Алгебра и анализ
Том 20 (2008), № 6

St. Petersburg Math. J.
Vol. 20 (2009), No. 6, Pages 955–981
S 1061-0022(09)01080-2
Article electronically published on October 2, 2009

OVERGROUPS OF F_4 IN E_6 OVER COMMUTATIVE RINGS

A. YU. LUZGAREV

Abstract. Overgroups of the elementary Chevalley group of type F_4 in the Chevalley group of type E_6 over an arbitrary commutative ring are described.

§1. Introduction

In the present paper we describe overgroups of the elementary Chevalley group of type F_4 in the Chevalley group of type E_6 over an arbitrary commutative ring. A systematic study of similar questions for the classical groups over fields was started by Roger Dye, Oliver King, and Li Shang Zhi; see [22, 23, 24, 29, 30, 33]. The description of overgroups for orthogonal, symplectic and unitary groups in the general linear group over a commutative ring was obtained by Vavilov and Petrov in [9, 10, 11, 35] and by Hong You in [27, 28].

The papers listed above deal with overgroups of classical groups in irreducible representations, or equivalently, with subgroups of $\mathrm{GL}(n, R)$ containing a given Chevalley group. It would be useful to transfer these results to the exceptional groups over commutative rings. In [12], the author presented some steps towards a description of overgroups for Chevalley groups of types E_6 and E_7 in their irreducible representations.

In [5, 8] it was observed that, in order to study Chevalley groups of type F_4, instead of the minimal 26-dimensional representation, it is often convenient to use the *reducible* 27-dimensional representation that arises as a result of twisting the minimal module of the group of type E_6. Thus, we have an inclusion $G_{\mathrm{sc}}(\mathrm{F}_4, R) \le G_{\mathrm{sc}}(\mathrm{E}_6, R)$, and it is natural to ask about intermediate subgroups in this setting.

Several technicalities arise when we try to adjust the proofs in [10] to exceptional groups, but the core remains the same, and so does the result: a "fan subgroup" descripton in the spirit of Borevich. More precisely, for any subgroup H lying between $E(\mathrm{F}_4, R)$ and $G(\mathrm{E}_6, R)$ (and viewed as subgroups in $\mathrm{GL}(27, R)$) there is a unique ideal A in R such that H lies between the group $\mathrm{E}(\mathrm{F}_4, R, A) = E(\mathrm{F}_4, R)E(\mathrm{E}_6, R, A)$ and its normalizer in $G(\mathrm{E}_6, R)$.

Theorem 1. *Let R be any commutative ring. For any subgroup H in $G = G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$, there exists a unique ideal $A \trianglelefteq R$ such that*

$$\mathrm{E}(\mathrm{F}_4, R, A) \le H \le N_G(\mathrm{E}(\mathrm{F}_4, R, A)).$$

Moreover, we compute the normalizer in question. Consider the extended Chevalley group

$$\bar{G}(\mathrm{F}_4, R) = G(\mathrm{F}_4, R)\,\mathrm{Cent}(G(\mathrm{E}_6, R))$$

2000 *Mathematics Subject Classification.* Primary 20H25.

Key words and phrases. Chevalley group, exceptional groups over commutative rings, localization-completion method.

Supported by the joint program "Mikhail Lomonosov" of DAAD and the Russian Ministry of Education, and by INTAS (grant no. 03-51-3251).

(where $\mathrm{Cent}(G)$ denotes the center of a group G).

For any two subgroups E, F in a group G, let $\mathrm{Tran}_G(E,F)$ denote the *transporter* of E to F:

$$\mathrm{Tran}_G(E,F) = \{g \in G \mid E^g \leq F\}.$$

Another result of this paper is the following theorem.

Theorem 2. *Under the hypothesis of Theorem* 1*, we have*

$$N_G(E(\mathrm{F}_4,R)) = N_G(G(\mathrm{F}_4,R)) = \mathrm{Tran}_G(E(\mathrm{F}_4,R), G(\mathrm{F}_4,R)) = \bar{G}(\mathrm{F}_4,R).$$

Now, consider the reduction homomorphism $\rho_A^{\mathrm{E}_6} : G(\mathrm{E}_6,R) \to G(\mathrm{E}_6,R/A)$ induced by a homomorphism of the corresponding general linear groups, and let $\mathrm{CG}(\mathrm{F}_4,R,A)$ denote the full preimage of $\overline{G}(\mathrm{F}_4,R/A)$ under $\rho_A^{\mathrm{E}_6}$.

Theorem 3. *Under the hypothesis of Theorem* 1*, for any ideal* $A \trianglelefteq R$ *we have*

$$N_G(\mathrm{E}(\mathrm{F}_4,R,A)) = \mathrm{CG}(\mathrm{F}_4,R,A).$$

The matrices in $\mathrm{CG}(\mathrm{F}_4,R,A)$ are determined by certain congruences modulo A. We make these congruences explicit in Proposition 2.

Technically, this work combines the *localization-completion* method, introduced by A. Bak in [19] and simplified by Hazrat and Vavilov in [25, 26, 10], with explicit computations in exceptional groups in their minimal representations, as developed by Vavilov and his students; see [4, 5, 6, 7, 8, 12].

This paper is organized as follows. In §§2 and 3 we recall the basic definitions pertaining to Chevalley groups of types E_6 and F_4 in their 27-dimensional representation. In §4 we reproduce some general facts about these groups. In §5 we prove technical results concerning the equations that define $G(\mathrm{E}_6,R)$. In §6 we describe certain parabolic subgroups in $G(\mathrm{E}_6,R)$ and $G(\mathrm{F}_4,R)$, together with their unipotent radicals. In §7 we prove Theorem 2 and describe equations on elements of the extended Chevalley group $\overline{G}(\mathrm{F}_4,R)$. The notion of the lower level for the subgroups in question is introduced in §8. In §9 we prove Theorem 3 and describe equations that determine the normalizer appearing in that theorem. The technical core of the proof of Theorem 1 is §10: localization helps us to simplify the extraction of a root element. The next three sections are devoted to this exrtraction: we gradually weaken conditions that ensure the existence of a root element. After that, the proof of Theorem 1 is finished easily in §14.

I am grateful to Nikolaĭ Aleksandrovich Vavilov, without whom this paper would not have been written, and to Antony Bak for hospitality and valuable support.

§2. Chevalley group of type E_6

In the present section we recall the basic notions and facts pertaining to Chevalley groups of type E_6 over commutative rings. Further information and references can be found in [1, 13, 34, 36, 15, 40, 41, 42, 4, 6, 7]. However, we wish to discuss more thoroughly the inclusion of a group of type F_4 in a group of type E_6, which is far less covered in the literature.

We consider the root systems E_6 and F_4 with fixed sets of fundamental roots $\Pi(\mathrm{E}_6)$ and $\Pi(\mathrm{F}_4)$. Our numbering of roots follows [2]. From now on, let R be a commutative ring with 1.

Our computations use an action of the Chevalley group $G = G(\mathrm{E}_6,R)$ of type E_6 on a module $V = V(\varpi_1)$ with the highest weight $\omega = \varpi_1$. Let Λ be the set of weights for this module. Then V has a crystalic base v^λ, $\lambda \in \Lambda$. This means (see [34]) that each v^λ

is a weight vector, and the action of the elementary root unipotents $x_\alpha(\xi)$ for $\alpha \in \mathrm{E}_6$, $\xi \in R$, can be written as follows:

$$x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha}.$$

Moreover, every structure constant $c_{\lambda\alpha}$ of this action is equal to ± 1, and $c_{\lambda\alpha} = +1$ for $\alpha \in \pm\Pi(\mathrm{E}_6)$. Let $T(\mathrm{E}_6, R)$ be a split maximal torus in $G(\mathrm{E}_6, R)$; in our representation, a matrix in $G(\mathrm{E}_6, R)$ belongs to $T(\mathrm{E}_6, R)$ if and only if it is diagonal. The group $T(\mathrm{E}_6, R)$ is Abelian and is generated by the following elements:

$$h_\alpha(\varepsilon) = e + \sum_{\lambda,\lambda+\alpha\in\Lambda} ((\varepsilon - 1)e_{\lambda+\alpha,\lambda+\alpha} + (\varepsilon^{-1} - 1)e_{\lambda,\lambda})$$

for every $\alpha \in \Pi(\mathrm{E}_6)$, $\varepsilon \in R^*$.

It is easily seen that

$${}^{h_\alpha(\varepsilon)}x_\beta(\xi) = x_\beta(\varepsilon^{\langle\alpha,\beta\rangle}\xi)$$

for all $\alpha \in \Pi$, $\beta \in \Phi$, $\varepsilon \in R^*$, $\xi \in R$. Recall that $\langle\alpha, \beta\rangle = 2(\alpha, \beta)/(\beta, \beta)$, where $(\cdot, \cdot)$ is a $W(\mathrm{E}_6)$-invariant scalar product on $P \otimes_{\mathbb{Z}} \mathbb{R}$, and in our case, $P = P_{\mathrm{sc}}$ is a weight lattice.

The paper [7] was devoted to a detailed study of the module V. In particular, the tables of signs of the structure constants in the crystalic base can be found there. Those tables are used extensively in our computations. As was noted in [17, 18, 6], a principal instrument for studying the module V is the invariant trilinear form. This form was described in [7]. One can look at $G(\mathrm{E}_6, R)$ from several different viewpoints: the classical definition says that it is the group of points of a certain affine group scheme (the Chevalley–Demazure scheme). But for specific matrix computations it is easier to use the fact that for any commutative ring R the group $G(\mathrm{E}_6, R)$ coincides with the group of isometries of a trilinear form T. The description of this form, and of the associated cubic form together with its partial derivatives can be found in [7] (see also [40, 41, 42, 8, 6]). We reproduce the cubic form and its partial derivatives in §5.

Moreover, $G(\mathrm{E}_6, R)$ coincides with the group of isometries of the *cubic form* Q mentioned above. This result is nontrivial (see [17]), especially if the invertibility of 2 and 3 in R is not assumed: note that $T(u, u, u) = 6Q(u)$.

In Figure 1 we reproduce the *weight diagram* of the module V. Its vertices are labeled by the weights of V, while each edge connects a pair of weights if their difference is a fundamental root. The edge is then labeled by this fundamental root, and the greatest of two weights is put on the left-hand side.

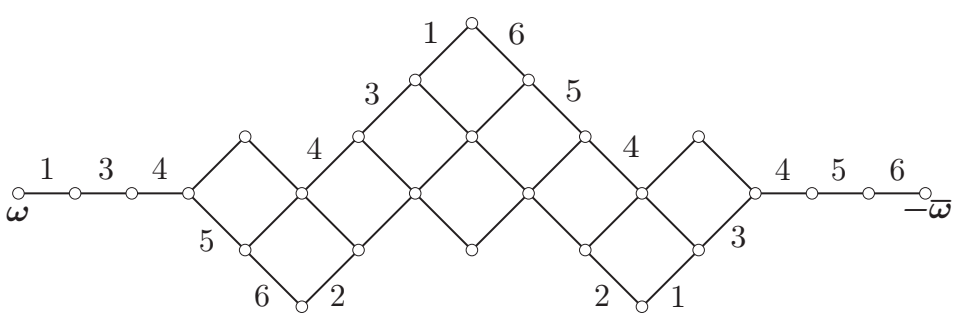

FIGURE 1

We view a vector $a \in V$, $a = \sum a_\lambda v^\lambda$, as a *column* of coordinates in the crystalic base: $a = (a_\lambda)$, $\lambda \in \Lambda$. At the same time, an element b of the contragredient module V^* can naturally be regarded as a *row* $b = (b_\lambda)$, $\lambda \in \Lambda$. Note that the coordinates of a vector

in V^* are indexed by weights of the module V; that is why they are written as rows. We identify an element $g \in G(\mathrm{E}_6, R)$ with its image with respect to the representation π and write it as a matrix $(g_{\lambda\mu})$ belonging to $\mathrm{GL}(V) \simeq \mathrm{GL}(27, R)$ with rows and columns indexed by the set of weights Λ. We denote by $g_{*\mu}$ the column of this matrix with index $\mu \in \Lambda$. In other words, $g_{*\mu} = gv^\mu$. Similarly, we denote by $g_{\lambda *}$ the row of the matrix g with index $\lambda \in \Lambda$. The entries of the inverse matrix of g are denoted as follows: $g^{-1} = (g'_{\lambda\mu})$, $\lambda, \mu \in \Lambda$.

The following lemma is a simple reformulation of the definition of a crystalic base, and we shall use it in the computations without any special mention.

Lemma 1. *For any $g \in \mathrm{GL}(27, R)$, $\alpha \in \mathrm{E}_6$, $\xi \in R$, we have*

$$(x_\alpha(\xi)g)_{\lambda\mu} = g_{\lambda\mu} + c_{\lambda-\alpha,\alpha}\xi g_{\lambda-\alpha,\mu}, \quad (gx_\alpha(\xi))_{\lambda\mu} = g_{\lambda\mu} + c_{\mu\alpha}\xi g_{\lambda,\mu+\alpha}.$$

Often we do not need to know the exact signs of the constants $c_{\lambda\alpha}$, and then we use the symbol '$\pm$' instead of the sign.

§3. Chevalley group of type F_4

From now on we assume (unless otherwise stated) that $\Phi = \mathrm{F}_4$, Φ_l is the set of long roots, and Φ_s is the set of short roots in Φ. We realize the root system F_4 as the projection of the root system E_6 to the four-dimensional subspace spanned by the vectors $\begin{smallmatrix}00000\\1\end{smallmatrix}$, $\begin{smallmatrix}00100\\0\end{smallmatrix}$, $\begin{smallmatrix}01010\\0\end{smallmatrix}$, $\begin{smallmatrix}10001\\0\end{smallmatrix}$. Then the long roots of F_4 are precisely the roots of E_6 lying in this subspace. Such a root must have the form $\begin{smallmatrix}abcba\\d\end{smallmatrix} \in \mathrm{E}_6$, and from the F_4 point of view it is the root $d\alpha_1 + c\alpha_2 + 2b\alpha_3 + 2a\alpha_4 \in \Phi_l$ (where the α_i, $1 \le i \le 4$, are the fundamental roots of F_4). Therefore, we may assume that $\Phi_l \subset \mathrm{E}_6$ (note that Φ_l is a root system of type D_4). On the other hand, a short root of F_4 is the projection of two roots of E_6 to our four-dimensional subspace: the roots $\begin{smallmatrix}abcb'a'\\d\end{smallmatrix}$ and $\begin{smallmatrix}a'b'cba\\d\end{smallmatrix}$ are projected to

$$d\alpha_1 + c\alpha_2 + (b + b')\alpha_3 + (a + a')\alpha_4 \in \Phi_s.$$

Let β_i, $1 \le i \le 6$, denote the short roots of E_6; we recall that our numbering follows [2]. Consider the outer automorphism $\alpha \mapsto \overline{\alpha}$ of order 2 of the root system E_6 that permutes β_1 with β_6 and β_3 with β_5, leaving β_2 and β_4 invariant. The one-element orbits of this automorphism consist precisely of the long roots of F_4, while each two-element orbit contains two roots that project to a short root of F_4. Note that the roots $\beta \ne \overline{\beta}$ in a two-element orbit are orthogonal to each other and form angles of $\pi/4$ with the corresponding short root $(\beta + \overline{\beta})/2 \in \Phi_s$. We shall identify the set of orbits with the set of roots F_4.

We denote by $x_\beta(\xi)$ the elementary root unipotents of the group $G(\mathrm{E}_6, R)$, and by $X_\beta(\xi)$ the elementary root unipotents of the group $G(\mathrm{F}_4, R)$. Each element $X_\beta(\xi)$ is equal to $x_\beta(\xi)$ for $\beta = \overline{\beta}$ (long root unipotent), or to $x_\beta(\xi)x_{\overline{\beta}}(\pm\xi)$ for $\beta \ne \overline{\beta}$ (short root unipotent). We shall use the explicit signs in the short root unipotents:

$$X_{0001}(\xi) = x_{\substack{10000\\0}}(\xi)x_{\substack{00001\\0}}(\xi), \qquad X_{0010}(\xi) = x_{\substack{01000\\0}}(\xi)x_{\substack{00010\\0}}(\xi),$$

$$X_{0011}(\xi) = x_{\substack{11000\\0}}(\xi)x_{\substack{00011\\0}}(-\xi), \qquad X_{0110}(\xi) = x_{\substack{01100\\0}}(\xi)x_{\substack{00110\\0}}(-\xi),$$

$$X_{1110}(\xi) = x_{\substack{01100\\1}}(\xi)x_{\substack{00110\\1}}(-\xi), \qquad X_{0111}(\xi) = x_{\substack{11100\\0}}(\xi)x_{\substack{00111\\0}}(\xi),$$

$$X_{1111}(\xi) = x_{\underset{1}{11100}}(\xi)x_{\underset{1}{00111}}(\xi), \qquad X_{0121}(\xi) = x_{\underset{0}{11110}}(\xi)x_{\underset{0}{01111}}(-\xi),$$

$$X_{1121}(\xi) = x_{\underset{1}{11110}}(\xi)x_{\underset{1}{01111}}(-\xi), \qquad X_{1221}(\xi) = x_{\underset{1}{11210}}(\xi)x_{\underset{1}{01211}}(-\xi),$$

$$X_{1231}(\xi) = x_{\underset{1}{12210}}(\xi)x_{\underset{1}{01221}}(-\xi), \qquad X_{1232}(\xi) = x_{\underset{1}{12211}}(\xi)x_{\underset{1}{11221}}(\xi).$$

The split maximal torus $T(\mathrm{F}_4, R)$ of the group $G(\mathrm{F}_4, R)$ is generated by the following diagonal elements:

$$H_{1000}(\varepsilon) = h_{\underset{1}{00000}}(\varepsilon), \qquad H_{0100}(\varepsilon) = h_{\underset{0}{00100}}(\varepsilon),$$

$$H_{0010}(\varepsilon) = h_{\underset{0}{01000}}(\varepsilon)h_{\underset{0}{00010}}(\varepsilon), \qquad H_{0001}(\varepsilon) = h_{\underset{0}{10000}}(\varepsilon)h_{\underset{0}{00001}}(\varepsilon).$$

When we restrict the representation π from $G(\mathrm{E}_6, R)$ to $G(\mathrm{F}_4, R)$, we obtain the 27-dimensional representation with the weight diagram depicted in Figure 2. Here the edges are labeled by the fundamental roots of F_4.

