Алгебра и анализ
Том 21 (2009), № 2

St. Petersburg Math. J.
Vol. 21 (2010), No. 2, Pages 217–230
S 1061-0022(10)01091-5
Article electronically published on January 21, 2010

ON THE ASYMPTOTICS OF POLYNOMIALS ORTHOGONAL ON A SYSTEM OF CURVES WITH RESPECT TO A MEASURE WITH DISCRETE PART

V. A. KALYAGIN AND A. A. KONONOVA

Abstract. Consider an absolutely continuous measure on a system of Jordan arcs and (closed) curves in the complex plane, assuming that this measure satisfies the Szegő condition on its support and that the support of the measure is the boundary of some (multiply connected) domain Ω containing infinity. Adding to the measure a finite number of discrete masses lying in Ω (off the support of the measure), we study the strong asymptotics of the polynomials orthogonal with respect to the perturbed measure. For this, we solve an extremal problem in a certain class of multivalued functions. Our goal is to give an explicit expression for the strong asymptotics on the support of the perturbed measure, as well as on the domain Ω.

§1. Introduction

Let σ be a positive measure with compact support in the complex plane. We introduce the sequence of polynomials $Q_n(z) = z^n + \cdots$ orthogonal with respect to the measure σ:

$$(Q_n, z^k)_\sigma := \int Q_n(\zeta;\sigma)\overline{\zeta}^k d\sigma = 0, \quad k = 0, 1, 2, \dots, n-1.$$

A fundamental problem of the theory of orthogonal polynomials is the study of their asymptotic behavior as $n \to \infty$.

As usual, by the *weak asymptotics* of orthogonal polynomials we mean the asymptotics of $|Q_n(z;\sigma)|^{1/n}$. The weak asymptotics is closely related to the distribution of zeros of $Q_n(z)$ and is determined by the support of the measure σ and by the regularity properties of σ on its support. The main tool of the investigation in this case is the logarithmic potential theory [20].

The *ratio asymptotics* is that of the ratio $Q_{n+1}(z;\sigma)/Q_n(z;\sigma)$. This aspect has attracted a lot of attention recently. The main progress was due to the varying weight approach, developed be G. Lopez Lagomasino and his collaborators [9]. On an interval of the real line, the ratio asymptotics of the orthogonal polynomials can be exhibited under various conditions, of which the best by now is due to *Rakhmanov*: $\sigma' > 0$ almost everywhere on the interval.

The *strong asymptotics problem* is the problem of uniform asymptotic behavior of the polynomials $Q_n(z)$ outside of the support of the measure and on this support. In the classical case where $\mathrm{supp}(\sigma) = [-1, 1]$, the strong asymptotics of orthogonal polynomials was investigated by S. N. Bernstein and G. Szegő [19]. The Bernstein–Szegő theory was extended to the case of the measure supported on a finite system of complex arcs

2000 *Mathematics Subject Classification.* Primary 42C05; Secondary 30D55, 30E15.
Key words and phrases. Orthogonal polynomials, strong asymptotics, multivalued functions, Hardy spaces.
The first author was supported in part by Scientific Schools grant 3906.2008.1 and by RFBR grant 08-01-00179.

and curves by H. Widom in his fundamental paper [21]. The main tools in this case are multivalued functions of a complex variable and Hardy spaces of analytic functions. Widom's results were improved by Aptekarev in [1], where explicit formulas for the strong asymptotics were obtained in terms of Riemann theta functions on the Riemann surface determined by the geometry of the curves and arcs in question. The main condition on the measure in this case is the so-called *Szegő condition* (see below). This condition makes it possible to construct the Hardy spaces associated with the weight function and to obtain the asymptotics from the extremal properties of orthogonal polynomials. Recently, new progress in the study of the asymptotics of orthogonal polynomials has been achieved by using the matrix Riemann–Hilbert approach (see [15, 10]).

For the first time, the asymptotic properties of polynomials orthogonal with respect to a measure with discrete masses added was investigated in an implicit form in the paper [4] by A. Gonchar for the case of one real interval. Rakhmanov [8] considered (also implicitly) the case of a measure supported on a finite number of disjoint real intervals; he used the quasiorthogonality property for the associated orthogonal polynomials. Later, Nikishin studied this problem explicitly for the unit circle; this case is closely related to the scattering problem for the second order Sturm–Liouville difference operator [6]. Kaliaguine and Benzine [12] and Kaliaguine [11] obtained asymptotic formulas for the case of one complex curve and one complex arc. Li and Pan [16] proved sharper results for the case of the unit circle. Marcellán and Maroni [17] developed a perturbation theory for polynomials obtained by adding a Dirac function to a moment linear functional in the space of polynomials. Aptekarev [1] used the strong asymptotics for the system of real intervals with finitely many mass points to obtain an explicit formula for the periodic motions of Toda lattices. Peherstorfer and Yuditskii [18] considered the case of the Szegő measure on a real interval with countably many mass points. Later, in [2], Peherstorfer, Vol′berg, and Yuditskiĭ considered the same problem for the unit circle.

In the present paper we study the strong asymptotics problem for the polynomials orthogonal with respect to a measure supported on a system of arcs and curves with Szegő condition and on a finite number of mass points off that system. Our main result is explicit strong asymptotic formulas for the orthogonal polynomials. This paper is based on the preprint [13] by the authors. The methods and results of the preprint were used in some papers published after its appearance. The case where infinitely many discrete masses are added was considered in [14] (see Remark 2 below). In the present paper we give complete proofs of all statements and correct some inaccuracies of [13].

§2. Preliminaries

2.1. Necessary definitions.

A class of curves and arcs. Let $E := \bigcup_{k=1}^{p} E_k$ be a union of complex, mutually exterior, and rectifiable Jordan curves and arcs: $E_j \cap E_k = \varnothing$, $j \neq k$. In what follows we shall always assume that each E_k is of class C^{2+}; this means that, in the canonical parametrization $z(s)$ of E_k, the second derivative of the function $z(s)$ satisfies the Lipschitz condition with some positive exponent. We denote by $E^{(1)}$ the union of (closed) curves of E and by $E^{(2)}$ the union of arcs.

A class of measures. Let Ω denote the connected component of $\mathbb{C} \setminus E$ such that $\infty \in \Omega$, and let $\rho(\zeta) \geq 0$ be a weight function on E with

$$\int_E \rho(\zeta)\,|d\zeta| < +\infty. \tag{1}$$

We shall study the strong asymptotics problem for the monic polynomials $Q_n(z;\sigma^0)$ orthogonal with respect to the measure $\sigma^0 = \sigma + \alpha$, where σ is a measure absolutely continuous on E, $d\sigma = \rho(\zeta)|d\zeta|$, and α is a discrete measure with masses $\lambda_j > 0$ at points z_j, $j = 1, 2, \dots, l$, $z_j \in \Omega$. Our investigation is based on an extremal property of the polynomials $Q_n(z, \sigma^0)$.

