Алгебра и анализ
Том 20 (2008), № 6

St. Petersburg Math. J.
Vol. 20 (2009), No. 6, Pages 1003–1022
S 1061-0022(09)01082-6
Article electronically published on October 2, 2009

CATEGORIES OF MOTIVES FOR ADDITIVE CATEGORIES. II

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Abstract. This is a continuation of the paper by the same author published in this journal, v. 19 (2007), no. 6.

This paper is a continuation of the paper [4] and makes a whole with it. We retain the notation, the definitions, and the assumptions of [4], and we continue the numeration of sections, theorems and lemmas. The proofs of Theorems 2–4 in §6 of [4] were in fact missing; it was said only that one can argue as in [3]. Next, there were some inaccuracies in that section. For this reason, the first section of the present paper is again §6, and it replaces §6 of the first part.

§6. Category $\overline{M}$

In this section, the study of arbitrary objects of the category M will be reduced largely to the study of objects whose additive groups of the endomorphism rings are torsion-free. For this purpose we shall introduce a new category $\overline{M}$. We start with some comments on one of the notions introduced earlier, and we give a new definition.

In §5 we defined homomorphisms that factor through the torsion, and we proved some important properties of such homomorphisms. But even the notion of a homomorphism of this sort depends on the structure of the groups of homomorphisms from arbitrary objects of M to infinite direct sums of l-periodic objects (such direct sums can exist in the category M). Nothing was said about these groups of homomorphisms, and they can be different for two categories, even if these categories are identical in all other respects. For example, it can happen that in our initial category no infinite direct sums of periodic objects are allowed; we can adjoin them to the category artificially, but the groups of homomorphisms in such infinite direct sums (which in general are not direct products!) can be defined in various ways. But not many properties of such homomorphisms were in fact used in our proofs; for this reason it is natural to consider not only homomorphisms that factor through the torsion, but also homomorphisms that *could factor through the torsion* for suitable groups of homomorphisms to periodic objects and could be described in terms of the initial category. Recall that any homomorphism that factors through the torsion is divisible modulo the periodic part of the group of homomorphisms (Lemma 5); moreover, if a homomorphism $a : A \to B$ factors through $t(B)$, then, for any object C, in the group $\mathrm{Hom}_M(A, C)$ there exists the product of a with every homomorphism from $t(B)$ to $t(C)$ (the argument in §5 was based precisely on this property). Roughly speaking, we say that a homomorphism potentially factors through the torsion if it satisfies the above conditions.

Before giving the precise definition, we recall some notation. For an object A of the category M, we denote by A_l the l-periodic component of A and by $i_l^A : A_l \to A$,

2000 *Mathematics Subject Classification.* Primary 18E05.
Key words and phrases. Category, torsion, l-periodic objects, potential factoring through the periodic part.

$p_l^A : A \to A_l$ the corresponding injection and projection. For objects A and B of M we denote by $T(A,B)$ the group of periodic elements of the group $\mathrm{Hom}_M(A,B)$ and by $D(A,B)$ the group of all elements of the group $\mathrm{Hom}_M(A,B)$ that are divisible modulo $T(A,B)$ by all nonzero integers. The following property of the elements of the group D will play a significant role below.

Lemma 8. *If $d \in D(A,B)$ and $b \in \mathrm{Hom}(B,C)$, then $bd \in D(A,C)$, and moreover, $p_l^C(bd) = p_l^C b e_l^B d$ for every $l \in \mathcal{P}$.*

Proof. The first statement is obvious. Next, the left multiplication by $1 - e_l^B$ is an endomorphism of the group $D = D(A,B)$, and the kernel of this endomorphism coincides with the Sylow l-component $\mathrm{Hom}_M(A,B_l)$ of D. Therefore, the periodic part of the group $(1-e_l^B)D$ is an l-divisible group, and consequently, its factor group over the periodic part is also l-divisible (and even divisible). Thus, $(1-e_l^B)D$ is an l-divisible group. In particular, the element $(1-e_l^B)d$ is l-divisible, and the element $b(1-e_l^B)d = bd - be_l^B d$ is l-divisible together with $(1-e_l^B)d$. Hence, $p_l^C(bd - be_l^B d) = 0$. □

Obviously, the mapping φ_B^A that sends any element $a \in D(A,B)$ to the element $\prod_{l\in\mathcal{P}} p_l^B a$ of the direct product $\prod_{l\in\mathcal{P}} \mathrm{Hom}_M(A,B_l)$ is a homomorphism of groups. Moreover, it is a monomorphism. Indeed, since all elements of $\mathrm{Ker}\,\varphi_B^A$ are divisible modulo $T(A,B)$ and the intersection of $\mathrm{Ker}\,\varphi_B^A$ with $T(A,B) = \bigoplus \mathrm{Hom}_M(A,B_l)$ is trivial, it follows that $\mathrm{Ker}\,\varphi_B^A$ is a divisible subgroup of the group $\mathrm{Hom}_M(A,B)$, which implies that $\mathrm{Ker}\,\varphi_B^A = 0$, because of condition $(*{*}*)$.

We shall say that a homomorphism $h : A \to B$ potentially factors through the periodic part $t(B)$ of the object B if it belongs to the group $D(A,B)$, and the element $\prod_{l\in\mathcal{P}} x_l p_l^B h$ belongs to the image of the homomorphism φ_C^A for every object C of the category M and every homomorphism $x_l : B_l \to C_l$ ($l \in \mathcal{P}$). We denote by $\mathrm{Hom}^t(A,B)$ the set of all homomorphisms $h \in \mathrm{Hom}_M(A,B)$ that potentially factor through the periodic part $t(B)$ of the object B. Thus, if $h \in \mathrm{Hom}^t(A,B)$ and $x_l : B_l \to C_l$ are arbitrary homomorphisms, then there exists a homomorphism $g : A \to C$ such that

$$\prod_{l\in\mathcal{P}} x_l p_l^B h = \varphi_C^A(g) = \prod_{l\in\mathcal{P}} p_l^C g.$$

Moreover, this g obviously belongs to the group $\mathrm{Hom}^t(A,C)$.

Lemma 9. *The set $\mathrm{Hom}^t(A,B)$ is a subgroup of the group $\mathrm{Hom}_M(A,B)$. If $h \in \mathrm{Hom}^t(A,B)$, $c \in \mathrm{Hom}_M(C,A)$, and $f \in \mathrm{Hom}_M(B,F)$, then $hc \in \mathrm{Hom}^t(C,B)$ and $fh \in \mathrm{Hom}^t(A,F)$. In other terms, the sets $\mathrm{Hom}^t(A,B)$ constitute an ideal of the category M.*

Proof. The first statement is obvious. Next, it is clear that $fh \in D(A,F)$ and $hc \in D(C,B)$. The products $p_l^F f i_l^B$ are homomorphisms from B_l to F_l. Using Lemma 8 and the definition of the group $\mathrm{Hom}^t(A,B)$, we see that for any homomorphism $y_l : F_l \to C_l$, the element

$$\prod_{l\in\mathcal{P}} y_l p_l^F fh = \prod_{l\in\mathcal{P}} y_l p_l^F f e_l^B p_l^B h = \prod_{l\in\mathcal{P}} y_l (p_l^F f i_l^B) p_l^B h$$

is contained in the image of the homomorphism φ_C^A, which means that $fh \in \mathrm{Hom}^t(A,F)$. Finally, for any $z_l : B_l \to F_l$ there exists $g \in D(A,F)$ such that $\varphi_F^A g = \prod_{l\in\mathcal{P}} z_l p_l^B h$; but then

$$\prod_{l\in\mathcal{P}} z_l p_l^B hc = \left(\prod_{l\in\mathcal{P}} z_l p_l^B h\right) c = (\varphi_F^A g)c = \left(\prod_{l\in\mathcal{P}} p_l^F g\right) c = \prod_{l\in\mathcal{P}} p_l^F gc = \varphi_F^C(gc),$$

so that $hc \in \mathrm{Hom}^t(C,B)$. □

Now we define the category $\overline{M}$. Its objects are the same as those of M, and the homomorphism group $\mathrm{Hom}_{\overline{M}}(A,B)$ is defined as the factor group of $\mathrm{Hom}_M(A,B)$ over the subgroup $\mathrm{Hom}^t(A,B)$ of homomorphisms that potentially factor through the torsion. We denote by $P : M \to \overline{M}$ the canonical functor that maps any object A of M to the same object A viewed as an object of $\overline{M}$.

The following statements show that the category $\overline{M}$ is a good approximation to M, but at the same time it seems to be simpler than M. In the formulation of some of these statements we use the term "periodic objects"; we say that an object A of M is *periodic* if it coincides with its periodic part, which means that the inclusion $i^A : t(A) \to A$ is an isomorphism. Clearly, an object A is periodic if and only if it is the direct sum of l-periodic objects A_l.

Theorem 2. *Let A, B be objects of the category M. The objects $P(A)$ and $P(B)$ are isomorphic in the category $\overline{M}$ if and only if in M there exist direct decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that the objects A_1 and A_2 are isomorphic in M, and B_1 and B_2 are periodic objects.*

Corollary. *Let A be an object of M. The object $P(A)$ is isomorphic in $\overline{M}$ to the zero object if and only if A is a periodic object of M.*

Indeed, the "only if" part follows immediately from Theorem 2, and the "if" part is obvious, because the identity automorphism $1_A : A \to A$ of the periodic object A coincides with the homomorphism $i^A : A = t(A) \to A$, so that it factors through the torsion, and consequently, in $\overline{M}$ this automorphism becomes the zero automorphism of the object $P(A)$.

