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ELEMENTARY SUBGROUPS OF ISOTROPIC REDUCTIVE GROUPS

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Abstract. Let G be a not necessarily split reductive group scheme over a commutative ring R with 1. Given a parabolic subgroup P of G, the elementary group $\mathrm{E}_P(R)$ is defined to be the subgroup of $G(R)$ generated by $\mathrm{U}_P(R)$ and $\mathrm{U}_{P^-}(R)$, where U_P and U_{P^-} are the unipotent radicals of P and its opposite P^-, respectively. It is proved that if G contains a Zariski locally split torus of rank 2, then the group $\mathrm{E}_P(R) = \mathrm{E}(R)$ does not depend on P, and, in particular, is normal in $G(R)$.

§1. Introduction

Let G be a reductive algebraic group over a commutative ring R with identity. Our aim is to give a definition of an *elementary subgroup* $\mathrm{E}(R)$ of the group of points $G(R)$, generalizing the notion of the elementary subgroup of a split reductive group and other similar concepts, and to show that under some natural restrictions, $\mathrm{E}(R)$ is normal in $G(R)$.

The notion of the elementary subgroup $\mathrm{E}_n(R)$ of the general linear group $\mathrm{GL}_n(R)$ was introduced by Bass [7] (while before it had been used implicitly by Whitehead in the study of homotopy types of CW-complexes) and served as a basis for his construction of algebraic K-theory. In particular, the nonstable K_1-functor is defined as the quotient $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$, and K_2 as the kernel of a certain central extension of $\mathrm{E}_n(R)$. The definition of the elementary subgroup involves a fixed basis in R^n, but by the Suslin theorem [26], if R is commutative and $n \geq 3$, then $\mathrm{E}_n(R)$ does not depend on the choice of a basis, or, in other words, is normal in $\mathrm{GL}_n(R)$. Various approaches to this result were discussed, for example, in [25, 35].

Later on, the elementary subgroup was defined for arbitrary split semisimple groups over R as the subgroup generated by all elementary root unipotents $x_\alpha(\xi)$ or, what is the same, by the R-points of the unipotent radical of a Borel subgroup B in G and of the unipotent radical of the opposite Borel subgroup B^- (see, e.g., [1, 21]). In the same way as in the case of $G = \mathrm{GL}_n$, it turns out that when the ranks of all irreducible components of the root system of G are at least 2, the elementary subgroup does not depend on the choice of a Borel subgroup, i.e., is normal in $G(R)$. For the orthogonal and symplectic groups, this fact was proved by Suslin and Kopeĭko [27, 18] and by Fu An Li [19], and for arbitrary Chevalley groups by Abe [1] in the case of local rings and by Taddei [29] in the general case (cf. [2]). A simpler proof was given by Hazrat and Vavilov in [15]. The normality of the elementary subgroup in twisted Chevalley groups was proved by Suzuki [28], and by Bak and Vavilov [5].

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For classical groups, there are versions of the definition of the elementary subgroup that involve an involution and a "form parameter" (in the sense of Bak). In that case normality was proved by Vaserstein and Hong You [34], and by Bak and Vavilov [6]; see also the paper of the first author on the case of "odd" unitary groups [22]. Certainly, not all classical groups in the sense of Bak can be presented as groups of points of reductive group schemes, but as for the methods, these works are direct generalizations of those mentioned above.

For nonsplit almost simple groups over a field k, the following analog of the elementary subgroup, introduced by Tits [30], is often considered. Namely, Tits defined the group $G^+(k)$ (originally G_k^0) as the subgroup generated by the k-points of the unipotent radicals of *all* parabolic subgroups in G defined over k. This definition is usually preferred, because it makes normality obvious, but in fact $G^+(k)$ is generated by the points of any two opposite unipotent radicals; see [10, Proposition 6.2]. Note that $G^+(k)$ is projectively simple in almost all cases [30], and the description of normal subgroups in $G(k)$ is reduced to the study of the so-called Whitehead group $G(k)/G^+(k)$, which is a natural analog of the K_1-functor. The famous Kneser–Tits problem asks whether the quotient is trivial in the case of a simply connected group G. It has an affirmative solution for number fields (the last step was recently done by Gille [14]), but in general the answer is negative even for groups of type A_l (the Platonov counterexample; see [14, 23]).

For nonsplit classical groups over rings, Vaserstein [32, 33] defined the elementary subgroup as the subgroup generated by *all* Eichler–Siegel–Dickson transvections. Normality is again obvious, but Vaserstein showed that the elementary subgroup is generated by transvections of a certain kind. Essentially, he fixed a parabolic subgroup of type P_1 and considered points of its unipotent radical and of the unipotent radical of an opposite parabolic subgroup.

Finally, we mention another definition of an elementary group that arises in the Jordan theory [3, 20]. The elementary group corresponding to a Jordan or Kantor pair is the group generated by all "exponents" of its elements taken in the adjoint representation. Morally, these are subgroups of suitable adjoint semisimple groups generated by points of two opposed unipotent radicals of nilpotency class 1 or 2.

This naturally leads us to the following definition generalizing all the definitions mentioned above.

Let P be a parabolic subgroup of a reductive group G over R, and let U_P be its unipotent radical. Since the base $\operatorname{Spec} R$ is affine, the group P has a Levi subgroup L_P (see [12, Exp. XXVI, Cor. 2.3]). There is a unique parabolic subgroup P^- in G that is opposite to P with respect to L_P (that is, $P^- \cap P = \mathrm{L}_P$; see [12, Exp. XXVI, Th. 4.3.2]).

We define the *elementary subgroup* $\mathrm{E}_P(R)$ corresponding to P as the subgroup of $G(R)$ generated as an abstract group by $\mathrm{U}_P(R)$ and $\mathrm{U}_{P^-}(R)$.

Note that if L'_P is another Levi subgroup of P, then L'_P and L_P are conjugate by some element $u \in \mathrm{U}_P(R)$ [12, Exp. XXVI, Cor. 1.8]; hence $\mathrm{E}_P(R)$ does not depend on the choice of a Levi subgroup or, respectively, of an opposite subgroup P^-. We shall show that, under some natural restrictions, $\mathrm{E}_P(R)$ does not depend on the choice of P as well, and, in particular, is normal in $G(R)$.

Recall that the main invariant of a split reductive group G over an algebraically closed field (as well as over a commutative ring; see [12, Exp. XXII]) is its root system Φ with respect to a split maximal torus T. Every parabolic subgroup P of a split group is characterized up to conjugacy by its type $J \subseteq \Pi$, where Π is a system of simple roots in Φ. A classical way to generalize these notions to the case of a nonsplit reductive group over an arbitrary field k (or over a local ring; see [12, Exp. XXVI, §7]) is to replace the root system Φ by the relative root system ${}_k\Phi$ in the sense of Borel and Tits [9, 31] and to adjust appropriately the definition of the type of a parabolic subgroup (cf. §2).

We return to the case of an arbitrary reductive group G over a ring R. Let G^{ad} denote the corresponding adjoint algebraic group. We say that a parabolic subgroup P in G is *strictly proper* if for every maximal ideal M in R the image of P_{R_M} in G_i under the projection map is a proper subgroup in G_i, where $G^{ad}_{R_M} = \prod_i G_i$ is the decomposition of the semisimple group $G^{ad}_{R_M}$ into a product of simple groups. In the language of Borel and Tits (and of [12, Exp. XXVI, §7]) this condition can be restated as follows: the type of the parabolic subgroup P_{R_M} meets every irreducible component of the relative root system of G_{R_M}.

Our main result is the following.

Theorem 1. *Let G be a reductive algebraic group over a commutative ring R. Assume that for any maximal ideal M in R all irreducible components of the relative root system of G_{R_M} are of rank at least 2. Then $\mathrm{E}_P(R)$ does not depend on the choice of a strictly proper parabolic subgroup P. In particular, $\mathrm{E}(R) = \mathrm{E}_P(R)$ is normal in $G(R)$.*

Remark 1. The condition that the ranks of irreducible components of the relative root system of G_{R_M} are at least 2 is equivalent to the existence of split tori of rank at least 2 in every simple factor of the adjoint group $G^{ad}_{R_M}$.

Remark 2. In essence, the theorem says that if P and P' are strictly proper parabolic subgroups in G, then $\mathrm{E}_{P'}(R) = \mathrm{E}_P(R)$ in the following cases:
- when P and P' are (locally) conjugate;
- when $P \leq P'$ are comparable with respect to inclusion.

In the second case the condition on the ranks of irreducible components may be omitted (Lemma 12).

The key point in the proof of Theorem 1 is to apply an analog of the Quillen–Suslin lemma (Lemma 17), which essentially reduces the problem to the case of a local ring R. A K_0-analog of that lemma appeared in Quillen's solution of the Serre problem [24], while a K_1-version that we use was proposed by Suslin [26]. Over a local ring the assertion of the theorem remains true even without the restriction on the rank of the relative root system; it is readily implied by the local conjugacy of minimal parabolic subgroups ([12, Exp. XXVI, §5]).

Our main technical tool is *relative root subschemes* of G. In §§3–4 we define the *system of relative roots* Φ_P of G with respect to a parabolic subgroup P, generalizing the classical definition of the relative roots by Borel and Tits [9] mentioned above. Unlike the classical case, now Φ_P is not necessarily a root system. Next, by using faithfully flat descent, for any relative root $A \in \Phi_P$, we construct (§4, Theorem 2) a projective R-module V_A and a closed embedding of *schemes* (but not of group schemes in general)

$$X_A\colon \mathrm{W}(V_A) \to G,$$

where $\mathrm{W}(V_A)$ is the affine group scheme corresponding to V_A. The elements $X_A(v)$, $A \in \Phi_P$, $v \in V_A$, of $G(R)$ play the same role as elementary root unipotents in split groups. In particular, they generate $\mathrm{E}_P(R)$ and are subject to certain commutator relations that generalize Chevalley commutator formulas:

$$[X_A(v), X_B(u)] = \prod_{i,j>0} X_{iA+jB}(N_{ABij}(v,u)),$$

where $N_{ABij}\colon V_A \times V_B \to V_{iA+jB}$ are certain polynomial maps homogeneous of degree i in the first argument and of degree j in the second argument (Lemmas 9, 10). Under certain restrictions, these maps N_{ABij} are surjective, which corresponds to the invertibility of coefficients in the split case.