The set E and the boundary $\partial\Omega$. The boundary $\partial\Omega$ of the domain Ω is the set E with any arc of E taken twice. Any weight function defined on E extends to the boundary $\partial\Omega$ in a natural way. We put $\rho_+(\zeta) = \rho_-(\zeta) = \rho(\zeta)$ in the case of an arc. Below we use the following notation for an integral over the oriented boundary $\partial\Omega$:

$$\oint_E f(\zeta)\,|d\zeta| := \int_{\partial\Omega} f(\zeta)\,|d\zeta|.$$

It is easily seen that condition (1) is equivalent to

$$\oint_E \rho(\zeta)\,|d\zeta| < +\infty.$$

An extremal property. Let $m_n(\sigma^0) := (Q_n(\zeta,\sigma^0), Q_n(\zeta,\sigma^0))_{\sigma^0} = \|Q_n(\zeta,\sigma^0)\|^2_{\sigma^0}$; then

$$m_n(\sigma^0) = \min_{P_n = z^n + \cdots} \left(\int_E |P_n(\zeta)|^2 \rho(\zeta)\,|d\zeta| + \sum_{k=1}^{l} \lambda_k |P_n(z_k)|^2 \right).$$

This property is well known (see, e.g., [20]).

Note that here we cannot replace the integral over E by the integral over $\partial\Omega$ (this leads to a different extremal problem).

The complex Green function. Let $g(z, z_0)$ be the real Green function for Ω with singularity at the point z_0; this means that

1) $g(z, z_0)$ is harmonic in $\Omega \setminus \{z_0\}$;
2) the function $[g(z, z_0) - \log(1/|z - z_0|)]$ is harmonic near z_0 (for $z_0 = \infty$ the function $[g(z) - \log|z|]$ is harmonic near ∞);
3) $\lim_{z\to\zeta} g(z, z_0) = 0$ for $\zeta \in \partial\Omega$ a.e. (nontangentially).

For a real harmonic function $u(z)$, we denote by $\widetilde{u}(z)$ its harmonic conjugate. The function $G(z, z_0) = g(z, z_0) + i\widetilde{g}(z, z_0)$ is called the complex Green function of the domain Ω with pole at z_0. We denote by $g(z)$ and $G(z)$ the real and complex Green functions with pole at infinity ($z_0 = \infty$).

A principal role in our investigation is played by the function $\Phi(z) = \exp[G(z)]$. This function is locally analytic in Ω, has no zeros there, and has a pole at infinity. For the boundary values of $\Phi(z)$ on $\partial\Omega$, we have $|\Phi(\zeta)|_{\zeta\in\partial\Omega} = 1$. We shall also employ the function $\Phi(z, z_0) := \exp[G(z, z_0)]$.

If $p > 1$ (i.e., if Ω is not simply connected), then $\Phi(z)$ and $\Phi(z, z_0)$ are multivalued in Ω. More precisely, $|\Phi(z)| = \exp(g(z))$ (and $|\Phi(z, z_0)| = \exp(g(z, z_0))$ is single-valued, and $\arg(\Phi(z))$ $(\arg(\Phi(z, z_0)))$ is multivalued.

By the Carathéodory theorem [3], the complex derivative $\Phi'(z)$ has nontangential limit values a.e. on E (everywhere except the endpoints of the arcs of $E \in C^{2+}$). It is possible to calculate the function $\Phi(z)$ for the simplest cases; for example, this is possible if $p = 1$ and E is a real interval, the unit circle, or an arc of the unit circle. Indeed, for $p = 1$ the domain Ω is simply connected, the function $\Phi(z)$ is single-valued and realizes the conformal mapping of the $\Omega = \mathrm{Ext}(E)$ onto the exterior of the unit circle.

It is not hard to show that

1) $\Phi(z) = z$ in the case of the unit circle;
2) $\Phi(z) = z + \sqrt{z^2 - 1}$ in the case of the interval $[-1, 1]$;

3) $\Phi(z) = -i(1 - \overline{w_c}w)/(w - w_c)$, $\quad w = u + \sqrt{u^2 - 1}$,
$u = (i/c)(z+1)/(z-1)$, $\quad c = \cot(\theta_1/2)$, $\quad w_c = w(i/c)$,
in the case of the arc $E = \{\zeta = e^{i\theta}, \theta_1 \le \theta \le 2\pi - \theta_1\}$;

4) $\Phi^2(z) = (z^2 - 5 + \sqrt{(z^2-1)(z^2-9)})/4$
for the system of intervals $[-3,-1] \cup [1,3]$.

The logarithmic capacity. The logarithmic capacity of the set E is the positive number $C(E) := \exp(-\gamma)$, where γ is the so-called Robin constant for Ω: $\gamma = \lim_{z\to\infty}[g(z) - \log|z|]$. If $p = 1$, we have $C(E) = \lim_{z\to\infty}[z/\Phi(z)]$. Thus, the capacity of the interval $[-1,1]$ is equal to $1/2$, the capacity of the unit circle is equal to 1, and the capacity of an arc of the unit circle (see above) is equal to $\cos(\theta_1/2)$.

The Szegő condition. We say that a weight function $\rho(\zeta)$ satisfies the Szegő condition on E (see [21]) if

$$\oint_E \log\rho(\zeta)|\Phi'(\zeta)|\,|d\zeta| = \oint_E \log\rho(\zeta)\frac{\partial g(\zeta)}{\partial n_\zeta}\,|d\zeta| > -\infty. \tag{2}$$

For the simplest cases mentioned above, the Szegő condition (2) is equivalent to the following:

1) $E = \{|\zeta| = 1\}$: $\displaystyle\int_0^{2\pi} \log\rho(e^{i\theta})\,d\theta > -\infty$;

2) $E = [-1,1]$: $\displaystyle\int_{-1}^{1} \frac{\log\rho(x)\,dx}{\sqrt{1-x^2}} > -\infty$;

3) $E = \{\zeta = e^{i\theta}, \theta_1 \le \theta \le 2\pi - \theta_1\}$: $\displaystyle\int_{\theta_1}^{2\pi-\theta_1} \frac{\log\rho(e^{i\theta}))\sin(\theta/2)\,d\theta}{\sqrt{\cos^2(\theta_1/2) - \cos^2(\theta/2)}} > -\infty$.

The Szegő function. Suppose that the Szegő condition (2) is satisfied for a weight function $\rho(\zeta)$. Then there exists a real function $h(z)$ harmonic in Ω with the following boundary condition on E: $h(\zeta)|_{\zeta\in\partial\Omega} = \log(\rho(\zeta))$ (see [21]). Then, the complex function $R(z) = \exp[h(z) + i\widetilde{h}(z)]$ is locally analytic in Ω, has nontangential limit values on $\partial\Omega$, and $|R(\zeta)|_{\zeta\in\partial\Omega} = \rho(\zeta)$. The function $D(z) = \sqrt{R(z)} = \exp[(1/2)(h + i\widetilde{h})]$ is usually referred to as the Szegő function associated with the weight $\rho(\zeta)$.