Theorem 3. *Let objects A, B_1, B_2 of M be such that the object $P(A)$ is isomorphic in $\overline{M}$ to the direct sum of $P(B_1)$ and $P(B_2)$. Then in M there exists a direct decomposition $A = B_1' \oplus B_2'$ such that the object $P(B_1')$ is isomorphic in $\overline{M}$ to $P(B_1)$, and the object $P(B_2')$ is isomorphic to $P(B_2)$.*

Corollary. *Let A be an object of M. The object $P(A)$ is indecomposable in $\overline{M}$ if and only if one of the summands X, Y in any direct decomposition $A = X \oplus Y$ of A in M is a periodic object.*

Theorem 4. *For any objects A, B of M, the group $\mathrm{Hom}_{\overline{M}}(A,B)$ is a torsion-free Abelian group. Moreover, if the rank of the factor group $D(A,B)/\mathrm{Hom}^t(A,B)$ is finite, then the rank of the group $\mathrm{Hom}_{\overline{M}}(A,B)$ is also finite.*

Theorem 4 follows immediately from Lemma 5 of the paper [4] and the assumption $(*{*}*)$, which states that the group $\mathrm{Hom}_M(A,B)$ contains no nonzero divisible elements. As to Theorems 2 and 3, they will be proved in the following sections.

§7. The category N

Till the end of the following section, we fix an object A of the category M. We consider the category $N = N(A)$ whose objects are A, $t(A)$, and all their direct summands in M. In particular, all objects A_l are objects of N. To complete the description of the category N, we need to describe the homomorphism groups and give the rules of multiplication of homomorphisms. We set:

$$\mathrm{Hom}_N(A,A) = \mathrm{Hom}_M(A,A), \qquad \mathrm{Hom}_N(t(A),t(A)) = \mathrm{Hom}_M(t(A),t(A)),$$
$$\mathrm{Hom}_N(t(A),A) = \mathrm{Hom}_M(t(A),A), \qquad \mathrm{Hom}_N(A,t(A)) = \varphi_A^A(\mathrm{Hom}^t(A,A)).$$

The products of elements of the first three groups are the same as in M. In particular, the endomorphism rings of A and $t(A)$ remain the same as in M; therefore, any object of

N is a direct summand of A or $t(A)$ not only in M (which is true by the definition of N), but also in N. The endomorphism rings of the direct summands of these objects (in particular, the endomorphism rings of the objects A_l) do not change either. As above, we denote by E, E_l the endomorphism rings of the objects A, A_l, which are the same for both categories. Then

$$\operatorname{End}_N(t(A)) = \operatorname{End}_M(t(A)) = \operatorname{End}_M\left(\bigoplus_{l\in\mathcal{P}} A_l\right) = \prod_{l\in\mathcal{P}} E_l.$$

The above argument shows that it suffices to define homomorphisms and their products only for the objects A and $t(A)$; then they will be defined automatically for all objects of N. Hence, to complete the definition of the category N, it remains to define the products in which one of the factors b' belongs to $\operatorname{Hom}_N(A, t(A))$. Moreover, since, by Lemma 4 in [4], every homomorphism of $t(A)$ to A is the product of an automorphism of the object $t(A)$ and the inclusion $i : t(A) \to A$, it suffices to define the products $b'a$, $b'u$, xb', ib', where

$$a : A \to A, \quad u : t(A) \to A, \quad x_l : A_l \to A_l, \quad x = \prod_{l\in\mathcal{P}} x_l : t(A) \to t(A)$$

are homomorphisms in M, and consequently, in N.

Since b' is an element of $\operatorname{Hom}_N(A, t(A)) = \varphi_A^A(\operatorname{Hom}^t(A, A))$, there is a unique homomorphism $b \in \operatorname{Hom}^t(A, A)$ such that $b' = \varphi_A^A(b)$. Since the groups $\operatorname{Hom}^t(*, *)$ constitute an ideal of the category M, we have $ba \in \operatorname{Hom}^t(A, A)$ and $bu \in \operatorname{Hom}^t(t(A), A) \subseteq \operatorname{Hom}_M(t(A), A)$. By [4, Lemma 4] there exists a unique homomorphism $x : t(A) \to t(A)$ such that $bu = iv$. We set

$$b'a = \varphi_A^A(ba), \quad b'u = v, \quad ib' = b.$$

Finally, since $b \in \operatorname{Hom}^t(A, A)$, the element $\prod_{l\in\mathcal{P}} x_l p_l b$ belongs to the image of the homomorphism φ_A^A (by the definition of the group $\operatorname{Hom}^t(A, A)$); moreover, this element belongs to the group

$$\varphi_A^A\big(\operatorname{Hom}^t(A, A) = \operatorname{Hom}_N(A, t(A)\big),$$

and we take it for xb'.

Let $a \in \operatorname{Hom}^t(A, A)$ be a homomorphism of the category M that factors through the torsion only potentially; as a homomorphism of the category N, the homomorphism a indeed factors through the periodic part $t(A)$ of the object A. In §5 we defined weakly nilpotent endomorphisms of objects of the category M. We say that an endomorphism a of an object A of M is potentially weakly nilpotent if there exists an integer $n > 0$ such that the homomorphism a^n potentially factors through the periodic part $t(A)$ of the object A and for each prime integer l there exists an integer $n_l > 0$ such that the homomorphism $p_l a^{n_l} i_l$ (which is the restriction of a^{n_l} to A_l) is actually the zero homomorphism. It is obvious that any potentially weakly nilpotent endomorphism of M is a weakly nilpotent endomorphism of N.

Lemma 10. *Suppose that an endomorphism a of an object A of M potentially factors through the periodic part $t(A)$, and let b be an endomorphism of A that commutes with a. Then there exists an idempotent endomorphism d of A with the following properties: d potentially factors through the periodic part $t(A)$ of A; d commutes with b; and $a(1-d)$ is a potentially weakly nilpotent endomorphism of A.*

Proof. In terms of the category N, this lemma is Lemma 7 in [4]. □

§8. Sums of infinite series of weakly nilpotent endomorphisms

In the category M, not only finite sums of endomorphisms of an object make sense, but also certain infinite sums can be calculated. More precisely, let $f(x) = c_0 + c_1x + c_2x^2 + \cdots$ be a formal series with integral coefficients, and let h be a potentially weakly nilpotent endomorphism of A; in this situation we shall introduce the endomorphism $f(h) \in E(A) = \mathrm{End}_M(A)$. First, we look at the structure of this endomorphism in the case where the formal series is in fact a polynomial, which means that almost all coefficients of the series are equal to 0. For convenience, we restrict ourselves to the category $N = N(A)$ defined in the preceding section and containing all necessary objects and homomorphisms.

Lemma 11. *Let $h \in E(A)$ and let $h' : A \to t(A)$ be a homomorphism of the category $N(A)$ such that $h^n = ih'$ for an integer $n \geq 1$. Then for each $m \geq 1$ we have*

$$h^{n+m} = i\biggl(\prod_{l\in\mathcal{P}} p_l h^m i_l\biggr)h'.$$

Proof. By [4, Lemma 4], we have $h^m i = i\prod_{l\in\mathcal{P}} p_l h^m i_l$, whence

$$h^{m+n} = h^m h^n = h^m ih' = i\biggl(\prod_{l\in\mathcal{P}} p_l a^m i_l\biggr)h'. \qquad \square$$

Corollary. *Let $m, n \geq 1$, and let $f(x) = c_0 + c_1x + \cdots + c_{n+m}x^{n+m}$ be a polynomial with integral coefficients. If $h \in E(A)$ and $h' \in \mathrm{Hom}_N(A, t(A))$ are homomorphisms such that $h^n = ih'$, then*

$$f(h) = \sum_{s=0}^{n} c_s h^s + i\biggl(\prod_{l\in\mathcal{P}} \Bigl(\sum_{t=1}^{m} c_{n+t} p_l h^t i_l\Bigr)\biggr)h'.$$

Now, let $f(x) = c_0 + c_1x + c_2x^2 + \cdots$ be a formal power series with integral coefficients, and let h be a potentially weakly nilpotent endomorphism of A; then there exists $n \geq 1$ and $h' \in \mathrm{Hom}_N(A, t(A))$ such that $h^n = ih'$; moreover, for each $l \in \mathcal{P}$ the homomorphism $p_l a^{n+m} i_l$ becomes zero if m is sufficiently large, so that the sum $\sum_{m=1}^{\infty} c_{n+m} p_l h^m i_l$ is in fact finite. Therefore, the endomorphism

$$\sum_{s=0}^{n} c_s h^s + i\biggl(\prod_{l\in\mathcal{P}} \Bigl(\sum_{m=1}^{\infty} c_{n+m} p_l h^m i_l\Bigr)\biggr)h'$$

of the object A is well defined; we take this endomorphism for $f(h)$. The above corollary shows that, in the case where the formal power series $f(x)$ degenerates into a polynomial, this definition gives the usual value to $f(h)$. It is not difficult to prove that the values of the sum and the product of two integral formal power series at any potentially weakly nilpotent point are equal, respectively, to the sum and the product of the values of the series at this point.

The proofs of the following lemmas are based on the possibility of summation of infinite power series of weakly nilpotent endomorphisms.

