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EXTENSION OF MATRICES WITH ENTRIES IN H^∞ ON COVERINGS OF RIEMANN SURFACES OF FINITE TYPE

A. BRUDNYI

Abstract. The paper continues an earlier work of the author. An extension theorem is proved for matrices with entries in the algebra of bounded holomorphic functions defined on an unbranched covering of a Carathéodory hyperbolic Riemann surface of finite type.

§1. Introduction

Let X be a complex manifold, and let $H^\infty(X)$ be the Banach algebra of bounded holomorphic functions on X, equipped with the supremum norm. We assume that X is Carathéodory hyperbolic, that is, the functions in $H^\infty(X)$ separate the points of X. The maximal ideal space $\mathcal{M} = \mathcal{M}(H^\infty(X))$ is the set of all nonzero linear multiplicative functionals on $H^\infty(X)$. Since the norm of each $\phi \in \mathcal{M}$ is at most 1, $\mathcal{M}$ is a subset of the closed unit ball of the dual space $(H^\infty(X))^*$. It is a compact Hausdorff space in the Gelfand topology (i.e., in the weak $*$ topology induced by $(H^\infty(X))^*$). Also, there is a continuous embedding $i : X \hookrightarrow \mathcal{M}$ taking $x \in X$ to the evaluation homomorphism $f \mapsto f(x)$, $f \in H^\infty(X)$. The complement to the closure of $i(X)$ in $\mathcal{M}$ is called the *corona*. The *corona problem* is as follows: given X, to determine whether the corona is empty. For example, Carleson's celebrated Corona theorem [C] states that this is true if X is the open unit disk in $\mathbb{C}$. (This was conjectured by Kakutani in 1941.) Also, there are nonplanar Riemann surfaces for which the corona is nontrivial (see, e.g., [JM, G, BD, L] and the references therein). This is due to a structure that in a sense makes the surface seem higher-dimensional. So there is some hope that the restriction to the Riemann sphere might prevent this obstacle. However, the general problem for planar domains is still open, as is the problem in several variables for the ball and polydisk. (In fact, there are no known examples of domains in $\mathbb{C}^n$, $n \geq 2$, without a corona.) At present, the strongest corona theorem for planar domains is due to Moore [M]. It states that the corona is empty for any domain with boundary contained in the graph of a $C^{1+\epsilon}$-function. This result is an extension of an earlier result of Jones and Garnett [GJ] for a Denjoy domain (i.e., a domain with boundary contained in $\mathbb{R}$).

The corona problem can be reformulated as follows; see, e.g., [Ga].

A collection $f_1, \dots, f_n$ of functions in $H^\infty(X)$ satisfies the *corona condition* if

$$\max_{1\leq j\leq n} |f_j(x)| \geq \delta > 0 \quad \text{for all} \quad x \in X. \tag{1.1}$$

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The corona problem being solvable (i.e., the corona is empty) means that the Bézout equation

$$f_1 g_1 + \cdots + f_n g_n \equiv 1 \tag{1.2}$$

has a solution $g_1, \dots, g_n \in H^\infty(X)$ for any $f_1, \dots, f_n$ satisfying the corona condition. We refer to $\max_{1 \le j \le n} \|g_j\|_\infty$ as a "bound on the corona solutions". (Here $\|\cdot\|_\infty$ is the norm on $H^\infty(X)$.)

In [Br4, Theorem 1.1], using an L^2 cohomology technique, we proved the following statement.

Theorem 1.1. *Let $r : X \to Y$ be a connected unbranched covering of a Carathéodory hyperbolic Riemann surface of finite type Y (i.e., the fundamental group of Y is finitely generated). Then X is Carathéodory hyperbolic and for any $f_1, \dots, f_n \in H^\infty(X)$ satisfying* (1.1) *there are solutions $g_1, \dots, g_n \in H^\infty(X)$ of* (1.2) *with the bound*

$$\max_{1 \le j \le n} \|g_j\|_\infty \le C\Big(Y, n, \max_{1 \le j \le n} \|f_j\|_\infty, \delta\Big).$$

