Алгебра и анализ
Том 20 (2008), № 4

St. Petersburg Math. J.
Vol. 20 (2009), No. 4, Pages 569–591
S 1061-0022(09)01062-0
Article electronically published on June 1, 2009

THE $\mathbb{Z}_p$-RANK OF A TOPOLOGICAL K-GROUP

O. YU. IVANOVA

ABSTRACT. A complete two-dimensional local field K of mixed characteristic with finite second residue field is considered. It is shown that the rank of the quotient $U(1)K_2^{\mathrm{top}}K/T_K$, where T_K is the closure of the torsion subgroup, is equal to the degree of the constant subfield of K over $\mathbb{Q}_p$. Also, a basis of this quotient is constructed in the case where there exists a standard field L containing K such that L/K is an unramified extension.

INTRODUCTION

In the present paper, we consider the second topological K-group $K_2^{\mathrm{top}}K$ of a complete two-dimensional local field K of mixed characteristic. We assume that the field K contains a primitive pth root of unity, where p is the characteristic of the residue field of K, and that the second residue field of K is finite.

In the paper [4], it was proved that if K is a standard field, k is the subfield of constants of K, A is a topological $\mathbb{Z}_p$-basis of V_k modulo torsion, t is a second local parameter of K, and T_K is the closure of the torsion of $U(1)K_2^{\mathrm{top}}K$, then the set $\{\{u,t\} \mid u \in A\}$ is a topological basis of the group $U(1)K_2^{\mathrm{top}}K/T_K$; in particular, this group is a $\mathbb{Z}_p$-module of rank $|k : \mathbb{Q}_p|$. In the present paper, we prove the statement concerning the basis of $U(1)K_2^{\mathrm{top}}K/T_K$ in the case where $p \nmid e(K/k)$ (see Lemma 4.7 and Corollary 5.4).

Also in the paper [4], it was proved that if K is a nonstandard field, then the rank of $U(1)K_2^{\mathrm{top}}K$ is finite. In [13], an estimate for the index of the subgroup $B + T_K$ in $U(1)K_2^{\mathrm{top}}K$, where $B = \{\{u,t\} \mid u \in V_k\}$, was obtained. However, since errors have been found in the proofs in [13], it is still unknown whether the statements are true. In the present paper, we consider the factor group by another subgroup: in the case of a standard field, T_K means, as before, the closure of the torsion, and, in the case of an arbitrary field, T_K is defined as the intersection of $U(1)K_2^{\mathrm{top}}K$ and the closure of the torsion of the K-group of the standard field containing K. The fact that the group T_K is well defined is proved in §4.

In §5, we prove that the index of the subgroup $B + T_K$ in $U(1)K_2^{\mathrm{top}}K$ is finite and obtain an upper estimate for that index.

In §6, we consider almost standard fields, i.e., fields some unramified extensions of which are standard fields. For such fields, we obtain lower estimates for the index of $B + T_K$. In particular, we prove that if the field K is almost standard and $p \mid e(K/k)$, then $B+T_K \neq U(1)K_2^{\mathrm{top}}K$. Furthermore, we construct an example of an almost standard field. Namely, we prove the following: if l/k is a totally ramified Galois extension such that k and l are finite extensions of $\mathbb{Q}_p$ and the group $\mathrm{Gal}(l/k)$ is the direct product of a group of order prime to p and the group $(\mathbb{Z}/p\mathbb{Z})^n$ for some n, then there exist fields K

2000 *Mathematics Subject Classification.* Primary 11S70.
Key words and phrases. Second topological K-group, local field, torsion.

and L with subfields of constants k and l, respectively, such that the field L is standard and the extension L/K is unramified.

In §7, we consider fields from which a standard field can be obtained by successive normal extensions that are totally ramified and ferocious. For such fields, we calculate the index of $B+T_K$ exactly and prove that $U(1)K_2^{\mathrm{top}}K$ is generated by the norms of the elements of $U(1)K_2^{\mathrm{top}}(lK)$, where l runs over all finite totally ramified extensions of the subfield of constants in K such that the field lK is standard.

§1. Basic definitions

1.1. Notation. Let p be a fixed odd prime.

We denote by $v_p(x)$ the p-adic valuation of a p-adic number x.

We denote by ζ_m a primitive mth root of unity and by μ_{p^∞} the set of all p^nth roots of unity.

We assume that the set $\mathbb{Z}^2$ is equipped with the following lexicographic ordering: $(a,b)<(c,d)$ if $b<d$ or $b=d$ and $a<c$.

For an Abelian group G, we denote by $T(G)$ its torsion.

1.2. Discrete valuation fields. For a local field k, we denote by v_k its valuation and by $\overline{k}$ its residue field, and we put $e_k=v_k(p)$, where $\operatorname{char}\overline{k}=p$.

Let K be a two-dimensional local field. We put $K^{(1)}=\overline{K}$ and $K^{(0)}=\overline{K^{(1)}}$. For all two-dimensional fields K under consideration, we assume that $\operatorname{char}K=0$, $\operatorname{char}\overline{K}=p$, the field $K^{(0)}$ is finite, and K contains a primitive pth root of unity. We use the following notation:

$\overline{v}_K=(v_K^{(1)},v_K^{(2)}):K\to\mathbb{Z}^2$ is the rank 2 valuation of K;

O_K is the ring of integers in K;

$\mathfrak{M}_K$ is the maximal ideal of O_K;

$V_K=\{1+a\mid\overline{v}_K(a)>0\}$;

$U_K(1)=\{1+a\mid v_K^{(2)}(a)\ge 1\}$;

$\mathfrak{R}_K$ is the canonical subgroup of K^* consisting of representatives of all nonzero elements of the last residue field;

$[\theta]$ is an element of $\mathfrak{R}_K$ that represents an element θ of the last residue field.

The valuation $\overline{v}_K$ gives rise to the following local parameters: an element π such that $\overline{v}_K(\pi)=(0,1)$, which we call a uniformizing element of K, and an element t such that $\overline{v}_K(t)=(1,0)$, which we call the second local parameter of K.

Definition 1.1. By a field of constants of a two-dimensional field K, we mean the maximal subfield k of K that is an algebraic extension of $\mathbb{Q}_p$.

We consider the following types of extensions.

Definition 1.2. Let L/K be a finite extension of two-dimensional fields. We say that this extension is

constant if $L=lK$, where l is the subfield of constants in L;

weakly unramified if $e(L/K)=1$;

unramified if $e(L/K)=1$ and the extension $\overline{L}/\overline{K}$ is separable;

purely unramified if it is unramified and $e(\overline{L}/\overline{K})=1$;

totally ramified if $e(L/K)=|L:K|$;

ferocious if $e(L/K)=1$ and the extension $\overline{L}/\overline{K}$ is purely unseparable;

completely ramified if there exist intermediate fields $K=L_0\subset L_1\subset\cdots\subset L_n=L$ such that each extension L_i/L_{i-1} is totally ramified or ferocious.

Definition 1.3. A two-dimensional field K is said to be *standard* if $e(K/k) = 1$, where k is the subfield of constants in K.

By [7], any standard two-dimensional field K is isomorphic to $k\{\{t\}\}$.

Definition 1.4. A two-dimensional field K is *almost standard* if there exists an unramified extension L/K such that the field L is standard.

Definition 1.5. Let K be a discrete valuation field, and let L/K be a finite Galois extension. Then, for $i \geq -1$, we define the ith *ramification subgroup* as

$$G_i = G_i(L/K) = \{\sigma \in \mathrm{Gal}(L/K) \mid v_L(\sigma a - a) \geq i + 1 \text{ for all } a \in O_L\}.$$

We have $G_{-1} = \mathrm{Gal}(L/K)$ and $G_{i+1} \subset G_i$ for all i. In [1, Chapter 2, §4], it was proved that $G_{i+1} \triangleleft \mathrm{Gal}(L/K)$, and if $\mathrm{char}\,\overline{K} = p$ and the extension $\overline{L}/\overline{K}$ is separable, then G_0/G_1 is a cyclic group of order prime to p and the G_i/G_{i+1} are Abelian p-groups if $i \geq 1$.

Definition 1.6. By the ramification jumps of a Galois field extension L/K such that $\mathrm{char}\,\overline{K} = p$, we mean the numbers

$$h^{(i)} = h^{(i)}(L/K) = \min\{j \geq 1 \mid p^i \nmid |G_j|\} - 1$$

for $i \geq 1$.

If $|L : K| = p$, then the only jump $h^{(1)}(L/K)$ is denoted by $h(L/K)$.

Definition 1.7. Let L/K be a finite separable extension of complete discrete valuation fields. By the ramification depth of L/K, we mean the number

$$d(L/K) = \frac{1}{e_L} \min\{v_L(\mathrm{Tr}_{L/K}\, a/a) \mid a \in L^*\}.$$

Definition 1.8. An infinite field extension L/K is said to be *deeply ramified* if the extension $\overline{L}/\overline{K}$ is finite and L/K contains finite separable subextensions of arbitrarily large depth, i.e., the set

$$\{d(K'/K) \mid K \subset K' \subset L,\ K'/K \text{ is separable}\}$$

is unbounded.

1.3. Topological groups.

Definition 1.9. Suppose F is an arbitrary field and $n \in \mathbb{N}$. We denote by I_n the subgroup of $F^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^*$ generated by the symbols $\alpha_1 \otimes \cdots \otimes \alpha_n$ such that $\alpha_i + \alpha_j = 1$ for some $i \neq j$. We put $K_0 F = \mathbb{Z}$ and

$$K_n F = F^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^* / I_n$$

if $n \in \mathbb{N}$. The group $K_n F$ is called the nth *Milnor K-group* of F.

If K is a discrete valuation field and $m \in \mathbb{N}$, we denote by $VK_m K$ the subgroup of $K_m K$ generated by the elements $\{u, a_1, \ldots, a_{m-1}\}$ with $u \in V_K$. We put

$$K'_m K = K_m K / \bigcap_{s \geq 1} s K_m K, \quad VK'_m K = VK_m K / \bigcap_{s \geq 1} s VK_m K.$$

Following [4] and [7], we define the canonical topologies on $\overline{K}$, K, K^*, and $VK_2 K$ for a two-dimensional field K.

The field $\overline{K}$ is isomorphic to $K^{(0)}((T))$. For every sequence $\{U_i\}_{i \in \mathbb{Z}}$ of subsets of $K^{(0)}$ such that $0 \in U_i$ for all i and $U_i = K^{(0)}$ for sufficiently large i, we put

$$W_{\{U_i\}} = \left\{ \sum a_i T^i \mid a_i \in U_i \right\}$$

and consider the topology on $\overline{K}$ in which the sets $W_{\{U_i\}}$ form a neighborhood base at zero.