Harmonic measures. The harmonic measure $\omega_k(z)$, $k = 1, \dots, p$, is the function harmonic in Ω including ∞ that has the boundary value 1 at E_k and 0 at E_j, $j \ne k$. By the definition of $\omega_k(z)$ and standard formulas of harmonic analysis, we have

$$\omega_k(z) = \oint_{E_k} \frac{\partial g(z,\zeta)}{\partial n_\zeta}\,|d\zeta|.$$

We denote by $\Omega_k(z)$ the following complex modification of the harmonic measure:

$$\Omega_k(z) := (1/2)[\omega_k(z) + i\widetilde{\omega}_k(z)].$$

2.2. Results by Widom–Aptekarev.

2.2.1. *Hardy spaces of multivalued functions.* If Ω is not simply connected ($p > 1$), then the functions $\Phi(z)$ and $R(z)$ defined above are multivalued. More precisely, their moduli are single-valued functions and their arguments are multivalued in Ω. Given a multivalued function $F(z)$ with single-valued modulus, we define

$$\gamma_k(F) = \frac{1}{2\pi}\Delta_{E_k}\arg F \pmod 1, \quad k = 1, \dots, p.$$

Let $\Gamma(F)$ be the vector with the coordinates γ_k: $\Gamma(F) = (\gamma_1, \dots, \gamma_p)$, $0 \le \gamma_k < 1$. Every such vector determines a class of functions with multivalued argument. We say that two functions F_1 and F_2 belong to the same class if and only if $\gamma_k(F_1) = \gamma_k(F_2)$, $k = 1, \dots, p$. Note that $\Gamma(F_1 F_2) = \Gamma(F_1) + \Gamma(F_2)$, and $\gamma(\Phi(z, z_0)) = (\omega_1(z_0), \omega_2(z_0), \dots, \omega_p(z_0))$, because

$$\gamma_k(\Phi(z, z_0)) = \frac{1}{2\pi}\Delta_{E_k}\widetilde{g}(z, z_0) = \frac{1}{2\pi}\int_{E_k}\frac{\partial\widetilde{g}(\zeta, z_0)}{\partial t_\zeta}\,|d\zeta|$$
$$= \frac{1}{2\pi}\int_{E_k}\frac{\partial g(\zeta, z_0)}{\partial n_\zeta}\,|d\zeta| = \omega_k(z_0).$$

Mostly, we shall consider the classes of multivalued functions associated with the vector

$$\Gamma_n = \Gamma(\Phi^{-n}(z)) = (-n\omega_1(\infty), -n\omega_2(\infty), \dots, -n\omega_p(\infty)).$$

For a given vector Γ, the Hardy space $H_2(\Omega, \rho, \Gamma)$ is the space of functions $F(z)$ locally analytic in Ω with single-valued modulus and multivalued argument, belonging to the class Γ, and such that $|F(z)^2 R(z)|$ has a harmonic majorant in Ω. Any function $F(z)$ in $H_2(\Omega, \rho, \Gamma)$ admits nontangential limit values a.e. on E, and

$$\|F\|_\rho^2 := \oint_E |F(\zeta)|^2\rho(\zeta)\,|d\zeta| < \infty.$$

The function V_Γ. In [21], Widom defined a *canonical function* V_Γ for any class Γ. This function is analytic in Ω, has no poles and zeros there, and enjoys the following property: if we divide some function of class Γ by V_Γ, then the quotient is single-valued. This function is defined by the formula

$$V_\Gamma(z) = \exp\left(\sum_{k=1}^{p}\lambda_k[\omega_k(z) + \widetilde{\omega}_k(z)]\right), \tag{3}$$

where the numbers λ_k are uniquely determined by the following system of equations:

$$\sum_{k=1}^{p}\lambda_k\Delta_{E_j}\widetilde{\omega}_k = \gamma_j,$$
$$\sum_{k=1}^{p}\lambda_k = 0.$$

2.2.2. *Extremal problems in Hardy spaces.* In [21], Widom studied the following extremal problem.

(I) Determine

$$\mu(\Omega, \rho, \Gamma) = \inf\oint_E |F(\zeta)|^2\rho(\zeta)\,|d\zeta|$$

in the class of functions $F(z) \in H_2(\Omega, \rho, \Gamma)$ satisfying $|F(\infty)| = 1$.

As was shown in [21], the solution of this problem is unique up to an (appropriately chosen) constant factor of modulus 1. We denote the extremal function for problem (I) by $\psi_\Gamma(z)$.

The reproducing kernel. The function $K_{\rho,n}(z, z_0) \in H_2(\Omega, \rho, \Gamma_n)$ is determined by the following reproducing property: for any function $f \in H_2(\Omega, \rho, \Gamma_n)$ we have

$$\oint_E f(\zeta)\overline{K_{\rho,n}(\zeta, z_0)}\rho(\zeta)\,|d\zeta| = f(z_0) \quad \text{for all } z_0 \in \Omega.$$

This function is called the Szegő reproducing kernel. The reproducing kernel $K_{\rho,n}(z, \infty)$ and the extremal function $\psi_n(z)$ are related to each other by the following formula [21]:

$$\psi_n(z) = \mu(\Omega, \rho, \Gamma_n)K_{\rho,n}(z, \infty). \tag{4}$$

2.2.3. *Widom's results.* Let $\psi_n(z)$ denote the extremal function of problem (I) for the class $\Gamma = \Gamma_n = \Gamma(\Phi^{-n})$. Let $Q_{n,\rho}(z)$ be the monic polynomials orthogonal with respect to the measure σ ($d\sigma(\zeta) = \rho(\zeta)|d\zeta|$ on E). Denote

$$\Psi_n(\zeta) = \begin{cases} \Phi^n(\zeta)\psi_n(\zeta) & \text{if } \ \zeta \in E^{(1)}, \\ \Phi^n_+(\zeta)\psi_{n+}(\zeta) + \Phi^n_-(\zeta)\psi_{n-}(\zeta) & \text{if } \ \zeta \in E^{(2)}, \end{cases}$$

where $E^{(1)}$ is the union of the (closed) curves and $E^{(2)}$ is the union of the arcs of E. Then the following strong asymptotic formula holds.

Theorem 1 ([21, Theorem 12.3])**.** *If $E \in C^{2+}$ and if $\rho(\zeta)$ satisfies conditions* (1) *and* (2) *on E, then*

(1) $m_n(\sigma) \sim C(E)^{2n}\mu(\Omega, \rho, \Gamma_n)$ *as* $n \to \infty$;
(2) $\int_E |C(E)^{-n}Q_n(\zeta;\rho) - \Psi_n(\zeta)|^2\rho(\zeta)\,|d\zeta| \to 0$ *as* $n \to \infty$;
(3) $Q_n(z;\rho) = C(E)^n\Phi^n(z)(\psi_n(z) + \alpha_n)$ *with* $\alpha_n \to 0$ *as* $n \to \infty$*, uniformly on the compact subsets of* Ω.

2.2.4. *Aptekarev's formulas.* The extremal function $\psi_n(z)$ admits a very interesting explicit representation given by Aptekarev in [1]. To formulate it, we introduce the Riemann surface associated with the domain Ω and define the standard Riemann θ-functions on this surface (see [5]).

The Riemann surface. Consider the surface $\overline{\Omega}$ symmetric to Ω and defined as Ω with all local parameters replaced by their complex conjugates. We glue Ω and $\overline{\Omega}$ together along their boundaries E and $\overline{E}$ obtaining a closed orientable surface $\Re$; this is a two-sheeted Riemann surface of genus $(p-1)$. The real Green function and the harmonic measures can be extended to $\Re$ in the following way:

$$g(z) = \begin{cases} g(z) & \text{if } \ z \in \Omega, \\ -g(\overline{z}) & \text{if } \ z \in \overline{\Omega}, \end{cases} \qquad \omega_k(z) = \begin{cases} \omega_k(z) & \text{if } \ z \in \Omega, \\ -\omega_k(\overline{z}) & \text{if } \ z \in \overline{\Omega}. \end{cases}$$

The differential $d\Omega_k(z)$ of the complex function $\Omega_k(z)$ can also be extended to the surface $\Re$: $d\Omega_k(\overline{z}) = -\overline{d\Omega_k(z)}$, $z \in \Omega$. The same is possible for the differential of the complex Green function: $dG(\overline{z}, z_0) = -\overline{dG(z, z_0)}$, $z \in \Omega$.