This result extends the class of Riemann surfaces for which the corona theorem is valid (see also [Br1]). On the other hand, the results of Lárusson [L] (sharpened in [Br3]) imply that the assumption of the Carathéodory hyperbolicity of Y cannot be dropped. Specifically, for any integer $n \ge 2$ there is a compact Riemann surface S_n and its regular covering $r_n : \widetilde{S}_n \to S_n$ such that

(a) $\widetilde{S}_n$ is a complex submanifold of an open Euclidean ball $\mathbb{B}_n \subset \mathbb{C}^n$;
(b) the embedding $i : \widetilde{S}_n \hookrightarrow \mathbb{B}_n$ induces an isometry $i^* : H^\infty(\mathbb{B}_n) \to H^\infty(\widetilde{S}_n)$.

In particular, the maximal ideal spaces of $H^\infty(\widetilde{S}_n)$ and $H^\infty(\mathbb{B}_n)$ coincide.

The main result of the present paper is the following noncommutative analog of the above theorem.

Theorem 1.2. *Let $r : X \to Y$ satisfy the assumptions of Theorem* 1.1*, and let $A = (a_{ij})$ be an $(n \times k)$-matrix, $k < n$, with entries in $H^\infty(X)$. Assume that the family of minors of order k of A satisfies the corona condition. Then there is an $(n \times n)$-matrix $\widetilde{A} = (\widetilde{a}_{ij})$, $\widetilde{a}_{ij} \in H^\infty(X)$, such that $\widetilde{a}_{ij} = a_{ij}$ for $1 \le j \le k$, $1 \le i \le n$, and* $\det \widetilde{A} = 1$.

Moreover, the corresponding norm of $\widetilde{A}$ is bounded by a constant depending only on the norm of A, on δ (from (1.1) *for the family of minors of order k of A), and on n and Y.*

Previously, a similar result was proved for matrices with entries in $H^\infty(U)$ for domains $U \hookrightarrow X$ such that the embedding induces an injective homomorphism of the corresponding fundamental groups and $r(U) \subset\subset Y$; see [Br2, Theorem 1.1]. The proof was based on a Forelli type theorem on projections in H^∞ (see [Br1]) and a Grauert type theorem for "holomorphic" vector bundles on maximal ideal spaces (which are not usual manifolds) of certain Banach algebras (see [Br2]), along with some ideas of Tolokonnikov [T] (see also that paper for further results and references on the extension problem for matrices with entries in various function algebras).

A remarkable class of Riemann surfaces X for which a Forelli type theorem and, hence, the corona theorem are valid was introduced by Jones and Marshall [JM]. The definition is in terms of an interpolating property for the critical points of the Green function on X. It is an interesting open question whether the result similar to Theorem 1.2 is valid for such X.

§2. Auxiliary results

2.1. For a set Λ of indices we consider the family $X_\Lambda := \{X_\lambda\}_{\lambda\in\Lambda}$, where each X_λ is a connected unbranched covering of Y. By $r_\lambda := X_\lambda \to Y$ we denote the corresponding projection. Viewing this family as the disjoint union of the sets X_λ, we introduce the natural complex structure on X_Λ. Thus, $r_\Lambda : X_\Lambda \to Y$ is an unbranched covering of Y, where $r_\Lambda|_{X_\lambda} := r_\lambda$.

We say that a function f on X_Λ belongs to $H^\infty(X_\Lambda)$ if $f|_{X_\lambda} \in H^\infty(X_\lambda)$, $\lambda \in \Lambda$, and $\sup_{\lambda\in\Lambda} \|f|_{X_\lambda}\|_\infty < \infty$.