Lemma 12. *Let an endomorphism e of the object A be such that the homomorphism $e^2 - e$ potentially factors through the torsion $t(A)$ of A. Then there exists an idempotent endomorphism e' of A such that $e - e'$ potentially factors through $t(A)$.*

Proof. Using Lemma 7 in [4] with $b = e$ and $a = e^2 - e$, we find an idempotent endomorphism d that commutes with e, potentially factors through $T(A)$, and is such that

$(e^2-e)(1-d)$ is a potentially weakly nilpotent endomorphism. Set $g=e(1-d)$; since e and d commute, and the endomorphism $1-d$ is idempotent together with d, we see that

$$g^2-g=e^2(1-d)^2-e(1-d)=(e^2-e)(1-d)$$

is a potentially weakly nilpotent endomorphism. Moreover, it is obvious that $g-e$ potentially factors through $T(A)$. Let $c_1y+c_2y^2+\cdots$ be the Taylor series for $(1-(1-4y)^{-1/2})/2$; it is well known that all coefficients of this series are integers (Catalan numbers). We denote by h the potentially weakly nilpotent endomorphism g^2-g and set

$$e'=(1-2h)(c_1h+c_2h^2+\cdots)+g$$

(the infinite sum is well defined because the endomorphism h is potentially weakly nilpotent). In a standard way we check that the endomorphism e' is idempotent (see, e.g., [5, Chapter 18]). Since the endomorphisms d and h potentially factor through the torsion, Lemma 9 implies that the endomorphism

$$e-e'=e-g-(1-2h)(c_1+c_2h+\cdots)h=de-(1-2h)(c_1+c_2h+\cdots)h$$

also potentially factors through the periodic part $t(A)$ of the object A. □

Lemma 13. *Let A be an object of the category M, and let $a\in E(A)$ be an endomorphism of A such that 1_A-a potentially factors through $t(A)$. Then there exists a direct decomposition $A=A_1\oplus A_2$ and an endomorphism b of the object A with the following properties. Let $j:A_1\to A$ and $q:A\to A_1$ be, respectively, the canonical injection of a direct summand into the direct sum and the canonical projection of the direct sum onto a summand, and let $d=jq$; then:*

1) *A_2 is a subobject (and moreover, a direct summand) of the periodic part $t(A)$ of the object A;*
2) *the endomorphisms 1_A-b and 1_A-d potentially factor through $t(A)$;*
3) *the restrictions qaj and qbj of the endomorphisms a,b to the direct summand A_1 are mutually inverse automorphisms of A_1.*

Proof. By [4, Lemma 7], there exists an idempotent endomorphism e of A that commutes with a and is such that e potentially factors through the periodic part $t(A)$ of A and $h=(1_A-a)(1_A-e)$ is a potentially weakly nilpotent endomorphism of A. The latter identity implies that

$$1_A-h=a+e-ae=a+e-ea.$$

Since h is a potentially weakly nilpotent endomorphism of A, the endomorphism $b=1_A+h+h^2+\cdots$ is well defined; observe that the endomorphism

$$1_A-b=h(-1_A-h-h^2-\cdots)$$

potentially factors through $t(A)$ together with h (see Lemma 9). Obviously, we have

$$b(1_A-h)=(1_A-h)b=1_A,$$

and keeping in mind that $ae=ea$, we can rewrite these identities in the form

$$b(1_A-e)a=1_A+be,\qquad a(1_A-e)b=1_A+eb.$$

Let $A=A_1\oplus A_2$ be the direct decomposition corresponding to the idempotent endomorphism e, and let

$$j:A_1\to A,\quad j_2:A_2\to A,\quad q:A\to A_1,\quad q_2:A\to A_2$$

be the canonical injections and projections for this decomposition, so that

$$d=jq=1_A-e,\quad j_2q_2=e,\quad qj=1_{A_1},\quad qe=0,\quad ej=0.$$

We have

$$(qaj)(qbj) = qa(1_A - e)bj = q(1_A + eb)j = q \cdot 1_A \cdot j + (qe)bj = 1_A,$$
$$(qbj)(qaj) = qb(1_A - e)aj = q(1_A + be)j = q \cdot 1_A \cdot j + qb(ej) = 1_A,$$

which means that qaj and qbj are mutually inverse automorphisms of A_1. Finally, since e potentially factors through the periodic part of A, there exists a homomorphism $e' \in \mathrm{Hom}_N(A, t(A))$ such that $e = ie'$; therefore, $j_2 = j_2 q_2 j_2 = ej_2 = i(e'j_2)$, whence A_2 is a subobject of $t(A)$. □

§9. On periodic objects

Recall that an object A of M is said to be periodic if it coincides with its periodic part, i.e., if the inclusion $i^A : t(A) \to A$ is an isomorphism. It is clear that an object A is periodic if and only if it decomposes into a direct sum of l-periodic objects A_l.

Lemma 14. *Let A be an object of the category M and let B be a direct summand of A and, at the same time, a subobject of $t(A)$. Then B is a periodic object.*

Proof. We start with a generalization of Lemma 4 in [4] (there it was assumed that $B = A$).

Lemma 15. *If $f \in \mathrm{Hom}_M(A, B)$, then*

$$fi^A = i^B \prod_{l \in \mathcal{P}} p_l^B f i_l^A.$$

Thus, the restriction fi^A of any homomorphism $f : A \to B$ to the torsion $t(A)$ of the object A factors through the torsion of the object B.

Proof. Since both sides of the identity to be proved are homomorphisms of the direct sum $t(A) = \bigoplus A_l$ to B, it suffices to verify that their restrictions to the direct summands coincide. The restriction fi_l^A of the left-hand side to A_l is contained in the group $\mathrm{Hom}_M(A_l, B)$, which is l-periodic and, consequently, coincides with the l-component of its periodic part; by condition $(**)$ the endomorphism e_l^B is a left unit of fi_l^A, so that $fi_l^A = e_l^B f i_l^A = i_l^B p_l^B f i_l^A$; it is obvious that the restriction of the right-hand side to A_l is equal to the same homomorphism. □

We return to the proof of Lemma 14. Let $j : B \to A$ and $q : A \to B$ be, respectively, the canonical injection of a summand into the direct sum and the canonical projection of the direct sum onto a summand. Then $qj = 1_B$. Moreover, since B is a subobject of $t(A)$, there exists a homomorphism $j' : B \to t(A)$ such that $j = i^A j'$. Next, by Lemma 15 there is a homomorphism $q' : t(A) \to t(B)$ such that $qi^A = i^B q'$. It follows that

$$i^B q' j' = q i^A j' = qj = 1_B.$$

Multiplying both sides of this identity from the right by i^B, we obtain

$$i^B q' j' i^B = 1_B \cdot i^B = i^B \cdot 1_{t(B)}.$$

Since i^B is a monomorphism, the above relation implies that $q'j'i^B = 1_{t(B)}$. Thus, $q'j'$ is the left and the right inverse of the homomorphism i^B. We conclude that i^B is an isomorphism, which means that B is a periodic object. □

§10. Proofs of Theorems 2 and 3

Proof of Theorem 2. Let objects A, B of the category M be such that the objects $P(A)$ and $P(B)$ are isomorphic in $\overline{M}$, and let $f : A \to B$, $g : B \to A$ be homomorphisms of M such that their classes $P(f)$, $P(g)$ modulo the homomorphisms that potentially factor through the torsion are mutually inverse isomorphisms in $\overline{M}$. Then the endomorphism gf of A and the endomorphism fg of B differ from the identities only in summands that potentially factor through the torsion. Let $A = A_1 \oplus A_2$ and $b \in E(A)$ be the direct decomposition and the endomorphism constructed as in Lemma 13 for the endomorphism $gf \in E(A)$; we recall their properties. Let, as above, $j : A_1 \to A$ and $q : A \to A_1$ be the canonical injection and projection, and let $d = qj$; in this notation, $qgfj$ and qbj are mutually inverse automorphisms of A_1, the endomorphisms $1_A - b$ and $1_A - d$ potentially factor through $t(A)$, and A_2 is a subobject of $t(A)$.

The product $qbjqgfj$ of the homomorphisms $qbjqg : B \to A_1$, $fj : A_1 \to B$ is the identity automorphism of A_1, and consequently, their product in the reverse order $e = fjqbjqg = fdbdg$ is an idempotent endomorphism of B. Let $B = B_1 \oplus B_2$ be the direct decomposition corresponding to the idempotent endomorphism e, and let $k_s : B_s \to B$, $r_s : B \to B_s$ be the canonical injections and projections for this decomposition ($s = 1, 2$); in particular, this implies that

$$r_1 k_1 = 1_{B_1}, \quad k_1 r_1 = e.$$

We show that the homomorphisms $r_1 fj : A_1 \to B_1$ and $qbdgk_1 : B_1 \to A_1$ are mutually inverse isomorphisms; indeed,

$$\begin{aligned}
(r_1 fj)(qbdgk_1) &= r_1 fdbdgk_1 = r_1 ek_1 = r_1(k_1 r_1)k_1 = 1_{B_1},\\
(qbdgk_1)(r_1 fj) &= (qbdg)(k_1 r_1)(fj) = (qbdg)e(fj)\\
&= (qbdg)(fdbdg)(fj) = ((qbj)(qgfj))^2 = 1_{A_1}^2 = 1_{A_1}.
\end{aligned}$$

Next, the endomorphisms b, d, and fg are congruent to the identical isomorphisms modulo homomorphisms that potentially factor through the torsion. This means that there exist homomorphisms $h_1, h_2 \in \operatorname{Hom}^t(A, A)$ and $h_3 \in \operatorname{Hom}^t(B, B)$ such that $b = 1_A + h_1$, $d = 1_A + h_2$, and $fg = 1_B + h_3$. Now Lemma 9 implies that

$$\begin{aligned}
1_B - e &= 1_B - fdbdg = 1_B - f(1_A + h_2)(1_A + h_1)(1_A + h_2)g\\
&= 1_B - fg - f(2h_2 + h_1 + h_2^2 + h_1 h_2 + h_2 h_1 + h_2 h_1 h_2)g\\
&= h_3 - f(2h_2 + h_1 + h_2^2 + h_1 h_2 + h_2 h_1 + h_2 h_1 h_2)g \in \operatorname{Hom}^t(B, B).
\end{aligned}$$

Thus, there exists a homomorphism $c' : B \to t(B)$ such that $k_2 r_2 = 1_B - e = i^B c'$, which shows that $k_2 = k_2 r_2 k_2 = i^B(c' k_2)$, and this means that B_2 is a subobject of the periodic part $t(B)$ of the object B.