For the second local parameter t of K, we construct the lift $h_t : \overline{K} \to O_K$ as in [5]. Let $H_t : \overline{K} \to O_K$ be a mapping such that $\overline{H_t(a)} = a$ and

$$H_t\biggl(\sum_{i=0}^{p-1} \overline{t}^i a_i^p\biggr) = \sum_{i=1}^{p-1} t^i (H_t(a_i))^p$$

for all $a, a_i \in \overline{K}$. We denote by k_0 the field of fractions of $W(K^{(0)})$ and put $K' = k_0\{\{T\}\}$. Then $\overline{K'}$ is isomorphic to $K^{(0)}((\overline{t}))$. Let H_T be a mapping similar to H_t, and let $h : \overline{K'} \to O_{K'}$ be a mapping such that

$$h\biggl(\sum_{\theta_i \in \mathbb{Z}} \theta_i \overline{T}^i\biggr) = \sum_{\theta_i \in \mathbb{Z}} [\theta_i] T^i.$$

We introduce mappings $l_i : K^{(0)}((\overline{t})) \to K^{(0)}((\overline{t}))$ such that

$$h(a) = H_T(a) + \sum_{i \geq 1} p^i H_T(l_i(a)).$$

The lift $h_t : \overline{K} \to O_K$ is defined by the formula

$$h_t(a) = H_t(a) + \sum_{i \geq 1} p^i H_t(l_i(a)).$$

Now, we define a topology on K. Let π and t be local parameters on K, and let $\{U_i\}_{i \in \mathbb{Z}}$ be a neighborhood system at zero in $\overline{K}$ such that $U_i = \overline{K}$ for sufficiently large i. We put

$$W_{\{U_i\}} = \Bigl\{\sum h_t(a_i)\pi^i \mid a_i \in U_i\Bigr\}$$

and introduce the topology on K in which $W_{\{U_i\}}$ is a neighborhood system at zero.

The topology on $K^* \cong V_K \oplus (K^*/V_K)$ is defined as the product of the topology on V_K induced from K and the discrete topology on (K^*/V_K).

By the canonical topology on VK_2K, we mean the strongest topology on VK_2K with the following properties:

1) the canonical mapping $V_K \times K^* \to VK_2K$ is sequentially continuous;

2) the operations in VK_2K are sequentially continuous.

By the topological space $VK_2^{\mathrm{top}}K$, we mean the set $VK_2'K$ with the topology induced by the canonical topology on VK_2K. For a two-dimensional field K, the topological space $VK_2^{\mathrm{top}}K$ is a topological group (see [12]).

We denote by $U(1)K_2^{\mathrm{top}}K$ the subgroup of $VK_2^{\mathrm{top}}K$ generated by the symbols $\{u, a\}$, where $u \in U_K(1)$.

Let L/K be a finite extension of two-dimensional fields, and let A be a subgroup of $U(1)\, K_2^{\mathrm{top}}L$. By $A \cap U(1)K_2^{\mathrm{top}}K$, we mean the set of elements $x \in U(1)K_2^{\mathrm{top}}K$ such that $i_{L/K}(x) \in A$, where

$$i_{L/K} : U(1)K_2^{\mathrm{top}}K \to U(1)K_2^{\mathrm{top}}L$$

is the homomorphism induced by the embedding of K in L.

§2. Embedding of a field in a standard field

Theorem 2.1. *For every two-dimensional field K, there exists a standard field L containing K and such that the extension L/K is finite, solvable, and constant.*

Proof. Let K be a two-dimensional field, let k be the subfield of constants of K, and let t be a second local parameter of K. We put $K_0 = k\{\{t\}\}$. By [6, Theorem 1], there exists a finite solvable extension l/k such that the extension lK/lK_0 is weakly ramified. The field lK_0 is standard, being a constant extension of the standard field K_0. The field lK is also standard, because it is a weakly ramified extension of the standard field lK_0. Thus, we can take the field lK as L. □

By an *appropriate* extension of K, we mean a standard field L that contains K and is such that the extension L/K is finite and constant.

For a field K and for an appropriate extension L of K, we put

$$m_f(L/K) = |\overline{L} : \overline{K}|_{\mathrm{insep}}, \quad m_u = \frac{|\overline{L} : \overline{K}|_{\mathrm{sep}}}{|\bar{l} : \bar{k}|},$$

where k and l are the subfields of constants in K and L, respectively. Suppose that the fields

$$K = L_0 \subset L_1 \subset \cdots \subset L_{N-1} \subset L_N = L$$

are such that the degrees $|L_i : L_{i-1}|$ are prime numbers. Then the product of the degrees of the extensions L_i/L_{i-1} that are ferocious is equal to $m_f(L/K)$, and the product of the degrees of unramified extensions that are not purely unramified is equal to $m_u(L/K)$. If $K \subset L \subset L'$ and L and L' are appropriate extensions of K, then

$$m_f(L'/K) = m_f(L'/L)m_f(L/K), \quad m_u(L'/K) = m_u(L'/L)m_u(L/K).$$

Lemma 2.2. *Let K be a two-dimensional field.*

1) *If K is standard, then, for every appropriate extension L, we have*

$$m_f(L/K) = m_u(L/K) = 1.$$

2) *The numbers $m_f(L/K)$ and $m_u(L/K)$ do not depend on the choice of an appropriate extension L.*

3) *If*

$$m_f(L/K) = m_u(L/K) = 1$$

for an appropriate extension L, then K is standard.

Proof. 1) Let k and l be the subfields of constants in K and L, respectively, and let t be the second local parameter of K. Then $K = k\{\{t\}\}$, $L = l\{\{t\}\}$, $\overline{K} = \bar{k}\{\{t\}\}$, and $\overline{L} = \bar{l}\{\{t\}\}$. Since the fields $\bar{k}$ and $\bar{l}$ are finite, we obtain

$$m_f(L/K) = |\overline{L} : \overline{K}|_{\mathrm{insep}} = |\bar{l} : \bar{k}|_{\mathrm{insep}} = 1,$$
$$m_u(L/K) = \frac{|\overline{L} : \overline{K}|_{\mathrm{sep}}}{|\bar{l} : \bar{k}|} = \frac{|\overline{L} : \overline{K}|}{|\bar{l} : \bar{k}|} = \frac{|\bar{l} : \bar{k}|}{|\bar{l} : \bar{k}|} = 1.$$

2) Let L_1 and L_2 be appropriate extensions of K. From the first statement of the lemma, it follows that, for $s = 1, 2$, we have the relation

$$m_f(L_s/K) = m_f(L_1L_2/L_s)m_f(L_s/K) = m_f(L_1L_2/K),$$

where the right-hand side does not depend on s.

The statement concerning $m_u(L/K)$ is proved similarly.

3) Let k and l be the subfields of constants for K and L, respectively. If $m_f(L/K) = m_u(L/K) = 1$, then

$$\frac{|L : K|}{e(L/K)} = |\overline{L} : \overline{K}| = m_f(L/K)m_u(L/K)|\bar{l} : \bar{k}| = \frac{|l : k|}{e(l/k)}.$$

The extension L/K is constant. Therefore, we have $|L : K| = |l : k|$, and consequently, $e(L/K) = e(l/k)$. We obtain the relation

$$e(K/k) = \frac{e(L/l)e(l/k)}{e(L/K)} = e(L/l) = 1,$$

and the field K is standard. $\square$

We denote the numbers $m_f(L/K)$ and $m_u(L/K)$ for an appropriate L by $m_f(K)$ and $m_u(K)$, respectively. Observe that

$$m_f(K)m_u(K) = e(K/k) \tag{1}$$

for every field K because, for every appropriate extension L with subfield of constants l, we have

$$e(K/k) = \frac{e(L/l)e(l/k)}{e(L/K)} = \frac{|l : k|/|\overline{l} : \overline{k}|}{|L : K|/|\overline{L} : \overline{K}|} = \frac{|\overline{L} : \overline{K}|}{|\overline{l} : \overline{k}|}.$$

We consider fields satisfying $m_f(K) = 1$. If K is almost standard, then the corresponding unramified extension can be chosen to be finite, constant, and solvable; i.e., the statement is true.

Lemma 2.3. *Let K be a two-dimensional field.*

1) *The field K is almost standard if and only if $m_f(K) = 1$.*

2) *If $m_f(K) = 1$, then there exists an appropriate solvable unramified extension of K.*

Proof. 1) Let K be almost standard, and let L be a standard field such that $L \supset K$ and the extension L/K is unramified. Then the field lK is standard because

$$e(lK/l) \le e(L/l) = 1.$$

We assume that $L = lK$.

Let k be the subfield of constants of K, and let l_1 be such that the extension l_1/k is finite and the field l_1K is standard. Such a field l_1 exists by Theorem 2.1. By Lemma 2.2, we have

$$1 = m_f(L) = |\overline{ll_1K} : \overline{lK}|_{\text{insep}} = |\overline{ll_1K} : \overline{K}|_{\text{insep}} \ge |\overline{l_1K} : \overline{K}|_{\text{insep}} = m_f(K).$$

2) Let L_0 be an arbitrary appropriate solvable extension of K, which exists by Theorem 2.1. We denote by L_u the maximal ramified subextension of L_0/K. Then the extension L_0/L_u is completely ramified, whence $m_u(L_u) = 1$. Moreover, $m_f(L_u) \le m_f(K) = 1$. By Lemma 2.2, the field L_u is standard; thus, it is the required extension. $\square$

Now, we consider fields satisfying $m_u(K) = 1$.

Lemma 2.4. *Let k be a complete discrete valuation field with uniformizing element π, suppose $\operatorname{char} k = 0$, $\operatorname{char} \overline{k} = p$, $\zeta_p \in k$, and let l/k be a Galois extension of degree p. Then $l = k(x)$ for some x such that $x^p = a \in k$, and one of the following conditions is satisfied*:

1) *$a = w\pi$, where $w \in U_k$;*

2) *$a = w$, where $w \in U_k$ and $\overline{w} \notin \overline{k}^p$;*

3) *$a = 1 + w\pi^s$, where $0 < s < \frac{pe_k}{p-1}$, $p \nmid s$, and $w \in U_k$;*

4) *$a = 1 + w\pi^s$, where $0 < s < \frac{pe_k}{p-1}$, $p \mid s$, $w \in U_k$, and $\overline{w} \notin \overline{k}^p$;*

5) *$a = 1 + w(1-\zeta_p)^p$, where $w \in U_k$ and $\overline{w} \notin \{\alpha^p - \alpha \mid \alpha \in \overline{k}\}$.*

In cases 1) and 2) we have

$$d(l/k) = 1 \quad \text{and} \quad h(l/k) = \frac{e_l}{p-1},$$

in cases 3) *and* 4) *we have*

$$d(l/k) = 1 - \frac{p-1}{e_l}s, \quad h(l/k) = \frac{e_l}{p-1} - s,$$

and in case 5) *we have* $d(l/k) = h(l/k) = 0$.

Moreover, in cases 1) *and* 3) *the extension* l/k *is totally ramified, in cases* 2) *and* 4) *the extension* l/k *is ferociously ramified, and in case* 5) *the extension* l/k *is unramified.*

Proof. See [6, Proposition 1.4] and [8, Lemma 2–10]. □

If m/k is a finite separable extension and $k \subset l \subset m$, then

$$d(m/k) = d(m/l) + d(l/k) \tag{2}$$

by [8, Lemma 2–4].

