^p \omega(x) dx\right)^{\frac{1}{p}}\end{aligned} \tag{2.23}$$

for any $f \in \operatorname{Lip}_0(B_0)$. *The constant* C *depends on* n, q, C_{pq}, *the constants* C, δ *occurring in* (2.3), *and also on the compact set* Ω *and the field* $\{X_j\}$.

Corollary 2.1. *Under the conditions of Theorem* 2.3, *let* $\omega \in A_p$, *and let condition* (2.5) *be fulfilled instead of* (2.4). *Then in* B_0 *we have*

$$\left(\frac{1}{v(B_0)} \int_{B_0} |f(x)|^q v(x) dx\right)^{1/q} \leq C r(B_0) \left(\frac{1}{\omega(B_0)} \int_{B_0} |Xf|^p \omega(x) dx\right)^{1/p}, \tag{2.24}$$

$$q \geq p \geq 1, \quad f \in \operatorname{Lip}_0(B_0).$$

As an easy consequence, we obtain the following statement.

Corollary 2.2. *Suppose* $1 \leq p < Q, \omega \in A_p(B_0, \rho, dx)$, *where* $B_0 \subset \Omega$ *is a fixed ball in the metric* (2.16). *Then*

$$\left(\frac{1}{\omega(B_0)} \int_{B_0} |f(x)|^{pQ'} \omega(x) dx\right)^{1/pQ'} \leq C r(B_0) \left(\frac{1}{\omega(B_0)} \int_{B_0} |Xf|^p \omega(x) dx\right)^{1/p}, \tag{2.25}$$

$$1 < Q < \infty, \quad f \in \operatorname{Lip}_0(B_0).$$

The constant C *only depends on* n, Q, p, C_p, *and on* $\Omega, \{X_j\}$.

In our Theorem 2.3 (and in Theorem 2.4 below), the left weight belongs to a class smaller than in [22, Theorem 2], but the right weight is almost free from condition A_p. For $q = p \geq 1$, our result is new even in the case of the Euclidean metric (cf. [5, Theorem 5]). In the corresponding results, the two weights v and $\omega^{-1/(p-1)}$ were assumed to belong to the A_∞-class [25], or to A_∞^β for some $\beta > n - 1$ (see [5, Theorem 5]). In the case where $\omega \in A_p$, the balance condition becomes somewhat better (in [22], the balance condition looked like this:

$$\frac{r(I)}{r(J)}\Big(\frac{v(I)}{v(J)}\Big)^{1/q} \le C\Big(\frac{\omega(I)}{\omega(J)}\Big)^{1/p} \tag{2.26}$$

for all metric balls I, J with $I \subset J$). These remarks pertain also to the following result on the Poincaré inequality.

Theorem 2.4. *Suppose* $1 \le p \le q < \infty, \Omega \Subset E_n$, *and* $B_0 = B(x_0, r_0)$, *where* $x_0 \in \Omega$, $0 < r_0 \le 2d(\Omega)$; *suppose* B_0 *is a fixed ball in the metric* (2.16) *corresponding to a vector field* $X = \{X_j\}$ *satisfying* (1.6). *Assume that* $v \in A_\infty(B_0, \rho, dx)$ *with respect to the same metric. Next, assume that the balance condition* (2.4) *is fulfilled in* B_0 (*i.e.,* $\Omega := B_0$). *Then inequality* (2.23) *holds true with the replacement of* f *by* $f - \bar{f}_{\nu,B_0}$ *on the left-hand side. The constant* C *only depends on* n, q, *and* C_{pq}, *on* C *and* δ *occurring in* (2.3), *and also on* Ω *and* $\{X_j\}$.

Remark 2.3. Theorem 2.4 implies corollaries similar to Corollaries 2.1 and 2.2, but now pertaining to the Poincaré inequality, with the replacement of f by $f - \bar{f}_{\nu,B_0}$ on the left-hand sides of (2.24) and (2.25).

§3. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Let $f \in \mathrm{Lip}_0(B_0)$; we put $B_0^+ = \{x \in B_0 : f(x) > 0\}$ and $B_0^- = B_0 \setminus \bar{B}_0^+$. Let D^i be a connected component of B_0^+ ($i = 1, 2, \dots$). We denote $D_\alpha = \{x \in D^i : f(x) > \alpha\}$, $\alpha > 0$.

Suppose that $D_{2\alpha}$ is nonempty. Then for any fixed $x \in D_{2\alpha}$ we can find $B(x, r(x))$ such that

$$|B(x, r(x)) \setminus D_\alpha| = \gamma |B(x, r(x))|, \tag{3.1}$$

where $\gamma \in (0, 1)$ is some number independent of α, x, and $r(x)$; this γ will be specified later. Indeed, to prove (3.1), it suffices to put

$$r(x) = \sup\{t > 0 : |B(x, t) \setminus D_\alpha| \le \gamma |B(x, t)|\}. \tag{3.2}$$

For a fixed $x \in D_{2\alpha}$, we denote $B = B(x, r(x))$ for simplicity. Two cases are possible.

1) If

$$|D_{2\alpha} \cap B| < \gamma |B|, \tag{3.3}$$

then, by (2.3), we have

$$v(D_{2\alpha} \cap B) \le C\gamma^\delta v(B). \tag{3.4}$$

Next, $v(B) = v(B \cap D_\alpha) + v(B \setminus D_\alpha) \le C\gamma^\delta v(B) + v(B \cap D_\alpha)$ by (2.3) and (3.1).