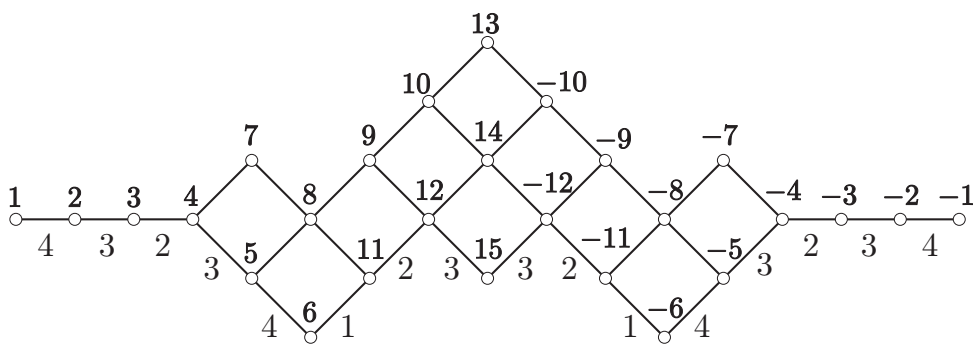

FIGURE 2

We shall use this numeration of weights in all our computations. For diversity, this numeration (and even its positive part) does not coincide with any of the three numerations listed in [7]. A weight labeled by an integer i on this diagram is denoted by λ_i or (if this does not cause ambiguity) simply by i. Note that after restriction to F_4 the weights 13, 14, and 15 become zero weights.

The representation $(\mathrm{E}_6, \varpi_1) \downarrow \mathrm{F}_4$ is reducible: it is the direct sum of a 26-dimensional representation on the short roots and a trivial 1-dimensional representation. Moreover, we need to consider the restriction $(\mathrm{E}_6, \varpi_1) \downarrow \mathrm{D}_4$. To visualize it, one should remove all edges labeled by 1 and 6 from the diagram (E_6, ϖ_1). Combining the restrictions to F_4 and to D_4, we obtain the restriction to B_3, which corresponds to removal of all edges labeled by 4. It is easily seen that the result is the direct sum of three one-dimensional representations (corresponding to the weights $\lambda_1 = \omega$, $\lambda_{-1} = -\overline{\omega}$, and λ_{13}), and three eight-dimensional representations (one of them is reducible and is the direct sum of a seven-dimensional and a one-dimensional representation). Let B, Γ, Δ be the sets of weights of these representations:

$$\mathrm{B} = \{2, 3, 4, 5, 7, 8, 9, 10\},$$
$$\Gamma = \{6, 11, 12, 14, 15, -12, -11, -6\},$$
$$\Delta = \{-10, -9, -8, -7, -5, -4, -3, -2\}.$$

As has already been noted, $G(\mathrm{E}_6, R)$ coincides with the group of transformations of the free right module R^{27} that preserve the trilinear form T. It is convenient to view the group $G(\mathrm{F}_4, R) < G(\mathrm{E}_6, R)$ as the group of transformations of class $G(\mathrm{E}_6, R)$ that stabilize a certain vector u with $Q(u) \neq 0$. Equivalently, $G(\mathrm{F}_4, R)$ is the group of transformations of class $G(\mathrm{E}_6, R)$ that preserve a bilinear form $B(x, y)$ defined by $B(x, y) = T(u, x, y)$. We can choose $u = v^{13} - v^{14} + v^{15}$, so that $Q(u) = -1$, while the bilinear form is expressed as

$$B(x,y) = \sum_{i=1}^{12} (-1)^{i+1}(x_i y_{-i} + x_{-i} y_i) + x_{13}y_{14} + x_{14}y_{13} + x_{14}y_{15} + x_{15}y_{14} - x_{13}y_{15} - x_{15}y_{13}.$$

Let F denote the Gram matrix of the bilinear form B. We see that a matrix $g = (g_{ij}) \in G(\mathrm{E}_6, R)$ belongs to $G(\mathrm{F}_4, R)$ if and only if $gFg^T = F$ (here, as usual, g^T is the transpose of g). This means that $G(\mathrm{F}_4, -)$ is a subscheme in $G(\mathrm{E}_6, -)$; a matrix $g \in G(\mathrm{E}_6, R)$ belongs to $G(\mathrm{F}_4, R)$ if and only if $(Fg^T)_{ij} = (g^{-1}F)_{ij}$ for all $i, j = 1, \dots, -1$. In particular, when $i, j = 1, \dots, 12, -12, \dots, -1$, these equations say that $g'_{ij} = \varepsilon_i \varepsilon_j g_{-j,-i}$.

We also need to consider the group of *similarities* of the bilinear form B, that is, the group of transformations $g \in G(\mathrm{E}_6, R)$ such that $B(gx, gy) = \lambda(g)B(x, y)$ for some $\lambda(g) \in R^*$. In terms of the Gram matrix, this condition can be rewritten as $gFg^T = \lambda(g)F$. This group is denoted by $\overline{G}(\mathrm{F}_4, R)$. Let $g \in \overline{G}(\mathrm{F}_4, R)$. For any $x, y \in R^{27}$ we have

$$T(u,x,y) = B(x,y) = \lambda(g)^{-1}B(gx,gy) = \lambda(g)^{-1}T(u,gx,gy) = \lambda(g)^{-1}T(g^{-1}u,x,y),$$

whence $T(\lambda(g)u - g^{-1}u, x, y) = 0$. Therefore, $g^{-1}u = \lambda(g)u$, which means that g maps the one-dimensional subspace $\langle u \rangle$ into itself. Since the converse is also true, $\overline{G}(\mathrm{F}_4, R)$ can be described as the group of matrices in $G(\mathrm{E}_6, R)$ that stabilize $\langle u \rangle$.

Lemma 2. *If $gu = \lambda u$ for some $g \in G(\mathrm{E}_6, R)$, $\lambda \in R$, then $\lambda^3 = 1$.*

Proof.

$$-1 = Q(u) = Q(gu) = Q(\lambda u) = \lambda^3 Q(u) = -\lambda^3. \qquad \square$$

Therefore, if R contains no nontrivial cubic roots of 1, then $\overline{G}(\mathrm{F}_4, R) = G(\mathrm{F}_4, R)$. But if $\lambda \in R$ and $\lambda^3 = 1$, then the matrix λI_{27} belongs to the center of $G(\mathrm{E}_6, R)$ and, at the same time, to $\overline{G}(\mathrm{F}_4, R)$; moreover, $\mathrm{Cent}(G(\mathrm{E}_6, R))$ consists of these scalar matrices. Hence,

$$\overline{G}(\mathrm{F}_4, R) = G(\mathrm{F}_4, R)\,\mathrm{Cent}(G(\mathrm{E}_6, R)). \tag{1}$$

We denote the diagonal subgroup in $\overline{G}(\mathrm{F}_4, R)$ by $\overline{T}(\mathrm{F}_4, R)$. Since this subgroup normalizes $E(\mathrm{F}_4, R)$, we can form the product

$$\overline{E}(\mathrm{F}_4, R) = E(\mathrm{F}_4, R)\overline{T}(\mathrm{F}_4, R) = E(\mathrm{F}_4, R)\,\mathrm{Cent}(G(\mathrm{E}_6, R)).$$

§4. General facts about Chevalley groups

In this section Φ is E_6 or F_4.

Let $A \trianglelefteq R$ be an ideal of the ring R. By definition, the group $E(\Phi, A)$ is generated by the root elements of level A:

$$E(\Phi, A) = \langle x_\alpha(\xi), \alpha \in \Phi, \xi \in A \rangle.$$

In particular, if $A = R$, the group $E(\Phi, R)$ is called an (absolute) *elementary group*. There are, of course, *relative elementary groups* $E(\Phi, R, A)$:

$$E(\Phi, R, A) = \langle x_a(\xi), \alpha \in \Phi, \xi \in A \rangle^{E(\Phi,R)}.$$

The following simple fact is well known (see, for example, [36, Corollary 4.4]).

Lemma 3. *If Φ is E_6 or F_4, then the elementary group $E(\Phi, R)$ is perfect.*

Consider the reduction homomorphism $\rho_A^\Phi : G(\Phi, R) \to G(\Phi, R/A)$, that is, the restriction of an obvious homomorphism $\mathrm{GL}(27, R) \to \mathrm{GL}(27, R/A)$ to the group $G(\Phi, R) \leq \mathrm{GL}(27, R)$. Let $G(\Phi, R, A)$ be the kernel of this homomorphism, and $C(\Phi, R, A)$ the preimage of the center of $G(\Phi, R/A)$.

The following equations are called the *standard commutator formulas*.

Lemma 4. *Suppose Φ is E_6 or F_4. For any ideal $A \trianglelefteq R$ the following equations hold true*:

$$[G(\Phi, R), E(\Phi, R, A)] = [E(\Phi, R), C(\Phi, R, A)] = E(\Phi, R, A).$$

In particular, $E(\Phi, R, A)$ is normal in $G(\Phi, R)$.

In the exceptional cases that we need here, this lemma was proved by Taddei [37] and Vaserstein [39]. One can find other proofs and further references in [40, 20, 26].

Let S be a multiplicative system in the ring R, i.e., a subset of R that contains 1 and is closed under multiplication. We denote by $S^{-1}R$ the *localization* of R with respect to S, and by $F_S : R \to S^{-1}R$ the canonical homomorphism. The following cases are of particular interest.

(1) Localization with respect to a maximal ideal: $S = R \setminus M$, where $M \in \mathrm{Max}(R)$ is a maximal ideal of R. In this case we write $(R \setminus M)^{-1}R = R_M$, and use the notation F_M instead of F_S. The Ring R_M is local with a unique maximal ideal $R_M F_M(M)$.
(2) Principal localization: $S = \langle s \rangle = \{1, s, s^2, \dots\}$ is the smallest multiplicative system that contains $s \in R$. In this case we write R_s instead of $\langle s \rangle^{-1}R$ and F_s instead of F_S.

Let X be an affine group scheme over $\mathbb{Z}$. The homomorphism $X(F_S) : X(R) \to X(S^{-1}R)$ induced by the localization homomorphism will also be denoted by F_S. Note that if X is one of $G(\mathrm{E}_6, R)$, $G(\mathrm{F}_4, R)$, $\overline{G}(\mathrm{F}_4, R)$, then the elementary root unipotents are mapped to elementary root unipotents: $F_S(x_\alpha(\xi)) = x_\alpha(F_S(\xi))$. Thus,

$$F_S(E(\mathrm{E}_6, R)) \leq E(\mathrm{E}_6, S^{-1}R),$$
$$F_S(E(\mathrm{F}_4, R)) \leq E(\mathrm{F}_4, S^{-1}R).$$

Since F_S maps tori to tori, $F_S(\overline{E}(\mathrm{F}_4, R)) \leq \overline{E}(\mathrm{F}_4, S^{-1}R)$. Hence, $E(\mathrm{E}_6, -)$, $E(\mathrm{F}_4, -)$, $\overline{E}(\mathrm{F}_4, -)$ are also functors from the category of commutative rings to the category of groups, but these functors are not representable.

It is well known that these functors commute with inductive limits. More precisely, if R_i, $i \in I$, is an inductive system of rings, and X is one of the functors $G(\mathrm{E}_6, -)$, $G(\mathrm{F}_4, -)$, $\overline{G}(\mathrm{F}_4, -)$, $E(\mathrm{E}_6, -)$, $E(\mathrm{F}_4, -)$, and $\overline{E}(\mathrm{F}_4, -)$, then $X(\varinjlim R_i) = \varinjlim X(R_i)$.

In particular, if R_i is the inductive system of all finitely generated subrings of R with respect to inclusion, then $X(R) = \varinjlim X(R_i)$, and we can restrict our attention to Noetherian rings.

Moreover, if S is a multiplicative system, then the rings R_s, $s \in S$, form an inductive system with respect to canonical localization homomorphisms $F_t : R_s \to R_{st}$. Therefore, $X(S^{-1}R) = \varinjlim X(R_s)$. This fact allows us to pass from arbitrary localizations (in particular, localizations with respect to maximal ideals) to principal localizations. Localizing with respect to maximal ideals, we obtain local rings. It is well known (see, for example, [16]) that for local (and even for semilocal) rings we have $G(\mathrm{E}_6, R) = E(\mathrm{E}_6, R)$, $G(\mathrm{F}_4, R) = E(\mathrm{F}_4, R)$; therefore, $\overline{G}(\mathrm{F}_4, R) = \overline{E}(\mathrm{F}_4, R)$.

§5. The study of equations on $G(\mathrm{E}_6, R)$

In the present section we collect some technical results concerning matrices in $G(\mathrm{E}_6, R)$. The proofs involve the explicit description of the trilinear form T, the cubic form Q, and its partial derivatives f_λ, $\lambda \in \Lambda$. With respect to our numeration of the weights, the cubic form looks like this:

$$\begin{aligned}
Q(x) = {} & x_1x_{13}x_{-1} - x_1x_{-10}x_{-2} + x_1x_{-9}x_{-3} - x_1x_{-8}x_{-4} + x_1x_{-5}x_{-7} \\
& - x_2x_{10}x_{-1} + x_2x_{14}x_{-2} - x_2x_{-12}x_{-3} + x_2x_{-11}x_{-4} - x_2x_{-6}x_{-7} \\
& + x_3x_9x_{-1} - x_3x_{12}x_{-2} + x_3x_{15}x_{-3} - x_3x_{-11}x_{-5} + x_3x_{-8}x_{-6} \\
& - x_4x_8x_{-1} + x_4x_{11}x_{-2} - x_4x_{15}x_{-4} + x_4x_{-12}x_{-5} - x_4x_{-9}x_{-6} \\
& + x_7x_5x_{-1} - x_7x_6x_{-2} + x_7x_{15}x_{-7} - x_7x_{-12}x_{-8} + x_7x_{-9}x_{-11} \\
& - x_5x_{11}x_{-3} + x_5x_{12}x_{-4} - x_5x_{14}x_{-5} + x_5x_{-10}x_{-6} + x_8x_6x_{-3} \\
& - x_8x_{12}x_{-7} + x_8x_{14}x_{-8} - x_8x_{-10}x_{-11} - x_6x_9x_{-4} + x_6x_{10}x_{-5} \\
& - x_6x_{13}x_{-6} + x_9x_{11}x_{-7} - x_9x_{14}x_{-9} + x_9x_{-10}x_{-12} - x_{11}x_{10}x_{-8} \\
& + x_{11}x_{13}x_{-11} + x_{10}x_{12}x_{-9} - x_{10}x_{15}x_{-10} - x_{12}x_{13}x_{-12} + x_{13}x_{14}x_{15}.
\end{aligned}$$