Lemma 2.5. *Let* K *be a two-dimensional field, let* $m_u(K) = 1$, *and let* k *be the subfield of constants of* K. *Then, every deeply ramified extension* l_0/k *has a finite subextension* l/k *such that the field* lK *is standard.*

Proof. By [6, Theorem 4], the extension l_0/k has a finite subextension l/k such that the field lK is almost standard. By Lemma 2.3, $m_f(lK) = 1$. Moreover, we have $m_u(lK) \le m_u(K) = 1$. Therefore, by Lemma 2.2, the field lK is standard. □

Corollary 2.6. *Let* K *be a two-dimensional field with* $m_u(K) = 1$. *Then the field* K *has an appropriate cyclic completely ramified extension.*

Proof. Let k be the subfield of constants of K, and π a uniformizing element of k. We put

$$l_0 = \bigcup_{n \in \mathbb{N}} k\big(\sqrt[p^n]{\pi}\big).$$

The extension l_0/k is cyclic, and, by Lemma 2.4, it is deeply ramified and has no unramified subextensions. By Lemma 2.5, the extension l_0/k has a finite subextension l/k such that the field lK is standard. We prove that lK can be taken as L. The extension l/k is cyclic, being a subextension of a cyclic extension. Therefore, the extension lK/K is also cyclic. Let L_u be the maximal unramified subextension of lK/K. Since $m_u(K) = 1$ and the extension l/k has no unramified subextensions, we obtain

$$|L_u : K| = |\overline{lK} : \overline{K}|_{\text{sep}} = |\overline{l} : \overline{k}| = 1.$$

Consequently, $L_u = K$, and the extension lK/K is completely ramified. □

§3. Definition and properties of the norm mapping

We follow [1, Chapter 9, §§2, 3] to define the norm mapping for the second Milnor K-group.

Definition 3.1. Let F be a field, let v be a discrete valuation on F, and let π be a uniformizing element of F corresponding to v. We denote by $\overline{F}_v$ the residue field of F with respect to v and by $U_{F,v}$ the group of units of F with respect to v. For $n \in \mathbb{N}$, we consider homomorphisms $\partial_v : K_nF \to K_{n-1}\overline{F}_v$ and $\sigma_\pi : K_nF \to K_n\overline{F}_v$ such that, for all $u_i \in U_{F,v}$ and $a, a_i \in \mathbb{N}_0$, the following relations are valid:

$$\partial_v\big(\{u_1, \dots, u_{n-1}, u_n\pi^a\}\big) = a\{\overline{u}_1, \dots, \overline{u}_{n-1}\},$$
$$\sigma_\pi\big(\{u_1\pi^{a_1}, \dots, u_n\pi^{a_n}\}\big) = \{\overline{u}_1, \dots, \overline{u}_n\}.$$

The homomorphism $\partial_{F,v}$ does not depend on the choice of a uniformizing element π, because $\pi'\pi^{-1} \in U_{F,v}$ for any uniformizing element π', and

$$\partial_v\bigl(\{u_1, \dots, u_{n-1}, u_n\}\bigr) = 0$$

for $u_i \in U_{F,v}$.

Since

$$\partial_v\Bigl(\bigcap_{s\geq 1} sK_nF\Bigr) \subset \bigcap_{s\geq 1} K_{n-1}\overline{F}_v, \quad \sigma_\pi\Bigl(\bigcap_{s\geq 1} sK_nF\Bigr) \subset \bigcap_{s\geq 1} K_n\overline{F}_v,$$

we can consider the mappings

$$\partial_v : K'_nF \to K'_{n-1}\overline{F}_v, \quad \sigma_\pi : K'_nF \to K'_n\overline{F}_v.$$

Definition 3.2. Let F be a field, and let $E = F(X)$. For all discrete valuations v of E trivial on F, we define homomorphisms $\mathrm{N}_{v/F} : K_n\overline{E}_v \to K_nF$ such that the sequence

$$K_{n+1}E \xrightarrow{\oplus\partial_v} \oplus K_n\overline{E}_v \xrightarrow{\oplus\,\mathrm{N}_{v/F}} K_nF \to 0$$

is exact and, for every valuation v_∞ corresponding to the uniformizing element $\frac{1}{X}$, the homomorphism $\mathrm{N}_{v_\infty/F}$ is the identity mapping.

For every field F, such a set of homomorphisms $\mathrm{N}_{v/F}$ exists and is uniquely determined; see [1, Chapter 9, 3.1].

Definition 3.3. Let E/F be a finite field extension, and let $E = F(\alpha_1, \dots, \alpha_s)$. We put $F_0 = F$ and $F_i = F_{i-1}(\alpha_i)$. Let v_i be the valuation on $F_{i-1}(X)$ corresponding to α_i. We define the homomorphism $\mathrm{N}_{E/F} : K_nE \to K_nF$ by the formula

$$\mathrm{N}_{E/F} = \mathrm{N}_{v_1/F_0} \circ \dots \circ \mathrm{N}_{v_s/F_{s-1}}.$$

By [1, Chapter 9, Theorem 3.8], the mapping $\mathrm{N}_{E/F}$ does not depend on the choice of $\alpha_1, \dots, \alpha_s$.

Lemma 3.4. *Let E/F be a finite field extension, and let* N *be the norm map for the nth Milnor K-groups.*

1) *If $F \subset F' \subset E$, then $\mathrm{N}_{E/F} = \mathrm{N}_{F'/F} \circ \mathrm{N}_{E/F'}$.*

2) *For all $x \in K_nF$ we have $\mathrm{N}_{E/F}\, x = |E : F| \cdot x$.*

3) *For all $x \in K_{n-1}F$ and $y \in E$ we have $\mathrm{N}_{E/F}\{x, y\} = \{x, y'\}$, where y' is the image of y under the norm map of the field extension E/F.*

4) *If F and E are discrete valuation fields and E/F is normal, then the diagram*

$$\begin{array}{ccc} K_nE & \xrightarrow{\partial} & K_{n-1}\overline{E} \\ {\scriptstyle \mathrm{N}_{E/F}}\downarrow & & {\scriptstyle \mathrm{N}_{\overline{E}/\overline{F}}}\downarrow \\ K_nF & \xrightarrow{\partial} & K_{n-1}\overline{F} \end{array}$$

commutes.

Proof. See [1, Chapter 9, 3.1, Theorem 3.8 and Exercise 3.3]. □

Let K be a two-dimensional field with local parameters π and t. We introduce the mappings

$$\alpha : K'_2K \to K'_0K^{(0)} = \mathbb{Z}, \quad \beta_1, \beta_2 : K'_2K \to K'_1K^{(0)} = \mathfrak{R}_K$$

as follows:

$$\alpha = \partial_{v_K^{(1)}} \circ \partial_{v_K^{(2)}}, \quad \beta_1 = \sigma_t \circ \partial_{v_K^{(2)}}, \quad \beta_2 = \partial_{v_K^{(1)}} \circ \sigma_\pi. \tag{3}$$

Lemma 3.5. *Put*

$$S = \{s\{\pi, t\} \mid s \in \mathbb{Z}\}, \quad S_\pi = \{\{\theta, \pi\} \mid \theta \in \mathfrak{R}_K\}, \quad \textit{and} \quad S_t = \{\{\theta, t\} \mid \theta \in \mathfrak{R}_K\}.$$

Then

$$K_2'K = S \oplus S_\pi \oplus S_t \oplus VK_2'K.$$

Moreover,

$$\ker\alpha = S_\pi + S_t + VK_2'K, \quad \ker\beta_1 = S + S_t + VK_2'K,$$
$$\ker\beta_2 = S + S_\pi + VK_2'K, \quad \ker(\partial_{v_K^{(2)}} \oplus \sigma_\pi) = U(1)K_2'K.$$

Proof. See [4, Propositions 1.2 and 2.5]. □

Let L/K be a finite Galois extension of two-dimensional fields. Then

$$\mathrm{N}_{L/K}\left(\bigcap_{s\geq 1} sK_2L\right) \subset \bigcap_{s\geq 1} sK_2K;$$

i.e., the mapping $\mathrm{N}_{L/K} : K_2'L \to K_2'K$ is well defined. We prove that the mappings

$$\mathrm{N}_{L/K} : VK_2'L \to VK_2'K \quad \text{and} \quad \mathrm{N}_{L/K} : U(1)K_2'L \to U(1)K_2'K$$

are also well defined.

Lemma 3.6. *We have*

$$\mathrm{N}_{L/K}(VK_2'L) \subset VK_2'K \quad \textit{and} \quad \mathrm{N}_{L/K}(U(1)K_2'L) \subset U(1)K_2'K.$$

Proof. Let $\alpha^{(L)}$, $\beta_1^{(L)}$, and $\beta_2^{(L)}$ be mappings defined for the field L by relations similar to (3). By Lemma 3.4, we have the following commutative diagrams:

$$\begin{array}{ccc} K_2'L & \xrightarrow{\alpha^{(L)}} & K_0L^{(0)} \\ {\scriptstyle \mathrm{N}_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{N}_{L^{(0)}/K^{(0)}}} \\ K_2'K & \xrightarrow{\alpha} & K_0K^{(0)} \end{array}, \qquad \begin{array}{ccc} K_2'L & \xrightarrow{\beta_i^{(L)}} & K_1\overline{L} \\ {\scriptstyle \mathrm{N}_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{N}_{\overline{L}/\overline{K}}} \\ K_2'K & \xrightarrow{\beta_i} & K_1\overline{K} \end{array},$$

where $i = 1, 2$. By Lemma 3.5, it follows that

$$\mathrm{N}_{L/K}\, VK_2'L = \mathrm{N}_{L/K}(\ker\alpha^{(L)} \cap \ker\beta_1^{(L)} \cap \ker\beta_2^{(L)})$$
$$\subset \ker\alpha \cap \ker\beta_1 \cap \ker\beta_2 = VK_2'K.$$

The second statement is deduced similarly from the properties of the mapping $\partial \oplus \sigma$. □

Lemma 3.7. *Let L/K be a finite Galois extension of two-dimensional fields.*

1) *If L/K is tame or unramified, then*

$$\mathrm{N}_{L/K}\, U(1)K_2^{\mathrm{top}}L = U(1)K_2^{\mathrm{top}}K.$$

2) *If L/K is a ferocious or completely ramified extension of degree p, then*

$$(U(1)K_2^{\mathrm{top}}K : \mathrm{N}_{L/K}\, U(1)K_2^{\mathrm{top}}L) = p.$$

Proof. We may assume that L/K has finite degree l. We choose local parameters π_K, t_K and π_L, t_L of K and L (respectively) so that $\pi_K = \pi_L$ if $e(L/K) = 1$, and $t_K = t_L$ if $e(\overline{L}/\overline{K}) = 1$. Let σ be a generator of $\mathrm{Gal}(L/K)$. For $L = K(\pi_L)$, let i and j be such that

$$\frac{\sigma(\pi_L)}{\pi_L} \equiv 1 + [\theta_0]\pi_L^i t_L^j \mod \pi_L^i t_L^j \mathfrak{M}_L,$$

and, for $L = K(t_L)$, let i and j be such that

$$\frac{\sigma(t_L)}{t_L} \equiv 1 + [\theta_0]\pi_L^i t_L^j \mod \pi_L^i t_L^j \mathfrak{M}_L.$$

Then, in the case of a totally ramified extension of degree p, we have $i > 0$, and in the case of an unramified but not purely unramified extension of degree p, we have $i = 0$.

In [9], it was proved that $K_2'K/N_{L/K}K_2'L$ is a group of order l with generator a, where

$a = \{1 + [\theta]\pi_K^i t_K^{pj}, t_K\}$ for some $\theta \in K^{(0)}$ if L/K is totally ramified and $l = p$;

$a = \{1 + [\theta]\pi_K^{pi} t_K^j, \pi_K\}$ for some $\theta \in K^{(0)}$ if L/K is weakly unramified but not purely unramified and $l = p$;

$a = \{[\theta], \pi_K\}$ or $a = \{[\theta], t_K\}$ for some $\theta \in K^{(0)}$ if L/K is not purely unramified and $l \neq p$;

$a = \{\pi_K, t_K\}$ if L/K is purely unramified.