Choosing γ so that $C\gamma^\delta < 1$, we have

$$v(B) \le \frac{1}{1 - C\gamma^\delta} v(B \cap D_\alpha).$$

Therefore, by (3.4), we obtain

$$v(B \cap D_{2\alpha}) \le \frac{C\gamma^\delta}{1 - C\gamma^\delta} v(B \cap D_\alpha). \tag{3.5}$$

2) If

$$|D_{2\alpha} \cap B| \ge \gamma |B|, \tag{3.6}$$

then, by (3.1) and (3.6), we have

$$\int_A \Big(\int_Z dy\Big)\, dx \ge \gamma^2 |B|^2, \quad \text{where } A = B \setminus D_\alpha, \quad Z = B \cap D_{2\alpha}.$$

Let points $x \in A$, $y \in Z$ be fixed arbitrarily. Clearly, the line passing through x and y lies in B and necessarily intersects the surfaces $\{x \in D^i : f(x) = \alpha\}$ and $\{x \in D^i : f(x) = 2\alpha\}$ at some points $x' = x + t_1(y - x)$ and $x'' = x + t_2(y - x)$, where $t_2 > t_1 > 0$ are numbers depending on x and y. Then $f(x') = \alpha$ and $f(x'') = 2\alpha$. Therefore, we have

$$\gamma^2 |B|^2 \le \int_A \left(\int_Z \frac{1}{\alpha} |f(x') - f(x'')|\, dy \right) dx,$$

whence

$$\gamma^2 |B|^2 \le \frac{1}{\alpha} \int_A \left(\int_Z \left(\int_{t_1(x,y)}^{t_2(x,y)} \left| \frac{\partial f}{\partial t}(x + t(y - x)) \right| dt \right) dy \right) dx.$$

The Fubini theorem yields

$$\gamma^2 |B|^2 \le \sum_{j=1}^n \frac{e_j(B)}{\alpha} \int_A \left(\int_0^1 \left(\int_{\{y \in B : x + t(y-x) \in G\}} \left| \frac{\partial f}{\partial x_j}(x + t(y - x)) \right| dy \right) dt \right) dx,$$

where $G = B \cap (D_\alpha \setminus D_{2\alpha})$. In the inner integral we put $z = x + t(y - x)$. Then $z \in G$ and

$$\gamma^2 |B|^2 \le \sum_{j=1}^n \frac{e_j(B)}{\alpha} \int \_A \left(\int_0^1 \left(\int_{\{z \in G : (z-x)/t + x \in B\}} \left| \frac{\partial f}{\partial z_j}(z) \right| dz \right) \frac{dt}{t^n} \right) dx.$$

Applying the Fubini formula once again, we obtain

$$\begin{aligned} \gamma^2 |B|^2 &\le \sum_{j=1}^n \frac{e_j(B)}{\alpha} \int_0^1 \left(\int_G \left| \frac{\partial f}{\partial z_j} \right| \left(\int_{\{x : |x_j - z_j|\} \le t e_j(B)} dx \right) dz \right) \frac{dt}{t^n} \\ &\le \sum_{j=1}^n \frac{|B| e_j(B)}{\alpha} \int_G \left| \frac{\partial f}{\partial z_j} \right| dz, \end{aligned}$$

whence

$$1 \le \sum_{j=1}^n \left(\frac{n^{\frac{q-1}{q}} e_j(B)}{|B| \alpha \gamma^2} \int_G \left| \frac{\partial f}{\partial z_j} \right| dz \right)^q. \tag{3.7}$$

By the Hölder inequality, this implies

$$1 \le \sum_{j=1}^n \left(\frac{n^{\frac{q-1}{q}} e_j(B)}{|B| \alpha \gamma^2} \right)^q \left(\int_B \omega_j^{-1/(p-1)}(x) dx \right)^{q(p-q)/p} \left(\int_G \left| \frac{\partial f}{\partial z_j} \right|^p \omega_j(z) dz \right)^{q/p}. \tag{3.8}$$

Estimates (2.7) and (3.8) show that

$$1 \le \frac{n^{q-1} A_{pq}^q}{\gamma^{2q} \alpha^q} \frac{1}{v(B)} \sum_{j=1}^n \left(\int_G \left| \frac{\partial f}{\partial z_j} \right|^p \omega_j(z) dz \right)^{q/p}. \tag{3.9}$$

Consequently,

$$v(B \cap D_{2\alpha}) \le \frac{n^{q-1} A_{pq}^q}{\gamma^{2q} \alpha^q} \sum_{j=1}^n \left(\int_G \left| \frac{\partial f}{\partial z_j} \right|^p \omega_j(z) dz \right)^{q/p}. \tag{3.10}$$

By (3.5) and (3.10), we have

$$\begin{aligned} v(B \cap D_{2\alpha}) &\le \frac{C\gamma^\delta}{1 - C\gamma^\delta} v(B \cap D_\alpha) \\ &\quad + \frac{n^{q-1} A_{pq}^q}{\gamma^{2q} \alpha^q} \sum_{j=1}^n \left(\int_{B \cap (D_\alpha \setminus D_{2\alpha})} \left| \frac{\partial f}{\partial z_j} \right|^p \omega_j(z) dz \right)^{q/p}. \end{aligned} \tag{3.11}$$

The system of balls $\{B = B(x, r(x)) : x \in D_{2\alpha}\}$ covers $D_{2\alpha}$. By the Guzmán–Besicovitch lemma on comparable intervals (parallelepipeds), see [14], the system $\{B\}$ admits a finite or countable subfamily $\{B_i\}_{i=1}^{\infty}$ that covers $D_{2\alpha}$ and has finite multiplicity, i.e.,

$$\sum_{i=1}^{\infty} \chi_{B_i}(x) \le C_n, \tag{3.12}$$

where $C_n > 0$ depends only on n. Summing over j all inequalities

$$v(B_j \cap D_{2\alpha}) \le \frac{C\gamma^\delta}{1 - C\gamma^\delta} v(B_j \cap D_\alpha) + \frac{n^{q-1} A_{pq}^q}{\gamma^{2q}\alpha^q} \sum_{k=1}^{n} \Big(\int_{G_j} \Big| \frac{\partial f}{\partial z_k} \Big|^p \omega_k(z)\, dz \Big)^{q/p},$$
$$G_j = B_j \cap (D_\alpha \setminus D_{2\alpha}),$$

obtained from (3.11) for $B = B_j$, and using (3.12), we arrive at

$$v(D_{2\alpha}) \le \frac{C_n C\gamma^\delta}{1 - C\gamma^\delta} v(D_\alpha) + \frac{C_n n^{q-1} A_{pq}^q}{\gamma^{2q}\alpha^q} \sum_{j=1}^{n} \Big(\int_{D_\alpha \setminus D_{2\alpha}} \Big| \frac{\partial f}{\partial z_j} \Big|^p \omega_j(z)\, dz \Big)^{q/p}. \tag{3.13}$$