Riemann θ-functions. For a given symmetric matrix $C = (C_{i,j})$ with positive definite imaginary part, the θ-function in h variables is defined by the relation

$$\theta(u_1, u_2, \ldots, u_h) = \sum_{n_1, n_2, \ldots, n_h \in \mathbb{Z}} \exp\left(\pi i \sum_{\mu=1}^{h}\sum_{\nu=1}^{h} C_{\mu,\nu} n_\mu n_\nu + 2\pi i \sum_{\nu=1}^{h} n_\nu u_\nu\right). \tag{5}$$

To construct the standard Riemann θ-function of the Riemann surface $\Re$, we calculate periods of the Abelian differentials $d\Omega_k(z)$,

$$\oint_{E_j} d\Omega_k(\zeta) = iB_{k,j}, \quad B_{k,j} \in R, \tag{6}$$

and choose an arbitrary complex vector $(b_1, b_2, \ldots, b_{p-1})$. Then the associated Riemann θ-function on $\Re$ is defined as follows:

$$\Theta(w) = \theta\biggl(\int_{z_0}^{w} d\Omega_1(\zeta) - b_1, \ldots, \int_{z_0}^{w} d\Omega_{p-1}(\zeta) - b_{p-1}\biggr),$$

where the function θ has the parameter matrix $C = (iB_{k,j})$.

2.2.5. *An explicit formula for the extremal function.* The following formula is valid (see [1]):

$$\psi_n = \chi(z)\frac{D(\infty)}{D(z)}\frac{\Theta_{n,\rho}(z)}{\Theta_n,\rho(\infty)}, \tag{7}$$

where the function $\chi(z)$ depends only on Ω (and neither on n nor on ρ), $D(z)$ is the Szegő function associated with the weight function ρ, and the function $\Theta_{n,\rho}(z)$ is defined by the formula

$$\Theta_{n,\rho}(z) = \theta\left(\left[\int_{z_0}^{z} d\Omega_j(\zeta)\right] - b_{j,n}\right), \quad j = 1, 2, \dots, p-1,$$

θ being the Riemann theta-function with the parameter matrix $C = i(B_{k,j})$, and

$$b_{\nu,n} = \Delta_\nu - n\omega_\nu(\infty) + \sum_{j=1}^{p-1}\int_{z_0}^{z_j^*} d\Omega_\nu(\zeta) + k_\nu, \quad \nu = 1, 2, \dots, p-1,$$

where $z_0 \in E_p$, the points $\{z_j^*\}_{j=1}^{p-1}$ are the zeros of $G'(z)$, the k_ν are constants depending only on Ω (the so-called Riemann constants of the surface $\Re$), and

$$\Delta_\nu = \frac{1}{4\pi}\oint_{E_\nu} \ln \rho_g(\zeta)\frac{\partial\omega_\nu}{\partial n_\zeta}\,|d\zeta|, \quad \rho_g(\zeta) = \rho(\zeta)/(\partial g/\partial n_\zeta).$$

§3. Extremal problem and mass points

In this section we introduce an extremal problem that plays the same role in the asymptotics of orthogonal polynomials for a measure with discrete part as problem (I) does in the case of an absolutely continuous measure. Let $H_2^0(\Omega,\rho,\Gamma)$ be the subspace of $H_2(\Omega,\rho,\Gamma)$ defined by

$$H_2^0(\Omega,\rho,\Gamma) = \bigl\{f \in H_2(\Omega,\rho,\Gamma);\ f(z_k) = 0,\ k = 1,\dots,l\bigr\}.$$

Consider the following extremal problem.

(II) For all $F(z) \in H_2^0(\Omega,\rho,\Gamma_n)$ satisfying $|F(\infty)| = 1$, determine

$$\mu^0(\Omega,\rho,\Gamma_n) = \inf_F \oint_E |F(\zeta)|^2\rho(\zeta)\,|d\zeta|.$$

We exhibit a solution of problem (II). The function

$$B(z) = \prod_{k=1}^{N}\frac{\Phi(\infty,z_k)}{\Phi(z,z_k)}$$

is analytic in Ω and has the following properties:

1) $B(z_k) = 0$;
2) $|B(\infty)| = 1$;
3) $|B(\zeta)|_{\zeta\in\partial\Omega}| = \prod_{k=1}^{l}|\Phi(z_k)|$;
4) $\gamma_j(B) = \sum_{k=1}^{l}\omega_j(z_k)$, $j = 1, 2, \dots, p$.

Thus, this function is a multivalued analog of a Blaschke product.

It is possible to reduce any extremal problem (II) to some extremal problem (I). Let $\mu^0(\Omega,\rho,\Gamma)$, $\psi_\Gamma^0(z) \in H_2^0(\Omega,\rho,\Gamma)$ denote the solution of the extremal problem (II) and $\mu(\Omega,\rho,\widetilde{\Gamma})$, $\psi_{\widetilde{\Gamma}}(z) \in H_2(\Omega,\rho,\widetilde{\Gamma})$ the solution of the extremal problem (I) associated with the class of multiplicity $\widetilde{\Gamma} = \Gamma - \Gamma(B)$.

Lemma 1. *We have*

$$1)\ \psi_{\widetilde{\Gamma}}(z)B(z) = \psi^0_{\Gamma}(z);$$
$$2)\ \mu^0(\Omega,\rho,\Gamma) = \mu(\Omega,\rho,\widetilde{\Gamma})\prod_{k=1}^{l}|\Phi(z_k)|^2.$$

Proof. 1) Since $\psi_{\widetilde{\Gamma}} \in H_2(\Omega,\rho,\widetilde{\Gamma})$, we have $|\psi_{\widetilde{\Gamma}}(\infty)| = 1$, $\Gamma(\psi_{\widetilde{\Gamma}}) = \widetilde{\Gamma}$. Let $f(z) = \psi_{\widetilde{\Gamma}}(z)B(z)$; then $f(z_k) = 0$, $k = 1,\dots,m$, $|f(\infty)| = 1$, and $\Gamma(f) = \widetilde{\Gamma} + \Gamma(B) = \Gamma$. Thus, $f \in H^0_2(\Omega,\rho,\Gamma)$ and

$$\mu^0(\Omega,\rho,\Gamma) = \|\psi^0_\Gamma\| = \oint_E |\psi^0_\Gamma(\zeta)|^2\rho(\zeta)\,|d\zeta| \le \int_E |\psi_{\widetilde{\Gamma}}(\zeta)B(\zeta)|^2\rho(\zeta)\,|d\zeta|$$
$$= \prod_{k=1}^{l}|\Phi(z_k)|^2\int_E |\psi_{\widetilde{\Gamma}}(\zeta)|^2\rho(\zeta)\,|d\zeta| = \mu(\Omega,\rho,\widetilde{\Gamma})\prod_{k=1}^{l}|\Phi(z_k)|^2.$$