By Lemma 3.5, the element a belongs to $U(1)K_2^{\mathrm{top}}K$ only in the case of a ferocious or totally ramified extension of degree p. □

Lemma 3.8. *Let L/K be a finite extension of two-dimensional fields. Then the mapping $\mathrm{N}_{L/K}$ is continuous.*

Proof. See [4, Corollary 4.4]. □

§4. Definition and properties of T_K

For a standard two-dimensional field K, we denote by T_K the closure of the torsion of $U(1)K_2^{\mathrm{top}}K$.

Theorem 4.1. *Let $K = k\{\{t\}\}$ be a standard two-dimensional field, and let A be a topological basis of $U_k/T(U_k)$ viewed as a $\mathbb{Z}_p$-module. Then the set*

$$B_0 = \{\{u, t\} \mid u \in A\}$$

is a topological basis of $U(1)K_2^{\mathrm{top}}K/T_K$.

This theorem was proved in [4] for a standard n-dimensional field of mixed characteristic with perfect residue field, but the proof of the fact that B_0 generates $U(1)K_2^{\mathrm{top}}K/T_K$ contained an error. We give the proof only for a two-dimensional field, though for an arbitrary field the proof is similar.

Proof. A) We prove that $\langle B_0\rangle + T_K = U(1)K_2^{\mathrm{top}}K$.

Let $a \in U(1)K_2^{\mathrm{top}}K$. We choose a uniformizing element π of K such that $\pi \in k$, and denote by e the absolute ramification index of K. By [4, Propositions 2.1 and 2.5], we have

$$a = \{u_t, t\} + \{u_\pi, \pi\} + \sum_{\substack{0<(j,i)<(0,\frac{pe}{p-1}) \\ p\nmid i,\ \theta \in K^{(0)}}} c_{i,j,\theta}\{1 + [\theta]\pi^i t^j, t\}$$

for some $u_t \in U_k$, $u_\pi \in U_K(1)$, and $c_{i,j,\theta} \in \mathbb{Z}_p$. The first term of the above expansion belongs to $\langle B_0\rangle$. The second term is a torsion element by [4, Corollary 7.1 and Proposition 7.9]. Moreover, we have

$$j\{1 + [\theta]\pi^i t^j, t\} = -i\{1 + [\theta]\pi^i t^j, \pi\} \in T(U(1)K_2^{\mathrm{top}}K).$$

Therefore, the third term also belongs to T_K.

B) By [4, Theorem 9.2], the elements of B_0 are linearly independent. □

Corollary 4.2. 1) *Let K be a standard two-dimensional field. Then the factor group $U(1)K_2^{\mathrm{top}}K/T_K$ has no torsion.*

2) *If L/K is a finite extension of standard fields, then $\mathrm{N}_{L/K}\, T_L \subset T_K$.*

3) *If L/K is a finite extension of standard two-dimensional fields, then*

$$T_L \cap U(1)K_2^{\mathrm{top}}K = T_K.$$

Proof. Statement 1) follows from Theorem 4.1.

Statement 2) follows from Lemma 3.8.

3) Let $x \in T_K$. Then there exists a sequence $\{x_n\}$ such that $x_n \in T(U(1)K_2^{\mathrm{top}}K)$ and $x_n \to x$ in $U(1)K_2^{\mathrm{top}}K$. Since $x_n \in T(U(1)K_2^{\mathrm{top}}L)$ and $x_n \to x$ in $U(1)K_2^{\mathrm{top}}L$, we have $x \in T_L \cap U(1)K_2^{\mathrm{top}}K$.

Let $x \in T_L \cap U(1)K_2^{\mathrm{top}}K$. The preceding statement implies that

$$|L : K| \cdot x = \mathrm{N}_{L/K}\, x \in T_K.$$

Applying statement 1), we see that $x \in T_K$. □

Let $K \subset L$ be two-dimensional fields such that L is standard and the extension L/K is finite. We put $T_K(L) = U(1)K_2^{\mathrm{top}}K \cap T_L$.

Lemma 4.3. *For every two-dimensional field K, the subgroup $T_K(L)$ does not depend on the choice of L.*

Proof. For a standard field K, the claim follows from Lemma 4.2, because $T_K(L) = T_K$ for every L.

Let K be nonstandard, and let L_1 and L_2 be standard fields such that the extensions L_1/K and L_2/K are finite. Applying the claim for the standard fields, we obtain the following relation for $s = 1, 2$:

$$\begin{aligned} T_K(L_s) &= U(1)K_2^{\mathrm{top}}K \cap T_{L_s} = U(1)K_2^{\mathrm{top}}K \cap T_{L_s}(L_1L_2) \\ &= U(1)K_2^{\mathrm{top}}K \cap U(1)K_2^{\mathrm{top}}L_s \cap T_{L_1L_2} = U(1)K_2^{\mathrm{top}}K \cap T_{L_1L_2}, \end{aligned}$$

where the latter expression does not depend on s. □

For a two-dimensional field K, we denote by T_K the group $T_K(L)$, where L is an arbitrary standard field such that the extension L/K is finite. Such a field L exists by Theorem 2.1. The group T_K is well defined because, by Lemma 4.3, it does not depend on the choice of a field L, and if K is standard, then the subgroup T_K coincides with the subgroup T_K defined previously.

Theorem 4.4. 1) *For every two-dimensional field K, the factor group $U(1)K_2^{\mathrm{top}}K/T_K$ has no torsion, and T_K is the smallest closed subgroup of $U(1)K_2^{\mathrm{top}}K$ such that this factor group has no torsion.*

2) *Let L/K be a finite extension of two-dimensional fields. Then*

$$\mathrm{N}_{L/K}\, T_L \subset T_K, \quad T_L \cap U(1)K_2^{\mathrm{top}}K = T_K.$$

Proof. For every two-dimensional field K, in $U(1)K_2^{\mathrm{top}}K$ there exists the smallest closed subgroup the factor group by which has no torsion. Indeed, let T_K^α be the set of all subgroups satisfying this condition. Then $\bigcap T_K^\alpha$ is a closed subgroup, and if x is a torsion element in $U(1)K_2^{\mathrm{top}}K/\bigcap T_K^\alpha$, then there exists $n \in \mathbb{N}$ such that $nx \in T_K^\alpha$ for all α. Therefore, $x \in T_K^\alpha$ for all α. We see that $\bigcap T_K^\alpha$ is the smallest subgroup satisfying the above conditions. We denote this subgroup by T_K^0.

We prove that

$$\mathrm{N}_{L/K}\, T_L^0 \subset T_K^0$$

for every finite extension L/K. By Lemma 3.8, the subgroup $\mathrm{N}_{L/K}^{-1} T_K^0$ is closed. Moreover, the factor group

$$U(1)K_2^{\mathrm{top}}L/\,\mathrm{N}_{L/K}^{-1} T_K^0$$

has no torsion. Indeed, if $nx \in \mathrm{N}_{L/K}^{-1} T_K^0$ for some $x \in U(1)K_2^{\mathrm{top}}L$ and $n \in \mathbb{N}$, then $n\,\mathrm{N}_{L/K}\,x \in T_K^0$, whence $\mathrm{N}_{L/K}\,x \in T_K^0$ and $x \in \mathrm{N}_{L/K}^{-1} T_K^0$. We conclude that $T_L^0 \subset \mathrm{N}_{L/K}^{-1} T_K^0$ and $\mathrm{N}_{L/K}\,T_L^0 \subset T_K^0$.

We prove that

$$T_L^0 \cap U(1)K_2^{\mathrm{top}}K = T_K^0$$

for every finite extension L/K. The inclusion $T_L^0 \cap U(1)K_2^{\mathrm{top}}K \subset T_K^0$ is checked as in the proof of Corollary 4.2. Namely, for every $x \in T_L^0 \cap U(1)K_2^{\mathrm{top}}K$ we have

$$|L:K| \cdot x = \mathrm{N}_{L/K}\,x \in T_K^0,$$

and so $x \in T_K^0$, because the group $U(1)K_2^{\mathrm{top}}K/T_K^0$ has no torsion. To obtain the converse inclusion, it suffices to prove that the factor group $U(1)K_2^{\mathrm{top}}K/U(1)K_2^{\mathrm{top}}K \cap T_L^0$ has no torsion. Let $x \in U(1)K_2^{\mathrm{top}}K$, and let $nx \in U(1)K_2^{\mathrm{top}}K \cap T_L^0$ for some $n \in \mathbb{N}$. Then $x \in T_L^0$, and consequently, $x \in U(1)K_2^{\mathrm{top}}K \cap T_L^0$.

To complete the proof of the theorem, it remains to prove that $T_K = T_K^0$ for a standard field K. The subgroup T_K is closed by definition, and the factor group $U(1)K_2^{\mathrm{top}}K/T_K$ has no torsion by Theorem 4.1. Consequently, $T_K^0 \subset T_K$. On the other hand, every subgroup the factor group by which has no torsion must contain the torsion of $U(1)K_2^{\mathrm{top}}K$, and a closed subgroup with this property must contain the closure of the torsion. Therefore, $T_K \subset T_K^0$. □

Corollary 4.5. *Let K be a two-dimensional field, and let T be the closure of the torsion of $U(1)K_2^{\mathrm{top}}K$. Then*:

1) $T \subset T_K$;
2) *the group T_K/T is a torsion p-group, and the orders of its elements are bounded.*

Proof. 1) This follows from the fact that the group T_K is closed and the factor group $U(1)K_2^{\mathrm{top}}K/T_K$ has no torsion.

2) Let L/K be a standard extension such that the field L is standard and $|L:K| = p^n q$, where $p \nmid q$. We prove that $p^n x \in T$ for each $x \in T_K$.

By the definition of T_K, there exists a sequence $\{x_n\}$ such that $x_n \in T(U(1)K_2^{\mathrm{top}}L)$ and $\lim x_n = x$. By Lemma 3.8,

$$p^n x = q^{-1}\,\mathrm{N}_{L/K}\,x = \lim \mathrm{N}_{L/K}\,q^{-1}x_n.$$

Since

$$\mathrm{N}_{L/K}\,q^{-1}x_n \in T(U(1)K_2^{\mathrm{top}}K),$$

we have $\lim \mathrm{N}_{L/K}\,q^{-1}x_n \in T$. □

Corollary 4.6. *Let K be a two-dimensional field, and let $\Psi_K : K_2^{\mathrm{top}}K \to \mathrm{Gal}(K^{ab}/K)$ be the reciprocity map. Then, for every $x \in T_K$ and every totally ramified Galois extension L/K such that $\mathrm{Gal}(L/K) \cong \mathbb{Z}_p$, we have $\Psi_K(x)|_L = id_L$.*

Proof. This is a consequence of Corollary 4.5 and [4, Lemma 1, 9.2]. □

Lemma 4.7. *Let $p \nmid e(K/k)$. Then*:

1) *there exists a finite, constant, solvable, and tame extension L/K such that the field L is standard*;
2) *the group T_K is the closure of the torsion of $U(1)K_2^{\mathrm{top}}K$.*

Proof. 1) Let L_1 be a standard field such that the extension L_1/K is finite, solvable, and constant. Let k and l_1 be the subfields of constants in K and L_1, respectively, and let l_0 be the maximal tame subextension of l_1/k. Then $p \nmid e(l_0K/l_0)$ because $e(l_0K/l_0) \mid e(K/k)$ and $e(l_1/l_0) = p^n$ for some n. Moreover,

$$e(l_0K/l_0) \mid e(l_1/l_0) = e(L_1/l_0).$$

Consequently, $e(l_0K/l_0) = 1$, and we can take the field l_0K as L.