We integrate (3.13) over $(0, \infty)$:

$$\begin{aligned} \int_0^\infty v(D_{2\alpha}) d\alpha^q &\le \frac{C_n C\gamma^\delta}{1 - C\gamma^\delta} \int_0^\infty v(D_\alpha) d\alpha^q \\ &\quad + \sum_{j=1}^{n} \frac{C_n n^{q-1} A_{pq}^q}{\gamma^{2q}} \int_0^\infty \frac{d\alpha}{\alpha} \Big(\int_{D_\alpha \setminus D_{2\alpha}} \Big| \frac{\partial f}{\partial z_j} \Big|^p \omega_j(z)\, dz \Big)^{q/p}. \end{aligned} \tag{3.14}$$

Since

$$\int_0^\infty v(D_{2\alpha}) d\alpha^q = \frac{1}{2^q} \int_{D^i} f^q(x) v(x)\, dx, \qquad \int_0^\infty v(D_\alpha) d\alpha^q = \int_{D^i} f^q(x) v(x)\, dx,$$

we can use the Minkowski inequality to show that

$$\begin{aligned} &\Big(\frac{1}{2^q} - \frac{C_n C\gamma^\delta}{1 - C\gamma^\delta} \Big) \int_{D^i} f^q(x) v(x) dx \\ &\qquad \le \frac{C_n q n^{q-1} A_{pq}^q}{\gamma^{2q}} \sum_{j=1}^{n} \Big(\int_{D^i} \Big| \frac{\partial f}{\partial z_j} \Big|^p \omega_j(z) \Big(\int_{f(z)/2}^{f(z)} \frac{d\alpha}{\alpha} \Big)^{p/q} dz \Big)^{q/p}. \end{aligned} \tag{3.15}$$

Choosing γ so small that

$$\frac{1}{2^q} - \frac{C_n C\gamma^\delta}{1 - C\gamma^\delta} > 0, \tag{3.16}$$

we see that (3.15) implies that

$$\begin{aligned} \int_{D^i} f^q(x) v(x)\, dx &\le \Big(\frac{1}{2^q} - \frac{C_n C\gamma^\delta}{1 - C\gamma^\delta} \Big)^{-1} \frac{C_n q n^{q-1} \ln 2}{\gamma^{2q}} A_{pq}^q \\ &\quad \times \sum_{j=1}^{n} \Big(\int_{D^i} \Big| \frac{\partial f}{\partial z_j} \Big|^p \omega_j(z)\, dz \Big)^{q/p}. \end{aligned} \tag{3.17}$$

Summing the inequalities (3.17) for all D^i, we obtain

$$\begin{aligned} \int_{B_0^+} f^q(x) v(x)\, dx &\le C_0^q A_{pq}^q \sum_{j=1}^{n} \Big(\int_{B_0^+} \Big| \frac{\partial f}{\partial z_j} \Big|^p \omega_j(z)\, dz \Big)^{q/p}, \\ C_0^q &= \Big(\frac{1}{2^q} - \frac{C_n C\gamma^\delta}{1 - C\gamma^\delta} \Big)^{-1} \frac{C_n q n^{q-1}}{\gamma^{2q}} \ln 2. \end{aligned} \tag{3.18}$$

A similar inequality holds true in B_0^- for the function $-f(x)$:

$$\int_{B_0^-} (-f(x))^q v(x)\,dx \le C_0^q A_{pq}^q \sum_{j=1}^n \left(\int_{B_0^-} \left| \frac{\partial f}{\partial z_j} \right|^p \omega_j(z)\,dz \right)^{q/p}. \tag{3.19}$$

Estimates (3.18) and (3.19) imply (2.8).

Theorem 2.1 is proved. □

Proof of Theorem 2.2. We can find $A \in E_1$ such that

$$|x \in B_0 : f(x) > A| \le \frac{1}{2}|B_0| \le |x \in B_0 : f(x) \ge A|. \tag{3.19$'$}$$

Put $B_0^+ = \{x \in B_0 : f(x) > A\}$. Let D^i $(i = 1, 2, \ldots)$ be a connected component of B_0^+. We denote $D_\alpha = \{x \in D^i : f(x) > A + \alpha\}$, $\alpha > 0$. Then $|B_0 \setminus B_0^+| \ge \frac{1}{2}|B_0|$ and $|B_0 \setminus B_0^-| \ge \frac{1}{2}|B_0|$. Let $\alpha > 0$ be such that $D_{2\alpha}$ is nonempty. For any fixed $x \in D_{2\alpha}$, there is a ball $B(x, r(x))$ such that

$$|(B(x, r(x)) \cap B_0) \setminus D_\alpha| = \gamma |B(x, r(x)) \cap B_0|, \tag{3.20}$$

where $0 < \gamma < \frac{1}{2}$ is a sufficiently small number to be specified later. Indeed, it suffices to take

$$r(x) = \sup\{t > 0 : |(B(x,t) \cap B_0) \setminus D_\alpha| \le \gamma |B(x,t) \cap B_0|\}. \tag{3.21}$$