Proposition 2.1. *The corona theorem is valid for $H^\infty(X_\Lambda)$.*

Proof. Let $f_1, \dots, f_n \in H^\infty(X_\Lambda)$ satisfy the corona condition (1.1). We set $f_{j\lambda} := f_j|_{X_\lambda}$. Then each family $f_{1\lambda}, \dots, f_{n\lambda} \in H^\infty(X_\lambda)$ satisfies (1.1) with the same δ as for $f_1, \dots, f_n$. By Theorem 1.1, there are functions $g_{1\lambda}, \dots, g_{n\lambda} \in H^\infty(X_\lambda)$ such that

$$f_{1\lambda}g_{1\lambda} + \cdots + f_{n\lambda}g_{n\lambda} \equiv 1$$

and

$$\max_{1\le j\le n} \|g_{j\lambda}\|_\infty \le C\Big(Y, n, \max_{1\le j\le n} \|f_j\|_{H^\infty(X_\Lambda)}, \delta\Big).$$

We define $g_1, \dots, g_n \in H^\infty(X_\Lambda)$ by the formulas

$$g_j|_{X_\lambda} := g_{j\lambda}.$$

Then $g_1 f_1 + \cdots + g_n f_n \equiv 1$. □

Let $\mathcal{M}_\Lambda$ be the maximal ideal space of the Banach algebra $H^\infty(X_\Lambda)$. In accordance with Theorem 1.1, $H^\infty(X_\Lambda)$ separates the points of X_Λ. Thus, X_Λ can be regarded as a subset of $\mathcal{M}_\Lambda$. Now, by Proposition 2.1, X_Λ is dense in $\mathcal{M}_\Lambda$ in the Gelfand topology.

We shall show that Theorem 1.2 follows directly from the statement below.

Theorem 2.2. *Let $A = (a_{ij})$ be an $(n \times k)$-matrix, $k < n$, with entries in $H^\infty(X_\Lambda)$. Assume that the family of minors of order k of A satisfies the corona condition. Then there is an $(n \times n)$-matrix $\widetilde{A} = (\widetilde{a}_{ij})$, $\widetilde{a}_{ij} \in H^\infty(X_\Lambda)$, so that $\widetilde{a}_{ij} = a_{ij}$ for $1 \le j \le k$, $1 \le i \le n$, and $\det \widetilde{A} = 1$.*

2.2. We recall some constructions and results presented in [Br2].

In accordance with a construction of [Br2, §2], the covering $r_\Lambda : X_\Lambda \to Y$ can be viewed as a fiber bundle over Y with a discrete fiber F_Λ, where F_Λ is the disjoint union of the fibers F_λ of the coverings $r_\lambda : X_\lambda \to Y$, $\lambda \in \Lambda$. Let $l^\infty(F_\Lambda)$ be the Banach algebra of bounded complex-valued functions f on the discrete set F_Λ with pointwise multiplication and with the norm $\|f\| = \sup_{x\in F_\Lambda} |f(x)|$. Let βF_Λ be the *Stone-Čech compactification* of F_Λ, i.e., the maximal ideal space of $l^\infty(F_\Lambda)$ equipped with the Gelfand topology. Then F_Λ is naturally embedded into βF_Λ as an open dense subset, and the topology on F_Λ induced by this embedding coincides with the original one, i.e., is discrete. Every function $f \in l^\infty(F_\Lambda)$ has a unique extension $\widehat{f} \in C(\beta F_\Lambda)$. Next, any homeomorphism $\phi : F_\Lambda \to F_\Lambda$ determines an isometric isomorphism of Banach algebras $\phi^* : l^\infty(F_\Lambda) \to l^\infty(F_\Lambda)$. Therefore, ϕ extends to a homeomorphism $\widehat{\phi} : \beta F_\Lambda \to \beta F_\Lambda$. Now, taking closures in βF_λ of the fibers of the bundle $r_\Lambda : X_\Lambda \to Y$, we obtain a fiber bundle $\widehat{r}_\Lambda : E(Y, \beta F_\Lambda) \to Y$ with fiber βF_Λ so that X_Λ is an open dense subset of $E(Y, \beta F_\Lambda)$ (in fact, an open subbundle of $E(Y, \beta F_\Lambda)$) and $\widehat{r}_\Lambda|_{X_\Lambda} = r_\Lambda$. Moreover, in [Br2, Proposition 2.1] it was proved that