2) Let T be the closure of the torsion of $U(1)K_2^{\mathrm{top}}K$. By Corollary 4.5, it suffices to show that $T_K \subset T$. We take L as in 1). Let $x \in T_K$. Then there exists a sequence $\{x_n\}$ such that $x_n \in T(U(1)K_2^{\mathrm{top}}L)$ and $\lim x_n = x$. We have

$$\mathrm{N}_{L/K}\, x_n \in T(U(1)K_2^{\mathrm{top}}K),$$

so that the element

$$x = \lim |L : K|^{-1}\, \mathrm{N}_{L/K}\, x_n$$

belongs to T. □

Lemma 4.8. *Let K be a two-dimensional field. Let k be the subfield of constants of K, and let $w \in U_K(1)$ and $u \in V_k$. Then $\{u, w\}$ is a torsion element in $VK_2^{\mathrm{top}}K$.*

Proof. See [4, Corollary 7.1]. □

We prove that an element $\{u, w\}$ belongs to T_K not only if $w \in U_K(1)$ but already if $w \in U_K$.

Lemma 4.9. *Let K be a two-dimensional field, let k be the subfield of constants of K, and let $w \in V_K$ and $u \in V_k$. Then $\{u, w\} \in T_K$.*

Proof. For a standard field, this statement was proved in [13, Lemma 5].

Now, let K be nonstandard. By Theorem 2.1, there exists a finite extension L/K such that L is standard. We have $w \in V_L$ and $u \in V_l$, where l is the subfield of constants of L. Applying the lemma to the standard field L, we obtain

$$\{u, w\} \in T_L \cap U(1)K_2^{\mathrm{top}}K = T_K.$$ □

§5. Subgroups of finite index

Let K be a two-dimensional field. We choose a standard field L such that the extension L/K is finite, solvable, and constant; such a field exists by Theorem 2.1. We denote by k and l the subfields of constants for K and L, respectively, and put

$$m = |L : K|, \quad m_0 = e(K/k), \quad m_1 = |\overline{L} : \overline{K}|_{\mathrm{sep}}. \tag{4}$$

Let

$$k = l_0 \subset l_1 \subset \cdots \subset l_{N-1} \subset l_N = l$$

be fields such that the l_i/l_{i-1} are normal extensions of prime degree. Then the $l_iK/l_{i-1}K$ are also normal extensions of prime degree. By (1), the product of the degrees of the extensions $l_iK/l_{i-1}K$ that are unramified is equal to m_1, and the product of the degrees of weakly ramified but not unramified extensions is equal to m_0. We choose second local parameters t and t_L of the fields K and L (respectively) so that we have $t^{-1}t_L^{m_0} \in U_L$. Put

$$r = |k : \mathbb{Q}_p|, \; n = v_p(m), \; n_0 = v_p(m_0), \quad n_1 = v_p(m_1), \tag{5}$$

and let n_2 be such that

$$(V_k : \mathrm{N}_{l/k}\, V_l \cdot T(V_k)) = p^{n_2}. \tag{6}$$

Let

$$B = \{\{u, t\} \mid u \in V_k\}. \tag{7}$$

By Lemma 4.9, the group $B + T_K$ does not depend on the choice of t.

Lemma 5.1. *The factor group $B + T_K/T_K$ is a free $\mathbb{Z}_p$-module of rank r, and the mapping $u \mapsto \{u, t\}$ from $V_k/T(V_k)$ to $B + T_K/T_K$ is an isomorphism.*

Proof. Let $u_1, \dots, u_r$ be a $\mathbb{Z}_p$-basis of $V_k/T(V_k)$. Then the elements $\{u_i, t\}$ generate $B/T(B)$, and therefore, they generate $B + T_K/T_K$. We prove that these elements are linearly independent in $B+T_K/T_K$. Let $K_0 = k\{\{t\}\}$. If $\sum c_i\{u_i, t\} = 0$ in $B+T_K/T_K$ for some $c_i \in \mathbb{Z}_p$, then, by Theorem 4.4,

$$\sum c_i\{u_i, t\} \in T_K \cap U(1)K_2^{\mathrm{top}}K_0 = T_{K_0}.$$

By Theorem 4.1, the elements $\{u_i, t\}$ are linearly independent in $B + T_{K_0}/T_{K_0}$. Consequently, $c_i = 0$ for all i. □

Lemma 5.2. *Let W be a subgroup of V_k of finite index, let*

$$A_0 = \{\{u, t\} \mid u \in W\},$$

and let A be a subgroup of $U(1)K_2^{\mathrm{top}}K$ such that

$$p^d A + T_K = A_0 + T_K$$

for some $d \in \mathbb{N}_0$. Then

$$(A + T_K : p^s A + T_K) = p^{sr}$$

for every $s \in \mathbb{N}$.

Proof. Let $u_1, \dots, u_r$ be a $\mathbb{Z}_p$-basis of $W/T(W)$. By Lemma 5.1, the elements $\{u_\nu, t\}$ form a basis of $A_0 + T_K/T_K$. We denote by w_i, where $1 \le i \le p^{sr}$, the sums of the form $\sum_{1\le\nu\le r} c_\nu\{u_\nu, t\}$, where $0 \le c_\nu \le p^s - 1$. Then the elements w_i belong to distinct cosets of $p^s A_0 + T_K$ in $A_0 + T_K$ and represent all cosets.

Let $x_i \in A$ be such that

$$p^d x_i \equiv w_i \mod T_K.$$

We prove that the elements x_i represent all cosets of $p^s A + T_K$ in $A + T_K$. Let $x \in A$. Then $p^d x \in A_0 + T_K$, and consequently, there exists $i = i(x)$ such that

$$p^d x - w_i \in p^s A_0 + T_K = p^s(p^d A) + T_K = p^d(p^s A) + T_K.$$

By Theorem 4.4, $x - x_i \in p^s A + T_K$.

We prove that the elements x_i belong to distinct cosets. Assuming the contrary, we see that $x_i - x_j \in p^s A + T_K$ for some $i \ne j$, whence

$$(w_i - w_j) + T_K = p^d(x_i - x_j) + T_K \subset p^d(p^s A) + T_K = p^s A_0 + T_K,$$

which contradicts the fact that all w_i belong to distinct cosets of $p^s A_0 + T_K$ in $A_0 + T_K$. □

Theorem 5.3. *The index of $B + T_K/T_K$ in $U(1)K_2^{\mathrm{top}}K/T_K$ is finite. Namely,*

$$(U(1)K_2^{\mathrm{top}}K : B + T_K) \le p^{n_0 r + n - n_1 - n_2}.$$

Proof. We put $\mathrm{N} = \mathrm{N}_{L/K}$,

$$B' = \{\{u, t_L\} \mid u \in V_k\}, \quad B^{(L)} = \{\{u, t_L\} \mid u \in V_l\}.$$

We have $t^{-m/m_0}\,\mathrm{N}\,t_L \in U_K$. Consequently, by Lemma 4.9, for every $u \in V_k$ we have

$$\frac{m}{m_0}\{u, t\} \equiv \{u, t^{m/m_0}\} \equiv \{u, \mathrm{N}\,t_L\} \equiv \mathrm{N}\{u, t_L\} \mod T_K.$$

Thus,

$$\mathrm{N}\, B' + T_K = p^{n-n_0} B + T_K. \tag{8}$$

Applying Lemma 5.2 to the subgroup B and Theorem 4.1 to the field L, we obtain

$$\begin{aligned}(U(1)K_2^{\mathrm{top}}K : B + T_K) &= p^{-(n-n_0)r}(U(1)K_2^{\mathrm{top}}K : \mathrm{N}\, B' + T_K)\\ &= p^{-(n-n_0)r}(U(1)K_2^{\mathrm{top}}K : \mathrm{N}\, U(1)K_2^{\mathrm{top}}L + T_K)\cdot(\mathrm{N}\, B^{(L)} + T_K : \mathrm{N}\, B' + T_K).\end{aligned}$$

Lemma 3.7 applied to the extensions $l_iK/l_{i-1}K$ shows that

$$(U(1)K_2^{\mathrm{top}}K : \mathrm{N}\, U(1)K_2^{\mathrm{top}}L + T_K) \le (U(1)K_2^{\mathrm{top}}K : \mathrm{N}\, U(1)K_2^{\mathrm{top}}L) \le p^{n-n_1}.$$

We find the index $(\mathrm{N}\, B^{(L)} + T_K : \mathrm{N}\, B' + T_K)$. Applying Lemma 4.9 and Theorem 4.4, we see that

$$m_0\, \mathrm{N}\{u, t_L\} \equiv \mathrm{N}\{u, t_L^{m_0}\} \equiv \mathrm{N}\{u, t\} \equiv \{\mathrm{N}_{l/k}\, u, t\} \mod T_K$$

for each $u \in V_l$. Consequently,

$$p^{n_0}\, \mathrm{N}\, B^{(L)} + T_K = \{\{u, t\} \mid u \in W\} + T_K,$$

where $W = \mathrm{N}_{l/k}\, V_l$. Since $(V_k : W) < \infty$, the subgroup $\mathrm{N}\, B^{(L)}$ satisfies the conditions of Lemma 5.2. Applying (8), we obtain

$$(\mathrm{N}\, B^{(L)} + T_K : \mathrm{N}\, B' + T_K) = N_1 N_2,$$

where

$$N_1 = \frac{(\mathrm{N}\, B^{(L)} + T_K : p^{n_0}\, \mathrm{N}\, B^{(L)} + T_K)(B + T_K : p^{n-n_0} B + T_K)}{(B + T_K : p^n B + T_K)}$$

and

$$N_2 = (p^{n_0}\, \mathrm{N}\, B^{(L)} + T_K : p^n B + T_K).$$

We have

$$N_1 = \frac{p^{n_0 r} p^{(n-n_0)r}}{p^{nr}} = 1$$

and

$$N_2 = (W \cdot T(V_k) : p^n V_k \cdot T(V_k)) = \frac{(V_k : p^n V_k \cdot T(V_k))}{(V_k : \mathrm{N}_{l/k}\, V_l \cdot T(V_k))} = p^{nr-n_2},$$

which proves the theorem. □

Corollary 5.4. *If $p \mid e(K/k)$, then $B + T_K = U(1)K_2^{\mathrm{top}}K$.*

Proof. This follows from Theorem 5.3 and Lemma 4.7. □

Corollary 5.5. *The $\mathbb{Z}_p$-rank of the factor group $U(1)K_2^{\mathrm{top}}K/T_K$ is equal to $|k : \mathbb{Q}_p|$.*

Proof. This follows from Theorem 5.3 and Lemma 5.1. □

§6. Almost standard fields

In the case of an almost standard field, we obtain a lower estimate for the index of the subgroup described in §5.

Theorem 6.1. *Let $K \subset L$ be two-dimensional fields such that the extension L/K is finite, solvable, constant, and unramified. Let the field L be standard, let k and l be the subfields of constants of K and L, and let B, r, n_0, and n_2 be as in* (7), (5), *and* (6).

1) *We have* $(U(1)K_2^{\mathrm{top}}K : B + T_K) \geq p^{rn_0 - n_0}$.