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§2. Local épinglages and parabolic subgroups

Let G be a reductive algebraic group over a commutative ring R with 1. Recall that an *épinglage* ([12, Exp. XXIII, Déf. 1.1]) $\mathcal{E}$ of G consists of the following data:

- the root datum $\mathcal{R} = (\mathrm{X}, \mathrm{X}^\vee, \Phi, \Phi^\vee, \Pi)$ of G, including a fixed set of simple roots $\Pi \subseteq \Phi$;
- a split maximal subtorus T of G, together with an isomorphism $\mathrm{X}^*(T) \simeq \mathrm{X}$, where $\mathrm{X}^*(T)$ is the character lattice of T;
- for any $\alpha \in \Pi$, an isomorphism $x_\alpha \colon \mathbf{G}_a \xrightarrow{\sim} X_\alpha$ between the additive group $\mathbf{G}_a$ and the corresponding root subgroup X_α of G, such that T acts on X_α by means of α.

Any *épinglage* can be extended to a *Chevalley system*, that is, a system of isomorphisms x_α for all $\alpha \in \Phi$, satisfying the Steinberg relations (in particular, the Chevalley commutator formulas). A reductive group G is *split* if and only if it admits an *épinglage*.

For any two *épinglages* $\mathcal{E}$ and $\mathcal{E}'$ of G, there exists a unique inner automorphism ι of G that takes, locally in the fpqc-topology, one *épinglage* into another. More precisely, this means that ι takes T to T', and there is an fpqc-covering

$$\coprod \operatorname{Spec} S_\mu \to \operatorname{Spec} R$$

together with certain elements $g_\mu \in G(S_\mu)$ and root data isomorphisms $\gamma_\mu \colon \mathcal{R} \to \mathcal{R}'$ (we require that $\gamma_\mu(\Pi) = \Pi'$) such that over any S_μ the morphism ι is the conjugation by g_μ, the isomorphism

$$\mathrm{X}'_{S_\mu} \simeq \mathrm{X}^*(T'_\mu) \simeq \mathrm{X}^*(T_\mu) \simeq \mathrm{X}_{S_\mu}$$

induced by ι coincides with γ_μ^{-1}, and $\iota \circ x_\alpha = x'_{\gamma_\mu(\alpha)}$ for all $\alpha \in \Pi$ (see Exp. XXIV, Lemme 1.5). Observe that if $\operatorname{Spec} S_\mu$ and $\operatorname{Spec} S_\nu$ have nontrivial intersection (i.e., $S_\mu \otimes_R S_\nu \neq 0$), then $\gamma_\mu = \gamma_\nu$. Therefore, the entire collection of isomorphisms $\{\gamma_\mu\}$ does not depend on a given covering; we shall call these isomorphisms the *patching symmetries* between $\mathcal{E}$ and $\mathcal{E}'$.

Let P be a parabolic subgroup ([12, Exp. XXVI, Déf. 1.1]) of a split reductive group G. An *épinglage* $\mathcal{E}$ is said to be *adapted* to P if there exists a parabolic set of roots Ψ, $\Pi \subseteq \Psi \subseteq \Phi$, such that P is (algebraically) generated by the torus T and the root subgroups X_γ, $\gamma \in \Psi$. In particular, this implies that the unipotent radical U_P of P is generated by X_α, $\alpha \in \Psi \setminus -\Psi$. If a Levi subgroup L_P of P is chosen, the *épinglage* is said to be *adapted to* P *and* L_P if L_P is generated by T and X_α, $\alpha \in \Psi \cap -\Psi$, i.e., if L_P is a unique Levi subgroup of P containing T (see [12, Exp. XXVI, Prop. 1.12]).

If $\mathcal{E}$ and $\mathcal{E}'$ are two *épinglages* of G adapted to P, then the corresponding elements g_μ belong to $P(S_\mu)$ ([12, Exp. XXVI, Prop. 1.15]). If, moreover, these *épinglages* are adapted to a Levi subgroup L_P of P, then each g_μ belongs to $\mathrm{L}_P(S_\mu)$ ([12, Exp. XXVI, Cor. 1.8]).

Assume now that G is an arbitrary (i.e., not necessarily split) reductive algebraic group. Then G is split locally in the *étale*, and hence also in the fpqc-topology ([12, Exp. XXII, Cor. 2.3]).

Let P be a parabolic subgroup of G. Then fpqc-locally on $\operatorname{Spec} R$, one can choose an *épinglage* of G adapted to P ([12, Exp. XXVI, Lemme 1.14]). Since G is a reductive group over an affine scheme $\operatorname{Spec} R$, P possesses a Levi subgroup L_P ([12, Exp. XXVI, Cor. 2.3]). Since, locally in the *étale* topology, L_P contains a maximal torus of G, we conclude that fpqc-locally one can choose an *épinglage* of G adapted to both P and L_P.

By a *local épinglage* of G we mean a triple τ consisting of

- an affine open subset $U_\tau \subseteq \operatorname{Spec} R$;
- its faithfully flat affine covering $\operatorname{Spec} S_\tau \to U_\tau$ such that G splits over S_τ;
- an *épinglage* $\mathcal{E}_\tau$ of G_{S_τ}.

Consider the category whose objects are local *épinglages* τ and whose morphisms are graph isomorphisms between the Dynkin diagrams D_τ arising in the *épinglages* $\mathcal{E}_\tau$. This category is the *symmetric groupoid* $\operatorname{Sym}(\{D_\tau\})$ determined by all Dynkin diagrams of all local épinglages τ.

Two objects τ and σ of $\operatorname{Sym}(\{D_\tau\})$ provide two *épinglages* of the group $G_{S_\tau \otimes S_\sigma}$. Hence, they induce patching symmetries from $\mathcal{R}_\tau$ to $\mathcal{R}_\sigma$, which in turn give rise to certain graph isomorphisms between the corresponding Dynkin diagrams D_τ and D_σ. Define the *patching groupoid* Γ to be the subgroupoid of $\operatorname{Sym}(\{D_\tau\})$ generated by all these isomorphisms. We denote by Γ_τ the group of automorphisms of an object τ of Γ. Clearly, Γ_τ is a subgroup of the group of automorphisms $\operatorname{Aut}(D_\tau)$. For example, if $R = k$ is a field and $S_\tau = K$ is a Galois extension, then Γ_τ coincides with the image of the Galois group $\operatorname{Gal}(K/k)$ in $\operatorname{Aut}(D_\tau)$ corresponding to the $*$-action of $\operatorname{Gal}(K/k)$ [31].

Consider a Γ-isomorphism class ξ of local *épinglages* and denote by U_ξ the union of all U_τ, $\tau \in \xi$. It is easy to see that the open subsets U_ξ form a partition of $\operatorname{Spec} R$; in particular, they are clopen affine subschemes. Since $\operatorname{Spec} R$ is quasicompact, their number is finite. Therefore, we can write $U_\xi = \operatorname{Spec} R_\xi$, where $R \simeq \prod_\xi R_\xi$.

Note that for any parabolic subgroup P of G, our definition of $\mathrm{E}_P(R)$ readily implies $\mathrm{E}_P(R) \simeq \prod_\xi \mathrm{E}_{P_{R_\xi}}(R_\xi)$. This allows us to reduce most questions on the elementary subgroup to the case where the groupoid Γ consists of a unique isomorphism class.

Now we define the type of a parabolic subgroup P. First, consider all local *épinglages* τ adapted to P. Every subgroup P_{S_τ} is a standard parabolic subgroup corresponding to a set $J_\tau \subseteq \Pi_\tau$ of simple roots, so that P_{S_τ} is generated by the respective torus T_τ and by the root subgroups corresponding to the roots in whose decomposition the simple roots from J_τ appear with nonnegative coefficients. The collection of all J_τ is invariant under the morphisms of Γ (in particular, every J_τ is invariant under Γ_τ). It is easy to see that, starting with this data, in *any* local *épinglage* τ we can choose a subset $J_\tau \subseteq \Pi_\tau$ so that Γ-invariance still occurs. The total collection $\{J_\tau\}$ will be called the *type of the parabolic subgroup* P.

In fact, the constant schemes D_τ over S_τ can be glued together along the patching isomorphisms to produce a twisted constant scheme over R, which is called the Dynkin scheme of G (see [12, Exp. XXIV, §3)]). In a similar way, their clopen subschemes J_τ can be glued to give a clopen subscheme of the Dynkin scheme, which is precisely what is called the type of a parabolic subgroup in [12]. However, we prefer to keep to the above set-theoretic notions.

Recall that if R is a local ring, there exists a unique maximal (with respect to inclusion) parabolic subgroup type, which comes from a *minimal* parabolic subgroup ([12, Exp. XXVI, Cor. 5.7]).

§3. Relative roots

Throughout this section, Φ is a reduced root system in an l-dimensional Euclidean space with the scalar product $(\ ,\ )$. We fix a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_l\}$ in Φ (when Φ is irreducible, our numbering follows [11]), and we identify the elements of Π with the corresponding vertices of the Dynkin diagram D of Φ.

Fix a subset $J \subseteq \Pi$, and let Δ be the subsystem of Φ spanned by $\Pi \setminus J$. Any root $\alpha \in \Phi$ has a unique decomposition of the form

$$\alpha = \sum_{1 \le r \le l} m_r(\alpha)\alpha_r.$$

We set

$$\alpha_J = \sum_{\substack{1 \le r \le l,\\ \alpha_r \in J}} m_r(\alpha)\alpha_r.$$

We call a linear combination a of the elements of J a *shape* (cf. [4]) if there exists a root $\alpha \in \Phi \setminus \Delta$ with $\alpha_J = a$. In this case we also say that α is a root of shape α_J.