(1) *for every $h \in H^\infty(X_\Lambda)$ there is a unique $\widehat{h} \in C(E(Y, \beta F_\Lambda))$ such that $\widehat{h}|_{X_\Lambda} = h$.*

Also, in [Br4, Theorem 1.5], it was proved that for every $x \in Y$ and every $\lambda \in \Lambda$ the sequence $r_\lambda^{-1}(x) \subset X_\lambda$ is interpolating for $H^\infty(X_\lambda)$ with the constant of interpolation bounded by a number depending on x and Y only. This immediately implies that

(2) *for each* $f \in l^\infty(r_\Lambda^{-1}(x))$ *there is a function* $\widetilde{f} \in H^\infty(X_\Lambda)$ *such that* $\widetilde{f}|_{r_\Lambda^{-1}(x)} = f$.

In particular, the continuous extension of the algebra $H^\infty(X_\Lambda)$ to $E(Y, \beta F_\Lambda)$ separates the points on $E(Y, \beta F_\Lambda)$. Thus, $E(Y, \beta F_\Lambda)$ can be regarded as a dense subset of $\mathcal{M}_\Lambda$.

Let $(U_i)_{i\in I}$ be a countable cover of Y by compact subsets $U_i \subset Y$ homeomorphic to a closed ball in $\mathbb{R}^2$. Then, by our construction, $\widehat{U}_i := \widehat{r}_\Lambda^{-1}(U_i)$ is homeomorphic to $U_i \times \beta F_\Lambda$. So, $E(Y, \beta F_\Lambda)$ is a countable union of compact subsets $\widehat{U}_i$. Since the covering dimension $\dim \widehat{U}_i$ of $\widehat{U}_i$ is 2, $i \in I$, this implies (cf. [Br2, Proposition 4.1]) that

(3) $\dim E(Y, \beta F_\Lambda) = 2$.

Taking now an open countable cover of Y by relatively compact subsets homeomorphic to an open ball in $\mathbb{R}^2$, and the corresponding open cover of $E(Y, \beta F_\Lambda)$ by their preimages with respect to $\widehat{r}_\Lambda$, we see that

(4) $E(Y, \beta F_\Lambda)$ *is an open dense subset of* $\mathcal{M}_\Lambda$, *and the restriction of the Gelfand topology on* $\mathcal{M}_\Lambda$ *to* $E(Y, \beta F_\Lambda)$ *coincides with the topology of* $E(Y, \beta F_\Lambda)$.

2.3. Since Y is a Riemann surface of finite type, the theorem of Stout (see [St, Theorem 8.1]) implies the existence of a compact Riemann surface R and a holomorphic embedding $\phi : Y \to R$ such that $R \setminus \phi(Y)$ consists of finitely many closed disks with analytic boundaries, together with finitely many isolated points. Since Y is Caratéodory hyperbolic, the set of disks in $R \setminus \phi(Y)$ is not empty. Also, without loss of generality, we may and shall assume that the set of isolated points in $R \setminus \phi(Y)$ is not empty. (For otherwise, $\phi(Y)$ is a bordered Riemann surface and the required result follows from [Br2, Theorem 1.1].) We shall naturally identify Y with $\phi(Y)$. Also, we set

$$R \setminus Y := \left(\bigsqcup_{1\le i\le k} \bar{D}_i \right) \cup \left(\bigcup_{1\le j\le l} \{x_j\} \right) \quad \text{and} \quad Z := Y \cup \left(\bigcup_{1\le j\le l} \{x_j\} \right), \tag{2.1}$$

where each D_i is biholomorphic to the open unit disk $\mathbb{D} \Subset \mathbb{C}$, and these biholomorphisms are extended to diffeomorphisms of the closures $\bar{D}_i \to \bar{\mathbb{D}}$. Then $Z \subset R$ is a bordered Riemann surface with a nonempty boundary. In particular, there is a bordered Riemann surface Z' containing $\bar{Z}$ and such that $\bar{Z}$ is a deformation retract of Z'. We set