Let a ball $B = B(x, r(x))$ in the system $\{B = B(x, r(x)) : x \in D_{2\alpha}\}$ be fixed. If

a)

$$|D_{2\alpha} \cap B| < \gamma |B \cap B_0|,$$

then

$$v(D_{2\alpha} \cap B) \le C\gamma^\delta v(B \cap B_0). \tag{3.21$'$}$$

On the other hand, (3.20) and (2.6) imply

$$v(B \cap B_0) = v((B \cap B_0) \setminus D_\alpha) + v(B \cap D_\alpha) \le C\gamma^\delta v(B \cap B_0) + v(B \cap D_\alpha).$$

Consequently, if γ satisfies $C\gamma^\delta < 1$, then

$$v(B \cap B_0) \le \frac{1}{1 - C\gamma^\delta} v(B \cap D_\alpha),$$

and (3.21′) shows that

$$v(B \cap D_{2\alpha}) \le \frac{C\gamma^\delta}{1 - C\gamma^\delta} v(B \cap D_\alpha). \tag{3.22}$$

Now, if

b)

$$|D_{2\alpha} \cap B| \ge \gamma |B \cap B_0|, \tag{3.23}$$

then we can repeat all the arguments in Theorem 2.1.

In this case, (3.20) and (3.23) imply that

$$|D_{2\alpha} \cap B| > \varepsilon\gamma |B|, \quad |(B \cap B_0) \setminus D_\alpha| > \varepsilon\gamma |B|, \tag{3.24}$$

where $\varepsilon \in (0, 1)$ is a number depending on n. Next, arguing as in Theorem 2.1 and using (3.24) and (2.9), we obtain

$$\left(\int_{B_0} |f(x) - A|^q v(x)\,dx \right)^{1/q} \le C_0 A_{pq} \sum_{j=1}^n \left(\int_{B_0} \left| \frac{\partial f}{\partial x_j} \right|^p \omega_j(x)\,dx \right)^{1/p}. \tag{3.25}$$

We show that

$$\|f - \bar{f}_{v,B_0}\|_{q,v} \le 2\|f - A\|_{q,v}. \tag{3.26}$$

Indeed, to obtain (3.26) it suffices to apply the Minkowski inequality,

$$\begin{aligned}\|f-\bar f_{v,B_0}\|_{q,v} &\le \|f-A\|_{q,v}+\|\bar f_{v,B_0}-A\|_{q,v}\\ &= \|f-A\|_{q,v}+|\bar f_{v,B_0}-A|(v(B_0))^{1/q},\end{aligned}$$

and the Hölder inequality,

$$|\bar f_{v,B_0}-A|\le \frac{1}{v(B_0)}\int_{B_0}|f-A|v(x)dx\le v(B_0)^{-1/q}\|f-A\|_{q,v}.$$

Theorem 2.2 is proved. □

Proof of the statement of Example 2.1. We apply Theorem 2.1 in the case where $p=2$, $v(x)\equiv 1$, $\omega_j(x)=|x|_{\bar\sigma}^{\beta_j}$, $\sigma_j=\frac{\beta_j+\eta}{2}$, $j=1,\dots,n$. It suffices to check condition (2.7), because (2.3) is fulfilled obviously. We consider two cases separately: 1) $\rho(a)\le CR$, and 2) $\rho(a)>CR$, where $C>1$ is a sufficiently large number independent of R and a. If 1) is fulfilled, then, for any ball $B=B(x,r(x))$, where $x\in B(a,R)$, $r<R$, there are two possibilities: a) $\rho(x)<Cr$, or b) $\rho(a)>Cr$ (C is as in 1)). In the case of 1a), we can verify (2.7) for $p=2$, $j=1,\dots,n$:

(3.27)

$$(v(B))^{1/q}|B|^{-1}e_j(B\biggl(\int_B\omega_j^{-1}(x)dx\biggr)^{1/2}\le |B(x,r)|^{1/q-1}r^{\sigma_j}\biggl(\int_{B(x,r)}|y|_{\bar\sigma}^{-\beta_j}dy\biggr)^{1/2}.$$

Observe that $\frac{1}{n}|y|_{\bar\sigma}\le\rho(y)\le|y|_{\bar\sigma}$; consequently,

$$\int_{B(x,r)}|y|_{\bar\sigma}^{-\beta_j}dy\le\int_{B(x,r)}\rho(y)^{-\beta_j}dy\le Cr^{\sum_{k=1}^n\sigma_k-\beta_j}.$$

Since $\sigma_j=\frac{\beta_j+\eta}{2}$, the right-hand side of (3.27) is dominated by

$$|B(x,r)|^{1/q-1/2+(2\sigma_j-\beta_j)/\left(2\sum_{k=1}^n\sigma_k\right)}=C|B(x,r)|^{1/q-1/2+\eta/\left(2\sum_{k=1}^n\sigma_k\right)}\le C_2.$$

In the case of 1b), the left-hand side of (3.27) does not exceed

$$r^{\sigma_j}|B(x,r)|^{1/q-1/2}|x|_{\bar\sigma}^{-\beta_j/2}\le C_2$$

because

$$|x|_{\bar\sigma}^{-\beta_j/2}\le\rho(x)^{-\beta_j/2}\le C_3r^{-\beta_j/2}.$$

Case 2) is similar to case b): for any ball $B(x,r)$, where $x\in B(a,R)$, $r<R$, the left-hand side of (3.27) is estimated by the expression

$$C|B(x,r)|^{1/q-1/2}R^{\sigma_j-\beta_j/2}\le C_3,$$

where C,C_2, and C_3 depend on n,β, and η. Using (2.11), we see that this implies (2.12). □

Proof of the statement of Example 2.2. We apply Theorem 2.2 to the case of $v(x)\equiv 1$, $\omega_j(x)=|x|_{\bar\sigma}^{\beta_j}$, $\sigma_j=\frac{\beta_j+\eta}{2}$, $p=2$. It suffices to check (2.6) and (2.9). Condition (2.6) is fulfilled obviously, and condition (2.9) was verified in Example 2.1. □

§4. Properties of metric balls of a vector field

Let $X = \{X_j\}$, $j = 1, \dots, n$, be a fixed vector field of the same form and satisfying the same assumptions as in §2. Let $\rho(x, y)$ be the metric corresponding to this field. We assume that condition (1.7) is fulfilled and that the coefficients of the field are infinitely differentiable.