Lemma 1. *Take $\alpha, \beta, \gamma \in \Phi$ such that none of them is opposite to another and $\alpha+\beta+\gamma$ is a root. Then at least two of the sums $\alpha+\beta, \alpha+\gamma, \beta+\gamma$ are roots.*

Proof. We can assume that Φ is irreducible. Set $\delta = \alpha+\beta+\gamma$. Since $(\delta, \alpha+\beta+\gamma) = (\delta,\delta) > 0$, one of the products (δ,α), (δ,β), (δ,γ) is positive; let it be (δ,α). Then $\delta - \alpha = \beta + \gamma$ is a root. Next, if $(\alpha, \beta+\gamma) < 0$, then one of the products (α,β), (α,γ) is negative, and hence either $\alpha+\beta$ or $\alpha+\gamma$ is a root. If $(\alpha,\beta+\gamma) \ge 0$, then $(\delta, \beta+\gamma) = (\alpha,\beta+\gamma) + (\beta+\gamma,\beta+\gamma) > 0$, which implies that one of (δ,β), (δ,γ) is positive; that is, $\delta-\beta$ or $\delta-\gamma$ is a root. □

Lemma 2. *Suppose that a, b, c are shapes and that $a + b = c$. Then for any root γ of shape c there exist roots α of shape a and β of shape b such that $\alpha+\beta = \gamma$.*

Proof. The relation $a + b = c$ implies that the shapes a, b, c are linear combinations of simple roots from the same irreducible component of Φ, so we assume that Φ is irreducible. We can represent γ as a sum $\gamma = \alpha_0 + \beta_0 + \lambda_1 + \cdots + \lambda_n$, where α_0 is a root of shape a, β_0 is a root of shape b, and $\lambda_i \in \Delta$. We proceed by induction on n. The case where $n = 0$ is obvious. If $(\gamma,\alpha_0) > 0$ or $(\gamma,\beta_0) > 0$, then $\gamma - \alpha_0$ or, respectively, $\gamma - \beta_0$ is a root; therefore, we can take $\alpha = \alpha_0$, $\beta = \gamma - \alpha_0$ or $\beta = \beta_0$, $\alpha = \gamma - \beta_0$. Otherwise $(\gamma,\gamma) > 0$ implies that there exists i such that $(\gamma,\lambda_i) > 0$; that is, $\gamma' = \gamma - \lambda_i$ is a root. By the inductive hypothesis we have $\gamma' = \alpha' + \beta'$, where α' is a root of shape a, and β' is a root of shape b. It remains to note that by Lemma 1, one of $\alpha' + \lambda_i$, $\beta' + \lambda_i$ is a root. □

Let Γ be a subgroup of $\mathrm{Aut}(D)$, and suppose that $J \subseteq \Pi$ is invariant under the action of Γ. Let Γ act trivially on $\mathbb{Z}$. Then the Abelian group $\mathrm{Map}_\Gamma(J,\mathbb{Z})$ of all Γ-invariant maps from J to $\mathbb{Z}$ is free, and its rank is equal to the number of Γ-orbits in J. We define a linear map

$$\pi = \pi_{J,\Gamma} \colon \mathbb{Z}\,\Phi \to \mathrm{Map}_\Gamma(J,\mathbb{Z}),$$

where $\mathbb{Z}\,\Phi$ is the root lattice, as follows: for $v = \sum_{\alpha_i \in \Pi} m_i(v)\alpha_i \in \mathbb{Z}\,\Phi$, we set

$$\pi(v)(\alpha_j) = \sum_{\alpha_i \in \Gamma(\alpha_j)} m_i(v) \quad \text{for any } \alpha_j \in \Pi.$$

The set $\pi(\Phi) \setminus \{0\}$ will be called the *set*, or the *system, of relative roots* and will be denoted by $\Phi_{J,\Gamma}$. The *rank* $\operatorname{rank}\Phi_{J,\Gamma}$ of a system of relative roots $\Phi_{J,\Gamma}$ is the rank of the group $\mathrm{Map}_\Gamma(J,\mathbb{Z})$.

Note that if R is a local ring, Φ is the root system of a reductive algebraic group G, J is the type of a minimal parabolic subgroup of G, and Γ is the group of automorphisms of any object of the patching groupoid, then $\Phi_{J,\Gamma}$ is indeed a root system (maybe a nonreduced one, i.e., of type BC_l). If the group G is semisimple, the rank of this root system is equal to the rank of a maximal split subtorus of G. See [12, Exp. XXVI, §7] or [9, §5] for the details. In general, however, $\Phi_{J,\Gamma}$ is not a root system.

It is clear that any relative root $A \in \Phi_{J,\Gamma}$ can be represented as a (unique) linear combination of relative roots from $\pi(\Pi)$. By the *level* $\operatorname{lev}(A)$ of a relative root A we mean the sum of the coefficients in this decomposition.

We say that $A \in \Phi_{J,\Gamma}$ is a *positive* (respectively, *negative*) relative root if it is a nonnegative (respectively, nonpositive) linear combination of the elements of $\pi(\Pi)$. The sets of positive and negative relative roots will be denoted by $\Phi^+_{J,\Gamma}$ and $\Phi^-_{J,\Gamma}$, respectively. It is seen immediately that $A \in \Phi^\pm_{J,\Gamma}$ if and only if $\pi^{-1}(A) \subseteq \Phi^\pm$, and, in particular, $\Phi_{J,\Gamma} = \Phi^+_{J,\Gamma} \cup \Phi^-_{J,\Gamma}$.

Observe that the group of automorphisms Γ acts on the set of irreducible components of the root system Φ. If this action is transitive, the system of relative roots $\Phi_{J,\Gamma}$ is said to be *irreducible*. Clearly, any system of relative roots $\Phi_{J,\Gamma}$ is a disjoint union of irreducible ones; we call them the *irreducible components* of $\Phi_{J,\Gamma}$.

Clearly, for $\alpha_i, \alpha_j \in J$ we have $\pi(\alpha_i) = \pi(\alpha_j)$ if and only if α_i and α_j are in the same Γ-orbit. Moreover, $\pi|_\Delta = 0$; that is, $\pi(\alpha) = \pi(\alpha_J)$ for any root α. If the group Γ is trivial, then the relative roots are in one-to-one correspondence with the shapes defined by J.

Lemma 3. *Let $\alpha, \beta \in \Phi$. Then $\pi(\alpha) = \pi(\beta)$ if and only if there exists $\sigma \in \Gamma$ such that $\sigma(\alpha_J) = \beta_J$.*

Proof. The case where $\Gamma = \{\mathrm{id}\}$ is clear. It is also easily seen that we can assume Φ is irreducible. Moreover, we can replace the subset J by any subset $J' \subseteq \Pi$ that differs from J by a union of one-element Γ-orbits. Then if $\Phi = D_l$, $l \geq 4$, everything reduces to the case where J consists of a unique orbit, and the claim is obvious. This leaves us with the cases where $\Phi = A_l$, $l \geq 1$, and $\Phi = E_6$.

It is easily seen that if $\Phi = A_l$, then the shapes with respect to a Γ-invariant subset J are in one-to-one correspondence with the roots of some root system $\Phi' = A_m$, $m \leq l$, so that the action of Γ coincides with the action of $\operatorname{Aut}(D')$, where D' is the Dynkin diagram of Φ'. Hence, we can assume $J = \Pi$. Then $\Phi_{J,\Gamma}$ can be identified with the relative root system ${}_k\Phi$ of a quasi-split algebraic group of type ${}^2A_m$ (defined over some field k) in the sense of Borel and Tits, and we can use the general theory of reductive groups over fields [9]. Namely, applying an element of the relative Weyl group ${}_kW$, we pass to the case where $\pi(\alpha) = \pi(\beta)$ is a simple root of ${}_k\Phi$, and the statement is clear.

Similarly, if $\Phi = E_6$ and $J \supseteq \{\alpha_1, \alpha_6\} \cup \{\alpha_3, \alpha_5\}$ contains two nontrivial Γ-orbits, we can assume that $J = \Pi$ and view $\Phi_{J,\Gamma}$ as a relative root system in the sense of Borel and Tits. But if J consists of a unique nontrivial Γ-orbit, that is, if $J = \{\alpha_1, \alpha_6\}$ or $J = \{\alpha_3, \alpha_5\}$, our statement is obvious. ☐

Lemma 4. *Suppose $A, B, C \in \Phi_{J,\Gamma}$ and $A + B = C$. Then for any $\gamma \in \pi^{-1}(C)$ there exist $\alpha \in \pi^{-1}(A)$ and $\beta \in \pi^{-1}(B)$ such that $\alpha + \beta = \gamma$.*

Proof. If Γ is trivial, then relative roots coincide with shapes with respect to J, and our statement is merely Lemma 2. In general, Lemma 2 implies that it suffices to find shapes a, b, c such that $\pi(a) = A$, $\pi(b) = B$, $\pi(c) = C$, and $a + b = c$. Next, transferring some of the roots A, B, C to the other side of the identity $A + B = C$, we may assume that $A, B, C \in \Phi^+_{J,\Gamma}$. As in the proof of Lemma 3, we are reduced to the situation where Φ is irreducible and J contains no one-element Γ-orbit. Then the case of $\Phi = D_l$, $l \geq 4$, is straightforward. To settle the other cases, let σ denote a unique nontrivial element of Γ.

If $\Phi = A_l$, again as in the proof of Lemma 3, we can assume that $J = \Pi$, and the system of relative roots is a relative root system ${}_k\Phi$ in the sense of Borel and Tits, corresponding to a quasi-split algebraic group of type ${}^2A_l$ over a field k, and Γ depicts the $*$-action of a Galois group [9, 31]. Since we can leave out any one-element orbit, it suffices to consider the case where $l = 2n$ is even. Then $\Phi_{J,\Gamma} = BC_n$. It is known [9] that any element of the Weyl group of $\Phi_{J,\Gamma}$ ("the relative Weyl group") can be lifted to an element of the Weyl group of Φ, so we can assume that one of the relative roots A, B, say, A, is a simple root of $\Phi_{J,\Gamma}$, that is, $A = \pi(\alpha_i)$, or a multiple of a short simple root, that is, $A = \pi(\alpha_n + \alpha_{n+1})$. Take some $\alpha \in \pi^{-1}(B)$, $\gamma \in \pi^{-1}(C)$, and set $J' = \Pi \setminus \pi^{-1}(A)$. Then $\alpha_{J'} = \gamma_{J'}$; hence by Lemma 3 we may assume that $\alpha_{J'} = \gamma_{J'}$. Now it is easy to show that $\pi(\alpha) + A = \pi(\gamma)$ implies $\alpha + \alpha_i = \gamma$ or $\alpha + \sigma(\alpha_i) = \gamma$ in the first case, and $\alpha + \alpha_n + \alpha_{n+1} = \gamma$ in the second.

Now, let $\Phi = E_6$, and let roots α, β, $\gamma \in \Phi^+$ be such that $\pi(\alpha) = A$, $\pi(\beta) = B$ and $\pi(\gamma) = C$. If the shapes α_J, β_J, γ_J have no coefficients greater than 1, the problem reduces to the case where $\Phi = A_5$, discussed above. Otherwise α_3, $\alpha_5 \in J$ and we may suppose that $m_3(\gamma) = 2$ without loss of generality. Then $m_5(\gamma) = 2$ or $m_5(\gamma) = 1$. If $J = \{\alpha_3, \alpha_5\}$, then the proof is finished by applying σ to one of α_J, β_J, so we consider $J = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$.