2) Since $\psi^0_\Gamma \in H^0_2(\Omega,\rho,\Gamma)$, we have $|\psi^0_\Gamma(\infty)| = 1$, $\psi^0(z_k) = 0$, $\Gamma(\psi^0) = \Gamma$. Let $f(z) = \psi^0_\Gamma(z)/B(z)$; then $f(z)$ is analytic in Ω and $|f(\infty)| = 1$, $\Gamma(f) = \Gamma(\psi^0) - \Gamma(B) = \widetilde{\Gamma}$. Thus, $f \in H_2(\Omega,\rho,\widetilde{\Gamma})$, whence

$$\mu(\Omega,\rho,\widetilde{\Gamma}) = \|\psi_{\widetilde{\Gamma}}\|^2 \le \oint_E |f(\zeta)|^2\rho(\zeta)\,|d\zeta| = \oint_E |\psi^0(\zeta)B^{-1}(\zeta)|^2\rho(\zeta)\,|d\zeta|$$
$$= \prod_{k=1}^{l}|\Phi^{-2}(z_k)|\oint_E |\psi^0_\Gamma(\zeta)|^2\rho(\zeta)\,|d\zeta|$$
$$= \mu^0(\Omega,\rho,\Gamma)\prod_{k=1}^{l}|\Phi^{-2}(z_k)|. \qquad \square$$

Example 1. In the case where Ω is simply connected ($p = 1$), all the functions under consideration are single-valued. The functions $\Phi(z,z_k)$ and $B(z)$ are given by the formulas

$$\Phi(z,z_k) = \frac{\Phi(z)\overline{\Phi(z_k)} - 1}{\Phi(z) - \Phi(z_k)}, \quad B(z) = \prod_{k=1}^{l}\frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1}\overline{\Phi(z_k)},$$

and Lemma 1 looks like this (see [12]):

$$\psi(z)B(z) = \psi^0(z), \quad \mu^0(\Omega,\rho) = \mu(\Omega,\rho)\prod_{k=1}^{l}|\Phi(z_k)|^2.$$

There is a different way to reduce the extremal problem (II) to (I). We can change the weight function instead of the class of multivaluedness (we shall need this in the next section to prove the asymptotic formula for orthogonal polynomials). Put

$$H(z) = \prod_{k=1}^{l}\frac{(z - z_k)}{C(E)\Phi(z)}.$$

Then $H \in \Gamma_l$, $|H(\infty)| = 1$, and $H(z_k) = 0$. We define a new weight function $\rho^*(\zeta)$ on E by the formula

$$\rho^*(\zeta) = \rho(\zeta)\prod_{k=1}^{l}|\zeta - z_k|^2/C(E)^2.$$

Let $\mu^0(\rho,\Gamma_n)$, $\psi^0_n(z) \in H^0_2(\Omega,\rho,\Gamma_n)$ solve the extremal problem (II), and let $\mu(\rho^*,\Gamma_{n-l})$, $\psi^*_{n-l}(z) \in H_2(\Omega,\rho^*,\Gamma_{n-l})$ solve the extremal problem (I) associated with the weight function $\rho^*(\zeta)$, in the class Γ_{n-l}.

Lemma 2. *We have*

$$1)\ \psi^*_{n-l}(z)H(z) = \psi^0_n(z);$$
$$2)\ \mu^0(\Omega, \rho, \Gamma_n) = \mu(\Omega, \rho^*, \Gamma_{n-l}).$$

Proof. 1) Since $\psi^*_{n-l} \in H_2(\Omega, \rho^*, \Gamma_{n-l})$, we have $|\psi^*_{n-l}(\infty)| = 1$, $\Gamma(\psi^*_{n-l}) = \Gamma_{n-l}$.

Let $f(z) = \psi^*_{n-l}H(z)$; then $f(z_k) = 0$, $k = 1, \dots, l$, $|f(\infty)| = 1$, and $\Gamma(f) = \Gamma(\psi_{n-l}) + \Gamma(H) = \Gamma_n$. So, f belongs to $H^0_2(\Omega, \rho, \Gamma_n)$, and

$$\begin{aligned}\mu^0(\Omega, \rho, \Gamma_n) \le \oint_E |\psi^*_{n-l}(\zeta)H(\zeta)|^2 \rho(\zeta)\,|d\zeta| &= \oint_E |\psi^*_{n-l}|^2 |H(\zeta)|^2 \rho(\zeta)\,|d\zeta| \\ &= \oint_E |\psi^*_{n-l}|^2 \rho^*(\zeta)\,|d\zeta| = \mu(\Omega, \rho^*, \Gamma_{n-l}).\end{aligned}$$

2) We have $\psi^0_n(z)/H(z) \in H_2(\Omega, \rho^*, \Gamma_{n-l})$. This implies that

$$\mu(\rho^*, \Gamma_{n-l}) \le \oint_E \left|\frac{\psi^0_n(\zeta)}{H(\zeta)}\right|^2 |H(\zeta)|^2 \rho(\zeta)\,|d\zeta| = \mu^0(\Omega, \rho; \Gamma_n). \qquad \square$$

§4. Asymptotics of orthogonal polynomials

In this section we obtain strong asymptotic formulas for the polynomials $Q_n(z, \sigma^0)$ orthogonal with respect to the measure σ^0 with discrete part. Our investigation is based on the extremal property of the polynomials $Q_n(z, \sigma^0)$ and on the reproducing property of the extremal functions $\psi_n(z)$. The extremal property in this case reads as follows:

$$\begin{aligned} m_n(\sigma^0) = \|Q_n(\sigma^0)\|^2_{\sigma^0} &= \int_E |Q_n(\zeta, \sigma^0)|^2 \rho(\zeta)\,|d\zeta| + \sum_{k=1}^{l} \lambda_k |Q_n(z_k, \sigma^0)|^2 \\ &= \min_{P_n = z^n + \dots} \left(\int_E |P_n(\zeta)|^2 \rho(\zeta)\,|d\zeta| + \sum_{k=1}^{l} \lambda_k |P_n(z_k)|^2 \right). \end{aligned}$$

Now, we state the main result of the paper.

Theorem 2. *Suppose that $E \in C^{2+}$ and $\rho(\zeta)$ satisfies conditions* (1) *and* (2). *Then*:

1) $m_n(\sigma^0) \sim C(E)^{2n} \mu^0(\Omega, \rho, \Gamma_n)$ *as* $n \to \infty$;
2) $\displaystyle\int_E |C(E)^{-n} Q_n(\zeta) - \Psi^0_n(\zeta)|^2 \rho(\zeta)\,|d\zeta| \to 0$;
3) $Q_n(z) = C(E)^n \Phi^n(z)[\psi^0_n(z) + \epsilon_n(z)]$, *where* $\epsilon_n \to 0$ *as* $n \to \infty$,

uniformly on the compact subsets of $\Omega \setminus \{z_1, z_2, \dots, z_l\}$, *and*

$$\Psi^0_n(\zeta) = \begin{cases} \Phi^n(\zeta)\psi^0_n(\zeta) & \text{if } \zeta \in E^{(1)}, \\ \Phi^n_+(\zeta)\psi^0_{n+}(\zeta) + \Phi^n_-(\zeta)\psi^0_{n-} & \text{if } \zeta \in E^{(2)}, \end{cases}$$

where $E^{(1)}$ is the union of the (*closed*) *curves in E, and $E^{(2)}$ is the union of the arcs.*

Proof. We start with the proof of 2), which implies 1), and then we prove 3). We split the integral in 2) into three integrals and evaluate each of them:

$$\begin{aligned} I_n &= \int_E |C(E)^{-n}Q_n(\zeta) - \Psi_n^0(\zeta)|^2 \rho(\zeta)\,|d\zeta| \\ &= \int_E |C(E)^{-n}Q_n(\zeta)|^2 \rho(\zeta)\,|d\zeta| + \int_E |\Psi_n^0(\zeta)|^2 \rho(\zeta)\,|d\zeta| \\ &\quad - 2\operatorname{Re}\int_E C(E)^{-n}Q_n(\zeta)\overline{\Psi_n^0(\zeta)\rho(\zeta)}\,|d\zeta| \\ &= I_n^{(1)} + I_n^{(2)} + I_n^{(3)}. \end{aligned}$$