2) *If* $\zeta \in \mathrm{N}_{l/k(\zeta)} V_l$ *for all* $\zeta \in T(V_l)$, *then*

$$(U(1)K_2^{\mathrm{top}}K : B + T_K) = p^{rn_0 - n_2}.$$

Proof. Let t be a second local parameter of K. Let $K_0 = k\{\{t\}\}$ and $L_0 = lK_0$. By Theorem 4.4, we have

$$\begin{aligned}(U(1)K_2^{\mathrm{top}}K : B + T_K) &\geq (\mathrm{N}_{K/K_0} U(1)K_2^{\mathrm{top}}K : \mathrm{N}_{K/K_0} B + \mathrm{N}_{K/K_0} T_K) \\ &\geq (\mathrm{N}_{K/K_0} U(1)K_2^{\mathrm{top}}K : p^{n_0} B + T_{K_0}) \\ &= \frac{(U(1)K_2^{\mathrm{top}}K_0 : p^{n_0} B + T_{K_0})}{(U(1)K_2^{\mathrm{top}}K_0 : \mathrm{N}_{K/K_0} U(1)K_2^{\mathrm{top}}K)}.\end{aligned} \tag{9}$$

By Theorem 4.1 and Lemma 5.2, the numerator of the above ratio is equal to p^{rn_0}. The extension L/K is unramified. Consequently, the mapping $\mathrm{N}_{L/K} : U(1)K_2^{\mathrm{top}}L \to U(1)K_2^{\mathrm{top}}K$ is surjective. The extension L/L_0 is also unramified, because

$$e(L/L_0) = e(L/l) = 1,$$

$$|\overline{L} : \overline{L}_0|_{\mathrm{insep}} \leq |\overline{L} : \overline{K}_0|_{\mathrm{insep}} = |\overline{L} : \overline{K}|_{\mathrm{insep}} \cdot |\overline{K} : \overline{K}_0|_{\mathrm{insep}} = 1.$$

Consequently, the mapping $\mathrm{N}_{L/L_0} : U(1)K_2^{\mathrm{top}}L \to U(1)K_2^{\mathrm{top}}L_0$ is surjective. Thus,

$$\mathrm{N}_{K/K_0} U(1)K_2^{\mathrm{top}}K = \mathrm{N}_{L/K_0} U(1)K_2^{\mathrm{top}}L = \mathrm{N}_{L_0/K_0} U(1)K_2^{\mathrm{top}}L_0.$$

Using (9) and Theorem 5.3, we see that it suffices to prove the inequality

$$(U(1)K_2^{\mathrm{top}}K_0 : \mathrm{N}_{L_0/K_0} U(1)K_2^{\mathrm{top}}L_0) \leq p^{n_0} \tag{10}$$

for all K and L; if the second condition is fulfilled, it suffices to check the inequality

$$(U(1)K_2^{\mathrm{top}}K_0 : \mathrm{N}_{L_0/K_0} U(1)K_2^{\mathrm{top}}L_0) \leq p^{n_2}.$$

Let t_L be a second local parameter of L, let m and m_0 be as in (4), and let the fields

$$k = l_0 \subset l_1 \subset \cdots \subset l_{N-1} \subset l_N = l$$

be such that the l_i/l_{i-1} are normal extensions of prime degrees. We put $M_i = l_iK_0$. By Lemma 3.7, we have

$$U(1)K_2^{\mathrm{top}}M_{i-1} = \mathrm{N}_{M_i/M_{i-1}} U(1)K_2^{\mathrm{top}}M_i$$

if the extension l_i/l_{i-1} is tame or unramified, and

$$(U(1)K_2^{\mathrm{top}}M_{i-1} : \mathrm{N}_{M_i/M_{i-1}} U(1)K_2^{\mathrm{top}}M_i) = p$$

otherwise. Since the extension L/K is unramified and the extension $\overline{l}/\overline{k}$ is separable, the product of the degrees of the extensions l_i/l_{i-1} that are not unramified is equal to

$$e(l/k) = \frac{e(K/k)e(L/K)}{e(L/l)} = m_0,$$

which proves (10).

Now, suppose the second condition is fulfilled. Then

$$T(V_{l_i}) \subset \mathrm{N}_{l/l_i} V_{l_i}, \quad T(V_{l_{i-1}}) \subset \mathrm{N}_{l_i/l_{i-1}} V_{l_i} \tag{11}$$

for each i. We prove that

$$T_{M_{i-1}} \subset \mathrm{N}_{M_i/M_{i-1}} U(1)K_2^{\mathrm{top}} M_i. \tag{12}$$

If i is such that the extension l_i/l_{i-1} is tame or unramified, then the claim follows from Lemma 3.7. Let l_i/l_{i-1} be a totally ramified extension of degree p. Then

$$\begin{aligned}(U(1)K_2^{\mathrm{top}} M_{i-1} : \mathrm{N}_{M_i/M_{i-1}} U(1)K_2^{\mathrm{top}} M_i) = p &= (V_{l_{i-1}} : \mathrm{N}_{l_i/l_{i-1}} V_{l_i}) \\ &= (V_{l_{i-1}} : \mathrm{N}_{l_i/l_{i-1}} V_{l_i} \cdot T(V_{l_{i-1}})).\end{aligned}$$

We choose an element $u_0 \in V_{l_{i-1}}$ generating $V_{l_{i-1}}$ over $\mathrm{N}_{l_i/l_{i-1}} V_{l_i} \cdot T(V_{l_{i-1}})$. Then $\{u_0, t\}$ generates $U(1)K_2^{\mathrm{top}} M_{i-1}$ over $\mathrm{N}_{M_i/M_{i-1}} U(1)K_2^{\mathrm{top}} M_i$. For every

$$x \in U(1)K_2^{\mathrm{top}} M_{i-1} \setminus \mathrm{N}_{M_i/M_{i-1}} U(1)K_2^{\mathrm{top}} M_i,$$

there exists s such that $1 \le s \le p-1$ and

$$x - s\{u_0, t\} \in \mathrm{N}_{M_i/M_{i-1}} U(1)K_2^{\mathrm{top}} M_i.$$

By Theorems 4.1 and 4.4, we have

$$x - s\{u_0, t\} \in \{\{\mathrm{N}_{l_i/l_{i-1}} u, t\} \mid u \in V_{l_i}\} + T_{M_{i-1}}.$$

Since $u_0^s \mathrm{N}_{l_i/l_{i-1}} u \notin T(V_{l_i})$ for $u \in V_{l_i}$, it follows that $x \notin T_{M_{i-1}}$.

We prove that $T_{M_i} \subset \mathrm{N}_{L_0/M_i} U(1)K_2^{\mathrm{top}} L$ for each i. This is true for $i = N$. We prove that if the claim is true for some i, then it is true for $i-1$. Let $x \in T_{M_{i-1}}$. By (12) and Theorem 4.1,

$$x = \mathrm{N}_{M_i/M_{i-1}} x_1 + \mathrm{N}_{M_i/M_{i-1}} x_2,$$

where $x_1 = \{u, t\}$ for some $u \in V_{l_i}$, $x_2 \in T_{M_i}$. We have

$$\begin{aligned}\mathrm{N}_{M_i/M_{i-1}} x_2 &\in \mathrm{N}_{M_i/M_{i-1}} T_{M_i} \\ &\subset \mathrm{N}_{M_i/M_{i-1}} \mathrm{N}_{L_0/M_i} U(1)K_2^{\mathrm{top}} L_0 = \mathrm{N}_{L_0/M_{i-1}} U(1)K_2^{\mathrm{top}} L_0.\end{aligned}$$

By Theorem 4.4, we have $\mathrm{N}_{M_i/M_{i-1}} x_2 \in T_{M_{i-1}}$, whence

$$\{\mathrm{N}_{l_i/l_{i-1}} u, t\} = \mathrm{N}_{M_i/M_{i-1}} x_1 = x - \mathrm{N}_{M_i/M_{i-1}} x_2 \in T_{M_{i-1}}.$$

By (11),

$$\mathrm{N}_{l_i/l_{i-1}} u \in T(V_{l_{i-1}}) \subset \mathrm{N}_{l/l_{i-1}} V_l,$$

whence $\mathrm{N}_{M_i/M_{i-1}} x_1 \in \mathrm{N}_{L_0/M_{i-1}} U(1)K_2^{\mathrm{top}} L_0$.

For $i = 0$, we have $T_{K_0} \subset \mathrm{N}_{L_0/K_0} U(1)K_2^{\mathrm{top}} L_0$. Let

$$B_0 = \{\{u, t\} \mid u \in V_l\}.$$

Applying Theorems 4.1 and 4.4, we obtain

$$\begin{aligned}&(U(1)K_2^{\mathrm{top}} K_0 : \mathrm{N}_{L_0/K_0} U(1)K_2^{\mathrm{top}} L_0) \\ &\quad = (B + T_{K_0} : \mathrm{N}_{L_0/K_0} U(1)K_2^{\mathrm{top}} L_0 + T_{K_0}) \\ &\quad = (B + T_{K_0} : \mathrm{N}_{L_0/K_0} B_0 + T_{K_0}) = (V_k : \mathrm{N}_{l/k} V_l \cdot T(V_k)) = p^{n_2}.\end{aligned}$$ □

Corollary 6.2. *If K is an almost standard field, $r > 1$, and $p \mid e(K/k)$, then*

$$B + T_K \ne U(1)K_2^{\mathrm{top}} K.$$

Remark 6.3. In the notation of §2, the first statement of Theorem 6.1 takes the form

$$(U(1)K_2^{\mathrm{top}} K : B + T_K) \ge p^{(r-1)v_p(m_u(K))} = p^{(r-1)v_p(e(K/k))},$$

where the right-hand side does not depend on L.

We have the following analog of Theorem 4.1.

Theorem 6.4. *Let K and L be two-dimensional fields such that L is standard and the extension L/K is unramified, finite, and constant. We denote by l the field of constants in L and by t_L the second local parameter of L. Then there exists a $\mathbb{Z}_p$-basis of $U(1)K_2^{\mathrm{top}}K/T_K$ consisting of elements of the form $\mathrm{N}_{L/K}\{u, t_L\}$, where $u \in V_l$.*

Proof. By Lemma 3.7, the mapping

$$\mathrm{N}_{L/K} : U(1)K_2^{\mathrm{top}}L \to U(1)K_2^{\mathrm{top}}K$$

is surjective. Applying Theorem 4.1 and Theorem 4.4, we obtain

$$U(1)K_2^{\mathrm{top}}K = \mathrm{N}_{L/K}\, U(1)K_2^{\mathrm{top}}L = \mathrm{N}_{L/K}\, B^{(L)} + \mathrm{N}_{L/K}\, T_L \subset \mathrm{N}_{L/K}\, B^{(L)} + T_K,$$

where $B^{(L)} = \{\{u, t_L\} \mid u \in V_l\}$. Thus, the set of elements $\mathrm{N}_{L/K}\{u, t_L\}$ with $u \in V_l$ generates the $\mathbb{Z}_p$-module $U(1)K_2^{\mathrm{top}}K/T_K$; consequently, it contains a $\mathbb{Z}_p$-basis of $U(1)K_2^{\mathrm{top}}K/T_K$. □

Using some results of [3], we construct an example of an almost standard field.