The symmetric trilinear form T is obtained from Q by polarization. For future references, we reproduce the explicit form of the partial derivatives in this numeration:

$$\begin{aligned}
f_1(x) &= x_{13}x_{-1} - x_{-10}x_{-2} + x_{-9}x_{-3} - x_{-8}x_{-4} + x_{-7}x_{-5}, \\
f_2(x) &= -x_{10}x_{-1} + x_{14}x_{-2} - x_{-12}x_{-3} + x_{-11}x_{-4} - x_{-6}x_{-7}, \\
f_3(x) &= x_9x_{-1} - x_{12}x_{-2} + x_{15}x_{-3} - x_{-11}x_{-5} + x_{-8}x_{-6}, \\
f_4(x) &= -x_8x_{-1} + x_{11}x_{-2} - x_{15}x_{-4} + x_{-12}x_{-5} - x_{-9}x_{-6}, \\
f_5(x) &= x_7x_{-1} - x_{11}x_{-3} + x_{12}x_{-4} - x_{14}x_{-5} + x_{-10}x_{-6}, \\
f_6(x) &= -x_7x_{-2} + x_8x_{-3} - x_9x_{-4} + x_{10}x_{-5} - x_{13}x_{-6}, \\
f_7(x) &= x_5x_{-1} - x_6x_{-2} + x_{15}x_{-7} - x_{-12}x_{-8} + x_{-9}x_{-11}, \\
f_8(x) &= -x_4x_{-1} + x_6x_{-3} - x_{12}x_{-7} + x_{14}x_{-8} - x_{-10}x_{-11}, \\
f_9(x) &= x_3x_{-1} - x_6x_{-4} + x_{11}x_{-7} - x_{14}x_{-9} + x_{-10}x_{-12}, \\
f_{10}(x) &= -x_2x_{-1} + x_6x_{-5} - x_{11}x_{-8} + x_{12}x_{-9} - x_{15}x_{-10}, \\
f_{11}(x) &= x_4x_{-2} - x_5x_{-3} + x_9x_{-7} - x_{10}x_{-8} + x_{13}x_{-11}, \\
f_{12}(x) &= -x_3x_{-2} + x_5x_{-4} - x_8x_{-7} + x_{10}x_{-9} - x_{13}x_{-12}, \\
f_{13}(x) &= x_1x_{-1} - x_6x_{-6} + x_{11}x_{-11} - x_{12}x_{-12} + x_{14}x_{15}, \\
f_{14}(x) &= x_2x_{-2} - x_5x_{-5} + x_8x_{-8} - x_9x_{-9} + x_{13}x_{15}, \\
f_{15}(x) &= x_3x_{-3} - x_4x_{-4} + x_7x_{-7} - x_{10}x_{-10} + x_{13}x_{14}, \\
f_{-12}(x) &= -x_2x_{-3} + x_4x_{-5} - x_7x_{-8} + x_9x_{-10} - x_{12}x_{13}, \\
f_{-11}(x) &= x_2x_{-4} - x_3x_{-5} + x_7x_{-9} - x_8x_{-10} + x_{11}x_{13}, \\
f_{-10}(x) &= -x_1x_{-2} + x_5x_{-6} - x_8x_{-11} + x_9x_{-12} - x_{10}x_{15}, \\
f_{-9}(x) &= x_1x_{-3} - x_4x_{-6} + x_7x_{-11} - x_9x_{14} + x_{10}x_{12}, \\
f_{-8}(x) &= -x_1x_{-4} + x_3x_{-6} - x_7x_{-12} + x_8x_{14} - x_{10}x_{11}, \\
f_{-7}(x) &= x_1x_{-5} - x_2x_{-6} + x_7x_{15} - x_8x_{12} + x_9x_{11}, \\
f_{-6}(x) &= -x_2x_{-7} + x_3x_{-8} - x_4x_{-9} + x_5x_{-10} - x_6x_{13}, \\
f_{-5}(x) &= x_1x_{-7} - x_3x_{-11} + x_4x_{-12} - x_5x_{14} + x_6x_{10}, \\
f_{-4}(x) &= -x_1x_{-8} + x_2x_{-11} - x_4x_{15} + x_5x_{12} - x_6x_9,
\end{aligned}$$

$$f_{-3}(x) = x_1x_{-9} - x_2x_{-12} + x_3x_{15} - x_5x_{11} + x_6x_8,$$
$$f_{-2}(x) = -x_1x_{-10} + x_2x_{14} - x_3x_{12} + x_4x_{11} - x_6x_7,$$
$$f_{-1}(x) = x_1x_{13} - x_2x_{10} + x_3x_9 - x_4x_8 + x_5x_7.$$

We shall often use the fact that every column v of a matrix in $G(\mathrm{E}_6, R)$ is a *singular* vector; hence, v satisfies the quadratic equations $f_\lambda(v) = 0$ for $\lambda \in \Lambda$.

Lemma 5. *Let v be a column of a matrix in $G(\mathrm{E}_6, R)$, and let the row $(v_2, \dots, v_{-1})$ be unimodular. If $v_j = 0$ for $j = 6, 11, 12, 13, -12, \dots, -1$ and $v_{14} + v_{15} = 0$, then $v_{14} = v_{15} = 0$.*

Proof. Let $\xi = v_{15} = -v_{14}$. Since v is a column of a matrix in $G(\mathrm{E}_6, R)$, we have $f_\lambda(v) = 0$ for every $\lambda \in \Lambda$. In particular,

$$\begin{aligned}
0 &= f_{-2}(v) = v_2v_{14} = -\xi v_2,\\
0 &= f_{-3}(v) = v_3v_{15} = \xi v_3,\\
0 &= f_{-4}(v) = -v_4v_{15} = -\xi v_4,\\
0 &= f_{-5}(v) = -v_5v_{14} = \xi v_5,\\
0 &= f_{-7}(v) = v_7v_{15} = \xi v_7,\\
0 &= f_{-8}(v) = v_8v_{14} = -\xi v_8,\\
0 &= f_{-9}(v) = -v_9v_{14} = \xi v_9,\\
0 &= f_{-10}(v) = -v_{10}v_{15} = -\xi v_{10},\\
0 &= f_{13}(v) = v_{14}v_{15} = \xi v_{14}.
\end{aligned}$$

Since the row $(v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}, v_{14})$ is unimodular, we have $\xi = 0$. □

§6. Parabolic subgroups

We split Λ into three parts: $\{\lambda_1\}$, $\mathrm{B} \cup \Gamma$, and $\{\lambda_{13}, \lambda_{-1}\} \cup \Delta$ (this partition corresponds to the removal of all edges labeled by 1 from the weight diagram of E_6).

If $g_{\lambda 1} = 0$ for all $\lambda \in \mathrm{B} \cup \Gamma$, and g_{11} is invertible, then the equations in the first column imply that it coincides with that of the identity matrix. Therefore, g lies in the parabolic subgroup of $G(\mathrm{E}_6, R)$, and the matrix g has the following block structure with respect to our partition of Λ:

$$g = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Here the diagonal blocks have sizes 1, 16, and 10. We denote this parabolic subgroup by $P_1(R)$. Its unipotent radical $U_1(R)$ looks like this:

$$\begin{pmatrix} 1 & A & * \\ 0 & I_{16} & * \\ 0 & 0 & I_{10} \end{pmatrix}.$$

The group $U_1(R)$ is Abelian and is isomorphic to R^{16} as an abstract group. Indeed, choosing any row A of length 16 consisting of elements of R, we have a unique way to construct a matrix in the unipotent radical $U_1(R)$. We describe this construction. Let Σ_1 be the set of all roots $\alpha \in \mathrm{E}_6$ such that $\lambda_1 - \alpha \in \Lambda$. When we subtract such roots α from λ_1, we precisely obtain the 16 weights in $\mathrm{B} \cup \Gamma$:

$$\Sigma_1 = \{\lambda_1 - \lambda \mid \lambda \in \mathrm{B} \cup \Gamma\} = \{{}^{1****}_{\ \ *} \in \mathrm{E}_6\}.$$

We choose any 16 elements $\xi_\alpha \in R$, $\alpha \in \Sigma_1$, and consider the matrix

$$\prod_{\alpha\in\Sigma_1} x_\alpha(\xi_\alpha) \in G(\mathrm{E}_6, R).$$

The product of these root elements can be taken in any order because they commute with one another. It is obvious that this matrix belongs to $U_1(R)$, and it has $\pm\xi_{\lambda_1-\lambda}$ at the intersection of the first row and the column v^λ for $\lambda \in \mathrm{B}\cup\Gamma$ (the sign here is in fact the sign of the structure constant $c_{\lambda,\lambda_1-\lambda}$). Moreover, any matrix in $U_1(R)$ can be expressed in this way uniquely.

The parabolic subgroup $P_6(R)$ and its unipotent radical $U_6(R)$ can be defined similarly. Consider the partition of Λ into the three subsets $\{\lambda_1, \lambda_{13}\} \cup \mathrm{B}$, $\Gamma \cup \Delta$, and $\{\lambda_{-1}\}$, corresponding to removal of all edges labeled by 6 from the weight diagram of E_6. The matrices in $P_6(R)$ and in $U_6(R)$ have the following block structure with respect to this partition:

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in P_6(R), \qquad \begin{pmatrix} I_{10} & * & * \\ 0 & I_{16} & * \\ 0 & 0 & 1 \end{pmatrix} \in U_6(R).$$

The group $U_6(R)$ is Abelian and can be expressed as the product of sixteen pairwise commuting subgroups. Indeed, let

$$\Sigma_6 = \{\alpha \in \mathrm{E}_6 \mid \lambda_{-1} + \alpha \in \Lambda\} = \{{}^{****1}_{\ \ *} \in \mathrm{E}_6\}.$$

We have $\lambda_{-1} + \alpha \in \Gamma \cup \Delta$ for $\alpha \in \Sigma_6$. Now we choose any $\xi_\alpha \in R$, $\alpha \in \Sigma_6$, and form the product

$$\prod_{\alpha\in\Sigma_6} x_\alpha(\xi_\alpha) \in G(\mathrm{E}_6, R).$$

It belongs to $U_6(R)$, and any element of $U_6(R)$ can be expressed in this way.

Now we consider the intersection $P_1(R) \cap P_6(R)$. We need to split Λ into six subsets:

$$\Lambda = \{\lambda_1\} \cup \mathrm{B} \cup \Gamma \cup \{\lambda_{13}\} \cup \Delta \cup \{\lambda_{-1}\}.$$

The block structure of the matrices in the intersection $P_1(R) \cap P_6(R)$ and in its unipotent radical is as follows:

$$\begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \in P_1(R) \cap P_6(R), \qquad \begin{pmatrix} 1 & 0 & * & 0 & * & * \\ 0 & I_8 & 0 & 0 & * & * \\ 0 & 0 & I_8 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in U_1(R) \cap U_6(R).$$

Let Ψ_{16} be the intersection $\Sigma_1 \cap \Sigma_6$, and let Ψ_1 and Ψ_6 be the complements of Ψ_{16} in Σ_1 and in Σ_6, respectively. It is easily seen that

$$\Psi_1 = \{\lambda_1 - \lambda \mid \lambda \in \mathrm{B}\} = \{{}^{1***0}_{\ \ *} \in \mathrm{E}_6\},$$
$$\Psi_6 = \{\lambda - \lambda_{-1} \mid \lambda \in \Delta\} = \{{}^{0***1}_{\ \ *} \in \mathrm{E}_6\},$$
$$\Psi_{16} = \{\lambda_1 - \lambda \mid \lambda \in \Gamma\} = \{\lambda - \lambda_{-1} \mid \lambda \in \Gamma\} = \{{}^{1***1}_{\ \ *} \in \mathrm{E}_6\}.$$

A matrix in $U_1(R) \cap U_6(R)$ can uniquely be expressed as a product

$$\prod_{\alpha\in\Psi_{16}} x_\alpha(\xi_\alpha) \in G(\mathrm{E}_6, R),$$

where $\xi_\alpha \in R$ for $\alpha \in \Psi_{16}$.

Lemma 6. *Suppose $g \in G(\mathrm{E}_6, R)$ is such that $g_{11} = 1$ and $g_{\lambda 1} = 0$ for $\lambda \in (\mathrm{B} \cup \Gamma) \setminus \{\lambda_{15}\}$. Then $g_{\lambda 1} = 0$ for all $\lambda \notin \{\lambda_1, \lambda_{15}\}$.*

Proof. We denote $g_{15,1}$ by ξ and consider the matrix

$$h = x_{-\substack{11221\\1}}(\xi) g.$$

We have $h_{15,1} = g_{15,1} - \xi g_{11} = 0$ and $h_{\lambda,1} = g_{\lambda,1}$ for every $\lambda \in (\mathrm{B} \cup \Gamma) \setminus \{\lambda_{15}\}$, because for such a λ the sum $\lambda + \begin{smallmatrix}11221\\1\end{smallmatrix}$ is not a weight. This implies that $h_{\lambda 1} = 0$ for every $\lambda \neq \lambda_1$, whence $h \in P_1(R)$. Therefore, since $g = x_{-\substack{11221\\1}}(-\xi) h$, we obtain

$$g_{\lambda,1} = h_{\lambda,1} = 0 \text{ for every } \lambda \in \Lambda \setminus \{\lambda_1\}, \lambda + \begin{smallmatrix}11221\\1\end{smallmatrix} \notin \Lambda.$$

Finally, if $\lambda, \lambda + \begin{smallmatrix}11221\\1\end{smallmatrix} \in \Lambda$ and $\lambda \neq \lambda_{15}$, then

$$g_{\lambda,1} = h_{\lambda,1} \pm \xi h_{\lambda + \substack{11221\\1}} = h_{\lambda,1} = 0. \qquad \square$$

Lemma 7. *Let $g = \prod_{\gamma \in \Psi_{16}} x_\gamma(\xi_\gamma) \in G(\mathrm{E}_6, R)$, where $\xi_\gamma \in R$ for all $\gamma \in \Psi_{16}$. The matrix g belongs to $\overline{G}(\mathrm{F}_4, R)$ if and only if $\xi_{\substack{12211\\1}} = \xi_{\substack{11221\\1}}$.*

Proof. If $\xi_{\substack{12211\\1}} = \xi_{\substack{11221\\1}} = \xi$, then

$$x_{\substack{12211\\1}}(\xi_{\substack{12211\\1}}) x_{\substack{11221\\1}}(\xi_{\substack{11221\\1}}) = X_{1232}(\xi),$$

and since all other roots of Ψ_{16} belong to Φ_l, we have $g \in E(\mathrm{F}_4, R)$. Conversely, if

$$g_{1,13} = 0, \quad g_{1,14} = -\xi_{\substack{12211\\1}}, \quad g_{1,15} = -\xi_{\substack{11221\\1}},$$

then

$$(gu)_1 = g_{1,13} - g_{1,14} + g_{1,15} = \xi_{\substack{12211\\1}} - \xi_{\substack{11221\\1}}.$$

By assumption, we have $gu = \lambda u$ for some $\lambda \in R^*$. Therefore,

$$\xi_{\substack{12211\\1}} - \xi_{\substack{11221\\1}} = (gu)_1 = \lambda u_1 = 0. \qquad \square$$

Lemma 8. *If $g \in P_1(R) \cap G(\mathrm{F}_4, R)$, then $g \in P_6(R)$.*

Proof. We have $g_{11} \in R^*$ and $g_{\lambda,1} = 0$ for every $\lambda \neq \lambda_1$. Choose any $\lambda \in \Delta \cup \{\lambda_{13}\}$. Then

$$0 = B(v^1, v^\lambda) = B(gv^1, gv^\lambda) = g_{11} g_{-1,\lambda},$$

and $g_{-1,\lambda} = 0$ for every such λ. On the other hand, for any $\lambda \in \mathrm{B} \cup \Delta \cup \{\lambda_1\}$ we have $g_{-1,\lambda} = 0$ because $g \in P_1(R)$. This means that the last row of g is proportional to the last row of the identity matrix, whence $g \in P_6(R)$. $\square$

§7. The normalizer of $E(\mathrm{F}_4, R)$ in $G(\mathrm{E}_6, R)$

Loosely speaking, Theorem 2 shows that $\overline{G}$ is not only a scheme-theoretic normalizer of F_4 in E_6, but also a pointwise normalizer: when we apply this functor to any commutative ring, we obtain the normalizer of the corresponding group in a group-theoretic sense. Note that our definition of $\overline{G}(\mathrm{F}_4, R)$ coincides with the definition of the *extended Chevalley group*, given originally in [14] for adjoint groups and later carried over in [21] to simply connected groups, which is the case of interest to us (see also [3]). Obviously, $G(\mathrm{F}_4, R)$ is a normal subgroup of $\overline{G}(\mathrm{F}_4, R)$.

By the very definition, $\overline{G}(\mathrm{F}_4, -)$ is an affine scheme over $\mathbb{Z}$. It is well known that the functor of points of any affine group scheme is determined by its values at all local rings. In particular, we have the following lemma.

Lemma 9. *Suppose $g \in G(\mathrm{E}_6, R)$ and $F_M(g) \in \overline{G}(\mathrm{F}_4, R)$ for every $M \in \mathrm{Max}(R)$. Then $g \in \overline{G}(\mathrm{F}_4, R)$.*

Lemma 10. *A matrix $g \in G(\mathrm{E}_6, R)$ belongs to $\overline{G}(\mathrm{F}_4, R)$ if and only if*

$$(Fg^T)_{ir}(g^{-1}F)_{js} = (g^{-1}F)_{ir}(Fg^T)_{js}$$

for all $i, j, r, s = 1, \dots, -1$.