□

4.1. The integral $I_n^{(1)}$. For the polynomials $Q_n = Q_n(z, \sigma^0)$ we have

$$\int_E |C(E)^{-n}Q_n(\zeta)|^2 \rho(\zeta)\,|d\zeta| = \frac{m_n(\sigma^0)}{C(E)^{2n}} - \sum_{k=1}^{l} \lambda_k \left|\frac{Q_n(z_k)}{C(E)^n}\right|^2 \le \frac{m_n(\sigma^0)}{C(E)^{2n}}.$$

Consider the weight function $\rho^*(\zeta) = \rho(\zeta)C(E)^{-2l}\prod_{k=1}^{l}|\zeta - z_k|^2$. Let $Q_{n-l}(z, \rho^*)$ denote the extremal polynomials for $m_{n-l}(\rho^*)$. Then $Q_{n-l}(z,\rho^*)\prod_{k=1}^{l}(z - z_k)$ is a monic polynomial of degree n. Obviously, we have

$$\begin{aligned} \frac{m_n(\sigma^0)}{C(E)^{2n}} &\le \frac{\|Q_{n-l}(z,\rho^*)\prod(z-z_k)\|_{\sigma^0}^2}{C(E)^{2n}} \\ &= \frac{1}{C(E)^{2n}} \int_E |Q_{n-l}(\zeta,\rho^*)|^2 \prod_{k=1}^{l} |\zeta - z_k|^2 \rho(\zeta)\,|d\zeta| \\ &= \frac{1}{C(E)^{2(n-l)}} \int_E |Q_{n-l}(\zeta,\rho^*)|^2 \rho^*(\zeta)\,|d\zeta| = \frac{m_{n-l}(\rho^*)}{C(E)^{2(n-l)}}. \end{aligned}$$

Theorem 1 and Lemma 2 show that

$$\frac{m_{n-l}(\rho^*)}{C(E)^{2(n-l)}} \sim \mu(\Omega, \rho^*, \Gamma_{n-l}) = \mu^0(\Omega, \rho, \Gamma_n), \quad n \to \infty.$$

Hence,

$$\int_E |C(E)^{-n}Q_n(\zeta)|^2 \rho(\zeta)\,|d\zeta| \le \mu^0(\Omega,\rho,\Gamma_n) + \alpha_n, \quad \alpha_n \to 0.$$

4.2. The integral $I_n^{(2)}$. For the union of curves we have

$$\int_{E^{(1)}} |\Psi_n^0(\zeta)|^2 \rho(\zeta)\,|d\zeta| = \oint_{E^{(1)}} |\psi_n^0(\zeta)|^2 \rho(\zeta)\,|d\zeta|,$$

and for the union of arcs we have

$$\begin{aligned} \int_{E^{(2)}} |\Psi_n^0(\zeta)|^2 \rho(\zeta)\,|d\zeta| = &\oint_{E^{(2)}} |\psi_n^0(\zeta)|^2 \rho(\zeta)\,|d\zeta| \\ &+ 2\operatorname{Re}\int_{E^{(2)}} \overline{\Phi_+^n(\zeta)\psi_{n+}^0(\zeta)}\Phi_-^n(\zeta)\psi_{n-}^0(\zeta)\rho(\zeta)\,|d\zeta|. \end{aligned}$$

The second integral tends to zero as $n \to \infty$ (see [21], Lemma 12.1). Finally, we obtain

$$\int_E |\Psi_n^0(\zeta)|^2 \rho(\zeta)\,|d\zeta| = \mu^0(\Omega,\rho,\Gamma_n) + \beta_n, \quad \beta_n \to 0.$$

4.3. The integral $I_n^{(3)}$. Since $|\Phi_n(\zeta)| = 1$ for $\zeta \in E$, it follows that

$$2\operatorname{Re}\int_E C(E)^{-n} Q_n(\zeta)\overline{\Psi_n^0(\zeta)}\rho(\zeta)\,|d\zeta| = 2\operatorname{Re}\oint_E \frac{Q_n(\zeta)}{C(E)^n\Phi^n(\zeta)}\overline{\psi_n^0(\zeta)}\rho(\zeta)\,|d\zeta|.$$

Let

$$\phi_n(\zeta) = \frac{Q_n(\zeta)}{C(E)^n\Phi^n(\zeta)}.$$

By Lemma 2, $\psi_n^0(z) = \psi_{n-l}^*(z)H(z)$. At the same time, the extremal function $\psi_{n-l}^*(z)$ is related by (4) with the reproducing kernel $K_{\rho^*,n-l}(\zeta,\infty)$. Consequently,

$$\begin{aligned}2\operatorname{Re}\oint_E \phi_n(\zeta)\overline{\psi_n^0(\zeta)}\rho(\zeta)\,|d\zeta| &= 2\operatorname{Re}\oint_E \phi_n(\zeta)\overline{\mu_{n-l}(\rho^*)K_{\rho^*,n-l}(\zeta,\infty)H(\zeta)}\rho(\zeta)\,|d\zeta|\\ &= 2\mu(\Omega,\rho^*,\Gamma_{n-l})\operatorname{Re}\oint_E \frac{\phi_n(\zeta)}{H(\zeta)}\overline{K_{\rho^*,n-l}(\zeta,\infty)}|H(\zeta)|^2\rho(\zeta)\,|d\zeta|\\ &= 2\mu(\Omega,\rho^*,\Gamma_{n-l})\operatorname{Re}\oint_E \frac{\phi_n(\zeta)}{H(\zeta)}\overline{K_{\rho^*,n-l}(\zeta,\infty)}\rho^*(\zeta)\,|d\zeta|.\end{aligned}$$

The function $\phi_n(z)/H(z)$ does not belong to the space $H_2(\Omega,\rho^*,\Gamma_{n-l})$, because it has poles at the points z_k. Consider the function

$$R_n(z) = \frac{\phi_n(z)}{H(z)V_{\Gamma_{n-l}}(z)},$$

where $V_{\Gamma_{n-l}}(z)$ is defined by (3); then $\dfrac{\phi_n(z)}{H(z)} = R_n(z)V_{\Gamma_{n-l}}(z)$. The function $R_n(z)$ is single-valued, takes the value 1 at infinity, and has poles at the points z_k. We have

$$R_n(z) = \sum_{k=1}^{l}\frac{\frac{\phi_n(z_k)}{V_{\Gamma_{n-l}}(z_k)}}{H'(z_k)(z-z_k)} + r_n(z).$$

The function r_n is single-valued and $r_n(\infty) = 1$.

Lemma 3.

$$\frac{\phi_n(z_k)}{V_{\Gamma_{n-l}}(z_k)} \to 0 \quad \textit{as } n\to\infty.$$

Proof. Since $|\Phi(z_k)| > 1$, it suffices to prove that the sequences $Q_n(z_k)C(E)^{-n}$ and $V_{\Gamma_{n-l}}(z_k)$ are bounded.