$$Y' := Z' \setminus \{x_1, \dots, x_l\}. \tag{2.2}$$

Then $Y \subset Y'$ and $\pi_1(Y) \cong \pi_1(Y')$ (here $\pi_1(M)$ stands for the fundamental group of M). This implies that for each $\lambda \in \Lambda$ there is a connected covering X'_λ of Y' such that X_λ is an open connected subset of X'_λ. Without loss of generality, we denote the covering projection $X'_\lambda \to Y'$ by the same symbol r_λ (as for X_λ). Now, we define $X'_\Lambda := \{X'_\lambda\}_{\lambda\in\Lambda}$ so that X_Λ is an open subset of X'_Λ and $r_\Lambda : X'_\Lambda \to Y'$, $r_\Lambda|_{X'_\lambda} := r_\lambda$.

Next, as in the constructions of Subsection 2.2, we determine the bundle $\widehat{r}_\Lambda : E(Y', \beta F_\Lambda) \to Y'$ so that $E(Y, \beta F_\Lambda)$ is an open subbundle of $E(Y', \beta F_\Lambda)$. Then X'_Λ and $E(Y', \beta F_\Lambda)$ possess properties similar to (1) and (3) for X_Λ and $E(Y, \beta F_\Lambda)$.

Let $cl(Y)$ denote the closure of Y in Y'. We set

$$E(cl(Y), \beta F_\Lambda) := \widehat{r}_\Lambda^{-1}(cl(Y)).$$

Then

(5) $\dim\, E(cl(Y), \beta F_\Lambda) = 2$ *and* $E(Y, \beta F_\Lambda) \subset E(cl(Y), \beta F_\Lambda)$ *is an open dense subset.*

2.4. By $H^\infty(E(Y,\beta F_\Lambda))$ we denote the extension of $H^\infty(X_\Lambda)$ to $E(Y,\beta F_\Lambda)$ described in Subsection 2.2. We shall also use the algebra $H^\infty(E(Y',\beta F_\Lambda))$ determined in a similar way (i.e., with Y and X_Λ in place of Y' and X'_Λ).

Next, we consider the Banach subalgebras $\mathcal{A}_1$, $\mathcal{A}_2$ of $H^\infty(E(Y,\beta F_\Lambda))$ defined as follows:

$$\mathcal{A}_1 := \{\widehat{r}_\Lambda^* f \in H^\infty(E(Y,\beta F_\Lambda)) \,:\, f \in H^\infty(Z')\}. \tag{2.3}$$

(Here $\widehat{r}_\Lambda^* f$ is the pullback of f with respect to $\widehat{r}_\Lambda$.)

To define $\mathcal{A}_2$, we choose a function $\phi \in H^\infty(Z')$ with the set of zeros $\{x_1,\dots,x_l\}$ so that each x_j is a zero of order 1 of ϕ. (Since $Z' \subset\subset R$ is a bordered Riemann surface with nonempty boundary, such ϕ exists by [Br2, Corollary 1.8].) Then $\mathcal{A}_2$ is the uniform closure of the algebra of functions $f \in H^\infty(E(Y,\beta F_\Lambda))$ of the form

$$f := g + ([\widehat{r}_\Lambda^*\phi]\cdot h)|_{E(Y,\beta F_\Lambda)}, \quad g \in \mathcal{A}_1,\ h \in H^\infty(E(Y',\beta F_\Lambda)). \tag{2.4}$$

The definition shows that the functions in $\mathcal{A}_2$ separate the points of $E(Y,\beta F_\Lambda)$ (because $H^\infty(E(Y',\beta F_\Lambda))$ separates the points of $E(Y',\beta F_\Lambda)$ and $\widehat{r}_\Lambda^*\phi$ is nonzero on the fibers of $\widehat{r}_\Lambda$).