Proof. Let X be an affine group subscheme of $G(\mathrm{E}_6, -)$ over $\mathbb{Z}$ defined by the above equations. It is clear that $\overline{G}(\mathrm{F}_4, R) \subset X(R)$. By Lemma 9, it suffices to establish the reverse inclusion for a local ring R. Let $M = R \setminus R^*$ denote the maximal ideal of R. First, we show that for $g \in X(R)$ there exist i and r such that $(Fg^T)_{ir}(g^{-1}F)_{ir} \in R^*$. Indeed, suppose $(Fg^T)_{ir}(g^{-1}F)_{ir} \in M$ for all i, r. Since Fg^T is invertible, for any i there exists an r such that $(Fg^T)_{ir} \notin M$. Since $g^{-1}F$ is invertible, for any j there exists an s such that $(g^{-1}F)_{js} \notin M$. Therefore, $(Fg^T)_{ir}(g^{-1}F)_{js} \in R^*$, but by assumption we have $(g^{-1}F)_{ir}(Fg^T)_{js} \in M$, which contradicts the fact that $g \in X(R)$.

Now we fix i, r so that $(Fg^T)_{ir}(g^{-1}F)_{ir} \in R^*$. Let

$$\lambda = (Fg^T)_{ir}((g^{-1}F)_{ir})^{-1} \in R^*.$$

Then the equations for g can be written as $(Fg^T)_{js} = \lambda(g^{-1}F)_{js}$. But this implies $Fg^T = \lambda g^{-1}F$, whence $gFg^T = \lambda F$ and $g \in \overline{G}(\mathrm{F}_4, R)$. □

The following theorem is a slightly stronger version of the Taddei theorem [37]. Strictly speaking, in [37] it was proved that $E(\Phi, R)$ is normal in the Chevalley group $G(\Phi, R)$, but later (see, for example, [9, 10, 11]) it was suggested that $G(\Phi, R)$ can be replaced by $\overline{G}(\Phi, R)$. In fact, in our case ($\Phi = \mathrm{F}_4$) this fact can easily be deduced from the Taddei theorem with the help of (1).

Lemma 11. *The elementary subgroup $E(\mathrm{F}_4, R)$ is normal in $\overline{G}(\mathrm{F}_4, R)$ for any commutative ring R.*

Proof of Theorem 2. Recall that G denotes the group $G(\mathrm{E}_6, R)$. Clearly, $\overline{G}(\mathrm{F}_4, R) \leq N_G(G(\mathrm{F}_4, R))$ (this follows directly from (1)). Lemma 11 implies that $\overline{G}(\mathrm{F}_4, R) \leq N_G(E(\mathrm{F}_4, R))$. Next, obviously,

$$N_G(E(\mathrm{F}_4, R)),\ N_G(G(\mathrm{F}_4, R)) \leq \mathrm{Tran}_G(E(\mathrm{F}_4, R), G(\mathrm{F}_4, R)).$$

To finish the proof we only need to establish the inclusion

$$\mathrm{Tran}_G(E(\mathrm{F}_4, R), G(\mathrm{F}_4, R)) \leq \overline{G}(\mathrm{F}_4, R).$$

Let $g \in \mathrm{Tran}_G(E(\mathrm{F}_4, R), G(\mathrm{F}_4, R))$. For some $\alpha \in \mathrm{F}_4$ and $\xi \in R$, consider the matrix $h = g^{-1}X_\alpha(\xi)g$. Since $h \in G(\mathrm{F}_4, R)$, we have $hu = u$, which implies $g^{-1}X_\alpha(\xi)gu = u$. We denote $gu = v$; then $X_\alpha(\xi)v = v$. Since $X_\alpha(\xi) = e + \xi e_\alpha$, we have $e_\alpha v = 0$ for all $\alpha \in \mathrm{F}_4$. This shows that if $\alpha \in \Phi_l$, $\lambda \in \Lambda$, and $\lambda + \alpha \in \Lambda$, then $v^\lambda = 0$. Therefore, the vector v has zeros in all entries except for 13, 14, and 15 (for any other entry we can easily choose a necessary root $\alpha \in \Phi_l$). Substituting $0001 \in \mathrm{F}_4$ in place of α, we obtain $v^{13} + v^{14} = 0$, while substituting $\alpha = 0010 \in \mathrm{F}_4$ yields $v^{14} + v^{15} = 0$. Thus, $v = \lambda u$ for some $\lambda \in R$, so that $gu = \lambda u$, and by Lemma 2 we have $\lambda^3 = 1$. Therefore, $g \in \overline{G}(\mathrm{F}_4, R)$. □

A simple group-theoretic argument allows us to refine the result of Theorem 2 as follows.

Corollary. *Under the hypothesis of Theorem* 2 *we have*

$$\mathrm{Tran}_G(E(\mathrm{F}_4, R), \overline{G}(\mathrm{F}_4, R)) = \overline{G}(\mathrm{F}_4, R).$$

Proof. Suppose that $[g, E(\mathrm{F}_4, R)] \le \overline{G}(\mathrm{F}_4, R)$ for some $g \in G(\mathrm{E}_6, R)$. Lemma 11 implies

$$[g, E(\mathrm{F}_4, R), E(\mathrm{F}_4, R)] \le E(\mathrm{F}_4, R).$$

But the group $E(\mathrm{F}_4, R)$ is perfect (Lemma 3), and applying the lemma on three subgroups, we obtain $g \in N_G(E(\mathrm{F}_4, R)) = \overline{G}(\mathrm{F}_4, R)$. □

§8. Relative groups and lower level

We recall the definition of the relative elementary groups. Let R be a commutative ring, let $A \trianglelefteq R$, and let Φ be any root system. Then

$$E(\Phi, R, A) = E(\Phi, A)^{E(\Phi, R)}.$$

The following proposition was proved by Tits [38].

Lemma 12. *The group $E(\mathrm{E}_6, R, A)$ is generated by the elements*

$$z_\alpha(\xi, \zeta) = {}^{x_{-\alpha}(\zeta)}x_\alpha(\xi), \quad \xi \in A,\ \zeta \in R,\ \alpha \in \mathrm{E}_6\,.$$

Lemma 13. *For any ideal $A \trianglelefteq R$ we have*

$$E(\mathrm{E}_6, A)^{E(\mathrm{F}_4, R)} = E(\mathrm{E}_6, R, A).$$

Proof. Obviously, the group on the left is included in that on the right. Let H denote the group on the left-hand side. By Lemma 12, it remains to check that $z_\alpha(\xi, \zeta) \in H$ for all $\alpha \in \mathrm{E}_6$, $\xi \in A$, $\zeta \in R$. This is obvious for $\alpha \in \Phi_l$. Otherwise, α and $\overline{\alpha} \ne \alpha$ project to a root $\beta \in \Phi_s$. Consider the element ${}^{X_{-\beta}(\zeta)}x_\alpha(\xi) = {}^{x_{-\alpha}(\pm\zeta)x_{-\overline{\alpha}}(\pm\zeta)}x_\alpha(\xi)$ in H. Since $\overline{\alpha}$ is orthogonal to α, this element equals ${}^{x_{-\alpha}(\pm\zeta)}x_\alpha(\xi) = z_\alpha(\xi, \pm\zeta)$, and the proof is finished. □

Lemma 14. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. For every $\alpha \in \mathrm{E}_6 \setminus \Phi_l$, let $I_\alpha = \{\xi \in R \mid x_\alpha(\xi) \in H\}$. Then for any $\beta \in \mathrm{E}_6 \setminus \Phi_l$ we have $I_\alpha = I_\beta = I$, and $I \trianglelefteq R$.*

Proof. Clearly, each set I_α is an additive subgroup in R. Suppose $\alpha \in \mathrm{E}_6 \setminus \Phi_l$ and $\xi \in I_\alpha$. We choose any $\zeta \in R$ and take $\beta \in \mathrm{E}_6 \setminus \Phi_l$ such that $\beta - \alpha \in \Phi_l$. Then

$$x_\beta(\pm\xi\zeta) = [x_\alpha(\xi), x_{\beta-\alpha}(\zeta)] = [x_\alpha(\xi), X_{\beta-\alpha}(\zeta)] \in H.$$

Hence, $I_\alpha R \subset I_\beta$. Moreover, for some choice of the signs, $x_\alpha(\pm\xi)x_{\overline{\alpha}}(\pm\xi)$ belongs to $E(\mathrm{F}_4, R)$, whence $I_\alpha = I_{\overline{\alpha}}$. We split the positive roots of $\mathrm{E}_6 \setminus \Phi_l$ into three sets:

$$\Theta_1 : \begin{matrix}10000\\0\end{matrix}, \begin{matrix}11110\\0\end{matrix}, \begin{matrix}11110\\1\end{matrix}, \begin{matrix}11210\\1\end{matrix}, \begin{matrix}00001\\0\end{matrix}, \begin{matrix}01111\\0\end{matrix}, \begin{matrix}01111\\1\end{matrix}, \begin{matrix}01211\\1\end{matrix}$$
$$\Theta_2 : \begin{matrix}11000\\0\end{matrix}, \begin{matrix}11100\\0\end{matrix}, \begin{matrix}11100\\1\end{matrix}, \begin{matrix}12210\\1\end{matrix}, \begin{matrix}00011\\0\end{matrix}, \begin{matrix}00111\\0\end{matrix}, \begin{matrix}00111\\1\end{matrix}, \begin{matrix}01221\\1\end{matrix}$$
$$\Theta_3 : \begin{matrix}01000\\0\end{matrix}, \begin{matrix}01100\\0\end{matrix}, \begin{matrix}01100\\1\end{matrix}, \begin{matrix}12211\\1\end{matrix}, \begin{matrix}00010\\0\end{matrix}, \begin{matrix}00110\\0\end{matrix}, \begin{matrix}00110\\1\end{matrix}, \begin{matrix}11221\\1\end{matrix}$$

It is easily seen that, in each of these sets, the difference of any two of the first four roots lies in Φ_l, while the last four roots are the images of the first four under the action of the automorphism $\alpha \mapsto \overline{\alpha}$. Therefore, for all roots α in each set Θ_i, the sets I_α coincide and form ideals. We denote these three ideals, corresponding to the roots in Θ_1, Θ_2, Θ_3, by I_1, I_2, I_3, respectively. Consider the commutator

$$[x_{\substack{10000\\0}}(\xi), X_{0010}(\zeta)] = [x_{\substack{10000\\0}}(\xi), x_{\substack{01000\\0}}(\pm\zeta)x_{\substack{00010\\0}}(\pm\zeta)] = x_{\substack{11000\\0}}(\pm\xi\zeta).$$

This implies that $I_1 R \subset I_2$. Acting similarly, we obtain $I_1 = I_2 = I_3$. Hence, the ideals I_α coincide for all positive roots $\alpha \in \mathrm{E}_6 \setminus \Phi_l$. Similar arguments can be used to show that the ideals I_α coincide for all negative roots $\alpha \in \mathrm{E}_6 \setminus \Phi_l$. It remains to note that the difference of the roots $\begin{matrix}10000\\0\end{matrix}$ and $-\begin{matrix}01111\\0\end{matrix}$ belongs to Φ_l, and we can repeat the

computation at the beginning of this proof to see that the ideals in question are identical for *all* roots in $\mathrm{E}_6 \setminus \Phi_l$. $\square$

Combining Lemmas 13 and 14, we arrive at the following statement.

Proposition 1. *Assume H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Then there exists a unique largest ideal $A \trianglelefteq R$ such that*

$$\mathrm{E}(\mathrm{F}_4, R, A) = E(\mathrm{F}_4, R)E(\mathrm{E}_6, R, A) \leq H.$$

Moreover, if $x_\alpha(\xi) \in H$ for some $\alpha \in \mathrm{E}_6 \setminus \mathrm{F}_4$, then $\xi \in A$.

This proposition shows that with every subgroup between $E(\mathrm{F}_4, R)$ and $G(\mathrm{E}_6, R)$ we can associate the *lower level*. In order to finish the proof of Theorem 1, it remains to show that this lower level coincides with the upper level; i.e., $\mathrm{E}(\mathrm{F}_4, R, A)$ is *normal* in H.

§9. Proof of Theorem 3

Lemma 15. *If R is a commutative ring and $A \trianglelefteq R$, then the group $\mathrm{E}(\mathrm{F}_4, R, A)$ is perfect.*

Proof. Lemma 13 implies that $\mathrm{E}(\mathrm{F}_4, R, A)$ is generated (as an abstract group) by the root elements $x_\alpha(\zeta)$ with $\alpha \in \mathrm{F}_4$, $\zeta \in R$ and the root elements $x_\alpha(\xi)$ with $\alpha \in \mathrm{E}_6 \setminus \mathrm{F}_4$, $\xi \in A$. We show that all these generators belong to the commutator subgroup of $\mathrm{E}(\mathrm{F}_4, R, A)$. For the root elements of F_4, this follows from the fact that the absolute elementary group is perfect (see Lemma 3). Now, consider $x_\alpha(\xi)$ for $\alpha \in \mathrm{E}_6 \setminus \mathrm{F}_4$ and $\xi \in A$. As in the proof of Lemma 14, we can find a root $\beta \in \mathrm{E}_6 \setminus \mathrm{F}_4$ such that $\alpha - \beta \in \mathrm{F}_4$. But

$$x_\alpha(\xi) = [x_\beta(\xi), x_{\alpha-\beta}(\pm 1)],$$

and the two root elements on the right-hand side belong to $\mathrm{E}(\mathrm{F}_4, R, A)$. $\square$

Consider the reduction homomorphism $\rho_A^{\mathrm{E}_6} : G(\mathrm{E}_6, R) \to G(\mathrm{E}_6, R/A)$ and denote by $\mathrm{CG}(\mathrm{F}_4, R, A)$ the full preimage of $\overline{G}(\mathrm{F}_4, R/A)$ under this reduction:

$$\mathrm{CG}(\mathrm{F}_4, R, A) = (\rho_A^{\mathrm{E}_6})^{-1}(\overline{G}(\mathrm{F}_4, R/A)).$$

We recall that $G(\mathrm{E}_6, R, A)$ denotes the kernal of $\rho_A^{\mathrm{E}_6}$. Note that $\overline{G}(\mathrm{F}_4, R)G(\mathrm{E}_6, R, A) \leq \mathrm{CG}(\mathrm{F}_4, R, A)$, but this inequality can be strict.

Lemma 10 immediately implies the following description of the group $\mathrm{CG}(\mathrm{F}_4, R, A)$.

Proposition 2. *A matrix $g \in G(\mathrm{E}_6, R)$ belongs to $\mathrm{CG}(\mathrm{F}_4, R, A)$ if and only if it satisfies the congruences*

$$(Fg^T)_{ir}(g^{-1}F)_{js} \equiv (g^{-1}F)_{ir}(Fg^T)_{js} \pmod{A}$$

for all $i, j, r, s = 1, \ldots, -1$.

Now everything is ready for the proof of Theorem 3.

Proof of Theorem 3. Recall that $G = G(\mathrm{E}_6, R)$. Clearly,

$$N_G(\mathrm{E}(\mathrm{F}_4, R, A)) \leq N_G(\mathrm{E}(\mathrm{F}_4, R, A)G(\mathrm{E}_6, R, A)).$$

Also, combining Theorem 2 applied to the ring R/A with the fundamental homomorphism theorem, we see that

$$N_G(\mathrm{E}(\mathrm{F}_4, R, A)G(\mathrm{E}_6, R, A)) = \mathrm{CG}(\mathrm{F}_4, R, A).$$

In particular,

$$[\mathrm{CG}(\mathrm{F}_4, R, A), \mathrm{E}(\mathrm{F}_4, R, A)] \leq \mathrm{E}(\mathrm{F}_4, R, A)G(\mathrm{E}_6, R, A).$$

It remains to show that $\mathrm{E}(\mathrm{F}_4, R, A)$ is normalized by $\mathrm{CG}(\mathrm{F}_4, R, A)$. Note that

$$[\overline{G}(\mathrm{F}_4, R, A)G(\mathrm{E}_6, R, A), \mathrm{E}(\mathrm{F}_4, R, A)] \leq \mathrm{E}(\mathrm{F}_4, R, A).$$

Indeed, consider the commutator

$$[xy, hg], \quad x \in \overline{G}(\mathrm{F}_4, R),\ y \in G(\mathrm{E}_6, R, A),\ h \in E(\mathrm{F}_4, R),\ g \in E(\mathrm{E}_6, R, A).$$

We have $[xy, hg] = {}^x[y, h] \cdot [x, h] \cdot {}^h[xy, g]$. Lemma 11 shows that the second commutator belongs to $E(\mathrm{F}_4, R)$. By Lemma 4, the commutators $[xy, g]$ and $[y, h]$ belong to $E(\mathrm{E}_6, R, A)$, so that ${}^h[xy, g] \in \mathrm{E}(\mathrm{F}_4, R, A)$ and, again by Lemma 4, we have ${}^x[y, h] \in E(\mathrm{E}_6, R, A)$.