The case of $J = \{\alpha_3, \alpha_5\}$ being settled, we can assume that $\alpha_{J'} + \beta_{J'} = \gamma_{J'}$, where $J' = \{\alpha_3, \alpha_5\}$. If $m_5(\gamma) = 2$, then $m_1(\gamma) = m_6(\gamma) = 1$. If one of the roots α, β has a coefficient ≥ 2, then, without loss of generality, $m_3(\alpha) = 2$ and $m_5(\alpha) = 1$, which implies $m_1(\alpha) = 1$ and $m_3(\beta) = 0$, $m_5(\beta) = 1$. Then $m_1(\beta) = 0$ and $m_6(\alpha) + m_6(\beta) = m_6(\gamma)$, so that $\alpha_J + \beta_J = \gamma_J$. If $m_3(\alpha) = m_5(\alpha) = m_3(\beta) = m_5(\beta) = 1$, we can use the case of $J = \{\alpha_1, \alpha_6\}$. The case where $m_3(\gamma) = 2$, $m_5(\gamma) = 1$ is completely similar. □

Lemma 5. *If a relative root $A \in \Phi_{J,\Gamma}$ is contained in an irreducible component of rank at least 2, then there exist noncollinear relative roots $B, C \in \Phi_{J,\Gamma}$ such that $A = B + C$. If $\Phi = G_2$, then B and C can be chosen so that $B - C \notin \Phi_{J,\Gamma}$.*

Proof. We can assume Φ is irreducible. First, consider the case where $\Phi = G_2$. Since $\operatorname{rank} \Phi_{J,\Gamma} \geq 2$, in this case $\Phi = \Phi_{J,\Gamma}$ and the relative roots coincide with the usual ones. Since the Weyl group transfers any root into a simple one, we can assume that A is a simple root of G_2. Then we take $B = \alpha_1 + \alpha_2$, $C = -\alpha_2$ if $A = \alpha_1$ is short, and $B = 3\alpha_1 + 2\alpha_2$, $C = -(3\alpha_1 + \alpha_2)$ if $A = \alpha_2$ is long.

Now, let $\Phi \neq G_2$. We can assume that A is a positive relative root, i.e., $\pi^{-1}(A) \subseteq \Phi^+$.

First, suppose that $A = k\pi(\alpha_r)$, where $\alpha_r \in \Pi$ is a simple root, $k > 0$. Let $\alpha_s \in J$ be a simple root that does not belong to the Γ-orbit of α_r and is the nearest to α_r on the Dynkin diagram among elements with this property. It is easily seen that for any $\alpha \in \pi^{-1}(A)$ there exists $\beta \in \pi^{-1}(\pi(\alpha_s))$ such that $(\alpha, \beta) < 0$ and thus $\alpha + \beta \in \Phi$. Indeed, for any $\alpha \in \pi^{-1}(A)$ we have $m_s(\alpha) = 0$ by the definition of π, so we can take β to be the sum of simple roots constituting the chain between α_s and the nearest simple root that appears in the decomposition of α with a nonzero coefficient. Now we can take $B = \pi(\alpha + \beta)$ and $C = \pi(-\beta)$. Since $\pi(\alpha) = k\pi(\alpha_r)$ and $\pi(-\beta) = -\pi(\alpha_s)$, the relative roots B and C are noncollinear.

Now, let $A \neq k\pi(\alpha_r)$. For any root $\alpha \in \pi^{-1}(A)$ there are roots $\beta_1, \dots, \beta_n \in \Pi$ such that $\alpha = \beta_1 + \cdots + \beta_n$ and for any $1 \leq i \leq n$ the sum $\beta_1 + \cdots + \beta_i$ is a root. Let i be the smallest possible index satisfying $\beta_{i+1}, \dots, \beta_n \in \Delta$. Then $\beta_i \in J$ and $\pi(\beta_1 + \cdots + \beta_{i-1} + \beta_i) = A$. Set $B = \pi(\beta_1 + \cdots + \beta_{i-1})$ and $C = \pi(\beta_i)$. The relative roots B and C are noncollinear, because otherwise we would have $A = k\pi(\beta_i)$ for some $k > 0$. □

§4. Relative root subschemes

Throughout this section, we assume that the patching groupoid consists of a unique isomorphism class; P is a fixed parabolic subgroup of G of type $\{J_\tau\}$. Then the maps $\pi_\tau \colon \mathrm{X}^*(T_\tau) \to \mathrm{Map}_\Gamma(J_\tau, \mathbb{Z})$ are transformed into each other by patching symmetries, so we can identify the corresponding systems of relative roots $\Phi_{J_\tau,\Gamma_\tau}$. We denote the resulting system by Φ_P.

Let $\Psi \subseteq \Phi_P$ be a *unipotent closed set* of relative roots, that is, a subset that contains the sum of any two of its elements (if this sum is a relative root) and does not contain any collinear oppositely directed relative roots. Then the set $\pi^{-1}(\Psi)$ is a unipotent closed subset of Φ in the usual sense.

We fix a Levi subgroup L_P of P. It is clear that the local *épinglages* τ adapted to P and L_P constitute an open covering of $\operatorname{Spec} R$. For any such τ we define $\mathrm{U}_{\Psi,\tau}$ to be the subgroup of G_{S_τ} generated by all $X_{\alpha,\tau}$, $\alpha \in \pi^{-1}(\Psi)$. Since any two *épinglages* τ and σ adapted to L_P are locally conjugate by an element of L_P, patching symmetries take $\mathrm{U}_{\Psi,\tau}$ to $\mathrm{U}_{\Psi,\sigma}$. Hence, the groups $\mathrm{U}_{\Psi,\tau}$ glue together into a global subgroup U_Ψ of G.

In particular, in this way we obtain closed subgroups $\mathrm{U}_{(A)}$ of G, where (A) is the set of all relative roots that are positive multiples of a relative root A. It is easily seen that $\mathrm{U}_{(iA)}$ is normal in $\mathrm{U}_{(A)}$ for any $i \geq 1$.

For any finitely generated projective R-module V, the functor $S \mapsto V \otimes_R S$ is represented by an affine group scheme $\mathrm{W}(V) = \operatorname{Spec} \operatorname{Sym}^*(V^*)$, where V^* is the dual R-module and Sym^* is the symmetric algebra. A morphism of *schemes* $\mathrm{W}(V_1) \to \mathrm{W}(V_2)$ is then determined by an element of $\operatorname{Sym}^*(V_1^*) \otimes_R V_2$. If this element lies in $\operatorname{Sym}^d(V_1^*) \otimes_R V_2$, we say that the morphism is *homogeneous of degree d*. In particular, the morphisms of degree 1 are linear morphisms.

Theorem 2. *For all relative roots $A \in \Phi_P$, there exist projective modules V_A over R and closed embeddings of schemes*

$$X_A\colon \mathrm{W}(V_A) \to G$$

such that, over any local épinglage τ adapted to P and L_P, the modules $V_A \otimes_R S_\tau$ are free, and if a basis $e_1, \dots, e_{k_A}$ of $V_A \otimes_R S_\tau$ is chosen, then the morphism X_A is given by

$$X_A(e_1a_1 + \cdots + e_{k_A}a_{k_A}) = \prod_{j=1}^{k_A} x_{\gamma_j}(a_j) \cdot \prod_{i\geq 2} \prod_{\pi(\beta)=iA} x_\beta(p^i_{\beta,\tau}(a_1, \dots, a_{k_A})), \tag{1}$$

where γ_j, $1 \leq j \leq k_A$, are all roots of $\pi^{-1}(A)$, and each $p^i_{\beta,\tau}$ is a homogeneous polynomial of degree i.

These morphisms enjoy the following properties.

1) $X_A(0) = 1$.
2) *For any $g \in \mathrm{L}_P$, we have*

$$gX_A(v)g^{-1} = \prod_{i\geq 1} X_{iA}(\varphi^i_{g,A}(v)),$$

where each $\varphi^i_{g,A}\colon \mathrm{W}(V_A) \to \mathrm{W}(V_{iA})$ is homogeneous of degree i.

3) *We have*

$$X_A(v)X_A(w) = X_A(v + w) \prod_{i>1} X_{iA}(q^i_A(v, w)),$$

where each $q^i_A\colon \mathrm{W}(V_A) \times_{\operatorname{Spec} R} \mathrm{W}(V_A) = \mathrm{W}(V_A \oplus V_A) \to \mathrm{W}(V_{iA})$ is homogeneous of degree i.

Proof. Over a local *épinglage* τ, we define $V_{A,\tau}$ as $S_\tau^{k_A}$, where $k_A = |\pi^{-1}(A)|$. Consider the morphisms of schemes

$$Y_{A,\tau}\colon \mathrm{W}(V_{A,\tau}) \to U_{A,\tau}$$

given by

$$Y_{A,\tau}(e_1a_1+\cdots+e_{k_A}a_{k_A})=\prod_j x_{\gamma_j}(a_j),$$

where the γ_j are all roots of $\pi^{-1}(A)$ in some order. The Chevalley commutator formula shows that over $\operatorname{Spec} S_\tau$ the morphisms $Y_{A,\tau}$ satisfy the analogs of properties 1 and 3, and also 2 for the elements of $(\mathrm{L}_P)_\tau$ belonging to the torus T_τ or to a root subgroup. Since over S_τ the big cell Ω_{L_P} is dense in L_P, property 2 holds true for all elements of $(\mathrm{L}_P)_\tau$.

Next, for any two local *épinglages* σ, τ we have

$$\iota_{\sigma\tau}(Y_{A,\tau}(v))=Y_{A,\sigma}(\varphi_{A,\sigma,\tau}(v)) \mod \mathrm{U}_{(2A)},$$

where $\varphi_{A,\sigma,\tau}\colon V_{A,\tau}\otimes_R S_\sigma\to V_{A,\sigma}\otimes_R S_\tau$ are linear maps. Clearly, they satisfy the cocycle condition, and hence glue the modules $V_{A,\tau}$ into a projective R-module V_A. This means that there are linear isomorphisms $\theta_{A,\tau}\colon V_A\otimes_R S_\tau\to V_{A,\tau}$ such that $\varphi_{A,\sigma,\tau}=\theta_{A,\sigma}\circ(\theta_{A,\tau})^{-1}$.