1) To prove the boundedness of $Q_n(z_k)C(E)^{-n}$, we use the weight function $\rho^*(\zeta)$ and the polynomials $Q_{n-l}(z,\rho^*)$ orthogonal with respect to $\rho^*(\zeta)$ on E. We have

$$\sum_{k=1}^{l}\frac{\lambda_k|Q_n(z_k)|^2}{C(E)^{2n}} \le \frac{\|Q_n\|^2_{\sigma^0}}{C(E)^{2n}} \le \frac{\|Q_{n-l}\prod_{k=1}^{l}(z-z_k)/C(E)\|^2_{\sigma^0}}{C(E)^{2(n-l)}} = \frac{m_{n-l}(\rho^*)}{C(E)^{2(n-l)}}.$$

The sequence on the right is bounded by Theorem 1.

2) Since the function $V_{\Gamma_{n-l}}$ has neither poles nor zeros in Ω, it attains its maximal and minimal values on the boundary. We have

$$|V_{\Gamma_{n-l}}(z)|_{E_k} = \exp(\lambda_k);$$

hence, it suffices to prove the boundedness of the sequence λ_k.

Recall that the numbers λ_k were defined as solutions of some nonhomogeneous system of linear equations with nonzero determinant. The right-hand sides of these equations are equal to the numbers $\gamma_k \in [0;1)$. Therefore, the solution of the system (the numbers λ_k) is also bounded. □

Using Lemma 3 and the relation

$$\oint_E r_n(\zeta)V_{\Gamma_{n-l}}(z)\overline{K_{\rho^*,n-l}(\zeta,\infty)}\rho^*(\zeta)\,|d\zeta| = r_n(\infty)V_{\Gamma_{n-l}}(\infty) = 1,$$

we get

$$2\operatorname{Re}\oint_E C(E)^{-n}Q_n(\zeta)\overline{\Psi^n(\zeta)}\rho(\zeta)\,|d\zeta| = 2\mu^0(\Omega,\rho,\Gamma_n)+\beta_n, \quad \beta_n\to 0.$$

Finally, we obtain

$$I = \int_E |C(E)^{-n}Q_n - \Psi_n^0|^2\rho(\zeta)\,|d\zeta| \le \alpha_n - \beta_n \to 0, \quad n\to\infty.$$

This completes the proof of items 2) and 1) of the theorem. To prove 3), we use the reproducing kernel $K_{\rho,n}(\zeta,z)$ (to simplify the notation, we write $K(z,\zeta)$ from now on):

$$\begin{aligned}
\frac{Q_n(z)}{C(E)^n\Phi^n(z)} &= \oint_E \frac{Q_n(\zeta)}{C(E)^n\Phi^n(\zeta)}\overline{K(\zeta,z)}\rho(\zeta)\,|d\zeta| \\
&= \int_{E^{(1)}} C(E)^{-n}Q_n(\zeta)\Phi^{-n}(\zeta)\overline{K(\zeta,z)}\rho(\zeta)\,|d\zeta| \\
&\quad + \int_{E^{(2)}} C(E)^{-n}Q_n(\zeta)\Big(\Phi_+^{-n}(\zeta)\overline{K_+(\zeta,z)} + \Phi_-^{-n}(\zeta)\overline{K_-(\zeta,z)}\Big)\rho(\zeta)\,|d\zeta| \\
&= \int_{E^{(1)}} (C(E)^{-n}Q_n(\zeta) - \Psi_n(\zeta))\Phi^{-n}(\zeta)\overline{K(\zeta,z)}\rho(\zeta)\,|d\zeta| \\
&\quad + \int_{E^{(2)}} \big(C(E)^{-n}Q_n(\zeta) - \Psi_n(\zeta)\big)\Big(\Phi_+^{-n}(\zeta)\overline{K_+(\zeta,z)} + \Phi_-^{-n}(\zeta)\overline{K_-(\zeta,z)}\Big)\rho(\zeta)\,|d\zeta| \\
&\quad + \int_{E^{(1)}} \Psi_n^0(\zeta)\Phi^{-n}(\zeta)\overline{K(\zeta,z)}\rho(\zeta)\,|d\zeta| \\
&\quad + \int_{E^{(2)}} \Psi_n(\zeta)\Big(\Phi_+^{-n}(\zeta)\overline{K_+(\zeta,z)} + \Phi_-^{-n}(\zeta)\overline{K_-(\zeta,z)}\Big)\rho(\zeta)\,|d\zeta|.
\end{aligned}$$

The first and second integrals tend to zero (see statement 2 of the theorem). We transform the last integral:

$$\begin{aligned}
&\int_{E^{(2)}} \Psi_n(\zeta)\Big(\Phi_+^{-n}(\zeta)\overline{K_+(\zeta,z)} + \Phi_-^{-n}(\zeta)\overline{K_-(\zeta,z)}\Big)\rho(\zeta)\,|d\zeta| \\
&\quad = \int_{E^{(2)}} \Big(\Phi_+^n(\zeta)\psi_{n+}^0(\zeta) + \Phi_-^n(\zeta)\psi_{n-}^0(\zeta)\Big)\Big(\Phi_+^{-n}(\zeta)\overline{K_+(\zeta,z)} + \Phi_-^{-n}(\zeta)\overline{K_-(\zeta,z)}\Big)\rho(\zeta)\,|d\zeta| \\
&\quad = \oint_{E^{(2)}} \psi_n^0(\zeta)\overline{K(\zeta,z)}\rho(\zeta)\,|d\zeta| \\
&\qquad + \int_{E^{(2)}} \Big(\Phi_+^n(\zeta)\psi_{n+}^0(\zeta)\Phi_-^{-n}(\zeta)\overline{K_-(\zeta,z)} + \Phi_-^n(\zeta)\psi_{n-}^0(\zeta)\Phi_+^{-n}\overline{K_+(\zeta,z)}\Big)\rho(\zeta)\,|d\zeta|.
\end{aligned}$$

In the last expression the second integral tends to zero by Lemma 12.1 in [21]. Finally, we obtain

$$\frac{Q_n(z)}{C(E)^n\Phi^n(z)} = \oint_{E^{(1)}\cup E^{(2)}} \psi_n^0(\zeta)\overline{K(\zeta,z)}\rho(\zeta)\,|d\zeta| = \psi_n^0(z) + \theta_n,$$

where $\theta_n \to 0$.

§5. Explicit formulas for extremal functions

Theorem 2, Lemma 1, and Aptekarev's formulas (7) allow us to calculate the extremal function $\psi_n^0(z)$:

$$\psi_n^0(z) = \chi(z)\frac{D(\infty)}{D(z)}\frac{\Theta_{n,\rho}^0(z)}{\Theta_{n,\rho}^0(\infty)}\prod_{k=1}^{l}\frac{\Phi(\infty, z_k)}{\Phi(z, z_k)}. \tag{8}$$

Here $\chi(z)$ is the function that depends only on Ω and not on Γ and the weight function ρ; $D(z)$ denotes the Szegő function associated with ρ,

$$\Theta_{n,\rho}^0(z) = \theta\left(\left[\int_{z_0}^{z} d\Omega_j(\zeta)\right] - b_j^0\right), \quad j = 1, 2, \dots, p-1,$$

where θ is the Riemann theta-function with the parameter matrix $C = (iB_{k,j})$, and

$$b_\nu^0 = \Delta_\nu - n\omega_\nu(\infty) + \sum_{k=1}^{l}\omega_\nu(z_k) + \sum_{j=1}^{p-1}\int_{z_0}^{z_j^*} d\Omega_\nu(\zeta) + k_\nu, \quad \nu = 1, 2, \dots, p-1,$$

where $z_0 \in E_p$, the points $\{z_j^*\}_{j=1}^{p-1}$ are the zeros of the function $G'(z)$, the k_ν are some constants depending only on the domain Ω (the Riemann constants of $\Re$), and

$$\Delta_\nu = \frac{1}{4\pi}\oint_{E_\nu} \ln \rho_g(\zeta)\frac{\partial \omega_\nu}{\partial n_\zeta}\,|d\zeta|, \quad \rho_g(\zeta) = \rho(\zeta)/(\partial g/\partial n_\zeta).$$

So, in the case of a measure with finitely many masses there are two changes in Aptekarev's formulas: the parameters b_ν are replaced by $b_\nu^0 = b_\nu + \sum_{k=1}^{l}\omega_\nu(z_k)$, and the extremal function is multiplied by the Blaschke product $B(z)$.