But $\mathrm{E}(\mathrm{F}_4, R, A)G(\mathrm{E}_6, R, A)$ is included in $\overline{G}(\mathrm{F}_4, R, A)G(\mathrm{E}_6, R, A)$, and *a fortiori*, we have

$$[\mathrm{E}(\mathrm{F}_4, R, A)G(\mathrm{E}_6, R, A), \mathrm{E}(\mathrm{F}_4, R, A)] \leq \mathrm{E}(\mathrm{F}_4, R, A).$$

Summarizing, we obtain

$$[[\mathrm{CG}(\mathrm{F}_4, R, A), \mathrm{E}(\mathrm{F}_4, R, A)], \mathrm{E}(\mathrm{F}_4, R, A)] \leq \mathrm{E}(\mathrm{F}_4, R, A).$$

Now we want to refine this result by showing that in fact

$$[[\mathrm{CG}(\mathrm{F}_4, R, A), \mathrm{E}(\mathrm{F}_4, R, A)], [\mathrm{CG}(\mathrm{F}_4, R, A), \mathrm{E}(\mathrm{F}_4, R, A)]] \leq \mathrm{E}(\mathrm{F}_4, R, A).$$

We have already proved that the left-hand side is generated by the commutators $[uv, [z, y]]$ for $u, y \in \mathrm{E}(\mathrm{F}_4, R, A)$, $v \in G(\mathrm{E}_6, R, A)$, and $z \in \mathrm{CG}(\mathrm{F}_4, R, A)$. But

$$[uv, [z, y]] = {}^u[v, [z, y]] \cdot [u, [z, y]],$$

and here the second commutator belongs to $\mathrm{E}(\mathrm{F}_4, R, A)$, while the first belongs to $[G(\mathrm{E}_6, R, A), E(\mathrm{E}_6, R)] \leq E(\mathrm{E}_6, R, A)$.

Now we can finish the proof. Recall that it remains to show that $\mathrm{E}(\mathrm{F}_4, R, A)$ is normalized by $\mathrm{CG}(\mathrm{F}_4, R, A)$. Lemma 15 says that the group $\mathrm{E}(\mathrm{F}_4, R, A)$ is perfect. Therefore, it suffices to show that $[z, [x, y]] \in \mathrm{E}(\mathrm{F}_4, R, A)$ for every $x, y \in \mathrm{E}(\mathrm{F}_4, R, A)$, $z \in \mathrm{CG}(\mathrm{F}_4, R, A)$. The Hall–Witt identity yields

$$[z, [x, y]] = {}^{xz}[[z^{-1}, x^{-1}], y] \cdot {}^{xy}[[y^{-1}, z], x^{-1}],$$

and the above implies that the second commutator belongs to $\mathrm{E}(\mathrm{F}_4, R, A)$. Note that

$${}^{xz}[[z^{-1}, x^{-1}], y] = {}^x[{}^z[z^{-1}, x^{-1}], {}^zy] = {}^x[[x^{-1}, z], [z, y]y].$$

Thus, it remains to check that $[[x^{-1}, z], [z, y]y] \in \mathrm{E}(\mathrm{F}_4, R, A)$. But

$$\begin{aligned}
[[x^{-1}, z], [z, y]y] &= [x^{-1}, z][z, y]y[z, x^{-1}]y^{-1}[y, z] \\
&= [[x^{-1}, z], [z, y]] \cdot [z, y][x^{-1}, z]y[z, x^{-1}]y^{-1}[y, z] \\
&= [[x^{-1}, z], [z, y]] \cdot {}^{[z,y]}[[x^{-1}, z], y],
\end{aligned}$$

and both $[[x^{-1}, z], [z, y]]$ and $[[x^{-1}, z], y]$ belong to $\mathrm{E}(\mathrm{F}_4, R, A)$, while the conjugating element $[z, y]$ of the second commutator belongs to $\mathrm{E}(\mathrm{F}_4, R, A)G(\mathrm{E}_6, R, A)$, and therefore, normalizes $\mathrm{E}(\mathrm{F}_4, R, A)$. □

§10. Localization functor

The following lemmas provide the technical base for localization. Lemma 16 is a particular case of [26, Theorem 5.3].

Lemma 16. *For any finite set of elements $g_1, \ldots, g_n \in \overline{E}(F_4, R)$ and any $k \geq 0$ there exists $m \geq 0$ such that*

$$[g_i, F_s(\overline{G}(\mathrm{F}_4, R, s^m R))] \leq E(\mathrm{F}_4, F_s(s^k R)).$$

Lemma 17. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Suppose that $X \leq G(\mathrm{E}_6, -)$ is a group subscheme and that for some $s \in R$ we have*

$$F_s(H)\overline{G}(\mathrm{F}_4, R_s) \cap X(R_s) \not\subseteq \overline{G}(\mathrm{F}_4, R_s).$$

Then there exists $t \in R$ such that

$$F_t(H)\overline{E}(\mathrm{F}_4, R_t) \cap X(R_t) \not\subseteq \overline{G}(\mathrm{F}_4, R_t).$$

Proof. Suppose that the element $F_s(g)x$, where $g \in H$, $x \in \overline{G}(\mathrm{F}_4, R_s)$, does not belong to $G(\mathrm{F}_4, R_s)$. By Lemma 9, there exists a maximal ideal $M \in \mathrm{Max}(R)$ such that $s \notin M$ and $F_M(g) \notin \overline{G}(\mathrm{F}_4, R_M)$. Since R_M is a local ring, we have $\overline{G}(\mathrm{F}_4, R_M) = \overline{E}(\mathrm{F}_4, R_M)$. On the other hand, since $\overline{E}(\mathrm{F}_4, R_M) = \varinjlim \overline{E}(\mathrm{F}_4, R_t)$, where the limit is taken over all $t \notin M$, there exists $t = sq \notin M$ such that $F_q(x) \in \overline{E}(\mathrm{F}_4, R_t)$. Therefore,

$$F_q(F_s(g)x) = F_t(g)F_q(x) \in F_t(H)\overline{G}(\mathrm{F}_4, R_t) \cap X(R_t)$$

and, by our choice of M, we have $F_t(g) \notin \overline{G}(\mathrm{F}_4, R_t)$. □

Lemma 18. *Under the assumptions of Lemma 17, if $y \in F_s(H)\overline{E}(\mathrm{F}_4, R_s)$, then there exists $n \in \mathbb{N}_0$ such that*

$$[y, X_\alpha(s^n/1)] \in F_s(H)$$

for all $\alpha \in \mathrm{F}_4$.

Proof. We write $y = gx$ for some $g \in F_s(H)$, $x \in \overline{E}(\mathrm{F}_4, R_s)$. For every n we have

$$[y, X_\alpha(s^n/1)] = {}^g[x, X_\alpha(s^n/1)][g, X_\alpha(s^n/1)].$$

By Lemma 16, we can choose n such that

$$[x, X_\alpha(s^n/1)] \in F_s(E(\mathrm{F}_4, R)) \subseteq F_s(H)$$

for all $\alpha \in \mathrm{F}_4$. The other factors on the right-hand side belong to $F_s(H)$. □

The next auxiliary result allows us to extract a root element from the group $F_s(H)$ by using not only elements of $F_s(E(\mathrm{F}_4, R))$, but also elements of $\overline{G}(\mathrm{F}_4, R_s)$. Thanks to that, we can finish the proof almost without the use of localization.

Proposition 3. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Suppose that there exists $s \in R$ such that $F_s(H)\overline{G}(\mathrm{F}_4, R_s)$ contains a nontrivial elementary root unipotent corresponding to a root in $\mathrm{E}_6 \setminus \Phi_l$. Then H contains a nontrivial elementary root unipotent $x_\alpha(\xi)$ for some $\alpha \in \mathrm{E}_6 \setminus \Phi_l$, $\xi \in R$.*

Proof. By Lemma 17, we may assume that

$$x_\alpha(a/s^k) \in F_s(H)\overline{E}(\mathrm{F}_4, R_s)$$

for some $\alpha \in \mathrm{E}_6 \setminus \Phi_l$, $a \in R$, $k \geq 0$, and $a/s^k \neq 0$. Choose a root $\beta \in \Phi_l$ such that $\alpha + \beta \in \mathrm{E}_6$ and consider the commutator

$$[x_\alpha(a/s^k), x_\beta(s^{n+k}/1)] = x_{\alpha+\beta}(\pm s^n a/1).$$

Lemma 18 implies the existence of n such that $x_{\alpha+\beta}(\pm s^n a/1) \in F_s(H)$, which means that there exists $g \in H$ with $F_s(g) = x_{\alpha+\beta}(\pm s^n a/1)$. On the other hand, we have $F_s(x_{\alpha+\beta}(\pm s^n a)) = x_{\alpha+\beta}(\pm s^n a/1)$, whence $g = x_{\alpha+\beta}(\pm s^n a)y$ for some $y \in \mathrm{Ker}(F_s)$. Therefore, there exists $m \in \mathbb{N}_0$ such that $y \in \mathrm{GL}(27, R, \mathrm{Ann}(s^m))$. Consider the commutator $z = [g, x_{-\beta}(s^m)] \in H$. Since $[y, x_{-\beta}(s^m)] = e$, we have

$$z = [x_{\alpha+\beta}(\pm s^n a), x_{-\beta}(s^m)] = x_\alpha(s^{n+m}a).$$

If $s^{m+n}a = 0$, then $a \in \mathrm{Ker}(F_s)$, which is impossible because we have assumed that $a/s^k \in R_s$ is nonzero. Therefore, $z = x_\alpha(s^{m+n}a) \in H$ is the required elementary root unipotent. □

§11. Extraction of a root unipotent from unipotent radicals

In the following propositions we *extract a root unipotent*, which is similar to the extraction of a transvection in the proofs of the standard descriptions of overgroups for the classical groups in the general linear group. Recall that $P_1(R)$ and $P_6(R)$ denote the maximal parabolic subgroups in $G(\mathrm{E}_6, R)$ that correspond to the roots α_1 and α_6, respectively; $U_1(R)$ and $U_6(R)$ denote their (Abelian) unipotent radicals. Now we show the existence of a root unipotent, first assuming the existence of a nontrivial element in the intersection of the unipotent radicals $U_1(R_s)$ and $U_6(R_s)$, then assuming the existence of a nontrivial element in their product, and finally, assuming the existence of a nontrivial element in the product of $U_1(R_s)$, $U_6(R_s)$ and the torus $T(\mathrm{E}_6, R_s)$. Thus, we relax our assumptions step-by-step.

Proposition 4. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Suppose that for some $s \in R$ we have*

$$F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \cap U_1(R_s) \cap U_6(R_s) \not\subseteq \overline{G}(\mathrm{F}_4, R_s).$$

Then H contains a nontrivial root unipotent corresponding to a root in $\mathrm{E}_6 \setminus \Phi_l$.

Proof. Every element of $U_1(R_s) \cap U_6(R_s)$ is a product of elementary root unipotents $x_\alpha(\xi_\alpha)$, where α has the form $\begin{smallmatrix} 1***1 \\ \quad * \end{smallmatrix}$. It is easily seen that all roots of this form, except for $\alpha = \begin{smallmatrix} 12211 \\ 1 \end{smallmatrix}$ and $\overline{\alpha} = \begin{smallmatrix} 11221 \\ 1 \end{smallmatrix}$, belong to Φ_l. Multipliying by the inverses to these root unipotents, we obtain

$$y = x_\alpha(\xi_\alpha) x_{\overline{\alpha}}(\xi_{\overline{\alpha}}) \in F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s),$$

where $y \notin \overline{G}(\mathrm{F}_4, R_s)$. The roots α and $\overline{\alpha}$ project to the short root $1232 \in \mathrm{F}_4$: $X_{1232}(\xi) = x_\alpha(\xi) x_{\overline{\alpha}}(\xi)$. Consider the element

$$z = y X_{1232}(-\xi_\alpha) = x_{\overline{\alpha}}(\xi_{\overline{\alpha}} - \xi_\alpha) \in F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s).$$

It is obvious that $z \notin \overline{G}(\mathrm{F}_4, R_s)$, whence $\xi_{\overline{\alpha}} - \xi_\alpha \neq 0 \in R_s$, and Proposition 3 shows that H contains the nontrivial root unipotent that we need. □

Proposition 5. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Suppose that for some $s \in R$ we have*

$$F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \cap U_1(R_s) \cdot U_6(R_s) \not\subseteq \overline{G}(\mathrm{F}_4, R_s).$$

Then H contains a nontrivial root unipotent corresponding to a root in $\mathrm{E}_6 \setminus \Phi_l$.

Proof. Every element y of the product of unipotent radicals $U_1(R_s) \cdot U_6(R_s)$ and the torus T can be expressed as follows:

$$y = \prod_{\gamma \in \Psi_6} x_\gamma(\xi_\gamma) \prod_{\gamma \in \Psi_1} x_\gamma(\xi_\gamma) \prod_{\gamma \in \Psi_{16}} x_\gamma(\xi_\gamma).$$

On the other hand, if we write y as

$$y = \prod_{\gamma \in \Psi_1} x_\gamma(\zeta_\gamma) \prod_{\gamma \in \Psi_6} x_\gamma(\zeta_\gamma) \prod_{\gamma \in \Psi_{16}} x_\gamma(\zeta_\gamma),$$

then $\xi_\gamma = \zeta_\gamma$ for every $\gamma \in \Psi_1 \cup \Psi_6$ (this follows immediately from the fact that for $\gamma \in \Psi_1$, $\delta \in \Psi_6$ we have $[x_\gamma(\xi), x_\delta(\zeta)] = x_{\gamma+\delta}(\pm\xi\zeta)$ if $\gamma + \delta \in \Psi_{16}$, and $[x_\gamma(\xi), x_\delta(\zeta)] = 1$ otherwise).

For every root $\gamma \in \Psi_6$ we can form the element $x_\gamma(-\xi_\gamma) x_{\overline{\gamma}}(\pm\xi_\gamma) \in E(\mathrm{F}_s)$ and multiply y on the left by the product of all such elements. Hence, we may assume that $\xi_\gamma = \zeta_\gamma = 0$ for all $\gamma \in \Psi_6$.

Choose a short root $\alpha \in \mathrm{F}_4$ such that $\alpha = {*}{*}{*}1$. The corresponding roots $\beta, \overline{\beta} \in \mathrm{E}_6$ have the following form: $\beta = \begin{smallmatrix} 1{*}{*}{*}0 \\ {*} \end{smallmatrix} \in \Psi_1$, $\overline{\beta} = \begin{smallmatrix} 0{*}{*}{*}1 \\ {*} \end{smallmatrix} \in \Psi_6$. Commuting y with the root unipotent $X_\alpha(\xi) = x_\beta(\xi) x_{\overline{\beta}}(\pm\xi)$, we obtain

$$[X_\alpha(\xi), y] = {}^{x_\beta(\xi)}[x_{\overline{\beta}}(\pm\xi), y] \cdot [x_\beta(\pm\xi), y].$$

Denote

$$y_6 = \prod_{\gamma \in \Psi_6} x_\gamma(\xi_\gamma), \quad y_1 = \prod_{\gamma \in \Psi_1} x_\gamma(\xi_\gamma), \quad y_{16} = \prod_{\gamma \in \Psi_{16}} x_\gamma(\xi_\gamma).$$

Then $y = y_6 y_1 y_{16}$. Note that $x_\beta(\xi)$ commutes with every $x_\gamma(\xi_\gamma)$ for $\gamma \in \Psi_1 \cup \Psi_{16}$; hence, it commutes with y_1 and y_{16}. Therefore,

$$\begin{aligned} [x_\beta(\pm\xi), y] &= [x_\beta(\pm\xi), y_6 y_1 y_{16}] \\ &= [x_\beta(\pm\xi), y_6] \cdot {}^{y_6}[x_\beta(\pm\xi), y_1] \cdot {}^{y_6 y_1}[x_\beta(\pm\xi), y_{16}] \\ &= [x_\beta(\pm\xi), y_6]. \end{aligned}$$

Similarly, denote

$$z_1 = \prod_{\gamma \in \Psi_1} x_\gamma(\zeta_\gamma), \quad z_6 = \prod_{\gamma \in \Psi_6} x_\gamma(\zeta_\gamma), \quad z_{16} = \prod_{\gamma \in \Psi_{16}} x_\gamma(\zeta_\gamma),$$

so that $y = z_1 z_6 z_{16}$, whence

$$[x_{\overline{\beta}}(\pm\xi), y] = [x_{\overline{\beta}}(\pm\xi), z_1].$$

Moreover, y_6 can be expressed as a product of pairwise commuting root unipotents $x_\gamma(\xi_\gamma)$, where $\gamma = \begin{smallmatrix} 0{*}{*}{*}1 \\ {*} \end{smallmatrix}$. If we commute $x_\beta(\pm\xi)$ with one such element, we obtain either e or a root unipotent corresponding to a root in Ψ_{16}; in either case, the result commutes with every root unipotent $x_\gamma(\xi_\gamma)$, $\gamma \in \Psi_1 \cup \Psi_6 \cup \Psi_{16}$. Therefore,

$$[x_\beta(\pm\xi), y_6] = [x_\beta(\pm\xi), \prod_{\gamma \in \Psi_6} x_\gamma(\xi_\gamma)] = \prod_{\gamma \in \Psi_6} [x_\beta(\pm\xi), x_\gamma(\xi_\gamma)].$$