Clearly, we have the embeddings

$$\mathcal{A}_1 \xrightarrow{i_1} \mathcal{A}_2 \xrightarrow{i_2} H^\infty(E(Y,\beta F_\Lambda)). \tag{2.5}$$

The transpose maps to these embeddings determine continuous surjective maps

$$\mathcal{M}_\Lambda \xrightarrow{i_2^*} M_2 \xrightarrow{i_1^*} M_1, \tag{2.6}$$

where M_2 is the closure in the Gelfand topology of the image of $E(Y,\beta F_\Lambda)$ in the maximal ideal space of $\mathcal{A}_2$, and M_1 is the closure in the Gelfand topology of the image of $E(Y,\beta F_\Lambda)$ in the maximal ideal space of $\mathcal{A}_1$. (Here we have used the fact that the closure in the Gelfand topology of $E(Y,\beta F_\Lambda) \subset \mathcal{M}_\Lambda$ is $\mathcal{M}_\Lambda$; see Proposition 2.1.)

Our definitions imply that $M_1 = \bar{Z}$ and $E(cl(Y),\beta F_\Lambda) \subset M_2$ (see Subsection 2.3). Moreover, the restriction of i_2^* to $E(Y,\beta F_\Lambda)$ is the identity map, and the restriction of i_1^* to $E(cl(Y),\beta F_\Lambda)$ can be identified naturally with $\widehat{r}_\Lambda$ so that $(i_1^*)^{-1}(cl(Y)) = \widehat{r}_\Lambda^{-1}(cl(Y)) = E(cl(Y),\beta F_\Lambda)$.

Lemma 2.3. *For each $x_j \in M_1$ the compact set $(i_1^*)^{-1}(x_j)$ consists of a unique point (which we identify naturally with x_j), $1 \le j \le l$.*

Proof. Let $\{\xi_{1,\alpha}\},\{\xi_{2,\alpha}\} \subset E(Y,\beta F_\Lambda)$ be nets converging to points $\xi_1,\xi_2 \in (i_1^*)^{-1}(x_j)$. Then for f as in (2.4) and $i = 1,2$ we have

$$f(\xi_i) = \lim_\alpha f(\xi_{i,\alpha}) = \lim_\alpha (g(\xi_{i,\alpha}) + (\widehat{r}_\Lambda^*\phi)(\xi_{i,\alpha})\cdot h(\xi_{i,\alpha})) := g(x_j). \tag{2.7}$$

(We have used the fact that the nets $\{i_1(\xi_{1,\alpha})\},\{i_1(\xi_{2,\alpha})\} \subset M_1$ converge to x_j.)

This implies that $\xi_1 = \xi_2$. □

Corollary 2.4.

$$\dim M_2 = 2.$$

Proof. By Lemma 2.3 and property (5) of Subsection 2.3, M_2 is the disjoint union of the zero-dimensional sets $\{x_j\}$, $1 \le j \le l$, and the two-dimensional set $E(cl(Y),\beta F_\Lambda)$. Hence, $\dim M_2 = 2$; see, e.g., [N, Chapter 2, Theorems 9–11]. □

2.5. We fix coordinate neighborhoods $N_j \subset\subset Z$ (see (2.1)) of the points x_j, $1 \le j \le l$, biholomorphic to $\mathbb{D}$, and a bordered Riemann surface $S \subset Y$ such that $N_i \cap N_j = \varnothing$ for $i \ne j$ and each $Y \cap N_j$ does not contain x_j, is biholomorphic to an annulus, and $\mathcal{U} := S \cup (\bigcup_{1\le j\le l} N_j^*)$ is an open cover of Y. Here $N_j^* := N_j \setminus \{x_j\}$ is biholomorphic to $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. We set

$$N_{j\Lambda}^* := r_\Lambda^{-1}(N_j^*), \quad 1 \le j \le l, \quad \text{and} \quad S_\Lambda := r_\Lambda^{-1}(S). \tag{2.8}$$

Let $V \subset X_\Lambda$ be either one of $N_{j\Lambda}^*$ or S_Λ, and let $H^\infty(V)$ denote the Banach algebra of bounded holomorphic functions on V defined as in Subsection 2.1 for X_Λ. Next, we set

$$\widehat{N}_j := (i_2^* \circ i_1^*)^{-1}(N_j), \quad 1 \le j \le l, \quad \widehat{S} := (i_2^* \circ i_1^*)^{-1}(S \cup \partial Z). \tag{2.9}$$

Here ∂Z is the boundary of the bordered Riemann surface Z, which can be regarded as the "outer boundary" of S.