Remark 1. Aptekarev [1] proved similar asymptotic formulas for the polynomials orthogonal with respect to a measure supported on a finite number of intervals and points lying on the real line. The paper [1] contains also the same formulas for the case of arbitrary curves. The proof employs Rakhmanov's results [8] based on some properties of quasiorthogonality; however, these polynomials are quasiorthogonal only in the case of measures supported on a subset of the real line.

Remark 2. The case of infinitely many mass points added to a measure supported on a system of curves and arcs was considered in [14]. The authors of [14] dealt with measures satisfying the following condition:

$$m_n(\sigma_N) \le \left(\prod_{k=1}^{\infty}|\Phi(z_k)|\right) m_n(\alpha), \tag{9}$$

where $\sigma = \alpha + \sum_{k=1}^{\infty} A_k\delta_{z_k}$ and $\sigma_N = \alpha + \sum_{k=1}^{N} A_k\delta_{z_k}$. We do not know of any general classes of measures for which this condition can be verified. Moreover, the main statement (Theorem 4.5) of [14] cannot be true in the case of multiply connected domains even when no point masses are added (see [21]).

References

[1] A. I. Aptekarev, *Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda chains*, Mat. Sb. (N.S.) **125(167)** (1984), no. 2, 231–258; English transl., Math. USSR-Sb. **53** (1986), no. 1, 233–260. MR0764479 (86g:35166)

[2] A. Vol′berg, F. Peherstorfer, and P. Yuditskiĭ, *Asymptotics of orthogonal polynomials in the case not covered by Szegő's theorem*, Funktsional. Anal. i Prilozhen. **40** (2006), no. 4, 22–32; English transl., Funct. Anal. Appl. **40** (2006), no. 4, 264–272. MR2307700 (2008c:42028)

[3] G. M. Goluzin, *Geometric theory of functions of a complex variable*, Nauka, Moscow, 1966; English transl., Transl. Math. Monogr., vol. 26, Amer. Math. Soc., Providence, RI, 1969. MR0219714 (36:2793); MR0247039 (40:308)

[4] A. A. Gonchar, *The convergence of Padé approximants for certain classes of meromorphic functions*, Mat. Sb. (N.S.) **97(139)** (1975), no. 4, 607–629; English transl. in Math. USSR-Sb. **26** (1975). MR0387552 (52:8392)

[5] B. A. Dubrovin, *Theta-functions and nonlinear equations*, Uspekhi Mat. Nauk **36** (1981), no. 2, 11–80; English transl., Russian Math. Surveys **36** (1981), no. 2, 11–92 (1982). MR0616797 (83i:35149)

[6] E. M. Nikishin, *The discrete Sturm–Liouville operator and some problems of function theory*, Trudy Sem. Petrovsk. No. 10 (1984), 3–77; English transl. in J. Soviet Math. **35** (1986). MR0778879 (86h:39007)

[7] E. A. Rakhmanov, *The asymptotic behavior of the ratio of orthogonal polynomials*, Mat. Sb. (N.S.) **103(145)** (1977), no. 2, 237–252; English transl. in Math. USSR-Sb. **32** (1977), no. 2. MR0445212 (56:3556)

[8] ———, *The convergence of diagonal Padé approximants*, Mat. Sb. (N.S.) **104(146)** (1977), no. 2, 271–291; English transl., Math. USSR-Sb. **33** (1977), no. 2, 243–260. MR0492292 (58:11430)

[9] M. Bello Hernández and G. Lopez Lagomasino, *Ratio and relative asymptotics of polynomials orthogonal on an arc of the unit circle*, J. Approx. Theory **92** (1998), 216–244. MR1604927 (99c:42041)

[10] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Comm. Pure Appl. Math. **52** (1999), no. 11, 1335–1425. MR1702716 (2001g:42050)

[11] V. A. Kaliaguine [Kalyagin], *A note on the asymptotics of orthogonal polynomials on a complex arc: the case of a measure with a discrete part*, J. Approx. Theory **80** (1995), no. 1, 138–145. MR1308599 (95k:42042)

[12] V. A. Kaliaguine and R. Benzine, *Sur la formule asymptotique des polynômes orthogonaux associés à une mesure concentrée sur un contour plus une partie discrète finie*, Bull. Soc. Math. Belg. Sér. B **41** (1989), no. 1, 29–46. MR1043998 (91f:42025)

[13] V. A. Kaliaguine and A. A. Kononova, *Strong asymptotics for polynomials orthogonal on a system of complex arcs and curves: Szegő condition on and a mass points off the system*, Publ. ANO-410, Univ. Sci. Techn. Lille, 2000.

[14] R. Khaldi and F. Aggoune, *On extremal polynomials on a system of curves and arcs*, J. Eng. Appl. Sci. **2** (2007), no. 2, 372–376.

[15] A. B. J. Kuijlaars, K. T.-R. McLaughlin, W. Van Assche, and M. Vanlessen, *The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on* $[-1,1]$, Adv. Math. **188** (2004), no. 2, 337–398. MR2087231 (2005i:33011)

[16] X. Li and K. Pan, *Asymptotic behavior of orthogonal polynomials corresponding to measure with discrete part off the unit circle*, J. Approx. Theory **79** (1994), 54–71. MR1294320 (96a:42029)

[17] F. Marcellán and P. Maroni, *Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique*, Ann. Math. Pura Appl. (4) **162** (1992), 1–22. MR1199643 (94e:33014)

[18] F. Peherstorfer and P. Yuditskii, *Asymptotics of orthonormal polynomials in the presence of a denumerable set of mass points*, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3213–3220. MR1844996 (2002h:42055)

[19] G. Szegő, *Orthogonal polynomials*, fourth edition, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975. MR0372517 (51:8724)

[20] H. Stahl and V. Totik, *General orthogonal polynomials*, Encyclopedia Math. Appl., vol. 43, Cambridge Univ. Press, Cambridge, 1992. MR1163828 (93d:42029)

[21] H. Widom, *Extremal polynomials associated with a system of curves in the complex plane*, Adv. Math. **3** (1969), no. 2, 127–232. MR0239059 (39:418)

Higher School of Economics, Nizhny Novgorod Branch, Lobachevsky State University, 25/12 Bolshaya Pecherskaya Street, Nizhny Novgorod 603155, Russia

E-mail address: `kalia@hse.nnov.ru`

Nizhny Novgorod State Technical University, 24 Minina Street, Nizhny Novgorod 603950, Russia

E-mail address: `anya.kononova@gmail.com`

Received 15/APR/2008

Translated by BY THE AUTHORS