To complete the proof of Theorem 2 it remains to apply Lemma 14. □

Proof of Theorem 3. Let objects A, B_1, B_2 of the category M be such that the object $P(A)$ is isomorphic in the category $\overline{M}$ to the direct sum of the objects $P(B_1)$ and $P(B_2)$, and let

$$\bar{j}_s : P(B_s) \to P(A), \quad \bar{q}_s : P(A) \to P(B_s) \qquad (s = 1, 2)$$

be the canonical injections and projections for this direct decomposition. Recall that the homomorphisms of the category $\overline{M}$ are classes of homomorphisms of the category M modulo homomorphisms that potentially factor through the torsion; let e be any endomorphism of the object A in the category M, the class of which is equal to $\bar{e} = \bar{q}_1 \bar{j}_1$. Since $\bar{e}$ is an idempotent endomorphism of the object $P(A)$ of the category $\overline{M}$, we see that e is idempotent modulo torsion, which means that the endomorphism $e^2 - e$ potentially factors through $t(A)$. By Lemma 12 there exists an idempotent endomorphism e' of A such that $e - e'$ potentially factors through the torsion, i.e., the class $P(e')$ of the endomorphism e' in the category $\overline{M}$ coincides with the class $\bar{e}$ of the endomorphism e. Let $A = B_1' \oplus B_2'$ be the direct decomposition corresponding to the idempotent endomorphism e' of the object A, and let $j_s' : B_s' \to A$ and $q_s' : A \to B_s'$ be the canonical injections and projections for this direct decomposition. We denote by $\bar{j}_s' = P(j_s')$, $\bar{q}_s' = P(q_s')$ the classes of j_s', q_s' in the category $\overline{M}$ $(s = 1, 2)$. Then the homomorphisms

$$(\bar{q}_1\bar{j}_1')(\bar{q}_1'\bar{j}_1) = \bar{q}_1\bar{e}\bar{j}_1 = \bar{q}_1\bar{j}_1, \qquad (\bar{q}_1'\bar{j}_1)(\bar{q}_1\bar{j}_1') = \bar{q}_1'\bar{e}\bar{j}_1' = \bar{q}_1'\bar{j}_1'$$

are the identity automorphisms of the objects B_1 and B_1' of $\overline{M}$, so that $\bar{q}_1'\bar{j}_1 : B_1 \to B_1'$ and $\bar{q}_1\bar{j}_1' : B_1' \to B_1$ are mutually inverse isomorphisms of $\overline{M}$. Similarly, the identities

$$(\bar{q}_2\bar{j}_2')(\bar{q}_2'\bar{j}_2) = \bar{q}_2(1-\bar{e})\bar{j}_2 = \bar{q}_2\bar{j}_2, \qquad (\bar{q}_2'\bar{j}_2)(\bar{q}_2\bar{j}_2') = \bar{q}_2'(1-\bar{e})\bar{j}_2' = \bar{q}_2'\bar{j}_2'$$

show that $P(B_2)$ and $P(B_2')$ are isomorphic objects of the category $\overline{M}$. □

§11. The categories $\overline{M}_l$, $\overline{M}_l^c$, and $\overline{M}_\infty$

The theory developed in the paper [2] for the category of torsion-free Abelian groups of finite rank made it possible to reduce many problems concerning such groups to similar problems for the "localizations" of this category and to a comparison of the images of groups in different localizations. But in fact all results of that paper describe the endomorphism rings of torsion-free Abelian groups of finite rank, rather than the groups themselves; it was important only that the additive groups of these endomorphism rings are torsion-free Abelian groups of finite rank. In accordance with Theorem 4, the endomorphism rings of objects of $\overline{M}$ satisfy the above condition if $D(A, B)/\operatorname{Hom}^t(A, B)$ are groups of finite rank; therefore, we can almost automatically extend the theory developed in [2] to the category $\overline{M}$, finally obtaining essential information about the category of motives M itself. We do this in this section.

Let Λ be a commutative associative ring with 1; denote by $\overline{M}_\Lambda'$ the category with the same objects as $\overline{M}$ and with the groups of homomorphisms equal to the tensor products of the groups of homomorphisms in $\overline{M}$ with Λ:

$$\operatorname{Hom}_{\overline{M}_\Lambda'}(A, B) = \operatorname{Hom}_{\overline{M}}(A, B) \otimes \Lambda.$$

Next, let $\overline{M}_\Lambda$ be the category of motives for the category $\overline{M}_\Lambda'$, i.e., the category obtained from $\overline{M}_\Lambda'$ by adding "imaginary" direct summands, as was described in §3 (see [4]). The objects of the category $\overline{M}_\Lambda$ are pairs (A, e), where A is an object of the category $\overline{M}$, and e is an idempotent endomorphism in the ring $\operatorname{End}_{\overline{M}_l'}(A) = \operatorname{End}_{\overline{M}}(A) \otimes \Lambda$, and

$$\operatorname{Hom}_{\overline{M}_\Lambda}((A, e), (B, d)) = d(\operatorname{Hom}_{\overline{M}_l'}(A, B))e.$$

Clearly, the category $\overline{M}$ coincides with the category $\overline{M}_\mathbb{Z}'$. Moreover, the category $\overline{M}_\mathbb{Z}$ coincides with $\overline{M}$. Indeed, by Lemma 12, every idempotent endomorphism $\bar{e}$ of $\overline{M}$ can be lifted to an idempotent endomorphism e of M which splits in M, and consequently, the initial endomorphism $\bar{e}$ splits in $\overline{M}$; therefore, we have no need to add "imaginary" direct summands to the category $\overline{M}_\mathbb{Z}' = \overline{M}$.

We shall shorten the notation in the following way:

$$\overline{M}_l = \overline{M}_{\mathbb{Z}_l}(\mathfrak{A}), \quad \overline{M}_0 = \overline{M}_\mathbb{Q}, \quad \overline{M}_l^c = \overline{M}_{\mathbb{Q}_l}, \quad \overline{M}_\infty = \overline{M}_\mathbb{C}$$

(as usual, $\mathbb{Z}, \mathbb{Z}_l, \mathbb{Q}, \mathbb{Q}_l, \mathbb{C}$ stay for the rings of integers and l-adic integers, and the fields of rational, l-adic, and complex numbers). Since the algebraic closures of the fields $\mathbb{Q}_l$ are isomorphic to the field of complex numbers, there exist (noncanonical) embeddings

$\mathbb{Q}_l \hookrightarrow \mathbb{C}$; these embeddings and the natural inclusions $\mathbb{Z} \hookrightarrow \mathbb{Q}, \mathbb{Z}_l$ and $\mathbb{Q}, \mathbb{Z}_l \hookrightarrow \mathbb{Q}_l$ induce functors represented by the arrows of the commutative diagram

$$\begin{tikzcd}
\overline{M} \arrow[r, "\bar{F}_0"] \arrow[d, "\bar{F}_l"'] & \overline{M}_0 \arrow[r, equal] \arrow[d, "\bar{G}_{0,l}"'] & \overline{M}_0 \arrow[d, "\bar{G}_0"'] \\
\overline{M}_l \arrow[r, "\bar{G}'_l"] & \overline{M}_l^c \arrow[r, "\bar{H}_l"] & \overline{M}_\infty
\end{tikzcd}$$

We denote by $G_l : \overline{M}_l \to \overline{M}_\infty$ the composition $\bar{H}_l \bar{G}'_l$ of functors.

We shall say that objects A, B of the category $\overline{M}$ belong to the same genus if the objects $\bar{F}_l(A)$ and $\bar{F}_l(B)$ are isomorphic in $\overline{M}_l$ for all $l \in \mathcal{P}_0$.

Theorem 5. 1. *The Krull–Schmidt theorem holds true in the categories $\overline{M}_0, \overline{M}_l$, and $\overline{M}_l^c$ (but not in $\overline{M}$!): every object of each of these categories decomposes into the direct sum of a finite number of indecomposable objects, and for any two such decompositions of the same object, there exists an automorphism of this object that transforms the first decomposition to the second.*

2. *If A, B are objects of the category $\overline{M}$ that belong to the same genus, and $A = X \oplus Y$, then there exists a direct decomposition $B = X' \oplus Y'$ such that X', X belong to the same genus and Y', Y belong to the same genus.*

3. *If A, B are objects of the category $\overline{M}$ that belong to the same genus, and the object A is indecomposable, then the object B is also indecomposable.*

4. *The functors $\bar{G}_{0,l}$, $\bar{G}_0$, $\bar{H}_l$ are injective in the following sense: under their action, nonisomorphic objects remain nonisomorphic.*

5. *For each $l \in \mathcal{P}_0$, choose an object X_l of the category $\overline{M}_l$. The existence of an object A in $\overline{M}$ such that $\bar{F}_l(A) \approx X_l$ for all $l \in \mathcal{P}_0$ is equivalent to the following:*

(i) *the objects $\bar{G}_l(X_l)$ and $\bar{G}_0(X_0)$ of $\overline{M}_\infty$ are isomorphic for all primes l;*

(ii) *there exists an object B of $\overline{M}$ such that $\bar{F}_l(B) \approx X_l$ for almost all (all but a finite number of) $l \in \mathcal{P}$.*

Remark. By statement 4, objects of $\overline{M}_l^c$ become isomorphic under the action of $\bar{H}_l$ only if they are isomorphic already in $\overline{M}_l^c$. Since $\bar{G}_0 = \bar{H}_l \bar{G}_{0,l}$ and $\bar{G}_l = \bar{H}_l \bar{G}'_l$, condition (i) in statement 5 is equivalent to the following condition that involves only canonical functors: for every prime integer l the objects $\bar{G}'_l(X_l)$ and $\bar{G}_{0,l}(X_0)$ of the category $\overline{M}_l^c$ are isomorphic.

Proof. Actually, this theorem states some properties of the endomorphism rings of objects of the categories under consideration, rather than properties of the objects. The statements are essentially almost the same as the principal results of the paper [2], though they were proved in [2] only for the category of torsion-free Abelian groups of finite rank. Unfortunately, Abelian groups themselves, not their endomorphism rings, appear often in the arguments in [2]. For example, an Abelian group satisfying the requirement of statement 5 was defined in [2] as the intersection of certain subgroups of the additive group of some rational vector space, and such argumentation is impossible in a more general situation such as ours. For this reason, here we reproduce the proof from [2], making the necessary changes in it.

