Алгебра и анализ
Том 20 (2008), № 3

St. Petersburg Math. J.
Vol. 20 (2009), No. 3, Pages 465–471
S 1061-0022(09)01056-5
Article electronically published on April 8, 2009

ON SYMMETRIZABILITY OF HYPERBOLIC MATRIX SPACES

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Abstract. A new symmetrizability criterion for linear matrix spaces is proposed, with applications to the theory of first order conservation laws.

Let $L \subset \mathrm{Mat}(n,k)$ be a real linear subspace of the space of $(n \times n)$-matrices with coefficients from the field $k = \mathbb{R}$ or $\mathbb{C}$.

Definition 1. The family L is said to be *hyperbolic* if

$$A^2 \in L \quad \text{for all} \quad A \in L, \tag{1}$$

and all matrices in L have a simple real spectrum (i.e., the eigenvalues of any matrix $A \in L$ are real and there is a basis consisting of the corresponding eigenvectors).

Condition (1) implies the relation

$$AB + BA = (A+B)^2 - A^2 - B^2 \in L \quad \text{for all} \quad A, B \in L$$

(this means that L is a special Jordan algebra). In particular, we can define linear operators S_A on L by the rule $S_A B = AB + BA$.

If L is a hyperbolic space, then its extension $\{A + \lambda E \mid \lambda \in \mathbb{R}\}$, obtained by adding the unit matrix E, is also a hyperbolic space.

The hyperbolicity condition admits the following reformulation.

Proposition 1. *A space L that satisfies* (1) *and contains the unit matrix E is hyperbolic if and only if the linear operators S_A have a simple real spectrum in L for all $A \in L$.*

Proof. Let L be a hyperbolic space. Then the spectrum $\sigma(A)$ of every matrix $A \in L$ is simple and real. We consider the symmetric bilinear form $(A, B) = \mathrm{Tr}\, AB$. Then $(A, A) = \sum_{\lambda \in \sigma(A)} \lambda^2 > 0$ for $A \neq 0$. Therefore, the form $(\cdot, \cdot)$ is positive definite and determines a scalar multiplication on L. Direct verification shows that the operators S_A are symmetric with respect to this scalar multiplication (i. e., $(S_A B, C) = (B, S_A C)$ for all $B, C \in L$); consequently, they have simple real spectra.

Conversely, suppose that each operator S_A has a simple real spectrum in L. Relation (1) and the condition $E \in L$ imply that L contains all powers A^n, $n \geq 0$, for $A \in L$, so that $f(A) \in L$ for every real polynomial $f(z)$. Let $A \in L$. By our assumptions, the operator S_A has a simple real spectrum. Clearly, the simplicity of the spectrum of a matrix (or an operator) A means that there exists a polynomial $p(z) = \prod_{k=1}^{m} (z - \lambda_k)$ with distinct real roots λ_k, $k = 1, \ldots, m$, such that $p(A) = 0$. Therefore, there exists a polynomial $p(z)$ with distinct real roots such that $p(S_A) = 0$. Since $p(2A) = p(S_A)E = 0$ and the polynomial $p(2z)$ also has distinct real roots, the spectrum of A is real and simple. The proof is complete. □

2000 *Mathematics Subject Classification.* Primary 15A30, 15A06.

Key words and phrases. Hyperbolic matrix space, spectrum, symmetrizable system.

Supported by RFBR (grant no. 06-01-00289) and by Deutsche Forschungsgemeinschaft (DFG project no. 436 RUS 113/895/0-1).

We note that the assumption $E \in L$ is essential for the inverse statement of Proposition 1. Indeed, let J be any nontrivial matrix such that $J^2 = 0$ and $L = \{\lambda J \mid \lambda \in \mathbb{R}\}$. Obviously, L satisfies (1), but $E \notin L$. It is easily seen that $S_A = 0$ for all $A \in L$, but L is certainly not a hyperbolic space.