Now we use induction on j to construct sections $X^j_{A,\tau}\colon V_{A,\tau}\to \mathrm{U}_{A,\tau}$ such that

$$\iota_{\sigma\tau}(X^j_{A,\tau}(v))=X^j_{A,\sigma}(\varphi_{A,\sigma,\tau}(v)) \mod \mathrm{U}_{((j+1)A)},$$

where the $X^j_{A,\tau}$ are determined by (1). Then, for j sufficiently large, since the morphisms $X^j_{A,\tau}$ are affine, they glue into a morphism $X_A\colon \mathrm{W}(V_A)\to \mathrm{U}_{(A)}$ defined globally, and all the properties we need follow by descent from the Chevalley commutator formula.

Set $X^1_{A,\tau}=Y_{A,\tau}$. Suppose we have already defined $X^j_{A,\tau}$. We are looking for a map $X^{j+1}_{A,\tau}$ of the form

$$X^{j+1}_{A,\tau}(v)=X^j_{A,\tau}(v)Y_{(j+1)A,\tau}(\chi_\tau(v)),$$

where $\chi_\tau\colon V_{A,\tau}\to V_{(j+1)A,\tau}$ is a homogeneous polynomial map. We also want it to satisfy the relation

$$\begin{aligned}&\iota_{\sigma\tau}(X^j_{A,\tau}(v)Y_{(j+1)A,\tau}(\chi_\tau(v)))\\ &\qquad=X^j_{A,\sigma}(\varphi_{A,\sigma,\tau}(v))Y_{(j+1)A,\sigma}(\chi_\sigma(\varphi_{A,\sigma,\tau}(v))) \mod \mathrm{U}_{((j+2)A)},\end{aligned}$$

or equivalently,

$$\begin{aligned}&\iota_{\sigma\tau}(X^j_{A,\tau}(v))^{-1}X^j_{A,\sigma}(\varphi_{A,\sigma,\tau}(v))\\ &\qquad=Y_{(j+1)A,\sigma}(\varphi_{(j+1)A,\sigma,\tau}(\chi_\tau(v))-\chi_\sigma(\varphi_{A,\sigma,\tau}(v))) \mod \mathrm{U}_{((j+2)A)}.\end{aligned}$$

Let $\psi_{\sigma\tau}(v)$ denote a unique element of $V_{(j+1)A,\sigma}\otimes_R S_\tau$ satisfying

$$\iota_{\sigma\tau}(X^j_{A,\tau}(v))^{-1}X^j_{A,\sigma}(\varphi_{A,\sigma,\tau}(v))=Y_{(j+1)A,\sigma}(\psi_{\sigma\tau}(v)) \mod \mathrm{U}_{((j+2)A)}.$$

Then routine computations give

$$\psi_{\rho\tau}=\varphi_{(j+1)A,\rho,\sigma}\circ\psi_{\sigma\tau}+\psi_{\rho\sigma}\circ\varphi_{A,\sigma,\tau}.$$

Taking $a_{\sigma\tau}=\theta^{-1}_{(j+1)A,\sigma}\circ\psi_{\sigma\tau}\circ\theta_{A,\tau}$, we rewrite this as

$$a_{\rho\tau}=a_{\rho\sigma}+a_{\sigma\tau}.$$

Our covering is acyclic with coefficients in $\mathrm{W}(V_{(j+1)A})$, and $\mathrm{H}^1(\operatorname{Spec} R,\mathrm{W}(V_{(j+1)A}))=0$. Therefore, there exist functions b_τ such that

$$a_{\sigma\tau}=b_\tau-b_\sigma.$$

Now we set $\chi_\tau = \theta_{(j+1)A,\tau} \circ b_\tau \circ \theta_{A,\tau}^{-1}$, obtaining

$$\psi_{\sigma\tau} = \varphi_{(j+1)A,\sigma,\tau} \circ \chi_\tau - \chi_\sigma \circ \varphi_{A,\sigma,\tau}, \tag{2}$$

as we wanted.

It remains to prove that the maps χ_τ can be chosen so that they will be polynomial and homogeneous of degree $j+1$. The Chevalley commutator relations imply that this is so for $\psi_{\sigma\tau}$. We extend the base to the polynomial ring $R[Z_1, \dots, Z_{k_A}]$ and set $v_Z = e_1 Z_1 + \dots + e_{k_A} Z_{k_A} \in S_\tau[Z_1, \dots, Z_{k_A}]^{k_A}$. We have $\chi_\tau(v_Z) \in S_\tau[Z_1, \dots, Z_{k_A}]^{k_{(j+1)A}}$. We define $\chi'_\tau(v_Z)$ to be the element of $S_\tau[Z_1, \dots, Z_{k_A}]^{k_{(j+1)A}}$ whose nth component, $1 \le n \le k_{(j+1)A}$, is the homogeneous summand of the nth component of $\chi_\tau(v_Z)$ of degree $j+1$. Then for any $v = e_1 a_1 + \dots + e_{k_A} a_{k_A}$ we define $\chi'_\tau(v)$ to be the image of $\chi'_\tau(v_Z)$ under the specialization homomorphism

$$S_\tau[Z_1, \dots, Z_{k_A}] \to S_\tau, \qquad Z_1 \mapsto a_1, \dots, Z_{k_A} \mapsto a_{k_A}.$$

It is easily seen that identity (2) remains true with χ'_τ instead of χ_τ. This finishes the proof. □

Lemma 6. *The map*

$$X_\Psi \colon \mathrm{W}\Big(\bigoplus_{A\in\Psi} V_A\Big) \to \mathrm{U}_\Psi, \qquad (v_A)_A \mapsto \prod_A X_A(v_A),$$

where the product is taken in any fixed order respecting the level, is an isomorphism of schemes.

Proof. The statement is verified easily over any local *épinglage* with the help of (1), and the general case follows by descent. □

Lemma 7. *For any $A \in \Psi$, let $f_1^A, \dots, f_{n_A}^A$, $n_A \ge 1$, be a system of generators for V_A over R. Then for any ring extension $R \to S$ the group of points $U_\Psi(S)$ is generated as an abstract group by the elements $X_A(\xi f_i^A)$, $\xi \in S$, $A \in \Psi$, $1 \le i \le n_A$.*

Proof. This follows from item 3 in Theorem 2 and Lemma 6. □

From now on, we fix an ordering of the system of relative roots that respects the level. Then, for any unipotent closed set $\Psi \subseteq \Phi_P$, Lemma 6 allows us to define certain morphisms

$$p_{\Psi,A} \colon \mathrm{U}_\Psi \to \mathrm{W}(V_A)$$

("the coefficient" at the relative root A).

Lemma 8. *For any $g \in \mathrm{U}_\Psi(R)$ there exists $g(X) \in \mathrm{U}_\Psi(R[X])$ such that $g(0) = 1$ and $g(1) = g$.*

Proof. If $g = \prod_A X_A(v_A)$, we take $g(X) = \prod_A X_A(v_A X)$. □

§5. Chevalley commutator formulas

We keep the assumption that the patching groupoid consists of a unique isomorphism class, and that L_P is a fixed Levi subgroup of the parabolic subgroup P.

For any relative roots $A, B \in \Phi_P$, we denote by (A, B) the unipotent closed set of relative roots consisting of all linear combinations $iA + jB$, $i, j > 0$, that are in Φ_P.

Lemma 9. *Let A, B be relative roots satisfying $mA \ne -kB$ for any $m, k \ge 1$. Then the commutator subgroup $[X_A(V_A), X_B(V_B)]$ is contained in*

$$U_{(A,B)} = \prod_{i,j>0} X_{iA+jB}(V_{iA+jB}),$$

and each map

$$\begin{aligned} N_{ABij}\ :\ & V_A \times V_B \to V_{iA+jB}, \\ & (v_A, v_B) \mapsto p_{(A,B),iA+jB}([X_A(v_A), X_B(v_B)]) \end{aligned}$$

is homogeneous of degrees i and j in the first and the second arguments, respectively.

Proof. This follows by descent from Theorem 2 and the Chevalley commutator formula in the split case. □

Lemma 10. *Assume that $A, B, A+B \in \Phi_P$ and A, B are noncollinear.*

1) *If $A - B \notin \Phi_P$, then*

$$\operatorname{Im} N_{AB11} = V_{A+B}.$$

2) *If $A - B \in \Phi_P$ and $\Phi \neq G_2$, then*

$$\operatorname{Im} N_{AB11} + \operatorname{Im} N_{A-B,2B,1,1} + \sum_{v \in V_B} \operatorname{Im}(N_{A-B,B,1,2}(-, v)) = V_{A+B},$$

where $\operatorname{Im} N_{A-B,2B,1,1} = 0$ *if* $2B \notin \Phi_P$.

Proof. 1) By Lemma 4, the assumption implies that any $\gamma \in \pi^{-1}(A+B)$ decomposes as $\gamma = \alpha + \beta$, $\alpha \in \pi^{-1}(A)$, $\beta \in \pi^{-1}(B)$, where $\alpha - \beta$ is not a root. Then the commutator $[X_A(e_\alpha), X_B(e_\beta)]$, taken modulo U_Ψ, where $\Psi = (A, B) \setminus \{A+B\}$, is of the form $x_{\alpha+\beta}(\pm 1)$ over any member S_τ of the covering. Hence, $\operatorname{Im}(N_{AB11})_\tau = V_{A+B} \otimes S_\tau$. Since $\operatorname{Im} N_{AB11}$ is a submodule of V_{A+B} defined over the base ring, we have $\operatorname{Im} N_{AB11} = V_{A+B}$.