We pass to the proof of the theorem. Statement 3 is a particular case of statement 2. Next, the Krull–Schmidt theorem for an object is equivalent to this theorem for the left ideals of the endomorphism ring of that object. But the endomorphism rings of objects of the categories $\overline{M}_0$ and $\overline{M}_l^c$ are finite-dimensional algebras over fields, and the classical Krull–Schmidt theorem is valid for them. As to the category $\overline{M}_l$, the endomorphism ring of any object of this category is an algebra of finite rank over $\mathbb{Z}_l$, and in this case the

Krull–Schmidt theorem is also well known (for the details, see, e.g., the proof of Theorem 3 in [2]). Thus, statement 1 of our theorem is true.

Statement 4 is a collection of three particular cases of Lemma 6 in [2]; in fact, it is the classical Deuring–Noether theorem, which says that if two representations of a finite-dimensional algebra over a field become isomorphic over an extension of the field, then they are isomorphic already over the ground field. The proof of this statement in [2] was given in terms of categories and does not involve any specific properties of Abelian groups.

It remains to prove statements 2 and 5; we shall do this in §13. But before this we need some further considerations. □

§12. The set $\mathfrak{E}$

We fix an object C of the category $\overline{M}$, and till the end of the section denote by E, E_0, E_l, $\widetilde{E}_l$ the endomorphism rings of C in the categories $\overline{M}$, $\overline{M}_0$, $\overline{M}_l$, $\overline{M}_l^c$. Thus, $E \subseteq E_0$ and $E_l \subseteq \widetilde{E}_l$.

Let $\mathfrak{E}$ be the set of all collections $\mathfrak{e} = \{e_l\}_{l\in\mathcal{P}_0}$ of idempotent elements of the rings E_l such that $e_l = e_0$ for almost all (all but a finite number of) $l \in \mathcal{P}$ and the elements e_l and e_0 are conjugate in $\widetilde{E}_l$ for all $l \in \mathcal{P}$ (this means that in the rings $\widetilde{E}_l$ there are invertible elements $\widetilde{u}_l$ such that $\widetilde{u}_l e_0 \widetilde{u}_l^{-1} = e_l$). We say that two such collections $\mathfrak{e} = \{e_l\}$ and $\mathfrak{e}' = \{e_l'\}$ are equivalent if the elements e_l and e_l' are conjugate in E_l for all $l \in \mathcal{P}_0$; i.e., in the rings E_l there exist invertible elements v_l such that $e_l' = v_l e_l v_l^{-1}$ for all $l \in \mathcal{P}_0$.

Our purpose in this section is to prove the following statement.

Lemma 16. *Each collection $\mathfrak{e} \in \mathfrak{E}$ is equivalent to a collection all components of which are equal to one and the same idempotent element e; this element e is automatically contained in the ring $E = \bigcap_{l\in\mathcal{P}_0} E_l$.*

We start with some auxiliary statements.

Lemma 17. *Let q be a prime integer, and let u be an invertible element of the ring $\widetilde{E}_q$. Then in the rings E_q, E_0 there exist invertible elements u_q, u_0 such that $u = u_0 u_q$.*

Proof. This is Lemma 9 in [2]; for completeness we reproduce here its short and standard proof. Let $v_1, \dots, v_n$ be a maximal linearly independent set of elements of E. Then all products $v_i v_j$ are linear combinations of the elements $v_1, \dots, v_n$ with rational coefficients; we denote by M any common denominator of all these coefficients. Let V be the additive subgroup of E generated by the elements $qMv_1, \dots, qMv_n$; it is obvious that $V^2 \subseteq qV$. The rings $E_q = E \otimes \mathbb{Z}_q$ and $\widetilde{E}_q = E \otimes \mathbb{Q}_q$ are topological rings (with the q-adic topology). The closure $V_1 = V \otimes \mathbb{Z}_q$ of the group V in E_q is a neighborhood of 0 both in E_q and in $\widetilde{E}_q$. Since, obviously, $\bigcap_{i=1}^{\infty} V_1^i = 0$, the sets V_1^i constitute a basis of the q-adic topology of the rings E_q and $\widetilde{E}_q$. For any element $v \in V_1$, the series $1 - v + v^2 - \cdots$ converges in E_q to the element inverse to $1 + v$, whence $W = 1 + V_1$ is a neighborhood of 1 in E_q and in $\widetilde{E}_q$ all elements of which are invertible in the ring E_q. Observe that $W^2 = W$ and $W^{-1} = W$.

Let u be an invertible element of the ring $\widetilde{E}_q$; in the neighborhood uW of u, we can choose an element $u_0 = uw$ that belongs to the everywhere dense subset E_0 of $\widetilde{E}_q$. Here w is an element of W, and, like all elements of W, the element w is invertible in E_q; setting $u_q = w^{-1}$, we have $u = u_0 u_q$, $u_0 \in E_0$, and $u_q \in E_q$. Since $u_q = w^{-1}$ is invertible in E_q and $u_0 = uw$ is invertible in $\widetilde{E}_q$, the left annihilator of u_0 in $\widetilde{E}_q$ is trivial, and consequently, the left annihilator of u_0 in the finite-dimensional $\mathbb{Q}$-algebra $E_0 \subset \widetilde{E}_q$ is also trivial. It follows that u_0 is an invertible element of E_0. □

Lemma 18. *Let e be an idempotent element of the ring E_0, and let u be an invertible element of E_0. Then u can be represented in the form $w_1 \cdots w_r v$, where $v \in E_0$ commutes with e, and each of w_s has the form $1 + x$ with $x \in E_0$, $x^2 = 0$.*

Proof. We continue the decomposition $1 = e + (1 - e)$ of the identity of the finite-dimensional algebra E_0 in the sum of two orthogonal idempotents to obtain a decomposition in the sum of pairwise orthogonal indecomposable idempotents:

$$1 = d_1 + \cdots + d_m + d_{m+1} + \cdots + d_n, \quad e = d_1 + \cdots + d_m, \quad 1 - e = d_{m+1} + \cdots + d_n,$$
$$d_i^2 = d_i, \quad d_i d_j = 0 \text{ if } i \neq j \quad (1 \leq i, j \leq n).$$

Obviously, if $s \leq m$, then $ed_s = d_s e = d_s$, and if $s > m$, then $ed_s = d_s e = 0$. The ring E_0 viewed as a right E_0-module decomposes into the direct sum of indecomposable right ideals $I_s = d_s E_0$, and the left multiplication by any element of E_0 is an endomorphism of this module. By Lemma 3 in [2] (which in its turn is none other than a categorical analog of the theorem asserting that a nondegenerate matrix can be transformed to a diagonal matrix by using only elementary transformations of rows), the automorphism u of the right E_0-module E_0 can be represented in the form $u = w_1 \cdots w_m v$, where v is a diagonal automorphism, i.e., an automorphism such that

$$v = d_1 v d_1 + \cdots + d_m v d_m + d_{m+1} v d_{m+1} + \cdots + d_n v d_n,$$

and each of w_s is a transvection, i.e., an automorphism of the form $1 + d_i y d_j$, $i \neq j$, $y \in E_0$. It is easily seen that the above decomposition of the element e satisfies the requirements of the lemma:

$$ve = ev = d_1 v d_1 + \cdots + d_m v d_m, \qquad (d_i y d_j)^2 = d_i y (d_j d_i) y d_j = 0.$$

□

Let $\mathfrak{E}_0$ denote the subset of $\mathfrak{E}$ formed by all collections $\{e_l\}$ with the following property: for each prime integer l such that $e_l \neq e_0$ there exist invertible elements $w_1, \ldots, w_r \in E_0$ for which the squares of all elements $w_i - 1$ are equal to 0 and

$$e_l = (w_r \cdots w_1) e_0 (w_r \cdots w_1)^{-1}.$$

From Lemmas 17 and 18 it follows that $\mathfrak{E}_0$ is a sufficiently representative subset of $\mathfrak{E}$. More precisely, the following statement is true.

Lemma 19. *For any collection $\mathfrak{e} \in \mathfrak{E}$ there exists a collection $\mathfrak{e}' \in \mathfrak{E}_0$ equivalent to $\mathfrak{e}$.*

Proof. Let $\mathfrak{e} = \{e_l\}$, and let P be the finite set of all prime integers l such that $e_l \neq e_0$. We construct a new collection $\mathfrak{e}' = \{e_l'\}$. If $l \neq P$, we set $e_l' = e_l$. Now, let $l \in P$. By the definition of the set $\mathfrak{E}$, there exists an invertible element $\widetilde{u}_l \in \widetilde{E}_l$ such that $e_l = \widetilde{u}_l e_0 \widetilde{u}_l^{-1}$, and by Lemma 17 there exist invertible elements $\hat{u}_l \in E_l$ and $v_l \in E_0$ with $\widetilde{u}_l = \hat{u}_l v_l$; we set $e_l' = \hat{u}_l^{-1} e_l \hat{u}_l$. It is clear that the resulting collection $\mathfrak{e}'$ is equivalent to $\mathfrak{e}$. By Lemma 18, for each $l \in P$ the element v_l can be written as $v_l = w_r \cdots w_1 v$, where $ve_0 = e_0 v$ and all $w_i - 1$ are elements of E_0, the squares of which are equal to 0; since

$$e_l' = v_l e_0 v_l^{-1} = (w_r \cdots w_1) v e_0 v^{-1} (w_r \cdots w_1)^{-1} = (w_r \cdots w_1) e_0' (w_r \cdots w_1)^{-1},$$

the collection $\mathfrak{e}'$ belongs to $\mathfrak{E}_0$. □

Now we are ready to prove the principal statement of this section.