As follows from Proposition 1, the hyperbolicity condition means that the system of conservation laws corresponding to the Burgers-like equation $U_t + (U^2)_x = 0$, $U = U(t,x) \in L$ is hyperbolic. In the papers [3, 4], general systems of the form

$$U_t + f(U)_x = 0 \tag{2}$$

were studied, in which the unknown function $U = U(t,x)$ takes its values in the space S_n of symmetric matrices of order n or in the space H_n of Hermitian matrices of order n, and $U \to f(U)$ is the functional calculus operator. In those papers it was shown that system (2) is hyperbolic. More generally, systems such as (2) can also be considered in the case where U takes its values in an arbitrary matrix linear space L invariant under the functional calculus operators: $f(U) \in L$ for all $U \in L$ and all real functions $f(u)$ admitting an analytic extension to the entire complex plane. In particular, the space L must satisfy (1). Our Theorem 1 (see below) shows that for a nonlinear function f, system (2) is hyperbolic only in the case, studied in [3, 4], where L consists of symmetric or Hermitian matrices (after an appropriate choice of a basis in k^n).

Before formulating our main result, we describe some useful constructions preserving the property of hyperbolicity.

Let L be a matrix space. We introduce the space L^* consisting of all conjugate matrices A^* with $A \in L$ (with respect to some scalar multiplication on k^n). Clearly, the space L^* is hyperbolic simultaneously with L. Now, suppose that $H \subset k^n$ is a linear subspace invariant under the action of L, i.e., $A(H) \subset H$ for all $A \in L$. Then, we can define matrix spaces L_H and $L_{/H}$ to consist of all matrices corresponding to the restricted operators $A|_H : H \to H$, $A \in L$, and to the factor operators $A/H : k^n/H \to k^n/H$, $A \in L$, respectively. Obviously, the orthogonal complement $H^\perp$ is an invariant space for L^* and $(L_{/H})^* = L^*|_{H^\perp}$. The following simple statement is true.

Lemma 1. *Suppose L is a hyperbolic matrix space and a subspace $H \subset k^n$ is invariant under the action of L. Then the matrix spaces L_H and $L_{/H}$ are hyperbolic.*

Proof. By the duality relation $L_{/H} = (L^*|_{H^\perp})^*$, it suffices to prove the lemma in the case of the matrix space L_H. Clearly, L_H satisfies (1). In the proof of Proposition 1 it was shown that for any matrix $A \in L$ there exists a polynomial $p(z)$ with distinct real roots such that $p(A) = 0$. Then $p(A|_H) = p(A)|_H = 0$. Therefore, the spectrum of $A|_H$ is real and simple. Hence, the space L_H is hyperbolic, as required. □

We are ready to formulate the main result.

Theorem 1. *A space L satisfying* (1) *is hyperbolic if and only if all matrices $A \in L$ are symmetric* (*Hermitian*) *with respect to some scalar multiplication in k^n*: $(Au, v) = (u, Av)$ *for all $u, v \in k^n$.*

Below it is assumed that L is a space of matrices over the field $k = \mathbb{C}$. The case of the real field $k = \mathbb{R}$ reduces to that of $k = \mathbb{C}$ by complexification. Indeed, if a real matrix family L consists of Hermitian matrices with respect to a scalar multiplication $(\cdot,\cdot)$ on $\mathbb{C}^n$, then the matrices in L are symmetric with respect to the real scalar multiplication $\mathrm{Re}(\cdot,\cdot)$ on $\mathbb{R}^n$. Observe that L remains hyperbolic after complexification.

To prove Theorem 1, we need some preliminary results and constructions.

Lemma 2. *For all A, B, $C \in L$ we have*

$$[S_A, S_B]C = [[A, B], C].$$

Here $[\cdot,\cdot]$ is the commutator of operators (matrices).