We need several auxiliary statements.

Lemma 4.1. *Suppose $\Omega \Subset E_n$ is a compact set, and $B_0 = B(x_0, r_0)$, where $x_0 \in \Omega, r_0 \le 5d(\Omega)$. Let $B = B(x, r)$ be another ball, $x \in B_0$, $r < 2r_0$. Then there exists $z \in B_0 \cap B$ such that $B(z, r/4) \subset B_0 \cap B$.*

Proof of Lemma 4.1. There exists a continuous curve $\{\gamma_{x_o x}\}$ that connects x_0 and x and satisfies (2.19). If i) $r/2 \le \rho(x_0, x)$, then we choose $z \in \{\gamma_{x_o x}\}$ so that $\rho(z, x) = r/2$, i.e., $z \in \partial B(x, r/2)$. Then we have $\rho(z, x_0) = r_0 - r/2$ by (2.19). We show that $B(z, r/2) \subset B \cap B_0$. Indeed, by the triangle inequality, for any $y \in \partial B_0$ we have

$$\rho(z, y) \ge \rho(y, x_0) - \rho(x_0, z) = r_0 - \left(r_0 - \frac{r}{2}\right) = \frac{r}{2}.$$

Therefore, $\rho(z, y) \ge \frac{r}{2}$ whenever $y \in \partial B$, and we have

$$\rho(z, t) \ge \rho(t, x) - \rho(x, z) = r - \frac{r}{2} = \frac{r}{2},$$

i.e., $\rho(z, t) \ge \frac{r}{2}$ for all $t \in \partial B$.

Now we show that $\rho(z, \partial(B_0 \cap B)) \ge r/2$. This means that

$$B(z, r/2) \subset B_0 \cap B. \tag{4.1}$$

If ii) $r/2 \ge \rho(x_0, x)$, then we choose $z \in \{\gamma_{x_o x}\}$ such that $\rho(z, x) = \rho(x_0, x)/2$ (i.e., $z \in \partial B(x, \rho(x_0, x)/2)$). Then $\rho(x_0, z) = \rho(x, z) = \rho(x_0, x)/2$ by (2.19). Now we check that $B(z, r/4) \subset B_0 \cap B$. For $y \in \partial B_0$, by the triangle inequality we have $\rho(z, y) \ge \rho(x_0, y) - \rho(x_0, x)/2 \ge r_0 - r/4 \ge r/4$, and also for $t \in \partial B$ we have $\rho(z, t) \ge \rho(x, t) - \rho(x, z) \ge r - \rho(x_0, z)/2 \ge r - r/4 \ge 3r/4$. Therefore, $\rho(z, \partial(B_0 \cap B)) \ge r/4$, which means that

$$B(z, r/4) \subset B_0 \cap B. \tag{4.2}$$

Relations (4.1) and (4.2) imply the claim.

Lemma 4.1 is proved. □

Lemma 4.2 (of Besicovitch type)**.** *Suppose $\Omega \Subset E_n$, and A is a bounded subset in Ω. If a ball $B(x, r(x))$ is given for any $x \in A$, i.e., $A \subset \bigcup_{x \in A} B(x, r(x))$, then there exists a finite or countable family of balls $\{B(x_j, r_j)\}_{j=1}^{\infty}$, where $r_j = r(x_j)$, such that*

$$A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), \quad \sum_{j=1}^{\infty} \chi_{B(x_j, r_j)}(x) \le C, \tag{4.3}$$

where C depends on n, the field $\{X_j\}$, and Ω.

Proof. As in [14], we denote

$$r_1^* = \sup\{r(B) : B = B(x, r(x)), x \in A\}.$$

Then there exists $B(x_1, r_1)$ with $r_1 > r_1^*/(1 + \delta)$, where $\delta > 0$ is an arbitrary fixed number. We put $B_1 = B(x_1, r_1)$. Suppose that m balls $B_1, \dots, B_m$ have already been chosen. We put

$$r_{m+1}^* = \sup\left\{r(B) : B = B(x, r(x)), x \in A \setminus \left(\bigcup_{k=1}^{m} B_k\right)\right\}.$$

Then there exists $B(x_{m+1}, r_{m+1})$ with $r_{m+1} \geq r^*_{m+1}/(1+\delta)$, and we set $B_{m+1} = B(x_{m+1}, r_{m+1})$.

We argue as in [14] to show that the system $\{B_j\}_{j=1}^{\infty}$ satisfies (4.3). Observe that the balls $B(x_j, \frac{1}{2}r_j)$ are pairwise disjoint. Therefore, by (2.18), we have

$$|A| \geq \sum_{j=1}^{\infty} \Big|\Big(B\Big(x_j, \frac{1}{3}r_j\Big)\Big)\Big| \geq C \sum_{j=1}^{\infty} r_j^Q,$$

whence we see that $r(B_j) \to 0$ as $j \to \infty$. Suppose $\bar{x} \in A$, but $\bar{x} \notin \bigcup_{j=1}^{\infty} B_j$. Then there exists $B(x_i, r_i)$ with $r(\bar{x}) > r_i$, where $B(\bar{x}, r(\bar{x}))$ is the ball in the initial covering. This means that the point $\bar{x}$ was missed in the process of choosing the points $\bar{x}$, which is impossible. Thus, $A \subset \bigcup_{j=1}^{\infty} B_j$.

Now we show that, at each point $z \in E_n$, only finitely many balls of the system can intersect at z, and moreover, their number is controlled in terms of the field, Ω, and n.