Proof of Lemma 16. By Lemma 19, it suffices to prove Lemma 16 for collections $\mathfrak{e} = \{e_l\}$ belonging to $\mathfrak{E}_0$.

For any collection $\mathfrak{e} = \{e_l\}$ in $\mathfrak{E}_0$, we define two integers $q(\mathfrak{e})$ and $r(\mathfrak{e})$. If all the components e_l of the collection $\mathfrak{e}$ are equal to e_0, then we set $q(\mathfrak{e}) = r(\mathfrak{e}) = 0$. Otherwise, we take for $q(\mathfrak{e})$ the greatest of the prime integers l for which $e_l \neq e_0$ (this definition is consistent because $e_l \neq e_0$ only for a finite set of primes). Let $q = q(\mathfrak{e})$. By the definition of the set $\mathfrak{E}_0$, the element e_q has the form $e_q = (w_r \cdots w_1)e'_0(w_r \cdots w_1)^{-1}$, where all w_i are contained in E_0 and $(1 - w_i)^2 = 0$. Certainly, this representation of e_q is not unique; we denote by $r(\mathfrak{e})$ the smallest possible number of factors involved in the above representation of e_q.

To prove the lemma, we need to check that for any collection $\mathfrak{e} \in \mathfrak{E}_0$ there exists an equivalent collection $\mathfrak{e}'$, all components of which are equal, i.e., a collection such that $q(\mathfrak{e}') = 0$. Clearly, it suffices to prove the following statement: if $\mathfrak{e} \in \mathfrak{E}_0$ and $q = q(\mathfrak{e}) > 0$, then there exists a collection $\mathfrak{e}' \in \mathfrak{E}_0$ equivalent to $\mathfrak{e}$ and such that $q(\mathfrak{e}') < q$ or $q(\mathfrak{e}') = q$, $r(\mathfrak{e}') < r(\mathfrak{e})$. But now it is easy to prove this. Let

$$e_q = (w_r \cdots w_1)e_0(w_r \cdots w_1)^{-1},$$

where $r = r(\mathfrak{e})$ and the w_i are elements of E_0 with $(w_i - 1)^2 = 0$. For the element $x = w_r - 1 \in E_0$ there exists a positive integer N such that $y = Nx$ belongs to E. Let $N = q^s N'$, where N' is not divisible by q. There are integers a, b with $aN' + bq^s = 1$; then

$$x = \frac{y}{N} = \frac{ay}{q^s} + \frac{by}{N'}.$$

Set

$$z_1 = 1 + \frac{ay}{q^s}, \qquad z_2 = 1 + \frac{by}{N'};$$

keeping in mind the relation $y^2 = (Nx)^2 = 0$, we see that

$$z_2 z_1 = \left(1 + \frac{by}{N'}\right)\left(1 + \frac{ay}{q^s}\right) = 1 + x = w_r, \quad z_1^{-1} = 1 - \frac{ay}{q^s}, \quad z_2^{-1} = 1 - \frac{by}{N'}.$$

Thus, the element $z_1 \in E_0$ and its inverse are contained in all rings E_l, $l \neq q$, and z_2 is an invertible element of E_l. Set $e'_q = z_2^{-1} e_q z_2$, $e'_l = z_1 e_l z_1^{-1}$ for all $l \in \mathcal{P}_0$, $l \neq q$, $w'_i = z_1 w_i z_1^{-1}$ for $1 \leq i < r$. Obviously, the elements $w'_i - 1 = z_1(w_i - 1)z_1^{-1}$ belong to E_0, and their squares are equal to 0. Moreover,

$$e'_q = (w'_{r-1} \cdots w'_1)e'_0(w'_{r-1} \cdots w'_1)^{-1}.$$

Thus, the collection $\mathfrak{e}' = \{e'_l\}$ belongs to $\mathfrak{E}_0$, and if $e'_q = e'_0$, then $q(\mathfrak{e}') < q$, while if $e'_q \neq e'_0$, then $q(\mathfrak{e}') = q$, but $r(\mathfrak{e}') \leq r - 1 < r(\mathfrak{e})$. This completes the proof of the lemma. □

§13. Completion of the proof of Theorem 5

Proof of statement 2. Let A be an object of the category $\overline{M}$ which belongs to the same genus as C, and let $A = X \oplus Y$ be a direct decomposition in $\overline{M}$. We denote by d the idempotent endomorphism of A that corresponds to this decomposition (in other words, d is the composition of the canonical projection $p : A \to X$ and the canonical injection $i : X \to A$). Since the objects $\bar{F}_0(A)$ and $\bar{F}_0(C)$ become isomorphic in the category $\overline{M}_0$, there exist mutually inverse isomorphisms $u_0 : \bar{F}_0(A) \to \bar{F}_0(C)$ and $v_0 : \bar{F}_0(C) \to \bar{F}_0(A)$. By the definition of $\overline{M}_0$, there exist homomorphisms $u : A \to C$, $v : C \to A$ of the category $\overline{M}$ and an integer $N > 0$ such that $u_0 = u/N$ and $v_0 = v/N$. If l is a prime integer not dividing N, then the homomorphisms u/N and v/N are mutually inverse isomorphisms not only in $\overline{M}_0$, but also in $\overline{M}_l$. For $l = 0$ and each prime integer l not dividing N, we set $e_l = (udv)/N^2 = (u/N)d(v/N)$; it is easily seen that e_l is an idempotent element of the endomorphism ring E_l of the object $\bar{F}_l(C)$ of the category $\overline{M}_l$. Now, suppose that l divides N; since A and C belong to the same genus, in the category $\overline{M}_l$ there exist mutually inverse isomorphisms $u_l : \bar{F}_l(A) \to \bar{F}_l(C)$, $v_l : \bar{F}_l(C) \to \bar{F}_l(A)$;

we set $e_l = u_l d v_l$, and, again, e_l is an idempotent element of the endomorphism ring E_l of the object $\bar{F}_l(C)$ of the category $\overline{M}_l$.

The resulting collection of idempotent elements $\{e_l\}$ belongs to the set $\mathfrak{E}$ defined above. Indeed, for almost all l (more precisely, for all l not dividing N) the component e_l is equal to e_0, and for all the remaining l we have

$$e_l = u_l d v_l = (u_l v_0) e_0 (u_0 v_l) = (u_l v_0) e_0 (u_l v_0)^{-1}.$$

By Lemma 16, in $\mathfrak{E}$ there exists a collection equivalent to the collection $\{e_l\}$ and such that all components of it are equal to one and the same element $e \in E$. The idempotent endomorphism e of the object C induces a direct decomposition $C = X' \oplus Y'$. It remains to show that for every $l \in \mathcal{P}_0$ the objects $\bar{F}_l(X')$ and $\bar{F}_l(X)$ of the category $\overline{M}_l$ are isomorphic (then the objects $\bar{F}_l(Y')$ and $\bar{F}_l(Y)$ are also isomorphic, because the Krull–Schmidt theorem is valid in all categories $\overline{M}_l$).

Let $p' : C \to X'$, $i' : X' \to C$ be the canonical projection and injection for the decomposition $C = X' \oplus Y'$, so that $e = i'p'$. By the definition of equivalence for collections from $\mathfrak{E}$, for each $l \in \mathcal{P}_0$ there exists an invertible element w_l of the ring E_l such that $w_l e w_l = e_l$; then

$$(p v_l w_l i')(p' w_l^{-1} u_l i) = p v_l (w_l e w_l^{-1}) u_l i = p v_l e_l u_l i = p d i = 1_{\bar{F}_l(X)},$$

and similarly $(p' w_l^{-1} u_l i)(p v_l w_l i') = 1_{\bar{F}_l(X')}$. Thus,

$$p v_l w i' : \bar{F}_l(X') \to \bar{F}_l(X), \qquad p' w^{-1} u_l i : \bar{F}_l(X) \to \bar{F}_l(X')$$

are mutually inverse isomorphisms in the category E_l (if l does not divide N, then we can take the homomorphisms v/N, u/N in place of v_l, u_l). $\square$

Proof of statement 5. Let X_l be objects of the categories $\overline{M}_l$ that are chosen for all $l \in \mathcal{P}_0$ and satisfy the following condition: for every prime integer l the objects $\bar{G}_l(X_l)$ and $\bar{G}_0(X_0)$ of the category M_∞ are isomorphic. Then from statement 4 it follows that for every $l \in \mathcal{P}$ the objects $\bar{G}'_l(X_l)$ and $\bar{G}_{0,l} X_0$ of the category $\overline{M}_l^c$ are also isomorphic (see also the remark after the formulation of Theorem 5). Let B be an object of $\overline{M}$ such that for all prime integers l off a finite set P, the object X_l is isomorphic to the object $\bar{F}_l(B)$. For each $l \in P$ we can find an object C_l of $\overline{M}$ such that X_l is a direct summand of the object $\bar{F}_l(C_l)$. Set $C = B \oplus (\bigoplus_{l \in P} C_l)$; now the objects X_l are direct summands of the objects $\bar{F}_l(C)$ for all $l \in \mathcal{P}_0$.