Proof. The claim follows directly from the identity

$$\begin{aligned}[S_A, S_B]C &= S_A S_B C - S_B S_A C\\ &= ABC + ACB + BCA + CBA - BAC - BCA - ACB - CAB\\ &= (AB - BA)C - C(AB - BA) = [[A, B], C].\end{aligned}$$ □

Corollary 1. 1) $[[A, B], C] \in L$ *for all* $A, B, C \in L$. 2) *Let* $[L, L]$ *be the linear hull of the set of commutators* $[A, B]$, A, $B \in L$. *Then* $[L, L]$ *is a Lie algebra* (*with the multiplication* $[\cdot,\cdot]$).

Proof. Statement 1) readily follows from Lemma 2. To prove 2), it suffices to verify that $[[A_1, B_1], [A_2, B_2]] \in [L, L]$ for all A_1, B_1, A_2, $B_2 \in L$. But this is implied by the identity

$$[[A_1, B_1], [A_2, B_2]] = [[[A_1, B_1], A_2], B_2] - [[[A_1, B_1], B_2], A_2] = [C_1, B_2] - [C_2, A_2],$$

where $C_1 = [[A_1, B_1], A_2]$, $C_2 = [[A_1, B_1], B_2] \in L$; see statement 1). □

We consider the linear subspace $\mathcal{A} = [L, L] \oplus L$ and define a multiplication on $\mathcal{A}$ by setting, for $x = X \oplus A$, $y = Y \oplus B$,

$$xy = ([X, Y] - [A, B]) \oplus ([X, B] - [Y, A]). \tag{3}$$

Observe that, by Corollary 1, $[X, Y]$, $[A, B] \in [L, L]$ and $[X, B]$, $[Y, A] \in L$ for any X, $Y \in [L, L]$ and any A, $B \in L$, so that this multiplication is well defined.

We define the subspaces

$$Z_1 = \{X \in [L, L] \mid [X, B] = 0 \text{ for all } B \in L\},$$
$$Z_2 = \{A \in L \mid [A, B] = 0 \text{ for all } B \in L\}$$

of the spaces $[L, L]$ and L, respectively.

Lemma 3. 1) *$\mathcal{A}$ is a Lie algebra, and its center $Z(\mathcal{A})$ coincides with $Z_1 \oplus Z_2$;* 2) *the maps*

$$f(X \oplus A)B = [X, B] + iS_A B, \qquad h(X \oplus A)v = Xv + iAv$$

are linear representations of $\mathcal{A}$ in the spaces $L \otimes \mathbb{C}$ and $\mathbb{C}^n$, respectively. Here $i^2 = -1$.

Proof. 1) The fact that $\mathcal{A}$ is a Lie algebra is verified directly with the help of the known properties of commutators. We omit the corresponding boring calculations. To describe the center $Z(\mathcal{A})$, suppose that $x = X \oplus A \in Z(\mathcal{A})$. Then $xy = 0$ for all $y \in \mathcal{A}$. Taking $y = 0 \oplus B$, we see that $[X, B] = [A, B] = 0$ for all $B \in L$. Therefore, $x \in Z_1 \oplus Z_2$. Conversely, if $x = X \oplus A \in Z_1 \oplus Z_2$, then X and A commute with all matrices in L, and therefore, they also commute with the matrices in $[L, L]$: $[X, Y] = [A, Y] = 0$ for all $Y \in [L, L]$ (this follows easily from the Jacobi identity). Now, (3) implies that $xy = 0$ for all $y \in \mathcal{A}$, i.e., $x \in Z(\mathcal{A})$.

2) For $X \in [L, L]$, we define an operator C_X acting in $L \otimes \mathbb{C}$ by the rule $C_X B = [X, B]$. Then $[C_X, C_Y] = C_{[X,Y]}$ and $[C_X, S_A] = S_{[X,A]}$ for $X, Y \in [L, L]$, $A \in L$. Let $x = X \oplus A$, and let $y = Y \oplus B \in \mathcal{A}$. Then, using the above relations and Lemma 2, we obtain

$$\begin{aligned}[f(x), f(y)] &= [C_X + iS_A, C_Y + iS_B]\\ &= [C_X, C_Y] - [S_A, S_B] + i([C_X, S_B] - [C_Y, S_A])\\ &= C_{[X,Y]-[A,B]} + iS_{[X,B]-[Y,A]} = f(xy),\end{aligned}$$