2) In the same way as in 1), we deduce that over any S_τ the module $\operatorname{Im}(N_{AB11})_\tau$ contains $\pm e_\gamma$, where γ is a short root in $\pi^{-1}(A+B)$. If γ is a long root in $\pi^{-1}(A+B)$, our assumptions and Lemma 4 imply the existence of $\alpha \in \pi^{-1}(A-B)$ and $\beta \in \pi^{-1}(B)$ such that $\gamma = \alpha + 2\beta$. Using item 3 of Theorem 2, we see that, over any S_τ, $[X_{A-B}(e_\alpha), X_B(e_\beta)]$ modulo U_Ψ, where $\Psi = (A-B, B) \setminus \{A+B\}$, equals

$$x_{\alpha+2\beta}(\pm 1) \cdot \prod_{\beta' \in 2B} [x_\alpha(1), x_{\beta'}(u_{\beta'})], \qquad u_{\beta'} \in S_\tau.$$

Consequently, $e_\gamma \in \operatorname{Im}(N_{A-B,B,1,2})_\tau(-, e_\beta) + \operatorname{Im}(N_{A-B,2B,1,1})_\tau$. We can represent e_β as

$$e_\beta = a_1 f_1 + \cdots + a_m f_m, \quad a_i \in S_\tau, \quad f_i \in V_B.$$

It is easily seen that, for any $\delta \in \pi^{-1}(A-B)$,

$$\begin{aligned} [X_{A-B}(e_\delta), X_B(e_\beta)] &= [X_{A-B}(e_\delta), \prod_i X_B(a_i f_i) \cdot X_{2B}(w)] \\ &= [X_{A-B}(e_\delta), \prod_i X_B(a_i f_i)] \cdot [X_{A-B}(e_\delta), X_{2B}(w)] \\ &= \prod_i [X_{A-B}(e_\delta), X_B(a_i f_i)] \prod_{i<j} [X_B(a_i f_i), [X_{A-B}(e_\delta), X_B(a_j f_j)]] \\ &\quad \cdot [X_{A-B}(e_\delta), X_{2B}(w)] \mod U_\Psi \end{aligned}$$

for some $w \in (V_{2B})_\tau$ (Theorem 2). Expanding commutators further, we see that $(N_{A-B,B,1,2})_\tau(e_\delta, e_\beta)$ lies in

$$\begin{aligned} &\sum_i (N_{A-B,B,1,2})_\tau(e_\delta, a_i f_i) + \operatorname{Im}(N_{AB11})_\tau + \operatorname{Im}(N_{A-B,2B,1,1})_\tau \\ &\qquad = \sum_i (N_{A-B,B,1,2})_\tau(a_i^2 e_\delta, f_i) + \operatorname{Im}(N_{AB11})_\tau + \operatorname{Im}(N_{A-B,2B,1,1})_\tau. \end{aligned}$$

Summarizing, we see that e_γ is in

$$\sum_i \operatorname{Im}(N_{A-B,B,1,2})_\tau(-,f_i) + \operatorname{Im}(N_{AB11})_\tau + \operatorname{Im}(N_{A-B,2B,1,1})_\tau,$$

which finishes the proof. □

Lemma 11. *Suppose that $A \in \Phi_P$ lies in an irreducible component of rank at least 2. Then for any $v \in V_A$ there exist relative roots $B_i, C_i \in \Phi_P$ noncollinear to A, elements $v_i \in V_{B_i}$, $u_i \in V_{C_i}$, and integers $k_i, l_i > 0$, $n_i \geq 0$ $(1 \leq i \leq m)$ such that for any $\xi, \eta \in R$ we have*[1]

$$X_A(\xi\eta^2 v) = \prod_{i=1}^{m} X_{B_i}(\xi^{k_i}\eta^{n_i}v_i)^{X_{C_i}(\eta^{l_i}u_i)}.$$

Proof. We view ξ, η as free variables generating a polynomial ring $R[\xi,\eta]$, and work with $R[\xi,\eta]$-points of the functors X_A, $A \in \Phi_P$, instead of R-points. The statement is then obtained by specializing ξ and η.

By Lemma 5, there are noncollinear relative roots $B, C \in \Phi_P$ such that $A = B + C$, and $B - C$ is not a relative root if $\Phi = G_2$. Then, by Lemmas 9 and 10 (the commutator formulas are still available over the extension $R[\xi,\eta]$ of R), the element $X_A(\xi\eta^2 v)$ is contained in the subgroup generated by

$$[X_B(\xi \cdot V_B), X_C(\eta^2 \cdot V_C)] \qquad \text{and} \qquad \prod_{\substack{i,j>0,\\(i,j)\neq(1,1)}} X_{iB+jC}(\xi^i\eta^{2j} \cdot V_{iB+jC}),$$

and also, if $B - C$ is a relative root, by

$$[X_{B-C}(\xi \cdot V_{B-C}), X_{2C}(\eta^2 \cdot V_{2C})], \qquad \prod_{\substack{i,j>0,\\(i,j)\neq(1,1)}} X_{i(B-C)+2jC}(\xi^i\eta^{2j} \cdot V_{i(B-C)+2jC}),$$

$$[X_{B-C}(\xi \cdot V_{B-C}), X_C(\eta \cdot V_C)], \qquad \prod_{\substack{i,j>0,\\(i,j)\neq(1,2)}} X_{i(B-C)+jC}(\xi^i\eta^{j} \cdot V_{i(B-C)+jC})$$

and, by property 3) in Theorem 2, by

$$\prod_{i>1} X_{iA}(\xi^i\eta^{2i} \cdot V_{iA}).$$

Since B and C are noncollinear, all relative roots involved are either noncollinear to A, or have the form iA, $i > 1$. Hence, we can use descending induction on $k = \operatorname{lev}(A)$. □

Now let $P \leq P'$ be two parabolic subgroups of G. Observe that if $\mathrm{L}_{P'}$ is a Levi subgroup of P', then P possesses a Levi subgroup L_P satisfying $\mathrm{L}_P \subseteq \mathrm{L}_{P'}$ ([12, Exp. XXVI, Prop. 1.20]). Suppose that τ is a local *épinglage* adapted to P and L_P and that P is of type $J = J_\tau \subseteq \Pi_\tau = \Pi$. Since $P \subseteq P'$, the local *épinglage* τ is *a fortiori* adapted to P' and $\mathrm{L}_{P'}$ (cf. [12, Exp. XXVI, Prop. 1.4]), and the parabolic subgroup P' is of type $J' \subseteq J$.

Lemma 12. *Let $P \leq P'$ be strictly proper parabolic subgroups of G. Then there exists $k > 0$ depending only on $\operatorname{rank}\Phi_P$ such that for any relative root $A \in \Phi_P$ and any $v \in V_A$ there exist relative roots $B_i, C_{ij} \in \Phi_{P'}$, elements $v_i \in V_{B_i}$, $u_{ij} \in V_{C_{ij}}$, and integers*

[1]We use exponential notation for conjugation, with $x^y = y^{-1}xy$.

$k_i, n_i, l_{ij} > 0$ $(1 \le i \le m,\ 1 \le j \le m_j)$ that satisfy

$$X_A(\xi\eta^k v) = \prod_{i=1}^{m} X_{B_i}(\xi^{k_i}\eta^{n_i}v_i)^{\prod_{j=1}^{m_i} X_{C_{ij}}(\eta^{l_{ij}}u_{ij})}.$$

for any $\xi, \eta \in R$. In particular, $\mathrm{E}_P(R) = \mathrm{E}_{P'}(R)$.

Proof. Let $\Theta \subseteq \Phi^+$ be the set of positive roots corresponding to the unipotent radical $\mathrm{U}_{P'}$. Clearly, $-\Theta$ corresponds to the unipotent radical $\mathrm{U}_{(P')^-}$. Then in the notation of §4 we have $\mathrm{U}_{P'} = \mathrm{U}_\Psi$, $\mathrm{U}_{(P')^-} = \mathrm{U}_{-\Psi}$, where $\Psi = \pi(\Theta) \subseteq \Phi_P^+$ is the corresponding set of relative roots.

Fix an order on Φ^+ in such a way that the induced order on $\Phi_{P'}^+$ respects the level. Without loss of generality, we take $A \in \Phi_P^+$. If $A \in \Psi$, then by Lemma 6 there are morphisms of schemes

$$\lambda_B = p_{\Phi_{P'}^+, B} \circ X_A : \mathrm{W}(V_A) \to \mathrm{W}(V_B), \qquad B \in \Phi_{P'}^+,$$

such that $X_A(u) = \prod_{B \in \Phi_{P'}^+} X_B(\lambda_B(u))$ for any $u \in V_A$, where the product is taken in the chosen order. The Chevalley commutator formulas and descent imply that the λ_B, $B \in \Phi_{P'}^+$, are homogeneous polynomial maps. Hence, for any $A \in \Psi$ (and similarly, for any $A \in (-\Psi)$) the statement of the lemma holds true with $u_{ij} = 0$, $1 \le i, j \le m$.

Now, consider the case where $A \notin \Psi$. The types J and J' of P and P' are both invariant under the group of automorphisms $\Gamma_\tau = \Gamma$, that is, are unions of some Γ-orbits of simple roots. Suppose first that $J \setminus J'$ consists of a unique Γ-orbit containing a simple root $\alpha_r \in \Pi$. Then $\Psi = \Phi_P^+ \setminus \mathbb{N}\,\pi(\alpha_r)$, so we can assume that $A \in \Phi_P^+$ is of the form $A = n\pi(\alpha_r)$, $n \in \mathbb{Z}$. Since P' is strictly proper, the rank of the irreducible component of Φ_P containing A is at least 2. Then our statement readily follows from Lemma 11 (with ξ replaced by $\xi\eta$) and the preceding case, because any root $B \in \Phi_P$ noncollinear to A automatically belongs to $\Psi \cup (-\Psi) = \Phi_P \setminus \mathbb{Z}\,\pi(\alpha_r)$.

Now if $J \setminus J'$ consists of more than one Γ-orbit, the proof is finished by induction, with the use of the fact that, for any Γ-invariant subset $J'' \subseteq \Pi$ such that $J' \subseteq J'' \subseteq J$, there exists a (strictly proper) parabolic subgroup P'' of G containing P and having type J'' ([12, Exp. XXVI, Lemma 3.8]). □

§6. Quillen–Suslin lemma and the proof of Theorem 1

We introduce some additional notation. For an ideal I of the ring R, we denote by $G(R, I)$ the kernel of the reduction homomorphism $G(R) \to G(R/I)$, by $\mathrm{U}_\Psi(R, I)$ the intersection $\mathrm{U}_\Psi(R) \cap G(R, I)$, by $\mathrm{E}_P(I)$ the subgroup generated by $\mathrm{U}_P(R, I)$ and $\mathrm{U}_{P^-}(R, I)$, and by $\mathrm{E}_P(R, I)$ the normal closure of $\mathrm{E}_P(I)$ in $\mathrm{E}_P(R)$. Also, for any maximal ideal M of the ring R, we denote by F_M the localization homomorphism $G(R) \to G(R_M)$.

Lemma 13. $\mathrm{E}_P(R[X]) \simeq \mathrm{E}_P(R[X], XR[X]) \rtimes \mathrm{E}_P(R)$.