As above, let E, E_0, E_l, $\widetilde{E}_l$ stand for the endomorphism rings of the object C in the categories $\overline{M}$, $\overline{M}_0$, $\overline{M}_l$, $\overline{M}_l^c$. Let $i : B \to C$ and $p : C \to B$ be, respectively, the canonical injection of a direct summand and the canonical projection onto a direct summand. Next, for each $l \in P$ we denote by i_l the injection of the direct summand X_l into $\bar{F}_l(C)$ and by p_l the projection of $\bar{F}_l(C)$ onto this direct summand. For $l \in \mathcal{P}_0$, $l \notin P$, we set $e_l = ip$; e_l is an idempotent element of the ring $E \subseteq E_l$. If $l \in P$, then the element $e_l = i_l p_l \in E_l$ is also an idempotent. The collection $\{e_l\}$ belongs to $\mathfrak{E}$; indeed, $e_l = e_0$ for all l not contained in the finite set P, and for every $l \in \mathcal{P}$ the idempotent endomorphisms e_l and e_0 are conjugate in $\widetilde{E}_l$, because the corresponding direct summands $\bar{G}'_l(X_l)$ and $\bar{G}_{0,l} X_0$ of the object C are isomorphic (here we regard C as an object of the category $\overline{M}_l^c$). By Lemma 16, in $\mathfrak{E}$ there is a collection equivalent to the collection $\{e_l\}$ and such that all components of it are equal to one and the same element $e \in E$. Let A be the direct summand of C that corresponds to the idempotent endomorphism e of C. Since for each $l \in \mathcal{P}_0$ the idempotent elements e, e_l are conjugate in E_l, the corresponding direct summands $\bar{F}_l(A)$, X_l of the object $\bar{F}_l(C)$ are isomorphic in the category $\overline{M}_l$. $\square$

§14. THE CATEGORIES M_0, M_l, M_l^c, M_∞

Let Λ be one of the rings $\mathbb{Q}$, $\mathbb{Z}_l$, $\mathbb{Q}_l$, $\mathbb{C}$; for each of them we shall introduce the category of Λ-motives $M_\Lambda = M_\Lambda(\mathfrak{A})$. The first idea that comes to mind is the following: consider the category with the same objects as the category $\mathfrak{A}$ and with the homomorphism groups equal to the tensor product of the homomorphism groups in $\mathfrak{A}$ with Λ, and then add "imaginary" direct summands to this category. But this method is not entirely satisfactory. The problem is that after tensor multiplication of the endomorphism groups by Λ, certain objects become zero objects. This would not be so bad should only finite direct sums exist in the category; but some infinite direct sums exist in M, and an unpleasant situation can occur: all direct summands become zero objects in the new category, but the sum remains a nonzero object. For example, if M is the category of Abelian groups and Λ is a field of characteristic 0, then for each $l \in \mathcal{P}$, the cyclic group of order l becomes the zero object in the new category, but the endomorphism ring of the direct sum of all these cyclic groups remains very large. Therefore, our approach must be more sophisticated.

We denote by M'_Λ the category with the same objects as M and with the groups of homomorphisms equal to the tensor products of the groups of homomorphisms in M with Λ:

$$\mathrm{Hom}_{M'_\Lambda}(A, B) = \mathrm{Hom}_M(A, B) \otimes \Lambda.$$

Some objects of M become zero objects in M'_Λ; this happens to the objects B for which $E_M(B) \otimes \Lambda = 0$ (as usual, we denote by $E_M(B)$ the endomorphism ring of the object B in the category M). If Λ is one of the rings mentioned above, then, obviously, only the objects of the form $\bigoplus_{q \in I} B_q$ become zero in M'_Λ, where I is a finite subset of $\mathcal{P}$ and each direct summand B_q is a q-periodic object (i.e., an object whose endomorphism ring is a finite q-group). If $\Lambda = \mathbb{Z}_l$, then the set I should not contain the prime integer l.

Let A be an object of M, and let $t_\Lambda(A)$ denote the direct sum of all objects A_q that vanish in M'_Λ ($q \in \mathcal{P}$); the object $t_\Lambda(A)$ will be called the Λ-periodic part of A. It is easy to understand that $t_\Lambda(A) = t(A)$ if Λ is any one of the fields $\mathbb{Q}$, $\mathbb{Q}_l$, $\mathbb{C}$, and that

$$t_{\mathbb{Z}_l}(A) = \bigoplus_{q \in \mathcal{P} \setminus \{l\}} A_q.$$

Precisely as in §6, we define the set $\mathrm{Hom}^t_\Lambda(A, B)$ of homomorphisms of the category M that potentially factor through the Λ-periodic part $t_\Lambda(B)$ of B (this set coincides with $\mathrm{Hom}^t(A, B)$ except for the case where $\Lambda = \mathbb{Z}_l$). Such sets constitute an ideal of the category M. We denote by M''_Λ the category with the same objects as M and with the groups of homomorphisms defined in the following way:

$$\mathrm{Hom}_{M''_\Lambda}(A, B) = \Big(\mathrm{Hom}_M(A, B) / \mathrm{Hom}^{t(\Lambda)}_M(A, B)\Big) \otimes \Lambda.$$

We can describe the category M''_Λ in other terms. Not going into details, we shall only say that, roughly speaking, it can be obtained from the category M'_Λ by a factorization that annihilates any infinite direct sum in M provided that all summands become the zero object in M'_Λ.

By the category of Λ-motives for the category $\mathfrak{A}$ we shall mean the category obtained from M''_Λ by adding direct summands that split idempotent endomorphisms of the objects of this category. The category of Λ-motives will be denoted by $M_\Lambda(\mathfrak{A})$ or simply by M_Λ.

We introduce an abbreviated notation for the categories of motives over the rings of interest for us:

$$M_0 = M_{\mathbb{Q}}(\mathfrak{A}), \quad M_\infty = M_{\mathbb{C}}(\mathfrak{A}), \quad M_l = M_{\mathbb{Z}_l}(\mathfrak{A}), \quad M_l^c = M_{\mathbb{Q}_l}(\mathfrak{A}) \quad (l \in \mathcal{P}).$$

The embeddings of rings (including the noncanonical embeddings $\mathbb{Q}_l \to \mathbb{C}$) induce functors depicted in the following commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{F_0} & M_0 & = & M_0 \\ {\scriptstyle F_l}\downarrow & & {\scriptstyle G_{0,l}}\downarrow & & {\scriptstyle G_0}\downarrow \\ M_l & \xrightarrow{G'_l} & M_l^c & \xrightarrow{H_l} & M_\infty \end{array}$$

We denote by $G_l : M_l \to M_\infty$ the composition $H_l G'_l$ of functors.

Among the categories introduced above, only the categories M_l are new, and

$$M_0 = \overline{M}_0, \quad M_l^c = \overline{M}_l^c, \quad M_\infty = \overline{M}_\infty.$$

The category M_l differs a little from $\overline{M}_l$, but there is an obvious canonical functor $P_l : M_l \to \overline{M}_l$. All categories and functors we have introduced are represented in the following commutative diagram as nodes and arrows:

$$\begin{array}{ccccccccc} M & \xrightarrow{P} & \overline{M} & \xrightarrow{\bar{F}_0} & \overline{M}_0 & = & M_0 & = & M_0 \\ {\scriptstyle F_l}\downarrow & & {\scriptstyle \bar{F}_l}\downarrow & & {\scriptstyle \bar{G}_{0,l}}\downarrow & & {\scriptstyle G_{0,l}}\downarrow & & {\scriptstyle G_0}\downarrow \\ M_l & \xrightarrow{P_l} & \overline{M}_l & \xrightarrow{\bar{G}_l} & \overline{M}_l^c & = & M_l^c & \xrightarrow{H_l} & M_\infty \end{array}$$

Observe for future use that the composition of all functors of the upper row is equal to F_0, and the composition of all functors of the lower row is equal to G_l.

Lemma 20. *Let A, B be objects of the category M. If A has no nonzero l-periodic direct summands, then the homomorphism*

$$\operatorname{Hom}_{M_l}(F_l(B), F_l(A)) \to \operatorname{Hom}_{\overline{M}_l}(P_l F_l(B), P_l F_l(A)) = \operatorname{Hom}_{\overline{M}_l}(\bar{F}_l(B), \bar{F}_l(A))$$

induced by the functor P_l is an isomorphism. If $A = A_l$ is an l-periodic object, then the functor F_l induces an isomorphism of finite l-groups:

$$\operatorname{Hom}_M(B, A_l) \to \operatorname{Hom}_{M_l}(F_l(B), F_l(A_l)).$$

Proof. The second statement is trivial: the tensor multiplication by $\mathbb{Z}_l$ does not change the finite l-group $\operatorname{Hom}_M(B, A_l)$. The first statement follows immediately from the above description of the M_l. Indeed, the absence of nonzero l-periodic direct summands of A means that $A_l = 0$, whence

$$t(A) = \bigoplus_{q \in \mathcal{P}} A_q = \bigoplus_{q \neq l} A_q = t_{\mathbb{Z}_l}(A).$$

Therefore, the group $\operatorname{Hom}_{M_l}(F_l(B), F_l(A))$ is obtained from the group $\operatorname{Hom}_M(B, A)$ by factorization over the subgroup of homomorphisms that potentially factor through the subobject $t_{\mathbb{Z}_l}(A) = t(A)$ of the object A, followed by tensor multiplication by $\mathbb{Z}_l$. But this is precisely the description of the group $\operatorname{Hom}_{\overline{M}_l}(\bar{F}_l(B), \bar{F}_l(A))$. $\square$

Lemma 21. *Let X be an object of the category M_l, and let $E_l^+(X)$ be the additive group of the endomorphism ring of X. The group of periodic elements of the group $E_l^+(X)$ is a finite l-group, and the factor group of $E_l^+(X)$ over this finite group is a torsion-free $\mathbb{Z}_l$-module of finite rank. If A is an object of M such that its l-component A_l is trivial, then the additive group of the endomorphism ring of the object $F_l(A)$ of M_l is a torsion-free $\mathbb{Z}_l$-module of finite rank.*

Proof. There exists an object A of M such that X is a direct summand of $F_l(A)$. Obviously, it suffices to prove the lemma for the case where $X = F_l(A)$. But in this case the claim follows immediately from Lemma 20. Indeed, let $A = A' \oplus A_l$, where A_l is the l-periodic component of A, and the object A' has no nonzero l-periodic direct summands. Then

$$\begin{aligned}\mathrm{End}_{M_l}(F_l(A)) &= \mathrm{Hom}_{M_l}(F_l(A), F_l(A)) \\ &= \mathrm{Hom}_{M_l}(F_l(A), F_l(A')) \oplus \mathrm{Hom}_{M_l}(F_l(A), F_l(A_l)),\end{aligned}$$

and by Lemma 20, the second summand is a finite l-group; now from the same lemma it follows that the first summand is isomorphic to the group $\mathrm{Hom}_{\overline{M}_l}(\bar{F}_l(A), \bar{F}_l(A'))$, which is a torsion-free $\mathbb{Z}_l$-module of finite rank. If the l-periodic component of A is trivial, then the second summand disappears. □

As for the category M, we say that an object of M_l is *l-periodic* if the additive group of its endomorphism ring is a finite l-group. We select two full subcategories of M_l: the category M_l^t of all l-periodic objects, and the category M_l^{tf} of objects that have no nontrivial l-periodic direct summands.