and the map f is a homomorphism of the Lie algebra $\mathcal{A}$ into the algebra $gl(L \otimes \mathbb{C})$ of linear operators in $L \otimes \mathbb{C}$, i.e., it is a linear representation of $\mathcal{A}$ in $L \otimes \mathbb{C}$. Next,

$$[h(x), h(y)] = [X + iA, Y + iB] = [X, Y] - [A, B] + i([X, B] - [Y, A]) = h(xy);$$

i.e., h is a representation of the algebra $\mathcal{A}$ in $\mathbb{C}^n$. The proof is complete. □

First, we prove Theorem 1 in the case where $Z_1 = \{0\}$.

Proposition 2. *Suppose L is a hyperbolic matrix space and $Z_1 = \{0\}$. Then all matrices in L are Hermitian with respect to some scalar multiplication in $\mathbb{C}^n$.*

Proof. If L is a hyperbolic matrix space, then its extension $\{A + \lambda E \mid \lambda \in \mathbb{R}\}$ obtained by adding the unit matrix E is also a hyperbolic space with the same algebra $[L, L]$. Therefore, there is no loss of generality in assuming that $E \in L$. It is easily seen that the operators $f(x)$ are skew-Hermitian in $L \otimes \mathbb{C}$ with respect to the scalar product $(A, B) = \operatorname{Tr} A\bar{B}$, where $B \to \bar{B}$ denotes complex conjugation on $L \otimes \mathbb{C}$. Therefore, the symmetric bilinear form $(x, y) = -\operatorname{Tr} f(x)f(y)$ is nonnegative definite. Moreover, if $x = X \oplus A \in \mathcal{A}$ and $(x, x) = 0$, then $f(x) = 0$. In particular, $A = -\frac{i}{2} f(x)E = 0$. Then $f(x)B = [X, B] = 0$ for all $B \in L$, i.e., $X \in Z_1$. Since $Z_1 = \{0\}$ by our assumptions, we have $X = 0$. Hence, $x = 0$, so that the form $(\cdot, \cdot)$ is nondegenerate. Therefore, this form determines a scalar product on $\mathcal{A}$. The operators $ad_x y = xy$ are skew-symmetric with respect to this scalar product. Indeed,

$$\begin{aligned}(ad_x y, z) &= -\operatorname{Tr} f(xy)f(z) = -\operatorname{Tr}[f(x), f(y)]f(z) \\ &= \operatorname{Tr} f(y)[f(x), f(z)] = \operatorname{Tr} f(y)f(xz) = -(y, ad_x z).\end{aligned}$$

The above property means that $\mathcal{A}$ is a compact Lie algebra (in the sense of [1, 2]). By the known properties of compact Lie algebras (see, e.g., [1, 2]), we have $\mathcal{A} = \mathcal{A}_1 \oplus Z(\mathcal{A})$, where $\mathcal{A}_1$ is a semisimple compact Lie algebra, which is the Lie algebra of a unique simply connected compact Lie group G. Moreover, the homomorphism $h : \mathcal{A}_1 \to gl(\mathbb{C}^n)$ induces a homomorphism of Lie groups $\tilde{h} : G \to \mathrm{GL}(\mathbb{C}^n)$. Here $\mathrm{GL}(\mathbb{C}^n)$ is the Lie group of nonsingular linear operators on $\mathbb{C}^n$ with the corresponding Lie algebra $gl(\mathbb{C}^n)$. In other words, G acts linearly on $\mathbb{C}^n$: $gv = \tilde{h}(g)v$. We decompose the space $\mathbb{C}^n$ into a direct sum of indecomposable subspaces invariant under the action of L: $\mathbb{C}^n = \bigoplus_{k=1}^m V_k$. If $x \in Z(\mathcal{A})$, then, by Lemma 3 and the condition $Z_1 = 0$, we have $x = 0 \oplus A$, where $[A, B] = 0$ for all $B \in L$. This implies that A acts trivially on the spaces V_k: $A = \lambda_k E$ on V_k. Indeed, otherwise V_k can be decomposed into a direct sum of proper subspaces that correspond to different eigenvalues of A (recall that the restriction $A|_{V_k}$ has a simple real spectrum, see Lemma 1), and these subspaces are invariant for all matrices $B \in L$, by the condition $[A, B] = 0$ for all $B \in L$. But this contradicts the fact that V_k is indecomposable. Clearly, all the subspaces V_k, $k = 1, \ldots, m$, are invariant subspaces for the matrices in $\mathcal{A}$; consequently, they are invariant under the action of the group G.