Proof. The group $\mathrm{E}_P(R[X], XR[X])$ is normal in $\mathrm{E}_P(R[X])$ by definition, and its intersection with $\mathrm{E}_P(R)$ is trivial. So it suffices to prove that $\mathrm{U}_P(R)$ and $\mathrm{U}_P(R[X], XR[X])$ generate $\mathrm{U}_P(R[X])$. Obviously, we can assume that the patching groupoid consists of a unique isomorphism class. Then the statement follows from Lemma 7. □

Corollary. $\mathrm{E}_P(R[X]) \cap G(R[X], XR[X]) = \mathrm{E}_P(R[X], XR[X])$.

Proof. Take an element $g(X)$ of $\mathrm{E}_P(R[X]) \cap G(R[X], XR[X])$; it can be presented as $g_1(X) \cdot g_2$, where $g_1(X) \in \mathrm{E}_P(R[X], XR[X])$, $g_2 \in \mathrm{E}_P(R)$. Then $g_2 = g_1(0) \cdot g_2 = g(0) = 1$. □

Lemma 14. *Let $g(Z)$, $h(Z) \in G(R[Z])$ be such that $F_M(g(Z)) = F_M(h(Z))$ and $g(0) = h(0)$. Then there exists $s \in R \setminus M$ such that $g(sZ) = h(sZ)$.*

Proof. The corresponding statement for $\mathbb{A}^n$ is clear, and G is a closed subscheme of $\mathbb{A}^n$ for some n. □

From now on we assume the hypothesis of Theorem 1.

Lemma 15. *For any $g(Z) \in \mathrm{E}_P(R_M[Z], ZR_M[Z])$ there exist $h(Z) \in \mathrm{E}_P(R[Z], ZR[Z])$ and $s \in R \setminus M$ such that $F_M(h(Z)) = g(sZ)$.*

Proof. We can assume that the patching groupoid of G (over R) consists of a unique isomorphism class. Indeed, if the closed point M of $\operatorname{Spec} R$ lies in an open subset $U_\xi = \operatorname{Spec} R_\xi$ (see §2), and $h'(Z) \in \mathrm{E}_P(R_\xi[Z], ZR_\xi[Z])$ is an element mapped to $g(sZ)$, we can take an $h(Z)$ that is equal to $h'(Z)$ over U_ξ and to 1 over $\coprod_{\eta \neq \xi} U_\eta$.

The proof of Lemma 13 shows that $\mathrm{E}_P(R_M[Z])$ is generated by $\mathrm{E}_P(ZR_M[Z])$ and $\mathrm{E}_P(R_M)$. Hence it suffices to consider elements $g(Z)$ of the form $g_1 g_2(Z) g_1^{-1}$, where $g_1 \in \mathrm{E}_P(R_M)$ and $g_2(Z) \in \mathrm{E}_P(ZR_M[Z])$. Set $S = R \setminus M$. It is easily seen that for any $s' \in S$ there exists $s \in S$ such that $g_2(sZ)$ belongs to $F_M(\mathrm{E}_P(s'ZR[Z]))$. It remains to prove that there exists $s' \in S$ satisfying

$$g_1 F_M(\mathrm{E}_P(s'ZR[Z])) g_1^{-1} \subseteq F_M(\mathrm{E}_P(R[Z], ZR[Z])).$$

Instead, we prove that for any $s'' \in S$ there exists $s' \in S$ such that

$$g_1 F_M(\mathrm{E}_P(s'ZR[Z]))^{F_M(\mathrm{E}_P(s'R[Z]))} g_1^{-1} \subseteq F_M(\mathrm{E}_P(s''ZR[Z]))^{F_M(\mathrm{E}_P(s''R[Z]))}. \tag{3}$$

Then we can assume that g_1 is a root generator of $\mathrm{E}_P(R_M)$.

Let $P_{\min}$ be the minimal parabolic subgroup of G_{R_M} contained in P_{R_M}, and let $\Phi_{P_{\min}}$ be the corresponding system of relative roots. Lemma 12 implies that $\mathrm{E}_P(R_M) = \mathrm{E}_{P_{R_M}}(R_M)$ coincides with $\mathrm{E}_{P_{\min}}(R_M)$, so we can take $g_1 = X_A(v)$ for some $A \in \Phi_{P_{\min}}$, $v \in V_A$. Moreover, by Lemma 12, we have $X_A(tv) \in F_M(\mathrm{E}_P(R))$ for some $t = t(g_1) \in S$.

By Lemma 7, the group $F_M(\mathrm{E}_P(s'R[Z]))$ (respectively, $F_M(\mathrm{E}_P(s'ZR[Z]))$) is generated by the elements h_0 of the form $X_C(\xi s' F_M(e_{C,i}))$ (respectively, $X_C(\xi s' Z F_M(e_{C,i}))$), where $C \in \Phi_P$, $\xi \in R[Z]$, and the elements $e_{C,i}$ span $V_C \otimes R[Z]$ over $R[Z]$. To prove (3), it suffices to show that $g_1 h_0 g_1^{-1} \in F_M(\mathrm{E}_P(s''R[Z]))$ (respectively, $g_1 h_0 g_1^{-1} \in F_M(\mathrm{E}_P(s''ZR[Z]))^{F_M(\mathrm{E}_P(s''R[Z]))}$) for all generators h_0 with $\xi = 1$, because the general statement follows readily if we replace Z by ξZ.

Taking, in Lemma 12, $\xi = 1$, $\eta = s'$ (respectively, $\xi = Z$, $\eta = s'$), and $s' = (s''')^k$ for some $s''' \in S$, we can represent h_0 as a (finite) product of elements h of the form $X_B(s'''u)$ (respectively, $X_B(s'''Zu)^{\prod_i X_{D_i}(s'''w_i)}$), where $B \in \Phi_{P_{\min}}$, $u \in V_B \otimes R_M[Z]$, $D_i \in \Phi_{P_{\min}}$, $w_i \in V_{D_i} \otimes R_M[Z]$. Clearly, we can restrict ourselves to the elements h of the form $X_B(s'''u)$ (respectively, $X_B(s'''Zu)$). As above, by Lemma 12 we have $h \in F_M(\mathrm{E}_P(R[Z]))$ (respectively, $F_M(\mathrm{E}_P(ZR[Z]))$) as soon as s''' is divisible by a certain $r = r(h) \in S$.

Next, in the case where $mB \neq -kA$ for any $m, k \geq 1$, Lemma 9 obviously implies $g_1 h g_1^{-1} \in F_M(\mathrm{E}_P(s''R[Z]))$ (respectively, $g_1 h g_1^{-1} \in F_M(\mathrm{E}_P(s''ZR[Z]))$) if s''' is divisible by s'' and by certain powers of t and r. Consider the case where A and B are collinear.

By the assumption of Theorem 1, the rank of any irreducible component of $\Phi_{P_{\min}}$ is at least 2. Then, by Lemma 11, for any $u \in V_B \otimes R_M[Z]$ we can find relative roots $B_1, \ldots, B_m, C_1, \ldots, C_m \in \Phi_{P_{\min}}$ noncollinear to B (and hence to A), elements $v_i \in V_{B_i} \otimes R_M[Z]$, $u_i \in V_{C_i} \otimes R_M[Z]$, and integers k_i, $l_i > 0$, $n_i \geq 0$ ($1 \leq i \leq m$) such that

$$X_B(\xi\eta^2 u) = \prod_{i=1}^{m} X_{B_i}(\xi^{k_i}\eta^{n_i} v_i)^{X_{C_i}(\eta^{l_i} u_i)}$$

for any $\xi, \eta \in R_M[Z]$. Set $\xi = s_1$ (respectively, Zs_1), $\eta = s_2$, where both $s_1, s_2 \in S$ are divisible by sufficiently large powers of t and s'', and also by $p \in S$ such that $pu_i, pv_i \in R[Z]$ for any $1 \le i \le m$. Now, if $s''' \in S$ is divisible by $s_1 s_2^2$, then by Lemmas 9 and 12 we obtain

$$g_1 h g_1^{-1} \in F_M(\mathrm{E}_P(s'' R[Z]))^{F_M(\mathrm{E}_P(s'' R[Z]))} = F_M(\mathrm{E}_P(s'' R[Z]))$$

(respectively,

$$g_1 h g_1^{-1} \in F_M(\mathrm{E}_P(s'' Z R[Z]))^{F_M(\mathrm{E}_P(s'' R[Z]))}),$$

as required. □

Lemma 16. *For any $g(X) \in G(R[X])$ such that $F_M(g(X))$ lies in $\mathrm{E}_P(R_M[X])$, there exists $s \in R \setminus M$ satisfying $g(aX)g(bX)^{-1} \in \mathrm{E}_P(R[X])$ for all $a, b \in R$ such that $a \equiv b \mod s$.*

Proof. Consider the element $f(Z) = g(X(Y+Z))g(XY)^{-1} \in G(R[X,Y,Z])$. Observe that $F_M(f(Z)) \in \mathrm{E}_P(R_M[X,Y,Z])$ and $f(0) = 1$. By the corollary to Lemma 13, $F_M(f(Z))$ belongs to $\mathrm{E}_P(R_M[X,Y,Z], ZR_M[X,Y,Z])$. Now, by Lemma 15, there exists $h(Z) \in \mathrm{E}_P(R[X,Y,Z], ZR[X,Y,Z])$ and $s_1 \notin M$ such that $F_M(h(Z)) = F_M(f(s_1 Z))$. By Lemma 14, there is $s_2 \notin M$ such that $h(s_2 Z) = f(s_1 s_2 Z)$. Set $s = s_1 s_2$; then $g(X(Y+sZ))g(XY)^{-1}$ lies in $\mathrm{E}_P(R[X,Y,Z])$. Now we specialize Y and Z to obtain the statement we need. □

Lemma 17. *Let $g(X) \in G(R[X])$ be such that $g(0) \in \mathrm{E}_P(R)$, and suppose $F_M(g(X)) \in \mathrm{E}_P(R_M[X])$ for all maximal ideals M. Then $g(X) \in \mathrm{E}_P(R[X])$.*