Let $\gamma_{xy}(t)$ be a geodesic that connects $x, y \in E_n$, i.e., $\gamma(t)$ satisfies (2.19). For $\varepsilon > 0$ and $x_0, z \in E_n$, we denote

$$K(z, \varepsilon, \{\gamma_{x_o z}\}) = \bigcup_{0<t<\rho(z,x_0)} [B(\gamma_{zx_0}(t), \varepsilon t) \cap S(z,t)]; \tag{4.4}$$

this set will be called the metric cone of opening ε, with vertex at z, and with ruling $\gamma_{zx_0}(t)$.

Let $z, x_0 \in \Omega$. We shall show that z can serve as the vertex for only finitely many cones (4.4) such that they cover the ball $B(z, \rho(z, x_0))$ and the cones of smaller opening $\frac{\varepsilon}{C}$ (C does not depend on ε) do not intersect (i.e., have no common points except for z). If this is not true, then

$$|B(x,t)| \geq \sum_{s=1}^{N} |K(z, \varepsilon/C, \{\gamma_{zx_s}\})|, \quad x_s \in B(z,t),$$

where N is the minimal number of such cones. The cone $K(z, \varepsilon/C, \{\gamma_{zx_s}\})$ contains a ball $B(a, t\varepsilon/2)$, where $a \in \{\gamma_{zx_s}\}$. By (2.17), we have $|B(a, \varepsilon t/2)| \geq (1/C(\varepsilon))|B(z,t)|$, so that $|B(z,t)| \geq (1/C(\varepsilon))N|B(z,t)|$. It follows that $N/C(\varepsilon) < 1$, i.e., the number of the cones is bounded by the number $C(\varepsilon)$.

Any fixed cone $K = K(z, \varepsilon, \{\gamma_{zx_0}\})$ possesses the following property: if $x_1, x_2 \in K$ and

$$\rho(x_1, z) \leq \rho(x_2, z) \leq (1+\delta)\rho(x_1, z),$$

then

$$\rho(x_1, x_2) \leq \rho(x_1, z).$$

Indeed, we can find $t_1, t_2 \in \{\gamma_{zx_0}\}$ with $\rho(x_1, z) = t_1$ and $\rho(x_2, z) = t_2$. Therefore,

$$\rho(x_1, x_2) \leq \rho(x_1, t_1) + t_2 - t_1 + \rho(x_2, t_2) \leq \varepsilon t_1 + t_2 - t_1 + \varepsilon t_2 \leq 2(\varepsilon + \delta)t_1.$$

Choosing $\delta = \varepsilon$ and $4\varepsilon < 1$, we have

$$\rho(x_1, x_2) \leq \min(\rho(x_1, z), \rho(x_2, z)). \tag{4.5}$$

Hence, any ball centered at x_1 (x_2) and containing z must also contain the point x_2 (x_1). This means that the point z lies in at most one of the balls in $\{B_j\}_{j=1}^{\infty}$ centered at a point of K. By construction, the center of any consequent ball belongs to none of the preceding balls. But, if x_1 is the center of the ith ball and x_2 is the center of the jth ball, then (4.5) implies that $x_1 \in B_j$ and $x_2 \in B_i$, a contradiction.

The lemma is proved. $\square$

§5. Proofs of Theorems 2.3 and 2.4

Proof of Theorem 2.3 (of Sobolev type). We argue as in Theorem 2.1, keeping the entire notation of that theorem concerning α, D_α, B_0, and B. Let $0 < \gamma < 1/C_1^3$ be fixed; here C_1 is the constant occurring in (2.17). By applying (2.17), it is easy to show that for any $x \in D_{2\alpha}$ there exists $B = B(x, r(x))$ such that $r(x) \in (0, 4r(B_0))$ and (3.1) is fulfilled. Indeed, the metric topology in question coincides with the Euclidean topology in E_n. Therefore, $B(x,t) \setminus D_\alpha = \varnothing$ for all sufficiently small t. Also, we have a continuous map $(E_n, \rho) \leftrightarrow (E_n, |\cdot|)$ between the metric space and the Euclidean space [18]. Let $x \in D_{2\alpha}$, let y be a point on the surface of the ball $B(x, 4r(B_0))$, and let $\{\gamma_{xy}\}$ be a geodesic connecting x and y (i.e., this curve satisfies (2.19)). Let z be a point of $\partial B(y, r(B_0)) \cap \{\gamma_{xy}\}$. Then $B(z, r(B_0)) \subset B(x, 4r(B_0)) \setminus B(x, 2r(B_0))$ because $\rho(z,x) = 3r(B_0)$. Consequently, putting $t = 4r(B_0)$ and using (2.17), we obtain $|B(x,t) \setminus D_\alpha| \geq |B(z, r(B_0))| \geq \frac{1}{C_1^3}|B(z, 8r(B_0))| \geq \frac{1}{C_1^3}|B(x,t)|$, because $B(z, 4r(B_0)) \subset B(z, 8r(B_0))$. Then $|B(x, 4r(B_0)) \setminus D_\alpha| \geq \gamma |B(x, 4r(B_0))|$. Thus, there exists $r(x) \in (0, 4r(B_0))$ such that (3.1) is fulfilled. Let $B = B(x, r(x))$. If 1) $|B \cap D_{2\alpha}| < \gamma|B|$, then, as in Theorem 2.1, we have

$$v(B \cap D_{2\alpha}) \leq \frac{C\gamma^\delta}{1 - C\gamma^\delta} v(B \cap D_\alpha). \tag{5.1}$$