We may assume that the scalar product in $\mathbb{C}^n$ is chosen in such a way that the spaces V_k, $k = 1, \ldots, m$, are pairwise orthogonal. We define a new scalar product $(u, v)_i$ in $\mathbb{C}^n$, invariant under the action of G, by setting $(u, v)_i = \int_G (gu, gv)\, d\mu(g)$, where μ is the Haar measure in G. Relative to this scalar product, $\tilde{h}$ takes values in the group $U(n)$ of unitary operators (matrices), and consequently, for $x \in \mathcal{A}_1$ the image $h(x)$ is contained in the corresponding Lie algebra $u(n)$ of skew-Hermitian matrices. It is easily seen that the spaces V_k, $k = 1, \ldots, m$, remain pairwise orthogonal under the new scalar product. Therefore, the matrices $h(x)$ are also skew-Hermitian for $x = 0 \oplus A \in Z(\mathcal{A})$, because $h(x) = iA = i\lambda_k E$ on the subspaces V_k, and $\lambda_k \in \mathbb{R}$, $k = 1, \ldots, m$. Thus, the image $h(\mathcal{A})$ belongs to $u(n)$, and since $h(0 \oplus A) = iA$, we see that all matrices $A \in L$ are Hermitian. The proof is complete. □

Now we prove that, in fact, our assumption $Z_1 = 0$ is always fulfilled.

Proposition 3. *Let L be a hyperbolic matrix space. Then $Z_1 = \{0\}$.*

Proof. We use induction on the dimension n. If $n = 0$ or 1, then $[L, L] = \{0\}$ and there is nothing to prove. Now, suppose that $n > 1$ and that our statement is true for all dimensions less than n. Suppose $X \in [L, L]$ and $[X, B] = 0$ for all $B \in L$. We need to check that $X = 0$. Let $\mu \in \mathbb{C}$ be an eigenvalue of X, and let $H \subset \mathbb{C}^n$ be the corresponding subspace of eigenvectors. If $H = \mathbb{C}^n$, then $X = \mu E = 0$ (this follows from the obvious relation $\operatorname{Tr} X = 0$), as required.

It remains to consider the case where H is a proper subspace of $\mathbb{C}^n$. Since $XAv = AXv = \mu Av$ for all $v \in H$, $A \in L$, we see that H is invariant under the action of L and, with it, of $[L, L]$. Therefore, we can define the homomorphisms of restriction $A \to A|_H$ of the spaces L and $[L, L]$ into the spaces L_H and $[L_H, L_H]$, respectively. By Lemma 1, L_H is a hyperbolic space of order $m = \dim H < n$, and the matrix $X|_H$ commutes with L_H. By the inductive hypothesis, we have $X|_H = 0$, that is, $\mu = 0$ and $H = \operatorname{Ker} X$. If $V \subset \mathbb{C}^n$ is a proper linear subspace invariant under the action of L, then V is also invariant under the action of $[L, L]$ and $X|_V \in [L_V, L_V]$, $[X|_V, A|_V] = 0$ for all $A \in L$. Again by the inductive hypothesis, we have $X|_V = 0$, i.e., $V \subset H$. Thus, H contains all proper invariant subspaces.