Proof. For any maximal ideal M, we choose $s_M \notin M$ as in Lemma 16. Since the ideal generated by all s_M's is not contained in any maximal one, there is a partition of unity $1 = \sum_{i=1}^{N} s_{M_i} t_i$. We apply the Abel method of summation by parts: if a_j denotes the partial sum $\sum_{i=1}^{N-j} s_{M_i} t_i$, then $a_{j+1} \equiv a_j \mod s_{M_{N-j}}$, and we have

$$g(X) = \Big(\prod_{j=0}^{N-1} g(a_j X) g(a_{j+1} X)^{-1} \Big) g(0),$$

where all factors are in $\mathrm{E}_P(R[X])$. □

Proof of Theorem 1. Let Q be a parabolic subgroup of G distinct from P. Let $g \in \mathrm{E}_Q(R)$; we need to prove that $g \in \mathrm{E}_P(R)$. We may assume that $g \in \mathrm{U}_Q(R)$. Choose $g(X) \in \mathrm{U}_Q(R[X])$ as in Lemma 8, and let M be a maximal ideal of R. By [12, Exp. XXVI, Cor. 5.2 and Cor. 5.7], over R_M both parabolic subgroups P and Q contain some minimal parabolic subgroups $P_{\min}$ and $Q_{\min}$, and these subgroups are conjugate by an element $h \in \mathrm{E}_{P_{\min}}(R_M)$. Now, $F_M(g(X))$ lies in $\mathrm{U}_Q(R_M[X])$, and *a fortiori* in $\mathrm{U}_{Q_{\min}}(R_M[X])$. Hence, $hF_M(g(X))h^{-1}$, and then also $F_M(g(X))$, are inside the group $\mathrm{E}_{P_{\min}}(R_M[X])$, which coincides with $\mathrm{E}_P(R_M[X])$ by Lemma 12. Since $g(0) = 1$, Lemma 17 implies that $g(X)$ is in $\mathrm{E}_P(R[X])$. But $g = g(1)$, so g lies in $\mathrm{E}_P(R)$, and the theorem is proved. □

§7. Examples

1. Let D be an Azumaya algebra over R, of degree d. The group $G = \mathrm{GL}_{r+1}(D)$ is a reductive algebraic group of type $A_{(r+1)d-1}$ (more precisely, the functor $S \mapsto \mathrm{GL}_{r+1}(D \otimes S)$ is represented by a reductive group scheme G). The subgroup $P \le G$ consisting of upper triangular matrices is a parabolic subgroup of type $\{d, 2d, \ldots, (r+1)d\}$. The system Φ_P of relative roots with respect to P is a root system of type A_r. The module V_A corresponding to relative roots $A \in \Phi_P$ can be identified with D, so that the maps $N_{\varepsilon_i - \varepsilon_j, \varepsilon_j - \varepsilon_k, 11} \colon D \times D \to D$ coincide with multiplication in D. The elementary subgroup

$\mathrm{E}_P(R)$ coincides with the subgroup $\mathrm{E}_{r+1}(D) \subseteq \mathrm{GL}_{r+1}(D)$ generated by elementary matrices.

2. Let V be a projective module of rank $2n$ endowed with a nondegenerate quadratic form Q. Consider the group $\mathrm{O}(V, Q)$ of Q-invariant automorphisms of V. One can define (see, e.g., [16]) the Dickson map from $\mathrm{O}(V, Q)$ to $(\mathbb{Z}/2\mathbb{Z})_R$; if $2 \in R^*$, this map coincides with the usual determinant map. Its kernel $\mathrm{O}^+(V, Q)$ is a reductive algebraic group of type D_n. Suppose that V contains $r < n$ pairwise orthogonal hyperbolic pairs $(e_1, f_1), \dots, (e_r, f_r)$ (i.e., $Q(e_i) = Q(f_i) = 0$ and $Q(e_i + f_j) = \delta_{ij}$). Then the subgroup $P \leq \mathrm{O}^+(V, Q)$ of automorphisms that preserve the flag

$$\langle e_1 \rangle \leq \langle e_1, e_2 \rangle \leq \cdots \leq \langle e_1, \dots, e_r \rangle$$

is a parabolic subgroup of type $\{1, \dots, r\}$. The respective relative roots form a root system of type B_r. The module V_A corresponding to a relative root A can be identified with R if A is long, and with the orthogonal complement to $\langle e_1, \dots, e_r, f_1, \dots, f_r \rangle \subseteq V$ if A is short. If A, B, and $A + B$ are relative roots, the map N_{AB11} looks like this:

- $(u, v) \mapsto \pm(Q(u + v) - Q(u) - Q(v))$ if A and B are short;
- $(a, b) \mapsto \pm ab$ if A and B are long;
- $(a, v) \mapsto \pm av$ if A is long and B is short.

If A is a long root and if B is a short root such that $A+2B$ is also a root, then the map N_{AB12} takes $(a, v) \in V_A \times V_B$ to $\pm aQ(v)$. The elementary subgroup $\mathrm{E}_P(R)$ coincides with the group generated by the so-called Eichler–Siegel–Dickson transvections.

3. Let S be a quadratic *étale* extension of R, i.e., a twisted form of the algebra $R \times R$. Then S possesses an involution $x \mapsto \bar{x}$ obtained by twisting the involution $(a, b) \mapsto (b, a)$. The set of $^-$-stable elements of S coincides with R. The map $\mathrm{tr}\colon S \to R$, $x \mapsto x + \bar{x}$, is called the *trace map*.

Let V be a projective S-module of rank $n + 1$ endowed with a nondegenerate form H. The group $\mathrm{U}(V, H)$ of H-invariant automorphisms of V is a reductive group over R, of type ${}^2A_n$ (index 2 means that the group is of outer type; that is, the automorphism group of an object of the patching groupoid consists of two elements). Suppose that V contains $r \leq \frac{n}{2}$ pairwise orthogonal hyperbolic pairs $(e_1, f_1), \dots, (e_r, f_r)$ (i.e., $H(e_i, e_i) = H(f_i, f_i) = 0$ and $H(e_i, f_j) = \delta_{ij}$). The subgroup $P \leq \mathrm{U}(V, H)$ of automorphisms that preserve the flag

$$\langle e_1 \rangle \leq \langle e_1, e_2 \rangle \leq \cdots \leq \langle e_1, \dots, e_r \rangle$$

is a parabolic subgroup of type $\{1, \dots, r, n-r+1, \dots, n\}$. The relative roots form a root system of type BC_r. The module V_A corresponding to a relative root A can be identified with S if A is short, with the orthogonal complement to $\langle e_1, \dots, e_r, f_1, \dots, f_r \rangle \subseteq V$ if A is extra short, and with $\ker \mathrm{tr}$ if A is long. If A, B, and $A + B$ are relative roots, the map N_{AB11} looks like this:

- $(u, v) \mapsto \pm H(u, v)$ if A and B are extra short;
- $(a, b) \mapsto \pm ab$ if A, B and $A + B$ are short;
- $(a, v) \mapsto \pm av$ if A is short, B is extra short;
- $(a, b) \mapsto \pm(ab - \bar{b}\bar{a})$ if A and B are short, $A + B$ is long;
- $(a, b) \mapsto \pm ab$ if A is long and B is short;
- $(u, v) \mapsto \pm(H(u, v) - H(v, u))$ if $A = B$ is extra short.

If $A + 2B$ is a relative root, then the map N_{AB12} looks like this:

- $(a, b) \mapsto \pm\bar{b}ab$ if A is long and B is short;
- $(a, v) \mapsto \pm a\sigma(H(v, v))$, where σ is a certain fixed section of tr, if A is short and B is extra short.

4. Recall that an algebra is said be *alternative* if any two elements generate an associative subalgebra. A *Cayley algebra* over a ring R is an alternative algebra C with 1 endowed with an involution $x \mapsto \bar{x}$ and such that C is a projective R-module of constant rank 8, and the *norm map* $n(x) = \bar{x}x = x\bar{x}$ takes values in R and is a nondegenerate quadratic form on C. Then the *trace map* $t(x) = x + \bar{x}$ on C also takes values in R.

Given a Cayley algebra C and three invertible scalars $\gamma_1, \gamma_2, \gamma_3 \in R$, we can construct the cubic Jordan algebra $J = \mathcal{H}_3(C, \gamma_1, \gamma_2, \gamma_3)$ consisting of the matrices

$$\begin{pmatrix} \xi_1 & c_3 & \gamma_1^{-1}\gamma_3\bar{c}_2 \\ \gamma_2^{-1}\gamma_1\bar{c}_3 & \xi_2 & c_1 \\ c_2 & \gamma_3^{-1}\gamma_2\bar{c}_1 & \xi_3 \end{pmatrix} \tag{4}$$

with $c_1, c_2, c_3 \in C$ and $\xi_1, \xi_2, \xi_3 \in R$. In particular, there is a *norm* on J, which is a cubic map $N\colon J \to R$; the norm of the matrix (4) equals

$$\xi_1\xi_2\xi_3 - \gamma_3^{-1}\gamma_2\xi_1 n(c_1) - \gamma_1^{-1}\gamma_3\xi_2 n(c_2) - \gamma_2^{-1}\gamma_1\xi_1 n(c_3) + t(c_1c_2c_3).$$

See [17] or [8] for the details.

Set

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that e_1, e_2, e_3 are pairwise orthogonal idempotents in J with sum 1. We denote by $c[ij]$, where $c \in C$ and $1 \leq i \neq j \leq 3$, the matrix of the form (4) with $\gamma_j c$ at the position $(i,\, j)$ and zeros at all positions distinct from $(i,\, j)$ and $(j,\, i)$.

The functor

$$S \mapsto \{g \in \mathrm{GL}(J \otimes S) \mid N(gx) = N(x) \text{ for all } x \in J \otimes S',\, S \subseteq S'\}$$

is represented by a semisimple group scheme G of type E_6. The subgroup $P \leq G$ of automorphisms that preserve the flag

$$\mathbf{0} \leq \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq J$$

is a parabolic subgroup of type $\{\alpha_1, \alpha_6\}$ (cf. [13]). The first nontrivial member of the flag is spanned by e_1, and the second coincides with the summand $J_0(e_3)$ of the Pierce decomposition induced by e_3 (i.e., with the set of elements of J cancelled by e_3).

The corresponding relative roots form a root system of type A_2, where $\Phi_P = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$ in the notation of [11]. The module V_A corresponding to a relative root A can be identified with C. Then the element $X_{\varepsilon_i - \varepsilon_j}(c)$ is the "algebraic transvection" $T_{\gamma_j^{-1}c[ij], e_j}$, and the map $N_{\varepsilon_i - \varepsilon_j, \varepsilon_j - \varepsilon_k, 11}\colon C \times C \to C$ coincides with multiplication in C (see, e.g., [8, (v)]). The elementary subgroup $\mathrm{E}_P(R)$ is the subgroup generated by all algebraic transvections.

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