If 2) $|D_{2\alpha} \cap B| \geq \gamma|B|$, then, by Federer's formula [26], we have

$$\int_{B \cap (D_\alpha \setminus D_{2\alpha})} |Xf| dx = \int_\alpha^{2\alpha} P_X(E_t; B) dt, \tag{5.2}$$

where $E_t = \{x \in B : f(x) > t\}, t \in (\alpha, 2\alpha)$, and $P_X(E_t; B)$ is the perimeter of E_t relative to B (see (2.22)). Next we use the following local isoperimetric inequality for the ball B:

$$P_X(E_t; B) \geq C\frac{|B|}{r(B)}, \quad t \in (\alpha, 2\alpha), \tag{5.3}$$

where C is independent of $B, r(B), t$, and E_t. Estimate (5.3) follows from inequality (2.20) (which was proved, e.g., in [18, 22]). Putting $E = E_t$ and using the estimates

$$|E_t \cap B| \geq \gamma|B|, \quad |B \setminus E_t| \geq \gamma|B|, \qquad t \in (\alpha, 2\alpha),$$

from (5.2) and (5.3) we deduce the inequality

$$\int_{B \cap (D_\alpha \setminus D_{2\alpha})} |Xf|\, dx \geq C\frac{|B|}{r(B)}\alpha. \tag{5.4}$$

This implies that

$$1 \leq \left(\frac{r(B)}{|B|\alpha} \int_{(D_\alpha \setminus D_{2\alpha}) \cap B} |Xf|\, dx \right)^q. \tag{5.5}$$

This inequality replaces estimate (3.7) in the proof of Theorem 2.1, and the rest of that proof needs no modification. By the Hölder inequality,

$$1 \leq \left(\frac{r(B)}{C|B|\alpha} \right)^q \left(\int_B \omega^{-1/(p-1)}(x)\, dx \right)^{(p-1)q/p} \left(\int_{(D_\alpha \setminus D_{2\alpha}) \cap B} |Xf|^p \omega(x)\, dx \right)^{q/p}, \tag{5.6}$$

and, by the balance condition (2.4),

$$v(B \cap D_{2\alpha}) \leq \left(\frac{1}{C\alpha} \right)^q \left(\frac{r(B_1)}{|B_1|} \right)^q v(B_1)(\omega^{-1/(p-1)}(B_1))^{q(p-1)/p}. \tag{5.7}$$

Now, using (2.17) and the properties of v and ω, we arrive at estimate (5.7) with B_1 replaced by B_0. Therefore, recalling (3.1), we obtain

$$\begin{aligned} v(B\cap D_{2\alpha}) &\le \frac{C\gamma^\delta}{1-C\gamma^\delta} v(B\cap D_\alpha) + \frac{CC_{pq}^q}{\alpha^q}(\omega^{-1/(p-1)}(B_0))^{q(p-1)/p} \\ &\quad \times \Big(\frac{r(B_0)}{|B_0|}\Big)^q v(B_0)\Big(\int_{B\cap(D_\alpha\setminus D_{2\alpha})}|Xf|^p\omega(x)\,dx\Big)^{q/p}. \end{aligned} \tag{5.8}$$

The system of metric balls $\{B=B(x,r(x)) : x\in D_{2\alpha}\}$ covers $D_{2\alpha}$. By Lemma 4.2, there exists a subfamily $\{B_i\}_{i=1}^\infty$ such that

$$\sum_{i=1}^\infty \chi_{B_i}(x)\le C,\quad D_{2\alpha}\subset\bigcup_{i=1}^\infty B_i, \tag{5.9}$$

where $C>0$ depends on n, Ω, and the field $\{X_j\}$. Arguing as in Theorem 2.1, we get

$$\begin{aligned} v(D_{2\alpha}) &\le \frac{C_nC\gamma^\delta}{1-C\gamma^\delta} v(D_\alpha) + C_nCC_{pq}^q(\omega^{-1/(p-1)}(B_0))^{q(p-1)/p} \\ &\quad \times \Big(\frac{r(B_0)}{|B_0|}\Big)^q v(B_0)\Big(\int_{D_\alpha\setminus D_{2\alpha}}|Xf|^p\omega(x)\,dx\Big)^{q/p}. \end{aligned} \tag{5.10}$$

Now we integrate (5.10) and choose γ sufficiently small, obtaining

$$\begin{aligned} \int_{B_0^+} f^q(x)v(x)\,dx &\le CC_{pq}^q\Big(\frac{r(B_0)}{|B_0|}\Big)^q v(B_0)(\omega^{-1/(p-1)}(B_0))^{\frac{q(p-1)}{p}} \\ &\quad \times \Big(\int_{B_0^+}|Xf|^p\omega(x)\,dx\Big)^{q/p}. \end{aligned} \tag{5.11}$$

A similar inequality is valid in B_0^- for the function $-f(x)$. As a result, we have

$$\int_{B_0}|f(x)|^q v(x)\,dx \le CC_{pq}^q\Big(\frac{r(B_0)}{|B_0|}\Big)^q v(B_0)(\omega^{-1/(p-1)}(B_0))^{\frac{q(p-1)}{p}}\Big(\int_{B_0}|Xf|^p\omega(x)\,dx\Big)^{q/p}.$$

The theorem is proved. □

Proof of Theorem 2.4. We modify somewhat the argument at the beginning of the proof of Theorem 2.2.

Let $r(x)$ and $B(x,r(x))$ be chosen as in (3.20). If a) $|D_{2\alpha}\cap B|<\gamma|B\cap B_0|$, then, since $v\in A_\infty$, we obtain $v(D_{2\alpha}\cap B)\le C\gamma^\delta v(B)$. Next, we have $v(B)\le C_2v(B\cap B_0)$, whence $v(B\cap D_{2\alpha})\le CC_2\gamma^\delta v(B\cap B_0)$, where $C_2>0$ depends on the constants C,δ in (2.3) and on n, Ω, $\{X_j\}$.