Acting similarly, we show that

$$[x_{\overline{\beta}}(\pm\xi), z_1] = [x_{\overline{\beta}}(\pm\xi), \prod_{\gamma \in \Psi_1} x_\gamma(\zeta_\gamma)] = \prod_{\gamma \in \Psi_1} [x_{\overline{\beta}}(\pm\xi), x_\gamma(\zeta_\gamma)],$$

and since the element $[x_{\overline{\beta}}(\pm\xi), y] = [x_{\overline{\beta}}(\pm\xi), z_1]$ is now a product of the root unipotent corresponding to roots in Ψ_{16}, it commutes with $x_\beta(\xi)$. Therefore,

$$\begin{aligned} z = [x_\alpha(\xi), y] &= [x_\beta(\pm\xi), y_6] \cdot [x_{\overline{\beta}}(\pm\xi), z_1] \\ &= \prod_{\gamma \in \Psi_6} [x_\beta(\pm\xi), x_\gamma(\xi_\gamma)] \prod_{\gamma \in \Psi_1} [x_{\overline{\beta}}(\pm\xi), x_\gamma(\zeta_\gamma)]. \end{aligned}$$

Each of these commutators is a root unipotent of the form $x_\gamma(*)$, $\gamma \in \Psi_{16}$, so that the entire product belongs to $U_1(R_s) \cap U_6(R_s)$. If we choose α in such a way that $z \notin \overline{G}(\mathrm{F}_4, R_s)$, then Proposition 4 applies, thus finishing the proof. We show that this is always possible. Recall that $\xi_\gamma = 0$ for all $\gamma \in \Psi_6$, and $\xi_\gamma = \zeta_\gamma$ for all $\gamma \in \Psi_1 \cup \Psi_6$. We have

$$z = \prod_{\gamma \in \Psi_1} [x_{\overline{\beta}}(\pm\xi), x_\gamma(\xi_\gamma)].$$

But Lemma 7 implies that the element $z = \prod_{\gamma\in\Psi_{16}} x_\gamma(\eta_\gamma)$ belongs to $\overline{G}(\mathrm{F}_4, R_s)$ if and only if $\eta_{\substack{12211\\1}} = \eta_{\substack{11221\\1}}$. We list all possible α's:

$$\begin{aligned}
&\alpha = 0001, \quad \overline{\beta} = \substack{00001\\0}, \quad \eta_{\substack{12211\\1}} = \pm\xi\xi_{\substack{12210\\1}}, \quad \eta_{\substack{11221\\1}} = 0;\\
&\alpha = 0011, \quad \overline{\beta} = \substack{00011\\0}, \quad \eta_{\substack{12211\\1}} = 0, \quad \eta_{\substack{11221\\1}} = \pm\xi\xi_{\substack{11210\\1}};\\
&\alpha = 0111, \quad \overline{\beta} = \substack{00111\\0}, \quad \eta_{\substack{12211\\1}} = 0, \quad \eta_{\substack{11221\\1}} = \pm\xi\xi_{\substack{11110\\1}};\\
&\alpha = 1111, \quad \overline{\beta} = \substack{00111\\1}, \quad \eta_{\substack{12211\\1}} = 0, \quad \eta_{\substack{11221\\1}} = \pm\xi\xi_{\substack{11110\\0}};\\
&\alpha = 0121, \quad \overline{\beta} = \substack{01111\\0}, \quad \eta_{\substack{12211\\1}} = \pm\xi\xi_{\substack{11100\\1}}, \quad \eta_{\substack{11221\\1}} = 0;\\
&\alpha = 1121, \quad \overline{\beta} = \substack{01111\\1}, \quad \eta_{\substack{12211\\1}} = \pm\xi\xi_{\substack{11100\\0}}, \quad \eta_{\substack{11221\\1}} = 0;\\
&\alpha = 1122, \quad \overline{\beta} = \substack{01211\\1}, \quad \eta_{\substack{12211\\1}} = \pm\xi\xi_{\substack{11000\\0}}, \quad \eta_{\substack{11221\\1}} = 0;\\
&\alpha = 1132, \quad \overline{\beta} = \substack{01221\\1}, \quad \eta_{\substack{12211\\1}} = 0, \quad \eta_{\substack{11221\\1}} = \pm\xi\xi_{\substack{10000\\0}}.
\end{aligned}$$

By our assumption, every such z belongs to $\overline{G}(\mathrm{F}_4, R_s)$. Therefore, $\xi_\gamma = 0$ for all $\gamma \in \Psi_1$ and all $y \in U_1(R_s) \cap U_6(R_s)$. Now we can apply Proposition 4. □

Proposition 6. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Suppose that for some $s \in R$ we have*

$$F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \cap U_1(R_s) \cdot U_6(R_s) \cdot T(\mathrm{E}_6, R_s) \not\subseteq \overline{G}(\mathrm{F}_4, R_s).$$

Then H contains a nontrivial root unipotent corresponding to a root in $\mathrm{E}_6 \setminus \Phi_l$.

Proof. Suppose $y \in F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \cap U_1(R_s) \cdot U_6(R_s) \cdot T(\mathrm{E}_6, R_s)$ and $y \notin \overline{G}(\mathrm{F}_4, R_s)$. After multiplying y by a suitable element of $T(\mathrm{F}_4, R_s)$, we may assume that $y = zd \in F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \setminus \overline{G}(\mathrm{F}_4, R_s)$, where

$$z \in U_1(R_s) \cdot U_6(R_s), \quad d = h_{\substack{10000\\0}}(\varepsilon) h_{\substack{01000\\0}}(\eta)$$

for some $\varepsilon, \eta \in R_s^*$.

Consider the roots $\beta = \substack{10000\\0}$, $\overline{\beta} = \substack{00001\\0}$, $\alpha = 0001 \in \mathrm{F}_4$ and put

$$\begin{aligned}
g = [X_\alpha(\xi), y] &= X_\alpha(\xi) z d x_\beta(-\xi) x_{\overline{\beta}}(-\xi) d^{-1} z^{-1}\\
&= X_\alpha(\xi) z x_\beta(-\varepsilon^2\eta\xi) x_{\overline{\beta}}(-\xi) z^{-1}.
\end{aligned}$$

Consider the commutator of z and the root unipotent $x_\beta(*)$. We know that z is a product of the root unipotents $x_\gamma(*)$ for $\gamma \in \Psi_1 \cup \Psi_6 \cup \Psi_{16}$. Since $\beta \in \Psi_1$, the element $x_\beta(*)$ commutes with $x_\gamma(*)$ for every γ such that $\beta + \gamma \notin \mathrm{E}_6$. On the other hand, if $\beta + \gamma \in \mathrm{E}_6$, then $\gamma \in \Psi_6$, $\beta + \gamma \in \Psi_{16}$, and $[x_\beta(*), x_\gamma(*)] = x_{\beta+\gamma}(*)$. Thus,

$$[z, x_\beta(*)] \in U_1(R_s) \cap U_6(R_s).$$

Arguing similarly, we obtain

$$[z, x_{\overline{\beta}}(*)] \in U_1(R_s) \cap U_6(R_s).$$

Hence,

$$g = X_\alpha(\xi) u x_\beta(-\varepsilon^2\eta\xi) x_{\overline{\beta}}(-\xi) z z^{-1} = u x_\beta((1 - \varepsilon^2\eta)\xi)$$

for some $u \in U_1(R_s) \cap U_6(R_s)$.

If $g \notin G(\mathrm{F}_4, R_s)$, we can apply Proposition 5. It is easily seen that $g_{12} = (1-\varepsilon^2\eta)\xi$ and $g_{-2,-1} = 0$. But if $g \in G(\mathrm{F}_4, R_s)$, then

$$0 = B(v^2, v^{-1}) = B(gv^2, gv^{-1}) = g_{12} - g_{-2,-1}.$$

Substituting $\xi = 1$ yields $\varepsilon^2\eta = 1$.

Now we repeat this argument for $\beta = \begin{smallmatrix}11000\\0\end{smallmatrix}$, $\overline{\beta} = \begin{smallmatrix}00011\\0\end{smallmatrix}$, $\alpha = 0011 \in \mathrm{F}_4$. In this case,

$$\begin{aligned} g = [X_\alpha(\xi), y] &= X_\alpha(\xi) z d x_\beta(-\xi) x_{\overline{\beta}}(\xi) d^{-1} z^{-1} \\ &= X_\alpha(\xi) z x_\beta(-\varepsilon\eta\xi) x_{\overline{\beta}}(\xi) z^{-1}. \end{aligned}$$

Commuting z with $x_\beta(*)$ and $x_{\overline{\beta}}(*)$, again we obtain elements in $U_1(R_s) \cap U_6(R_s)$. Therefore,

$$g = X_\alpha(\xi) u x_\beta(-\varepsilon\eta\xi) x_{\overline{\beta}}(\xi) z z^{-1} = u x_\beta((1-\varepsilon\eta)\xi)$$

for some $u \in U_1(R_s) \cap U_6(R_s)$.

If $g \notin G(\mathrm{F}_4, R_s)$, we can apply Proposition 4. As above, it is easily seen that $g_{13} = (1-\varepsilon\eta)\xi$, $g_{-3,-1} = g_{-2,-1} = 0$. But if $g \in G(\mathrm{F}_4, R_s)$, then

$$0 = B(v^3, v^{-1}) = B(gv^3, gv^{-1}) = g_{13} + g_{-2,-1}.$$

Substituting $\xi = 1$ yields $\varepsilon\eta = 1$. From the equations $\varepsilon\eta = \varepsilon^2\eta = 1$ we obtain $\varepsilon = \eta = 1$. Therefore, $d = 1$, $y \in U_1(R_s) \cdot U_6(R_s)$, and we could apply Proposition 5 from the very beginning. □

§12. Extraction of a root unipotent from the parabolic subgroups

We continue to relax the assumptions: in this section we extract a root element from parabolic subgroups (first from the intersection of $P_1(R_s)$ and $P_6(R_s)$, then from $P_1(R_s)$), reducing the problem to extraction from unipotent radicals, which was performed in the preceding section. Loosely speaking, here we destroy the Levi factors.

Proposition 7. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Suppose that for some $s \in R$ we have*

$$F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \cap P_1(R_s) \cap P_6(R_s) \not\subseteq \overline{G}(\mathrm{F}_4, R_s).$$

Then H contains a nontrivial root unipotent corresponding to a root in $\mathrm{E}_6 \setminus \Phi_l$.

Proof. Suppose $y \in F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \cap P_1(R_s) \cap P_6(R_s)$ and $y \notin \overline{G}(\mathrm{F}_4, R_s)$. Choose a root $\alpha \in \Sigma_1 \cap \Phi_l$, that is, a long root of F_4 such that $\omega - \alpha \in \Lambda$. Let $z = y^{-1} X_\alpha(1) y$. Note that $\omega - \alpha \in \mathrm{B}$. It is easily seen that $z \in U_1(R_s) \cap U_6(R_s)$. If $z \notin \overline{G}(\mathrm{F}_4, R_s)$, we can apply Proposition 4. Now we can assume that $z \in \overline{G}(\mathrm{F}_4, R_s)$, whence $z \in G(\mathrm{F}_4, R_s)$. For every $j \in \Gamma$ we have

$$z_{1j} = \sum_{\lambda, \lambda+\alpha \in \Lambda} c_{\lambda,\alpha} y'_{1,\lambda+\alpha} y_{\lambda,j} = c_{\omega-\alpha,\alpha} y'_{11} y_{\omega-\alpha,j},$$

because for each of the other five summands, the factor $y_{\lambda,j}$ equals 0: indeed, for four of them, $\lambda \in \Delta$, while $y_{\Delta,\Gamma} = 0$; and the fifth has $\lambda = -\omega$, and $y_{-1,\Gamma} = 0$. Furthermore,

$$z_{1,13} = \sum_{\lambda, \lambda+\alpha \in \Lambda} c_{\lambda,\alpha} y'_{1,\lambda+\alpha} y_{\lambda,13} = c_{\omega-\alpha,\alpha} y'_{11} y_{\omega-\alpha,13} = 0,$$

because $y_{\Delta,13} = 0$, $y_{-1,13} = 0$, and $y_{\mathrm{B},13} = 0$. By our assumption, $z \in G(\mathrm{F}_4, R_s)$; hence $zu = u$, which yields $z_{1,13} - z_{1,14} + z_{1,15} = 0$. Thus, $c_{\omega-\alpha,\alpha}\xi y'_{11}(-y_{\omega-\alpha,14} + y_{\omega-\alpha,15}) = 0$. Since $c_{\omega-\alpha,\alpha} = \pm 1$ and $y'_{11} \in R^*$, we have $-y_{\omega-\alpha,14} + y_{\omega-\alpha,15} = 0$. At the same time, $y_{\omega-\alpha,13} = 0$ because $\omega - \alpha \in \mathrm{B}$. Substituting all possible α's shows that for every $\lambda \in \Gamma \setminus \{14, 15\}$, we have $y_{\lambda,13} - y_{\lambda,14} + y_{\lambda,15} = 0$.

Now let $\alpha = 1232 \in \Phi_s$, $\beta = \begin{smallmatrix}12211\\1\end{smallmatrix} \in \mathrm{E}_6$, $\overline{\beta} = \begin{smallmatrix}11221\\1\end{smallmatrix} \in \mathrm{E}_6$, and

$$z = y^{-1}X_\alpha(1)y = y^{-1}x_\beta(1)x_{\overline{\beta}}(1)y.$$

The same argument as above shows that $z \in U_1(R_s) \cap U_6(R_s)$, and we may assume that $z \in G(\mathrm{F}_4, R_s)$. For every $j \in \Gamma$ we have

$$\begin{aligned} z_{1j} &= \sum_{\lambda,\lambda+\beta\in\Lambda} c_{\lambda,\beta} y'_{1,\lambda+\beta} y_{\lambda,j} + \sum_{\lambda,\lambda+\overline{\beta}\in\Lambda} c_{\lambda,\overline{\beta}} y'_{1,\lambda+\overline{\beta}} y_{\lambda,j} \\ &= c_{\omega-\beta,\beta} y'_{11} y_{\omega-\beta,j} + c_{\omega-\overline{\beta},\overline{\beta}} y'_{11} y_{\omega-\overline{\beta},j} = -y'_{11}(y_{14,j} + y_{15,j}), \end{aligned}$$

because $c_{\omega-\beta,\beta} = c_{\omega-\overline{\beta},\overline{\beta}} = -1$. Furthermore,

$$\begin{aligned} z_{1,13} &= \sum_{\lambda,\lambda+\beta\in\Lambda} c_{\lambda,\beta} y'_{1,\lambda+\beta} y_{\lambda,13} + \sum_{\lambda,\lambda+\overline{\beta}\in\Lambda} c_{\lambda,\overline{\beta}} y'_{1,\lambda+\overline{\beta}} y_{\lambda,13} \\ &= c_{\omega-\beta,\beta} y'_{11} y_{\omega-\beta,13} + c_{\omega-\overline{\beta},\overline{\beta}} y'_{11} y_{\omega-\overline{\beta},13} = 0. \end{aligned}$$

By assumption, $z \in G(\mathrm{F}_4, R_s)$; hence $z_{1,13} - z_{1,14} + z_{1,15} = 0$ and $y'_{11}(y_{14,14} + y_{15,14} - y_{14,15} - y_{15,15}) = 0$. We denote $-y_{15,14} + y_{15,15} = \xi$, $y_{13,13} = \zeta$; then $-y_{14,14} + y_{14,15} = -\xi$. Moreover, $y_{14,13} = y_{15,13} = 0$.

Now, consider the block-diagonal matrix g with the blocks y_{11}, $y_{\mathrm{B\,B}}$, $y_{\Gamma\Gamma}$, $y_{13,13}$, $y_{\Delta\Delta}$, $y_{-1,-1}$. Since g is a Levi factor of y with respect to the Levi decomposition for the parabolic subgroup $P_1(R_s) \cap P_6(R_s)$, we have $g \in G(\mathrm{E}_6, R_s)$.

Now we are going to multiply g by a certain diagonal matrix in $G(\mathrm{E}_6, R_s)$ and the result will belong to $G(\mathrm{F}_4, R_s)$. Consider the vector gu. The definition of g yields

$$(gu)_\lambda = g_{\lambda,13} - g_{\lambda,14} + g_{\lambda,15} = 0 \text{ for } \lambda \in \mathrm{B} \cup \Delta \cup \{1, -1\}.$$

Furthermore,

$$(gu)_\lambda = g_{\lambda,13} - g_{\lambda,14} + g_{\lambda,15} = y_{\lambda,13} - y_{\lambda,14} + y_{\lambda,15} = 0 = ru_\lambda$$

for $\lambda \in \Gamma \setminus \{14, 15\}$. Finally,

$$\begin{aligned} (gu)_{13} &= g_{13,13} - g_{13,14} + g_{13,15} = y_{13,13} = \zeta, \\ (gu)_{14} &= g_{14,13} - g_{14,14} + g_{14,15} = y_{14,13} - y_{14,14} + y_{14,15} = -\xi, \\ (gu)_{15} &= g_{15,13} - g_{15,14} + g_{15,15} = y_{15,13} - y_{15,14} + y_{15,15} = \xi. \end{aligned}$$

Note that $\zeta \in R_s^*$. We look closely at the invertible matrix $g_{\Gamma\Gamma} = y_{\Gamma\Gamma}$. If we subtract the column $y_{\Gamma,14}$ from the column $y_{\Gamma,15}$, then the matrix will remain invertible, while all of the entries in its column with index 15 will be 0, except for $-\xi$ in the row 14 and ξ in the row 15. Therefore, $\xi \in R_s^*$.