Now, observe that $H_1 = \operatorname{Im} X$ is an invariant subspace, which follows directly from the relation $AXv = XAv$ for all $A \in L$, $v \in \mathbb{C}^n$. Since $H_1 \neq \mathbb{C}^n$ (otherwise $H = \ker X = \{0\}$, which is not true), we see that H_1 is a proper invariant subspace, whence $H_1 \subset H$, i.e., $X^2 = 0$. As shown above, L_H is a hyperbolic space for which $Z_1 = 0$. By Proposition 2, there exists a scalar multiplication on H under which the matrices $A|_H$ are Hermitian for all $A \in L$. Let $H_2 = H \ominus H_1$ be the orthogonal complement to H_1 in H. Since the matrices $A \in L$ are Hermitian on H, if follows that H_2 is invariant under the actions of L and $[L, L]$. Hence, we can consider the space $L_{/H_2}$, which is hyperbolic by Lemma 1. It is clear that X/H_2 belongs to $[L_{/H_2}, L_{/H_2}]$ and commutes with $L_{/H_2}$. Assume that $H_2 \neq \{0\}$, i.e., $\dim \mathbb{C}^n/H_2 < n$. Then, by the inductive hypothesis, $X/H_2 = 0$. But this contradicts the fact that $\operatorname{Im} X/H_2 = H/H_2 \simeq H_1 \neq \{0\}$. Thus, $H_2 = 0$, i.e., $H_1 = H$. This implies that $\mathbb{C}^n = (\mathbb{C}^n/H) \oplus H$ and the operator X gives rise to an isomorphism $X : \mathbb{C}^n/H \to H$. Identifying $\mathbb{C}^n/H$ and H via this isomorphism, we see that $\mathbb{C}^n = H \oplus H$ and $X(u, v) = (0, u)$. Any operator $A \in L$ can be represented in the form $A(u, v) = (A_1 u, A_2 u + A_3 v)$, because the space $0 \oplus H$ is invariant. Since $0 = [X, A](u, v) = (0, A_1 u - A_3 u)$, we have $A_3 = A_1$. Next, if $A, B \in L$ and $A(u, v) = (A_1 u, A_2 u + A_1 v)$, $B(u, v) = (B_1 u, B_2 u + B_1 v)$, then a direct computation shows that $[A, B](u, v) = (C_1 u, C_2 u + C_1 v)$, where $C_1 = [A_1, B_1]$, $C_2 = [A_2, B_1] + [A_1, B_2]$. In particular, $\operatorname{Tr} C_1 = \operatorname{Tr} C_2 = 0$. Clearly, this property holds true for all matrices in $[L, L]$, because they are linear combinations of commutators $[A, B]$, $A, B \in L$. Since $X \in [L, L]$ and $X(u, v) = (0, Eu)$, where E is the unit matrix, we obtain the wrong relation $\operatorname{Tr} E = 0$. This contradiction shows that $X = 0$. The proof is complete. □

We are ready to finish the proof of Theorem 1. The direct statement of Theorem 1 immediately follows from Propositions 2 and 3. Conversely, if all matrices in a linear matrix space L are Hermitian, then they have simple real spectra, and consequently, the space L is hyperbolic. The proof of the theorem is complete.

Now, we apply our result to the problem of symmetrizability for the first order system

$$u_t + \sum_{k=1}^{m} A_k u_{x_k} = 0, \quad A_k = A_k(t, x, u) \in \operatorname{Mat}(n, \mathbb{R}),\ k = 1, \ldots, m. \tag{4}$$

Recall that system (4) is symmetrizable if, for fixed t, x, u, all matrices A_k, $k = 1, \dots, m$, can be symmetrized simultaneously by an appropriate choice of a basis or, equivalently, by the choice of a scalar product (Bu, v) given by some positive definite matrix B. Multiplying system (4) by the matrix B, we arrive at the following symmetric form of that system:

$$Bu_t + \sum_{k=1}^{m} C_k u_{x_k} = 0,$$

where the matrices B and C_k, $k = 1, \dots, m$, are symmetric and B is positive definite.