Similarly, (3.20) implies $v((B\cap B_0)\setminus D_{2\alpha})\le CC_2\gamma^\delta v(B\cap B_0)$. Then $v(B\cap B_0)=v((B\cap B_0)\setminus D_\alpha)+v(B\cap D_\alpha)\le CC_2\gamma^\delta v(B\cap B_0)+v(B\cap D_\alpha)$. Therefore, if γ satisfies $CC_2\gamma^\delta<1$, then

$$v(B\cap B_0)\le\frac{1}{1-CC_2\gamma^\delta}v(B\cap D_\alpha),$$

whence

$$v(B\cap D_{2\alpha})\le\frac{C\gamma^\delta}{1-C\gamma^\delta}v(B\cap D_\alpha). \tag{5.12}$$

In the case where b) $|D_{2\alpha}\cap B|\ge\gamma|B\cap B_0|$, we argue as in Theorem 2.1. By the isoperimetric estimate (2.20) with $E=E_t$, we have

$$P_X(E_t;B)\ge C\frac{|B|^{1/Q}}{r(B)}|B\cap B_0|^{\frac{Q-1}{q}}, \tag{5.13}$$

where $t \in (\alpha, 2\alpha)$. Thus, an estimate similar to (5.4) takes the form

$$\int_{B\cap(D_\alpha\setminus D_{2\alpha})} |Xf|\,dx \geq C\alpha \frac{|B|^{1/Q}}{r(B)} |B_0 \cap B|^{\frac{Q-1}{q}}. \tag{5.14}$$

Now, applying Lemma 4.1, we obtain $|B_0 \cap B| \geq C|B|$, where $C > 0$ is independent of B_0 and B. Plugging this into (5.14), we arrive at estimate (5.4). Next, we can repeat the arguments of the preceding proof (including the application of Lemma 4.2) to complete the proof of Theorem 2.4. □

References

[1] V. G. Maz′ya, *Sobolev spaces*, Leningrad. Gos. Univ., Leningrad, 1985; English transl., Springer-Verlag, Berlin, 1985. MR0807364 (87g:46055); MR0817985 (87g:46056)

[2] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591. MR0159138 (28:2356)

[3] B. Franchi and E. Lanconelli, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), 523–541. MR0753153 (85k:35094)

[4] F. Chiarenza and R. Serapioni, *A Harnack inequality for degenerate parabolic equations*, Comm. Partial Differential Equations **9** (1984), 719–749. MR0748366 (86c:35082)

[5] E. T. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874. MR1175693 (94i:42024)

[6] P. Pansu, *Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) **129** (1989), 1–60. MR0979599 (90e:53058)

[7] M. Gromov, *Carnot–Carathéodory spaces seen from within*, Sub-Riemannian Geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 79–323. MR1421823 (2000f:53034)

[8] B. Franchi, *Weighted Sobolev–Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations*, Trans. Amer. Math. Soc. **327** (1991), 125–158. MR1040042 (91m:35095)

[9] B. Franchi, C. E. Guttiérez, and R. L. Wheeden, *Weighted Sobolev–Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations **19** (1994), 523–604. MR1265808 (96h:26019)

[10] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171. MR0222474 (36:5526)

[11] L. Capogna, D. Danielli, and N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Partial Differential Equations **18** (1993), 1765–1794. MR1239930 (94j:35038)

[12] ______, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*, Comm. Anal. Geom. **2** (1994), 203–215. MR1312686 (96d:46032)

[13] D. Danielli, *Regularity at the boundary for solutions of nonlinear subelliptic equations*, Indiana Univ. Math. J. **44** (1995), 269–286. MR1336442 (97b:35028)

[14] M. de Guzmán, *Differentiation of integrals in R^n*, Lecture Notes in Math., vol. 481, Springer-Verlag, Berlin–New York, 1975. MR0457661 (56:15866)

[15] C. Fefferman and D. H. Phong, *Subelliptic eigenvalue problems*, Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vols. I, II (Chicago, Ill., 1981), Wadsworth, Belmont, CA, 1983, pp. 590–606. MR0730094 (86c:35112)

[16] P. Hajlasz and P. Strzelecki, *Subelliptic p-harmonic maps into spheres and the ghost of Hardy spaces*, Max-Planck-Inst. Mat. Naturwiss., Leipzig, Preprint no. 36, 1997, pp. 1–22. Math. Ann. **312** (1998), 341–362. MR1671796 (2000b:35033)

[17] A. Nagel, E. M. Stein, and S. Wainger, *Balls and metrics defined by vector fields.* I. *Basic properties*, Acta Math. **155** (1985), 103–147. MR0793239 (86k:46049)

[18] N. Garofalo and D. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081–1144. MR1404326 (97i:58032)

[19] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1975), 161–207. MR0494315 (58:13215)

[20] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), 503–523. MR0850547 (87i:35027)

[21] G. Lu, *The sharp Poincaré inequality for free vector fields: An endpoint result*, Preprint, 1992; Rev. Math. Iberoamericana **10** (1994), 453–466. MR1286482 (96g:26023)

[22] B. Franchi, G. Lu, and R. L. Wheeden, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier (Grenoble) **45** (1995), 577–604. MR1343563 (96i:46037)

[23] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices **1992**, no. 2, 27–38. MR1150597 (93d:58158)

[24] D. Danielli, *Formules de représentation et théorèmes d'inclusion pour des opérateurs sous elliptiques*, C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), 987–990. MR1168522 (93e:35020)

[25] C. Pérez, *Two weighted norm inequalities for Riesz potentials and uniform L^p-weighted Sobolev inequalities*, Indiana Univ. Math. J. **39** (1990), 31–44. MR1052009 (92a:42024)

[26] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., Bd. 153, Springer-Verlag New York, Inc., New York, 1969. MR0257325 (41:1976)

[27] W. Fleming and R. Rishel, *An integral formula for the total gradient variation*, Arch. Math. **11** (1960), 218–222. MR0114892 (22:5710)

Institute of Mathematics and Mechanics, National Academy of Sciences, Azerbaidzhan, and Dichle University, Diyarbakir, Turkey
E-mail address: farman-m@mail.ru

Institute of Mathematics and Mechanics, National Academy of Sciences, Azerbaidzhan
E-mail address: rabilamanov@hotmail.com

Received 14/JUN/2006

Translated by A. PLOTKIN