Theorem 6. 1. *The Krull–Schmidt theorem holds true in the category M_l: every object of this category decomposes into the direct sum of a finite number of indecomposable objects, and for any two such decompositions of the same object, there exists an automorphism of this object that transforms the first decomposition to the second.*

2. *Each object of M_l can be represented as the direct sum of an object of the subcategory M_l^{tf} and an object of the category M_l^t, and this representation is unique up to isomorphism.*

3. *The restriction of the functor F_l to the subcategory of M that consists of all l-periodic objects is an isomorphism of this category onto the category M_l^t, and the functor P_l annihilates all objects of M_l^t and induces an isomorphism of the category M_l^{tf} onto $\overline{M}_l$.*

4. *Let X, Y be objects of M_l, and let $X = X' \oplus X_l$, $Y = Y' \oplus Y_l$ be their direct decompositions such that X' and Y' are free of l-periodic direct summands, and X_l, Y_l are l-periodic objects. The objects $P_l(X)$ and $P_l(Y)$ of $\overline{M}_l$ are isomorphic if and only if the objects X' and Y' of M_l are isomorphic.*

Proof. 1. This follows immediately from the description of the endomorphism rings of objects of M_l given in Lemma 21.

2. Let X be an object of M_l; then there exists an object A of M and an object Y of M_l such that $F_l(A) = X \oplus Y$. On the other hand, A decomposes into the sum $A = A' \oplus A_l$, where A_l is an l-periodic object, and A' is an object free of l-periodic direct summands. So, we have another direct decomposition of $F_l(A)$:

$$F_l(A) = F_l(A') \oplus F_l(A_l).$$

Since the Krull–Schmidt theorem is valid in M_l, we can find objects X', X_l, Y', Y_l of M_l such that

$$X \approx X' \oplus X_l, \quad Y \approx Y' \oplus Y_l, \quad F_l(A') \approx X' \oplus Y', \quad F_l(A_l) \approx X_l \oplus Y_l.$$

The object X_l, which is a direct summand of the l-periodic object $F_l(A_l)$, is itself an l-periodic object, i.e., an object of the subcategory M_l^t. Next, the l-component of the object A' is trivial, whence, by Lemma 21, the additive group of the endomorphism ring $\mathrm{End}_{M_l}(F_l(A'))$ has no l-torsion. The same is true for the endomorphism ring of the direct summand X' of the object $F_l(A')$, so that X' is free of l-periodic direct summands; i.e.,

X' is an object of the subcategory M_l^{tf}. The uniqueness of the decomposition in question follows from the Krull–Schmidt theorem.

Observe that, in passing, we have proved the following fact: for any object X of the subcategory M_l^{tf} there is an object A' of M such that A' is free of l-periodic direct summands and X is a direct summand of $F(A')$. Indeed, in this case, $X_l = 0$, and $X \approx X'$ is a direct summand of $F(A')$ (here we have used the notation as above).

3. All statements that concern l-periodic objects are trivial. Now, let X, Y be objects of M_l^{tf}; in the preceding paragraph it was observed that there exist objects A and B of M such that X is a direct summand of $F_l(A)$, Y is a direct summand of $F_l(B)$, and the objects A, B are free of l-periodic direct summands. By Lemma 20, the functor P_l induces an isomorphism

$$\operatorname{Hom}_{M_l}(F_l(A), F_l(B)) \to \operatorname{Hom}_{\overline{M}_l}(\bar{F}_l(A), \bar{F}_l(B)) = \operatorname{Hom}_{\overline{M}_l}(P_lF_l(A), P_lF_l(B)).$$

Passing to direct summands, we see that the homomorphism induced by P_l,

$$\operatorname{Hom}_{M_l}(X, Y)) \to \operatorname{Hom}_{\overline{M}_l}(P_l(X), P_l(Y)),$$

is also an isomorphism.

4. This claim obviously follows from statement 3. $\square$

§15. Main theorem

Now we can return to the principal result of the paper, which was described (not quite precisely) in the Introduction (see [4]). Before formulating this result, we recall that we have constructed categories and functors represented by the nodes and arrows of the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "P"] \arrow[d, "F_l"'] & \overline{M} \arrow[r, "\bar{F}_0"] \arrow[d, "\bar{F}_l"'] & \overline{M}_0 \arrow[r, equal] \arrow[d, "\bar{G}_{0,l}"'] & M_0 \arrow[r, equal] \arrow[d, "G_{0,l}"'] & M_0 \arrow[d, "G_0"'] \\
M_l \arrow[r, "P_l"] & \overline{M}_l \arrow[r, "\bar{G}_l"] & \overline{M}_l^c \arrow[r, equal] & M_l^c \arrow[r, "H_l"] & M_\infty
\end{tikzcd}$$

also, we recall that the composition of all functors of the upper row is equal to F_0, and the composition of all functors of the lower row is equal to G_l. Retaining only the outer square of this diagram, we obtain the following commutative diagram of functors and categories:

$$\begin{tikzcd}
M \arrow[r, "F_0"] \arrow[d, "F_l"'] & M_0 \arrow[d, "G_0"] \\
M_l \arrow[r, "G_l"'] & M_\infty
\end{tikzcd}$$

We shall say that objects A, B of the category M belong to the same genus if there exist periodic objects U, V of M such that the objects $F_l(A \oplus U)$ and $F_l(B \oplus V)$ of M_l are isomorphic for all $l \in \mathcal{P}_0$. An object A of M is said to be *almost indecomposable* if for each direct decomposition $A = X \oplus Y$, one of the objects $F_0(X)$, $F_0(Y)$ is the zero object of M_0.

Theorem 1 (refined formulation)**.** *Let $\mathfrak{A}$ be an additive category satisfying conditions* $(*)$, $(**)$, $(***)$ *of* §2, *and suppose that its category of motives $M = M(\mathfrak{A})$ satisfies conditions* (i), (ii) *of* §3. *Also, assume that for any objects A, B of M, the rank of the factor group $D(A, B)/\operatorname{Hom}^t(A, B)$ is finite (as usual, $\operatorname{Hom}^t(A, B)$ stands for the subgroup of homomorphisms that potentially factor through the periodic part $t(B)$ of the object B). Then the following statements hold true.*

1. *The Krull–Schmidt theorem is valid in the categories* M_0, M_l, M_l^c *(but not in* M*!): every object of each of these categories decomposes into the direct sum of a finite number of indecomposable objects, and for any two such decompositions of the same object, there exists an automorphism of this object that transforms the first decomposition to the second.*

2. *If* A, B *are objects of* M *belonging to the same genus, and* $A = X \oplus Y$, *then there exists a direct decomposition* $B = X' \oplus Y'$ *such that* X', X *belong to the same genus and* Y', Y *belong to the same genus.*

3. *If* A, B *are objects of* M *that belong to the same genus, and if the object* A *is almost indecomposable, then the object* B *is almost indecomposable.*

4. *For each* $l \in \mathcal{P}_0$, *choose an object* X_l *of the category* M_l. *The existence of an object* A *of* M *and a periodic object* U *of* M *such that* $F_l(A \oplus U) \approx X_l \oplus F_l(U)$ *for all* $l \in \mathcal{P}_0$ *is equivalent to the following:*

(i) *the objects* $G_l(X_l)$ *and* $G_0(X_0)$ *of* M_∞ *are isomorphic for all primes* l*;*

(ii) *there exists an object* B *of* $\overline{M}$ *such that for almost all (all but a finite number of)* $l \in \mathcal{P}$ *the object* $\bar{F}_l(B)$ *is isomorphic to the direct sum of* X_l *and an* l*-periodic object of the category* M_l.

Proof. Statement 1 is contained in Theorems 5 and 6, and all other statements follow immediately from Theorem 5 and Theorems 2, 3, in which the properties of the functor P are described. □

Remark 1. As in Theorem 5, we can replace condition (i) in statement 4 with the following condition, in which only canonical functors are involved:

(i′) for every prime integer l the objects $\bar{G}_l P_l(X_l)$ and $G_{0,l}(X_0)$ of the category M_l^c are isomorphic.

Remark 2. Conditions $(*)$, $(**)$, $(***)$ are rather restrictive, and they may fail for some objects of the category (for example, for an exotic object, the l-component of the additive group of its endomorphism ring can be infinite). But our results can be useful even in this case. For instance, if these conditions are fulfilled for an object A of the category $\mathfrak{A}$, then so they are in the full subcategory of the category $\mathfrak{A}$ that consists of all direct summands of the direct sums of several copies of A.

Remark 3. If we make the "finiteness" condition $(*)$ on the objects of the category stronger by demanding that all components of the periodic part of the group $\mathrm{Hom}_{\mathfrak{A}}(A, B)$ be finite and that the factor-group of this group over its periodic part be a group of finite rank, then everything will be much simpler. Conditions $(**)$, $(***)$, (i), (ii) will no longer be necessary, and all proofs will be easier. Precisely this case was the subject of talks given by the author at several conferences. But such a simplification is not always appropriate: already for the class of mixed Abelian groups that was introduced in [3], the factor groups of the groups of homomorphisms over their periodic parts are usually very large. In this paper we tried to show that, under some suitable restrictions, the results remain valid even if the ranks of the factor groups are infinite, and our main efforts were focused on this.

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Received 6/JUN/2008

Translated by THE AUTHOR