Thus, we have proved that

$$gu = \zeta v^{13} - \xi v^{14} + \xi v^{15}, \text{ where } \xi, \zeta \in R_s^*.$$

Now it is easy to change g slightly via a diagonal matrix in $G(\mathrm{E}_6, R_s)$ to obtain a matrix in $G(\mathrm{F}_4, R_s)$. Note that

$$-1 = Q(u) = Q(gu) = -\zeta\xi^2,$$

whence $\zeta = \xi^{-2}$. Consider the product of weight elements

$$h = h_{\beta_6}(\xi^2) h_{\beta_5}(\xi).$$

Since $hgu = u$, we have $hg \in G(\mathrm{F}_4, R_s)$. Moreover, the product $y(hg)^{-1}$ belongs to $T \cdot U_1(R_s) \cdot U_6(R_s)$, and at the same time $y(hg)^{-1} \in F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s)$ and $y(hg)^{-1} \notin \overline{G}(\mathrm{F}_4, R_s)$. Therefore, we can apply Proposition 6, and the proof is finished. $\square$

Proposition 8. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Suppose that for some $s \in R$ we have*

$$F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \cap P_1(R_s) \not\subseteq \overline{G}(\mathrm{F}_4, R_s).$$

Then H contains a nontrivial root unipotent corresponding to a root in $\mathrm{E}_6 \setminus \Phi_l$.

Proof. Suppose $y \in F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s) \cap P_1(R_s)$ and $y \notin \overline{G}(\mathrm{F}_4, R_s)$. As in the proof of the preceding proposition, we choose a root $\alpha \in \Sigma_1 \cap \Phi_l$, that is, a long root in F_4 such that $\omega - \alpha \in \Lambda$. Let $z = y^{-1} X_\alpha(1) y$. It is easily seen that $z \in U_1$. If $z \notin \overline{G}(\mathrm{F}_4, R_s)$, we can apply Proposition 5 immediately and finish the proof. Otherwise, if $z \in \overline{G}(\mathrm{F}_4, R_s)$, then in fact $z \in G(\mathrm{F}_4, R_s)$. In this case, $B(zv^i, zv^j) = B(v^i, v^j)$ for all $i, j \in \Lambda$. In particular, taking $i \in \mathrm{B}$ and $j = -\omega$, we obtain $B(z_{*i}, z_{*,-1}) = 0$. This yields $z_{1i} z_{-1,-1} + z_{ii} z_{-i,-1} = 0$. But $z_{-1,-1} = 1$ and $z_{-i,-1} = 0$ because $-i \in \Delta$. Therefore, $z_{1i} = 0$. But

$$z_{1i} = \sum_{\lambda, \lambda+\alpha \in \Lambda} \pm y'_{1,\lambda+\alpha} y_{\lambda,i}.$$

It is easily seen that every root $\alpha \in \Sigma_1 \cap \Phi_l$ makes one addition from the weight $\omega - \alpha \in \Gamma$ to the weight ω, four additions from some weights in Δ, and one addition from $-\omega$. Since y belongs to P_1, the five additions mentioned change nothing: we have $y_{\lambda,i} = 0$ and $z_{1i} = \pm y'_{11} y_{\omega-\alpha,i}$. Moreover, the element y'_{11} is invertible. Thus, $y_{\omega-\alpha,i} = 0$.

Now we take a *short* root $\alpha = 1232$. The root unipotent $X_\alpha(1)$ is a product of two root unipotents of E_6, namely, $X_\alpha(1) = x_{\overset{1}{1}1221}(1) x_{\overset{1}{1}2211}(1)$, and both of them belong to U_1. Thus, we can apply the same argument, obtaining $y_{14,i} + y_{15,i} = 0$ for all $i \in \mathrm{B}$.

Therefore, if $v = y_{*i}$ is the column y with index $i \in \mathrm{B}$, then Lemma 5 yields $v_{14} = v_{15} = 0$. Thus, $y_{ij} = 0$ for all $i \in \Gamma$, $j \in \mathrm{B}$.

Let $i \in \Gamma$, $j \in \Delta$. Consider the equations on the *rows* z_{i*} and $z_{-j,*}$ of the matrix z. These equations are $B(z_{i*}, z_{-j,*}) = 0$. Since all entries of the row $z_{-j,*}$ with indices in $\mathrm{B} \cup \Gamma \cup \{\lambda_{13}\}$ are equal to 0, except for $z_{-j,-j} = 1$, this equation reduces to $z_{ij} = 0$. But $z_{ij} = \sum \pm y'_{i,\lambda+\alpha} y_{\lambda,j} = \pm y'_{i,-\omega+\alpha} y_{-\omega,j}$, because we know that $y'_{i,\lambda+\alpha} = 0$ for all $i \in \Gamma$, $\lambda + \alpha \in \{\omega\} \cup \mathrm{B}$. Now note that $-\omega + \alpha \in \Gamma$, and the row $y'_{i,\lambda_{-1}+\alpha}$ (where i ranges over Γ) is a row of an invertible matrix $y'_{\Gamma\Gamma}$, so that $y_{-1,j} = 0$.

Consider the column $z_{*,13}$. For $i \in \{6, 11, 12, -12, -11, -6\}$ we have $B(z_{*,13}, z_{*,-i}) = 0$, whence $z_{i,13} = 0$. At the same time, $B(z_{*,13}, z_{*,14}) = 1$, whence $z_{13,13} + z_{15,13} = 1$. Therefore, $z_{15,13} = 0$. Similarly, since $B(z_{*,13}, z_{*,15}) = -1$, we have $-z_{13,13} + z_{14,13} = -1$ and $z_{14,13} = 0$. We have proved that $z_{i,13} = 0$ for every $i \in \mathrm{B}$. Now we can repeat the argument of the preceding paragraph with 13 in place of j, obtaining $y_{-1,13} = 0$. It follows that the last row of y is proportional to the last row of the identity matrix, whence $y \in P_1 \cap P_6$, and we can apply Proposition 7. □

§13. Extraction of a root unipotent: The final part

It remains to get into the parabolic subgroup. Over a local ring the orbits of actions of $G(\mathrm{F}_4, R)$ and $G(\mathrm{E}_6, R)$ do not coincide (since our representation of F_4 is reducible), and we must relax the assumptions yet again: now we need a nontrivial element in the product of the parabolic subgroup and a certain root unipotent of E_6.

Proposition 9. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Suppose that for some $s \in R$ there exists $g \in F_s(H) \cdot \overline{G}(\mathrm{F}_4, R_s)$ such that $g \notin \overline{G}(\mathrm{F}_4, R_s)$ and the first column of g coincides with that of the identity matrix in all entries except possibly for λ_{15}. Then H contains a nontrivial root unipotent corresponding to a root in $\mathrm{E}_6 \setminus \Phi_l$.*

Proof. We denote $a = g_{15,1}$ and consider $h = x_{-\substack{11221\\1}}(a)g$. It is easily seen that h belongs to the parabolic subgroup $P_1(R_s)$. Choose $\alpha \in \mathrm{F}_4$ and $\xi \in R$, and consider the element

$$z = X_\alpha(\xi)^g = g^{-1}X_\alpha(\xi)g = h^{-1}x_{-\substack{11221\\1}}(a)X_\alpha(\xi)x_{-\substack{11221\\1}}(-a)h.$$

Suppose that $\alpha = {*}{*}{*}1 \in \Phi_s$; we have $X_\alpha(\xi) = x_{\alpha'}(\xi)x_{\alpha''}(\pm\xi)$, where $\alpha' = \substack{1{*}{*}{*}0\\*}$ and $\alpha'' = \substack{0{*}{*}{*}1\\*}$ are roots of E_6. We want to have $\alpha' - \substack{11221\\1} \in \mathrm{E}_6$ and $\alpha'' - \substack{11221\\1} \notin \mathrm{E}_6$ (in fact, these two conditions are equivalent). Now

$$\begin{aligned} x_{-\substack{11221\\1}}(a)X_\alpha(\xi)x_{-\substack{11221\\1}}(-a) &= x_{-\substack{11221\\1}}(a)x_{\alpha'}(\xi)x_{\alpha''}(\pm\xi)x_{-\substack{11221\\1}}(-a)\\ &= x_{-\substack{11221\\1}}(a)x_{\alpha'}(\xi)x_{-\substack{11221\\1}}(-a)x_{\alpha''}(\pm\xi)\\ &= x_{\alpha'}(\xi)[x_{\alpha'}(-\xi), x_{-\substack{11221\\1}}(a)]x_{\alpha''}(\pm\xi)\\ &= x_{\alpha'}(\xi)x_{\alpha'-\substack{11221\\1}}(\pm a\xi)x_{\alpha''}(\pm\xi). \end{aligned}$$

Since $\alpha' - \substack{11221\\1}$ has the form $-\substack{0{*}{*}{*}{*}\\*}$, the entire product belongs to $P_1(R_s)$. Thus, $z \in P_1(R_s)$, so that if $z \notin \overline{G}(\mathrm{F}_4, R_s)$, we can apply Proposition 8.

Otherwise, if $z \in \overline{G}(\mathrm{F}_4, R_s)$, then $z \in G(\mathrm{F}_4, R_s)$ and Lemma 8 yields $z \in P_1(R_s) \cap P_6(R_s)$. Next, $z_{11} = z_{-1,-1} = 1$. Thus, the last row of z coincides with that of the identity matrix. Let $w = h'_{-1,*} \in {}^{27}R$ denote the last row of h^{-1}. We have $zh^{-1} = h^{-1}x_{\alpha'}(\xi)x_{\alpha'-\substack{11221\\1}}(\pm a\xi)x_{\alpha''}(\pm\xi)$. The last row of the matrix on the left in this identity coincides with w. The matrix on the right-hand side is $h^{-1}(e+\xi e_{\alpha'} \pm a\xi e_{\alpha'-\substack{11221\\1}} \pm \xi e_{\alpha''})$. Therefore, the last row of the matrix $h^{-1}(\xi e_{\alpha'} \pm a\xi e_{\alpha'-\substack{11221\\1}} \pm \xi e_{\alpha''})$ is zero, whence $w(\xi e_{\alpha'} \pm a\xi e_{\alpha'-\substack{11221\\1}} \pm \xi e_{\alpha''}) = 0$. Now we can use explicit formulas: $(we_\gamma)_\lambda = \pm w_{\lambda+\gamma}$ if $\lambda + \gamma \in \Lambda$; $(we_\gamma)_\lambda = 0$ if $\lambda + \gamma \notin \Lambda$. Substituting $\xi = 1$, $\alpha = 0001, 0121, 1121, 1221$ and considering $w(\xi e_{\alpha'} \pm a\xi e_{\alpha'-\substack{11221\\1}} \pm \xi e_{\alpha''})_\lambda$ for $\lambda = \lambda_{-1}$ and $\lambda = \lambda_{-10}$, we obtain $w_{-1} = w_{-5} = w_{-8} = w_{-9} = w_{13} = 0$. Moreover, taking $\xi = 1$, $\alpha = 1221$, $\lambda = \lambda_{-3}$, we obtain $aw_{-1} = 0$

Now we choose $\alpha = {*}{*}{*}0 \in \Phi_l$ and argue as above: here $X_\alpha(\xi) = x_\alpha(\xi)$ for $\alpha = \substack{0{*}{*}{*}0\\*} \in \mathrm{E}_6$. Thus, $\alpha - \substack{11221\\1} \notin \mathrm{E}_6$. Hence, $x_{-\substack{11221\\1}}(a)X_\alpha(\xi)x_{-\substack{11221\\1}}(-a) = x_\alpha(\xi)$ and again z belongs to $P_1(R_s)$. If $z \notin \overline{G}(\mathrm{F}_4, R_s)$, we can apply Proposition 8. Otherwise, if $z \in \overline{G}(\mathrm{F}_4, R_s)$, then $z \in G(\mathrm{F}_4, R_s)$, and Lemma 8 yields $z \in P_1(R_s) \cap P_6(R_s)$, while $z_{11} = z_{-1,-1} = 1$. Therefore, the last row of z coincides with that of the identity matrix. Since $zh^{-1} = h^{-1}x_\alpha(\xi)$, we have $w = wx_\alpha(\xi)$, whence $we_\alpha = 0$ (we can plug $\xi = 1$). Thus, $w_{\lambda+\alpha} = 0$ whenever $\lambda, \lambda + \alpha \in \Lambda$. Substituting $\alpha = \pm 1000, \pm 0100, \pm 0120$, we obtain $w_{-3} = w_{-4} = w_{-7} = w_{-10} = 0$.

Thus, all the entries in the last row w of the matrix h^{-1} equal 0, except for $h'_{-1,-1}$, and moreover, $ah'_{-1,-1} = 0$. Since the matrix h^{-1} is invertible, we have $h'_{-1,-1} \in R^*$, $a = 0$, which means that we could apply Proposition 8 from the very beginning. □

If R is a local ring, the singular vectors in R^{27} form one orbit under the action of $E(\mathrm{E}_6, R)$. The next proposition describes the orbits into which it splits when we restrict the group action to $E(\mathrm{F}_4, R)$.

Proclaim 10. *Assume that R is a local ring and $g \in G(\mathrm{E}_6, R)$. There exists $x \in E(\mathrm{F}_4, R)$ such that the first column of xg coincides with the first column of the identity matrix in all entries, except possibly for λ_{15}.*

Proof. Let M denote the maximal ideal of R. First we show that there exists $x_1 \in E(\mathrm{F}_4, R)$ with $(x_1 g)_{11} = 1$.

Since R is local, we can choose $\lambda \in \Lambda$ such that $g_{\lambda 1}$ is invertible. We consider several cases separately.

(1) $\lambda \in \mathrm{B}$. Let $\alpha = \omega - \lambda \in \Phi_s$. Consider the element
$$h = X_\alpha((1 - g_{11})g_{\lambda 1}^{-1})g.$$
We have $X_\alpha(\xi) = x_{\alpha'}(\xi)x_{\alpha''}(\pm\xi)$, where $\alpha' = \begin{smallmatrix} 1 & * & * & * & 0 \\ & & & * & \end{smallmatrix}$, $\alpha'' = \begin{smallmatrix} 0 & * & * & * & 1 \\ & & & * & \end{smallmatrix}$. Hence, $h_{11} = g_{11} \pm (1 - g_{11})g_{\lambda 1}^{-1} g_{\lambda 1}$. By changing the sign of the argument of X_α if needed, we can arrange that $h_{11} = 1$.

(2) $\lambda \in \Gamma \setminus \{\lambda_{14}, \lambda_{15}\}$. Similarly, let $\alpha = \omega - \lambda$ (now $\alpha \in \Phi_l$). Consider the element $h = X_\alpha((1 - g_{11})g_{\lambda 1}^{-1})g$. As in case (1), by changing the sign of the argument of X_α if needed, we obtain $h_{11} = 1$.

(3) $\lambda = \lambda_1$. First, we get 1 in the entry 10 of the first column: put $\alpha = \lambda_{10} - \lambda_1 = 1231 \in \Phi_s$ and $h = X_\alpha((1 - g_{10,1})g_{11}^{-1})g$. By changing the sign of the argument, we may assume that $h_{10,1} = 1$, and then invoke case (1).

(4) $\lambda = \lambda_{-1}$. We can easily obtain 1 in the entry -6: put $\alpha = \lambda_{-6} - \lambda_{-1} = 0122 \in \Phi_l$, $h = X_\alpha((1 - g_{-6,1})g_{-1,1}^{-1})g$, and, by changing the sign of the argument if needed, we have $h_{-6,1} = 1$, which brings us to case (2) already discussed.

(5) $\lambda \in \Delta$. Similarly, it is easy to choose $\alpha \in \Phi_l$ such that $\lambda + \alpha \in \mathrm{B}$: for example, put $\alpha = 0122$ for $\lambda \in \{\lambda_{-10}, \lambda_{-9}, \lambda_{-8}, \lambda_{-7}\}$ and $\alpha = 2342$ for $\lambda \in \{\lambda_{-5}, \lambda_{-4}, \lambda_{-3}, \lambda_{-2}\}$. Then, consider $h = X_\alpha((1 - g_{\lambda+\alpha,1})g_{\lambda,1}^{-1})g$. By changing the sign of the argument if needed, we obtain $h_{\lambda+\alpha,1} = 1$, and now we can use case (1).