Definition 6.5. Let E/F be a finite Galois extension of discrete valuation fields such that the extension $\overline{E}/\overline{F}$ is separable. For a real $a \geq -1$, we denote by G_a the ramification subgroup $G_i(E/F)$, where i is the smallest integer such that $i \geq a$. The Herbrand function of the extension E/F is defined as follows:

$$\varphi(x) = \varphi_{E/F}(x) = \int_0^x \frac{da}{(G_0 : G_a)}.$$

If $m \leq x \leq m + 1$ for a nonnegative integer m, then

$$\varphi(x) = \frac{1}{|G_0|}(|G_1| + \cdots + |G_m| + (x - m)|G_{m+1}|). \tag{13}$$

Lemma 6.6. *Let F be a complete discrete valuation field, let* $\operatorname{char} F = p$, *and let E/F be a Galois extension of degree p such that the extension $\overline{E}/\overline{F}$ is separable. Then $E = F(x)$ for some x such that $\wp(x) = a \in F$ and either $v_F(a) = 0$, or $v_F(a) < 0$ and $p \nmid v_F(a)$. In the first case, the extension E/F is unramified and*

$$d(E/F) = h(E/F) = 0.$$

In the second case, the extension E/F is totally ramified and

$$d(E/F) = -\frac{p-1}{pe_F} v_F(a), \quad h(E/F) = -v_F(a).$$

Proof. See [6, Proposition 1.4] and [8, Lemma 2–10]. □

Lemma 6.7. *Let F be a complete discrete valuation field, let* $\operatorname{char} F = p$, *and let $\overline{F}$ be finite. Suppose that F_1/F and F_2/F are totally ramified Galois extensions of degree p, and $h(F_1/F) > h(F_2/F)$. Then F_1F_2/F is a totally ramified extension of degree p^2, and*

$$h(F_1F_2/F_1) = h^{(2)}(F_1F_2/F) = h(F_2/F).$$

Proof. Put $E = F_1F_2$, $h_i = h(F_i/F)$, and $h^{(i)} = h^{(i)}(E/F)$ for $i = 1, 2$. By (13), the function φ is piecewise linear for every field extension, and its derivative changes at the points corresponding to the jumps of the extension. By [3, Chapter 4, Proposition 15], we have

$$\varphi_{E/F} = \varphi_{F_2/F} \circ \varphi_{E/F_2}, \tag{14}$$

and a similar relation is valid for F_1. Consequently, the jumps of the extensions E/F_1 and E/F_2 are jumps of the extension E/F. Using (2) and Lemma 6.6, we obtain

$$h(E/F_1) + ph_1 = h(E/F_2) + ph_2. \tag{15}$$

Since $h_1 > h_2$, we have $h(E/F_2) > h(E/F_1)$, whence $h^{(1)} = h(E/F_2)$ and $h^{(2)} = h(E/F_1)$. We have $h(E/F_2) > 0$. Therefore, the extension E/F_2 is totally ramified by Lemma 6.6, and so the extension E/F is totally ramified.

Let $h^{(1)} < h_2$. If $x \geq h^{(1)}$ and $\varphi_{E/F_2}(x) \leq h_2$, then, by formula (13) applied to the extensions E/F_2 and F_2/F and by (14), we have

$$\varphi_{E/F}(x) = \frac{1}{p}x + \Big(1 - \frac{1}{p}\Big)h^{(1)}.$$

The element $x_0 = ph_2 - (p-1)h^{(1)}$ satisfies $\varphi_{E/F_2}(x_0) = h_2$; therefore,

$$h^{(2)} = ph_2 - (p-1)h^{(1)}.$$

Applying (15), we obtain $h_1 = h^{(1)}$, which contradicts the inequality $h_2 < h_1$.

If $h^{(1)} = h_2$, then, by (14), the derivative of $\varphi_{E/F}$ changes only at one point, and the jumps of E/F coincide. This contradicts the fact that the extension E/F has distinct jumps h_1 and h_2.

Thus, $h^{(1)} > h_2$. By (14), the derivative of $\varphi_{E/F}$ changes at $h^{(1)}$ and h_2, and the number h_2 is a jump of E/F. Consequently, $h_2 = h^{(2)}$. □

Lemma 6.8. *Let F be a complete discrete valuation field, let* $\operatorname{char} F = p$, *and let $\overline{F}$ be finite. Let F_0/F be a Galois field extension of degree q, $p \nmid q$, and let F_i/F, where $i = 1, \dots, nF_i/F$, be a totally ramified Galois extension of degree p with jumps $h_i = h(F_i/F)$, where*

$$h_1 < h_2 < \cdots < h_n.$$

Then

1) *for $1 \leq i \leq n-1$, we have*

$$h(F_iF_{i+1}\dots F_n/F_{i+1}\dots F_n) = h_i;$$

2) *$F_0F_1F_2\dots F_n/F$ is a totally ramified extension of degree qp^n.*

Proof. 1) We use induction on n. For $n = 2$, it suffices to apply Lemma 6.7. Assuming that the claim is valid for some n, we prove it for $n+1$. Put $E_i = F_iF_{n+1}$ for $i = 1, \dots, n$. By Lemma 6.7, $h(E_i/F_{n+1}) = h_i > 0$. Therefore, by Lemma 6.6, the extensions E_i/F_{n+1} are totally ramified. Since these extensions have distinct jumps, we can apply the inductive hypothesis. Thus, for $1 \leq i \leq n$, we have

$$\begin{aligned} &h(F_iF_{i+1}\dots F_nF_{n+1}/F_{i+1}\dots F_nF_{n+1}) \\ &\quad = h(E_iE_{i+1}\dots E_n/E_{i+1}\dots E_n) = h(E_i/F_{n+1}) = h_i. \end{aligned}$$

2) Let $E_0 = F_1F_2\dots F_n$ and $E = F_0F_1F_2\dots F_n$. The first statement of the lemma implies that the extensions $F_iF_{i+1}\dots F_n/F_{i+1}\dots F_n$ have nonzero jumps for $1 \leq i \leq n-1$. Consequently, by Lemma 6.6, these extensions are totally ramified and $e(E_0/F) = p^n$. We have $e(E_0/F) \mid e(E/F)$ and $e(F_0/F) \mid e(E/F)$, whence $qp^n \mid e(E/F)$. On the other hand, $|E : F| \leq qp^n$. □

We apply Lemma 6.8 to residue fields of two-dimensional fields.

Theorem 6.9. *Let k be a finite extension of $\mathbb{Q}_p$ containing a primitive pth root of unity, and let l/k be a totally ramified Galois extension such that* $\operatorname{Gal}(l/k) \cong (\mathbb{Z}/p\mathbb{Z})^n \times G_0$, *where $p \nmid |G_0|$. Then there exists a two-dimensional field K with subfield of constants k such that the field lK is standard and the extension lK/K is unramified.*

Proof. We put $q = |G_0|$. Let $k_0, k_1, \dots, k_n$ be fields such that the extensions k_i/k are totally ramified, let $|k_0 : k| = q$, $|k_i : k| = p$ for $1 \leq i \leq n$, and let $l = k_0k_1, \dots, k_n$. We choose elements $u_i \in O_k$ such that $k_0 = \sqrt[q]{u_0}$, $k_i = k(\sqrt[p]{u_i}\,)$ for $1 \leq i \leq n$, $v_k(u_0) = 1$ and, for $i \geq 1$, either $p \nmid v_k(u_i)$, or $p \nmid v_k(u_i - 1)$, and $0 < v_k(u_i - 1) < \frac{pe_k}{p-1}$. Put $K_0 = k\{\{t\}\}$ and $c = \frac{pe_k}{p-1}$. For every $A = \{a_i\}_{i=1}^n$ such that $a_i \in U_{K_0}(c) \setminus U_{K_0}(c+1)$, we denote

$$K_0^A = K_0(\sqrt[q]{u_0 t}), \quad K_i^A = K_{i-1}^A(\sqrt[p]{a_i u_i}) \quad \text{for} \quad 1 \leq i \leq n, \quad L^A = lK_n^A.$$

We prove that, for some A, the field K_n^A can be taken for the role of the required field K.

For any A, the field $lK_0 = K_0(\sqrt[q]{u_0}, \sqrt[p]{u_1}, \dots, \sqrt[p]{u_n})$ is standard, the extensions L^A/K_n^A and L^A/lK_0 are unramified because

$$L^A = K_n^A(\sqrt[q]{t}, \sqrt[p]{a_1}, \dots, \sqrt[p]{a_n}) = lK_0(\sqrt[q]{t}, \sqrt[p]{a_1}, \dots, \sqrt[p]{a_n}), \tag{16}$$

and the field L^A is standard, being an unramified extension of the standard field lK_0. Moreover,

$$e(K_n^A/K_0) = e(L^A/K_0) = e(lK_0/K_0) = e(l/k) = p^n q. \tag{17}$$

Lemma 6.10. *There exists a set A such that*
1) *k is the field of constants of $K_i = K_i^A$ for each $i \geq 1$;*
2) *l is the field of constants of $L = L^A$.*

Proof. Let π_k and π_l be uniformizing elements of k and l, respectively. The required elements a_i have the form $a_i = 1 + \pi_k^c b_i$ for some $b_i \in U_{K_0}$.

The field k is the subfield of constants of K_0^A. We prove that there exist a_i such that k is the subfield of constants of K_i for each i, and $v_{\overline{K}_0}(\overline{b}_i)$ are distinct negative integers not divisible by p. Assume that the numbers $a_1, \dots, a_m$ have already been found; we show how to find a_{m+1}. By [2, Chapter 2, Proposition 14], the field k has only finitely many extensions of degree p; we denote these extensions by $l_1, \dots, l_N$. For every $b \in U_{K_0}$ such that $v_{\overline{K}_0}(\overline{b}) < 0$ and $p \nmid v_{\overline{K}_0}(\overline{b})$, either k or one of the fields l_s is the subfield of constants of

$$K_m\Big(\sqrt[p]{(1 + \pi_k^c b) u_{m+1}}\Big).$$

For $s = 1, \dots, N+1$, we choose $b^{(s)} \in U_{K_0}$ with the following properties: the elements $\overline{b^{(s)}}$ belong to distinct cosets of $\wp(\overline{K_0})$ in $\overline{K_0}$ and their valuations in $\overline{K}_0$ are negative, not divisible by p, and distinct from the valuations of $\overline{b}_1, \dots, \overline{b}_m$. We put

$$a^{(s)} = 1 + \pi_k^c b^{(s)}, \quad K_{m+1,s} = K_m\Big(\sqrt[p]{a^{(s)} u_{m+1}}\Big)$$

and prove that the fields $K_{m+1,s}$ are distinct. Suppose the contrary. Let $K_{m+1,s} = K_{m+1,s'}$ for some $s \neq s'$. Then $\sqrt[p]{a^{(s)}/a^{(s')}} \in K_{m+1,s}$. We have

$$a^{(s)}/a^{(s')} \equiv 1 + \pi_k^c (b^{(s)} - b^{(s')}) \mod U_{K_m}(p^m c + 1),$$

and since $\overline{b^{(s)} - b^{(s')}} \notin \wp(\overline{K}_0) = \wp(\overline{K}_m)$, we obtain $\sqrt[p]{a^{(s)}/a^{(s')}} \notin K_m$. Therefore,

$$K_{m+1,s} = K_m\Big(\sqrt[p]{a^{(s)}/a^{(s')}}\Big),$$

and the extension $K_{m+1,s}/K_m$ is unramified, which contradicts (17). Thus,

$$K_{m+1,s} \cap K_{m+1,s'} = K_m$$

for all $s \neq s'$. Consequently, the field k is the subfield of constants of at least one of the fields $K_{m+1,s}$. This field can be taken as K_{m+1}, and the corresponding $a^{(s)}$ can be taken as a_{m+1}.