The symmetrizability of system (4) can be stated as the capability of symmetrizing all matrices in the real linear hull M of the matrices A_k, $k = 1, \dots, m$. Clearly, the hyperbolicity condition

$$A \text{ has a simple real spectrum for all } A \in M \tag{5}$$

is necessary for the symmetrizability of the real linear matrix subspace $M \subset \mathrm{Mat}(n, \mathbb{C})$. In the case of complex matrices, symmetrizability is understood as being able to reduce all matrices in M to Hermitian form. In the cases where $m = 1$ or $n = 2$, condition (5) is also sufficient for symmetrizability (see, e.g., [5]).

It turns out that this remains true only in the cases indicated. If $n > 2$, then condition (5) and even the stronger condition of strict hyperbolicity,

$$A \text{ has distinct and real eigenvalues for all } A \in M,\ A \neq 0, \tag{6}$$

does not suffice for the symmetrizability of a matrix space M with $\dim M > 1$. The corresponding example was constructed in [5]. For completeness, we present this example below.

Example. For $n = 3$, consider the two-dimensional linear matrix space M that consists of matrices of the form

$$A = \begin{pmatrix} 0 & 0 & a-b \\ 0 & 0 & b \\ a-b & a & 0 \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

The eigenvalues of A are computed easily: $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{(a-b)^2 + ab}$. They are real and distinct for $A \neq 0$, because the quadratic form $(a-b)^2 + ab$ is positive definite. Thus, condition (6) is satisfied. We prove that this "strictly hyperbolic" family cannot be symmetrized. Assuming the contrary, we find a scalar product (Px, y) corresponding to some positive definite matrix

$$P = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{pmatrix}$$

such that all matrices $A \in M$ are symmetric. Then $(PAx, y) = (x, PAy)$; i.e., the matrices PA are symmetric under the original scalar product. Writing the relations $(PA)_{12} = (PA)_{21}$, $(PA)_{13} = (PA)_{31}$, and $(PA)_{23} = (PA)_{32}$ explicitly, we see that $ap_3 = (a-b)p_5$, $(a-b)p_1 + bp_2 = (a-b)p_6$, $(a-b)p_2 + bp_4 = ap_6$ for all $a, b \in \mathbb{R}$. This implies that $p_i = 0$, $i = 1, \dots, 6$, i.e., $P = 0$. But this contradicts the condition $P > 0$. Therefore, the family M is not symmetrizable. Taking the basis matrices A_1, A_2 corresponding to $a = 1$, $b = 0$, and $a = b = 1$, we arrive at a strictly hyperbolic but not symmetrizable system $q_t = A_1 q_x + A_2 q_y = 0$, $q = (u, v, w)^\top$ of the form

$$\begin{cases} u_t = w_x, \\ v_t = w_y, \\ w_t = (u+v)_x + v_y. \end{cases} \tag{7}$$

One criterion for symmetrizability was found in [5], saying that a space M can be symmetrized if and only if all matrices in the minimal real Lie algebra containing iA, $A \in M$ (with $i^2 = -1$), have a simple imaginary spectrum.

Now we are able to introduce a new symmetrizability criterion, which is an easy consequence of Theorem 1. We denote by $L = L(M)$ the minimal linear matrix subspace that contains M and satisfies condition (1).

Theorem 2. *The family M is symmetrizable if and only if the space L is hyperbolic.*

Proof. If all matrices $A \in M$ are symmetric (Hermitian) under some scalar product, then the same is true for the matrices in L. Consequently, each of them has a simple real spectrum; i.e., the space L is hyperbolic.

The converse statement follows directly from Theorem 1. □

Observe that the result on nonsymmetrizability in the above example follows from Theorem 2. Indeed, let

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

be the basis matrices defined in this example. Then the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = A_2 A_1 A_2 = \frac{1}{2}\left[(S_{A_2})^2 A_1 - S_{A_2^2} A_1\right]$$

belongs to L, but its spectrum is not simple.

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Received 29/JAN/2007

Translated by THE AUTHOR