(6) Now we may assume that $g_{\lambda,1} \in M$ for any $\lambda \in \Lambda \setminus \{\lambda_{13}, \lambda_{14}, \lambda_{15}\}$. Note that the elements $g_{13,1}$, $g_{14,1}$, $g_{15,1}$ cannot be invertible simultaneously: otherwise $Q(g_{*1})$ is congruent to $\pm g_{13,1}g_{14,1}g_{15,1}$ modulo M, and hence, invertible. On the other hand, since the column g_{*1} is singular, we have $Q(g_{*1}) = 0$. Assume that $g_{14,1} \in R^*$. We know that at least one of the elements $g_{13,1}$, $g_{15,1}$ belongs to M. Suppose $g_{13,1} \in M$. Consider $\alpha = 0001 \in \Phi_s$, $\xi = (1 - g_{10,1})g_{14,1}^{-1}$, $h = X_\alpha(\xi)$. Since $X_\alpha(\xi) = x_{10000}(\xi)x_{00001}(\pm\xi)$, we have $h_{10,1} = g_{10,1} \pm \xi g_{14,1} \pm \xi g_{13,1} = g_{10,1} \pm (1 - g_{10,1}) \overset{0}{\pm} \xi g_{13,1}$. By changing the sign of ξ, we may assume that $h_{10,1} = 1 \pm \xi g_{13,1} \in R^*$, because $g_{13,1} \in M$. This means that we can apply case (1). Similarly, if $g_{15,1} \in M$, we take $\alpha = 0010 \in \Phi_s$ and $\xi = (1 - g_{12,1})g_{14,1}^{-1}$; hence, $h_{12,1} \in R^*$, where $h = X_\alpha(\pm\xi)g$, and we can apply case (2). The same argument applies to the last case where $g_{14,1} \in M$, because at least one of the elements $g_{13,1}$, $g_{15.1}$ is invertible.

Now we have $x_1 \in E(\mathrm{F}_4, R)$ and $y = x_1 g$ such that $y_{11} = 1$. First, we shall show that we can use downwards additions from the first element of the first column to put 0 into every entry except possibly λ_{15}. To start with, we obtain 0 in the entries from B: put $x_2 = \prod_{\lambda \in \mathrm{B}} X_{\lambda-\omega}(\pm y_{\lambda 1})$ and $z = x_2 y$. The signs here must be chosen so that the element $X_{\lambda-\omega}(\pm y_{\lambda 1})$ executes *subtraction* of the first row from the row λ with the coefficient $y_{\lambda 1}$ (with respect to the left action on matrices). Equivalently, the matrix entry $(X_{\lambda-\omega}(\pm y_{\lambda 1}))_{\lambda 1}$ must be equal to $-y_{\lambda 1}$, not to $y_{\lambda 1}$. It is clear that $z_{\lambda 1} = 0$ for all $\lambda \in B$.

Now we can put 0 in all entries in Γ except λ_{14} and λ_{15}: it suffices to consider $x_3 = \prod_{\lambda \in \Gamma} X_{\lambda-\omega}(\pm z_{\lambda 1})$ and $u = x_3 z$, again with a clever choice of the signs.

If $u_{14,1} \neq 0$, we consider $x_4 = X_{1232}(\pm u_{14,1})$ and $v = x_4 u$. We can choose the signs so as to obtain $v_{14,1} = 0$.

This proves that there exists a matrix x with $(xg)_{11} = 1$ and $(xg)_{\lambda 1} = 0$ for all $\lambda \in (\mathrm{B} \cup \Gamma) \setminus \{\lambda_{15}\}$. By Lemma 6, this implies that the other entries in the first column are also equal to 0. □

§14. Proof of Theorem 1

The next lemma summarizes the extraction of a root element.

Main lemma. *Assume that H is a subgroup in $G(\mathrm{E}_6, R)$ that contains $E(\mathrm{F}_4, R)$. Then either $H \le \overline{G}(\mathrm{F}_4, R)$, or H contains a nontrivial root unipotent $x_\alpha(\xi)$, where $\alpha \in \mathrm{E}_6 \setminus \Phi_l$ and $\xi \in R$.*

Proof. Suppose that $g \in H$ and $g \notin \overline{G}(\mathrm{F}_4, R)$. Lemma 9 shows that there exists a maximal ideal $M \in \mathrm{Max}(R)$ such that $F_M(g) \notin \overline{G}(\mathrm{F}_4, R_M)$. Since R_M is a local ring, Proposition 10 implies the existence of $x \in E(\mathrm{F}_4, R_M)$ such that the first column of $xF_M(g)$ coincides with that of the identity matrix in all entries except for λ_{15}. Since $E(\mathrm{F}_4, R_M) = \varinjlim E(\mathrm{F}_4, R_s)$, where the limit is taken over all $s \notin M$, there exists $s \in M$ and $x \in E(\mathrm{F}_4, R_s)$ such that the first column of $y = xF_s(g)$ coincides with that of the identity matrix in all entries except for λ_{15}. Obviously, $y \notin \overline{G}(\mathrm{F}_4, R_s)$. Now we can invoke Proposition 9 and finish the proof. □

Proof of Theorem 1. Suppose, as in Proposition 1, that A is the greatest ideal such that $\mathrm{E}(\mathrm{F}_4, R, A) \le H$. Let $\overline{H} = \rho_A^{\mathrm{E}_6}(H)$ be the image of H under the reduction homomorphism $\rho_A^{\mathrm{E}_6} : G(\mathrm{E}_6, R) \to G(\mathrm{E}_6, R/A)$. It is clear that $\overline{H}$ contains $E(\mathrm{F}_4, R/A)$, and the main lemma implies that either $\overline{H} \le \overline{G}(\mathrm{F}_4, R, A)$, or $\overline{H}$ contains a nontrivial elementary root unipotent $x_\alpha(\xi + A)$, where $\alpha \in \mathrm{E}_6 \setminus \mathrm{F}_4$, $\xi \in R$. We shall show that the latter is impossible. Indeed, express $x_\alpha(\xi) \in H\ G(\mathrm{F}_4, R, A)$ as a product $x_\alpha(\xi) = ab$ for some $a \in H$, $b \in G(\mathrm{F}_4, R, A)$. There exists a root $\beta \in \mathrm{E}_6 \setminus \mathrm{F}_4$ such that $\beta - \alpha \in \mathrm{F}_4$. Consider the commutator

$$[x_\alpha(\xi), x_{\beta-\alpha}(1)] = x_\beta(\pm\xi).$$

Substituting $x_\alpha(\xi) = ab$, we obtain

$$x_\beta(\pm\xi) = [ab, x_{\beta-\alpha}(1)] = {}^a[b, x_{\beta-\alpha}(1)][a, x_{\beta-\alpha}(1)].$$

The standard commutator formulas show that the first of the commutators on the right-hand side belongs to $E(\mathrm{E}_6, R, A)$, while the second belongs to H. Therefore, $x_\beta(\pm\xi) \in H$, and $\xi \notin A$. This contradicts the maximality of A. Hence, we always have $\overline{H} \le \overline{G}(\mathrm{F}_4, R/A)$, and Theorem 3 yields

$$H \le (\rho_A^{\mathrm{E}_6})^{-1}(\overline{G}(\mathrm{F}_4, R/A)) = \mathrm{CG}(\mathrm{F}_4, R, A) = N_G(\mathrm{E}(\mathrm{F}_4, R, A)).$$

This finishes the proof. □

References

[1] A. Borel, *Properties and linear representations of Chevalley groups*, Seminar on Algebraic Groups and Related Finite Groups (Inst. Adv. Study, Princeton, NJ, 1968/1969), Lecture Notes in Math., vol. 131, Springer-Verlag, Berlin–New York, 1970, pp. 1–55. MR0258838 (41:3484)

[2] N. Bourbaki, *Lie groups and Lie algebras*. Ch. 4–6, Springer-Verlag, Berlin, 2000. MR1890629 (2003a:17001)

[3] N. A. Vavilov, *Weight elements of Chevalley groups*, Algebra i Analiz **20** (2008), no. 1, 34–85; English transl., St. Petersburg Math. J. **20** (2009), no. 1, 23–57. MR2411968 (2009c:20086)

[4] N. A. Vavilov and M. R. Gavrilovich, *An A_2-proof of structure theorems for Chevalley groups of types E_6 and E_7*, Algebra i Analiz **16** (2004), no. 4, 54–87; English transl., St. Petersburg Math. J. **16** (2005), no. 4, 649–672. MR2090851 (2005m:20115)

[5] N. A. Vavilov, M. R. Gavrilovich, and S. I. Nikolenko, *The structure of Chevalley groups: A proof from the Book*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **330** (2006), 36–76; English transl., J. Math. Sci. (N. Y.) **140** (2007), no. 5, 626–645. MR2253566 (2007k:20108)

[6] N. A. Vavilov and A. Yu. Luzgarev, *Normalizer of the Chevalley group of type* E_6, Algebra i Analiz **19** (2007), no. 5, 37–64; English transl., St. Petersburg Math. J. **19** (2008), no. 5, 699–718. MR2381940 (2008m:20077)

[7] N. A. Vavilov, A. Yu. Luzgarev, and I. M. Pevzner, *A Chevalley group of type* E_6 *in the* 27-*dimensional representation*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **338** (2006), 5–68; English transl., J. Math. Sci. (N. Y.) **145** (2007), no. 1, 4697–4736. MR2354606 (2009b:20022)

[8] N. A. Vavilov and S. I Nikolenko, A_2-*proof of structure theorems for Chevalley groups of type* F_4, Algebra i Analiz **20** (2008), no. 4, 27–63; English transl. in St. Petersburg Math. J. **20** (2009), no. 4. MR2473743

[9] N. A. Vavilov and V. A. Petrov, *On overgroups of* $\mathrm{EO}(2l, R)$, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **272** (2000), 68–85; English transl., J. Math. Sci. (N. Y.) **116** (2003), no. 1, 2917–2925. MR1811793 (2001m:20078)

[10] ———, *On overgroups of* $\mathrm{Ep}(2l, R)$, Algebra i Analiz **15** (2003), no. 4, 72–114; English transl., St. Petersburg Math. J. **15** (2004), no. 4, 515–543. MR2068980 (2005d:20086)

[11] ———, *On overgroups* $\mathrm{EO}(n, R)$, Algebra i Analiz **19** (2007), no. 2, 10–51; English transl., St. Petersburg Math. J. **19** (2008), no. 2, 167–195. MR2333895 (2008d:20094)

[12] A. Yu. Luzgarev, *On overgroups* $E(\mathrm{E}_6, R)$ *and* $E(\mathrm{E}_7, R)$ *in minimal representations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **319** (2004), 216–243; English transl., J. Math. Sci. (N. Y.) **134** (2006), no. 6, 2558–2571. MR2117858 (2006k:20098)

[13] R. Steinberg, *Lectures on Chevalley groups*, Yale Univ., New Haven, Conn., 1968. MR0466335 (57:6215)

[14] C. Chevalley, *Sur certains groupes simples*, Tôhoku Math. J. (2) **7** (1955), 14–66. MR0073602 (17:457c)

[15] E. Abe, *Chevalley groups over commutative rings*, Radical Theory (Sendai, 1988), Uchida Rokakuho, Tokyo, 1989, pp. 1–23. MR0999577 (91a:20047)

[16] E. Abe and K. Suzuki, *On normal subgroups of Chevalley groups over commutative rings*, Tôhoku Math. J. (2) **28** (1976), no. 2, 185–198. MR0439947 (55:12828)

[17] M. Aschbacher, *The* 27-*dimensional module for* E_6. I–IV, Invent. Math. **89** (1987), no. 1, 159–195; J. London Math. Soc. **37** (1988), 275–293; Trans. Amer. Math. Soc. **321** (1990), 45–84; J. Algebra **131** (1990), 23–39. MR0892190 (88h:20045); MR0928524 (89a:20041); MR0986684 (90m:20044); MR1054997 (91f:20049)

[18] ———, *Some multilinear forms with large isometry groups*, Geom. Dedicata **25** (1988), no. 1–3, 417–465. MR0925846 (89c:20067)

[19] A. Bak, *Nonabelian* K-*theory*: *The nilpotent class of* K_1 *and general stability*, K-Theory **4** (1991), 363–397. MR1115826 (92g:19001)

[20] A. Bak and N. Vavilov, *Normality for elementary subgroup functors*, Math. Proc. Cambridge Philos. Soc. **118** (1995), no. 1, 35–47. MR1329456 (96d:20046)

[21] S. Berman and R. Moody, *Extensions of Chevalley groups*, Israel J. Math. **22** (1975), no. 1, 42–51. MR0390077 (52:10903)

[22] R. H. Dye, *Interrelations of symplectic and orthogonal groups in characteristic two*, J. Algebra **59** (1979), no. 1, 202–221. MR0541675 (81c:20028)

[23] ———, *On the maximality of the orthogonal groups in the symplectic groups in characteristic two*, Math. Z. **172** (1980), no. 3, 203–212. MR0581439 (81h:20060)

[24] ———, *Maximal subgroups of* $\mathrm{GL}_{2n}(K)$, $\mathrm{SL}_{2n}(K)$, $\mathrm{PGL}_{2n}(K)$, $\mathrm{PSL}_{2n}(K)$ *associated with symplectic polarities*, J. Algebra **66** (1980), no. 1, 1–11. MR0591244 (81j:20061)

[25] R. Hazrat, *Dimension theory and nonstable* K_1 *of quadratic modules*, K-Theory **27** (2002), no. 4, 293–328. MR1962906 (2004a:19005)

[26] R. Hazrat and N. A. Vavilov, K_1 *of Chevalley groups are nilpotent*, J. Pure Appl. Algebra **179** (2003), 99–116. MR1958377 (2004i:20081)

[27] Hong You, *Overgroups of symplectic group in linear group over commutative rings*, J. Algebra **282** (2004), no. 1, 23–32. MR2095570 (2005g:20076)

[28] ———, *Overgroups of classical groups over commutative rings in linear group*, Sci. China Ser. A **49** (2006), no. 5, 626–638. MR2250893 (2007d:20087)

[29] O. H. King, *On subgroups of the special linear group containing the special orthogonal group*, J. Algebra **96** (1985), no. 1, 178–193. MR0808847 (87b:20057)

[30] ———, *On subgroups of the special linear group containing the special unitary group*, Geom. Dedicata **19** (1985), no. 3, 297–310. MR0815209 (87c:20081)

[31] Shang Zhi Li, *Overgroups of* $\mathrm{SU}(n, K, f)$ *or* $\Omega(n, K, Q)$ *in* $\mathrm{GL}(n, K)$, Geom. Dedicata **33** (1990), no. 3, 241–250. MR1050412 (91g:11038)

[32] ———, *Overgroups of a unitary group in* $\mathrm{GL}(2, K)$, J. Algebra **149** (1992), no. 2, 275–286. MR1172429 (93e:20067)

[33] ———, *Overgroups in* $\mathrm{GL}(n, F)$ *of a classical group over a subfield of* F, Algebra Colloq. **1** (1994), no. 4, 335–346. MR1301157 (95i:20068)

[34] H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. (4) **2** (1969), no. 1, 1–62. MR0240214 (39:1566)

[35] V. A. Petrov, *Overgroups of unitary groups*, K-Theory **29** (2003), no. 3, 147–174. MR2028500 (2004m:20094)

[36] M. R. Stein, *Generators, relations and coverings of Chevalley groups over commutative rings*, Amer. J. Math. **93** (1971), no. 4, 965–1004. MR0322073 (48:437)

[37] G. Taddei, *Normalité des groupes élémentaires dans les groupes de Chevalley sur un anneau*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 693–710. MR0862660 (88a:20054)

[38] J. Tits, *Systèmes générateurs de groupes de congruence*, C. R. Acad. Sci. Paris Sér. A **283** (1976), 693–695. MR0424966 (54:12924)

[39] L. N. Vaserstein, *On normal subgroups of Chevalley groups over commutative rings*, Tôhoku Math. J. (2) **38** (1986), no. 2, 219–230. MR0843808 (87k:20081)

[40] N. A. Vavilov, *Structure of Chevalley groups over commutative rings*, Nonassociative Algebras and Related Topics (Hiroshima, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 219–335. MR1150262 (92k:20090)

[41] ———, *A third look at weight diagrams*, Rend. Sem. Mat. Univ. Padova **104** (2000), 201–250. MR1809357 (2001i:20099)

[42] ———, *An* A_3*-proof of structure theorems for Chevalley groups of types* E_6 *and* E_7, Internat. J. Algebra Comput. **17** (2007), no. 5–6, 1283–1298. MR2355697

Department of Mathematics and Mechanics, St. Petersburg State University, Universitetskiĭ Prospekt 28, Petrodvorets, 198504 St. Petersburg, Russia

E-mail address: mahalex@yandex.ru

Received 12/FEB/2007

Translated by THE AUTHOR