We prove that the second condition is valid for this choice of a_i. Let $L_0 = lK_0$. We have

$$[\theta] \pi_l^{p^n q} \pi_k^{-1} \in V_l$$

for some $\theta \in L_0^{(0)}$, and

$$a_i \equiv 1 + [\theta^c]\pi_l^{p^n qc} b_i \mod U_{L_0}(p^n qc + 1).$$

Consequently, $\overline{L_0(\sqrt[p]{a_i})} = \overline{L}_0(\overline{\beta}_i)$, where $\overline{\beta}_i^p - \overline{\beta}_i = \overline{[\theta^c]}\overline{b}_i$. Thus, the extensions $\overline{L_0(a_i)}/\overline{L}_0$ are totally ramified and, by Lemma 6.6, have distinct jumps,

$$h(\overline{L_0(\sqrt[p]{a_i})}/\overline{L}_0) = -v_{\overline{L}_0}(\overline{b}_i) = -v_{\overline{K}_0}(\overline{b}_i).$$

Moreover, $\overline{L}_0(\overline{\sqrt[q]{t}})/\overline{L}_0$ is a tame totally ramified extension. Consequently, by Lemma 6.8, the extension $\overline{L}/\overline{L}_0$ is totally ramified and has degree $p^n q$. By (16), it follows that $|L : L_0| \le p^n q$. Thus, L/L_0 is an unramified extension of degree $p^n q$ and $L^{(0)} = L_0^{(0)}$. Let l' be the subfield of constants of L. Then

$$\overline{l'} \cong L^{(0)} = L_0^{(0)} \cong \overline{l}.$$

Moreover,

$$l = k(\sqrt[q]{u_0}, \sqrt[p]{u_1}, \dots, \sqrt[p]{u_n}\,) \subset L,$$

whence, $l \subset l'$. We conclude that $l' = l$. □

The field K_n^A obtained in this way can be taken for the role of K. □

§7. Fields with $m_u(K) = 1$

In this section, we consider two-dimensional fields K such that $m_u(K) = 1$ in the notation of §2. We choose a second local parameter t of K and denote by k the subfield of constants of K. Let B, r, m_0, and n_0 be as in (7), (4), and (5).

Lemma 7.1. *For every $u \in V_k$, there exists a finite totally ramified extension l/k such that $u \in \mathrm{N}_{l/k} V_l$ and the field lK is standard.*

Proof. We construct a sequence of fields

$$k = k_0 \subset k_1 \subset k_2 \subset \cdots$$

satisfying the following conditions: $d(k_{i+1}/k_i) = 1$, the extensions k_{i+1}/k_i are totally ramified and have degree p, and there exist elements $u_i \in V_{k_i}$ such that

$$u_0 = u, \quad \mathrm{N}_{k_{i+1}/k_i} u_{i+1} = u_i \text{ for } i \ge 0.$$

Assume that the elements k_i and u_i have already been constructed. There exist $a, b \in V_{k_i}$ such that $u_i = a^p b$ and $p \nmid v_{k_i}(b-1)$. Indeed, we represent u_i in the form

$$u_i = \prod_{\substack{\nu > 0,\, p \nmid \nu \\ \theta \in \mathfrak{B}}} (1 + [\theta]\pi^\nu)^{\alpha_{\nu,\theta}},$$

where π is a uniformizing element of k_i, $\mathfrak{B}$ is a basis of $K^{(0)}$ over $\mathbb{F}_p$, and $\alpha_{\nu,\theta} \in \mathbb{Z}_p$. As a^p we can take the product of the terms of degrees divisible by p, and as b we can take the product of the other terms. We put $k_{i+1} = k_i(x)$, where $x^p = b - 1$. Then the extension k_{i+1}/k_i has the required depth by Lemma 2.4,

$$\mathrm{N}_{k_{i+1}/k_i}(1 + x) = b,$$

and we can put $u_{i+1} = a(1 + x)$.

By (2), we have $d(k_i/k) = i$ for all i. Therefore, the extension $\bigcup k_i/k$ is deeply ramified and has no unramified subextensions. By Lemma 2.5, this extension has a finite subextension l/k such that lK is standard. The extension l/k satisfies the required condition. □

Lemma 7.2. *We have*

$$(U(1)K_2^{\mathrm{top}}K : B + T_K) \le p^{rn_0}.$$

Proof. Put

$$l_0 = \bigcup_{n\in\mathbb{N}} k(\zeta_{p^n})/k.$$

The extension l_0/k is cyclic and deeply ramified. Consequently, by Lemma 2.5, there exists a finite cyclic extension l/k such that the field $L = lK$ is standard and $T(V_k) \subset \mathrm{N}_{l/k} V_l$. Let n, n_1, and n_2 be as in (4), (5), and (6) for the extension L/K, and let l'/k be the maximal unramified subextension of l/k. By Lemma 2.2, all unramified subextensions of L/K are purely unramified. Hence $n_1 = v_p(|l' : k|)$. Moreover,

$$n_2 = v_p((V_k : \mathrm{N}_{l/k} V_l)) = v_p(e(l/k)) = v_p(|l : l'|).$$

We obtain

$$n = v_p(|l : k|) = v_p(|l' : k|) + v_p(|l : l'|) = n_1 + n_2,$$

and the estimate in Theorem 5.3 takes the form

$$(U(1)K_2^{\mathrm{top}}K : B + T_K) \le p^{rn_0+n-n_1-n_2} = p^{rn_0}.$$ □

Now we prove that the estimate in Lemma 7.2 is sharp and obtain an analog of Theorem 6.4.

Theorem 7.3. *Let I be the set of finite totally ramified extensions l of k such that lK is standard and*

$$B_0 = \bigcup_{l\in I} \mathrm{N}_{lK/K} U(1)K_2^{\mathrm{top}}(lK).$$

Then $B_0 + T_K = U(1)K_2^{\mathrm{top}}K$.

Moreover,

$$(U(1)K_2^{\mathrm{top}}K : B + T_K) = p^{rn_0}.$$

Proof. For $l \in I$, we denote by t_l the second local parameter of lK. By Lemma 2.2 and (1), we have $e(\overline{lK}/\overline{K}) = m_f(K) = m_0$. Consequently, $t_l^{m_0}t^{-1} \in V_{lK}$. By Lemma 4.9 and Theorem 4.4, the relation

$$\{\mathrm{N}_{l/k} w, t\} \equiv \mathrm{N}_{lK/K}\{w, t\} \equiv m_0 \,\mathrm{N}_{lK/K}\{w, t_l\} \mod T_K \tag{18}$$

is valid for all $w \in V_l$. Applying Theorem 4.4 and Theorem 4.1 once again, we obtain

$$\begin{aligned} &m_0 \,\mathrm{N}_{lK/K} U(1)K_2^{\mathrm{top}}(lK) \\ &\quad= m_0 \,\mathrm{N}_{lK/K}\{\{w, t_l\} \mid w \in V_l\} + m_0 \,\mathrm{N}_{lK/K} T_{lK} \subset B + T_K. \end{aligned}$$

Thus, $p^{n_0}B_0 + T_K \subset B + T_K$.

We check that $B + T_K = p^{n_0}B_0 + T_K$. Let $x \in B + T_K$, and let $u \in V_k$ be such that $x - \{u, t\} \in T_K$. By Lemma 7.1, we can find $l_u \in I$ and $w \in V_{l_u}$ such that $\mathrm{N}_{l_u/k} w = u$. By (18),

$$x \equiv \{u, t\} \equiv m_0 \,\mathrm{N}_{l_uK/K}\{w, t_{l_u}\} \mod T_K.$$

Consequently, $x \in p^{n_0}B_0 + T_K$.

By Lemma 5.2, we have

$$(B_0 + T_K : p^{n_0}B_0 + T_K) = p^{rn_0}.$$

Using Lemma 7.2, we obtain

$$(U(1)K_2^{\mathrm{top}}K : B_0 + T_K) = \frac{(U(1)K_2^{\mathrm{top}}K : B + T_K)}{(B_0 + T_K : p^{n_0}B_0 + T_K)} \le \frac{p^{rn_0}}{p^{rn_0}} = 1.$$

Consequently, $U(1)K_2^{\mathrm{top}}K = B_0 + T_K$, and we see that the index of $B + T_K$ is equal to p^{rn_0}. □

References

[1] I. B. Fesenko and S. V. Vostokov, *Local fields and their extensions. A constructive approach*, Transl. Math. Monogr., vol. 121, Amer. Math. Soc., Providence, RI, 1993. MR1218392 (94d:11095)

[2] S. Lang, *Algebraic numbers*, Addison-Wesley Publ. Co., Inc., Reading, MA 1964. MR0160763 (28:3974)

[3] J.-P. Serre, *Local fields*, Grad. Texts in Math., vol. 67, Springer-Verlag, New York–Berlin, 1979. MR0554237 (82e:12016)

[4] I. B. Zhukov, *Milnor and topological K-groups of multidimensional complete fields*, Algebra i Analiz **9** (1997), no. 1, 98–147; English transl., St. Petersburg Math. J. **9** (1998), no. 1, 69–105. MR1458420 (98m:11131)

[5] I. B. Zhukov and A. I. Madunts, *Multidimensional complete fields: Topology and other basic constructions*, Trudy S.-Peterburg. Mat. Obshch. **3** (1995), 4–46; English transl., Amer. Math. Soc. Transl. Ser. 2, vol. 166, Amer. Math. Soc., Providence, RI, 1995, pp. 1–34. MR1363290 (97b:12009)

[6] I. B. Zhukov and M. V. Koroteev, *Elimination of wild ramification*, Algebra i Analiz **11** (1999), no. 6, 153–177; English transl., St. Petersburg Math. J. **11** (2000), no. 6, 1063–1083. MR1746073 (2002a:11134)

[7] I. Zhukov, *Higher dimensional local fields*, Invitation to Higher Local Fields (Münster, 1999), Geom. Topol. Monogr., vol. 3, Geom. Topol. Publ., Coventry, 2000, pp. 5–18. MR1804916 (2001k:11245)

[8] O. Hyodo, *Wild ramification in the imperfect residue field case*, Galois Representations and Arithmetic Algebraic Geometry (Kyoto, 1985/Tokyo, 1986), Adv. Stud. Pure Math., vol. 12, North-Holland, Amsterdam, 1987, pp. 287–314. MR0948250 (89j:11116)

[9] I. Fesenko, *Topological Milnor K-groups of higher local fields*, Invitation to Higher Local Fields (Münster, 1999), Geom. Topol. Monogr., vol. 3, Geom. Topol. Publ., Coventry, 2000, pp. 61–74. MR1804920 (2001k:11247)

[10] ———, *Theory of local fields. Local class field theory. Multidimensional local class field theory*, Algebra i Analiz **4** (1992), no. 3, 1–41; English transl., St. Petersburg Math. J. **4** (1993), no. 3, 403–438. MR1190770 (93i:11138)

[11] S. Vostokov, *Explicit formulas for the Hilbert symbol*, Invitation to Higher Local Fields (Münster, 1999), Geom. Topol. Monogr., vol. 3, Geom. Topol. Publ., Coventry, 2000, pp. 81–89. MR1804922 (2002a:11138)

[12] K. Kato, *A generalization of local class field theory by using K-groups.* I, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **26** (1979), 303–376. MR0550688 (81b:12016)

[13] O. Yu. Ivanova, *Topological K-groups of two-dimensional local fields*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **343** (2007), 206–221; English transl., J. Math. Sci. (N. Y.) **147** (2007), no. 5, 7088–1097. MR2469418

Department of Mathematics and Mechanics, St. Petersburg State University, Universitetskiĭ Prospekt 28, Staryĭ Peterhof, 198504 St. Petersburg, Russia

E-mail address: `olgaiv80@mail.ru`

Received 21/DEC/2007

Translated by B. M. BEKKER