By definition, the $\widehat{N}_j$, $1 \le j \le l$, and $\widehat{S}$ are open subsets of $\mathcal{M}_\Lambda$ forming a cover of this space. The main fact used in the proof of Theorem 1.1 is as follows.

Proposition 2.5. *Assume that $f \in H^\infty(V)$, where V is either one of $N_{j\Lambda}^*$ or S_Λ. Then f admits a continuous extension $\hat{f}$ to $\hat{V}$, where $\hat{V}$ stands for the corresponding $\widehat{N}_j$ or $\widehat{S}$.*

Proof. First, we prove the result for $V = N_{j\Lambda}^*$.

Let ρ_j be a C^∞-function on R equal to 1 in a neighborhood of x_j with $\mathrm{supp}(\rho) \subset\subset N_j$. We set

$$f_1 := (r_\Lambda^* \rho_j) \cdot f. \tag{2.10}$$

Then f_1 can be viewed as a C^∞-function on X'_Λ (defined as in Subsection 2.3). We introduce a $(0,1)$-form on X'_Λ by the formula

$$\omega := \frac{\bar{\partial} f_1}{\rho_\Lambda^* \phi}. \tag{2.11}$$

This definition makes sense because $\bar{\partial} f_1$ equals 0 on $\rho_\Lambda^{-1}(O)$ for some neighborhood O of x_j and on $X'_\Lambda \setminus N_{j\Lambda}^*$, and $\rho_\Lambda^* \phi \ne 0$ on $N_{j\Lambda}^*$. Thus, ω is a $\bar{\partial}$-closed 1-form on X'_Λ. Consider the form

$$\omega_\lambda := \omega|_{X'_\lambda} \quad \text{on} \quad X'_\lambda.$$

We assume that Z' is equipped with a Hermitian metric $h_{Z'}$ with the associated $(1,1)$-form $\omega_{Z'}$. Then we equip X'_λ with the Hermitian metric $h_{X'_\lambda}$ induced by the pullback $r_\lambda^* \omega_{Z'}$ of $\omega_{Z'}$ to X'_λ. Now, if η is a smooth $(0,1)$-form on X'_λ, $z \in X'_\lambda$, we denote by $|\eta|_z$, $z \in X'_\lambda$, the norm of η at z defined by the Hermitian metric $h^*_{X'_\lambda}$ on the fibers of the cotangent bundle $T^* X'_\lambda$ on X'_λ.

Next, since $f \in H^\infty(N_{j\Lambda}^*)$ and $r_\Lambda(\mathrm{supp}(\omega)) =: K \subset\subset Y'$, see (2.2), the definition (2.11) implies that

$$\|\omega\| := \sup_{\lambda \in \Lambda} \left\{ \sup_{z \in X_\lambda} |\omega_\lambda|_z \right\} < \infty. \tag{2.12}$$

Now, using [Br4, Theorem 1.6], we see that the equation $\bar{\partial} g_\lambda = \omega_\lambda$ has a smooth bounded solution g_λ on X'_λ such that

$$\|g_\lambda\|_{L^\infty} := \sup_{z \in X'_\lambda} |g_\lambda(z)| \le C \|\omega\| \tag{2.13}$$

with C depending only on K, Z', and $h_{Z'}$.

We define bounded functions g and f_2 on X'_Λ by the formulas

$$g|_{X'_\lambda} := g_\lambda, \quad \lambda \in \Lambda, \quad f_2 := (\rho_\Lambda^* \phi) \cdot g. \tag{2.14}$$

Then

$$\text{(2.15)} \qquad \text{(a)}\ \bar\partial f_2 = \bar\partial f_1 \quad \text{on} \quad X'_\Lambda \quad \text{and} \quad \text{(b)}\ \lim_\alpha f_2(\xi_\alpha) = 0$$

for each net $\{\xi_\alpha\} \subset X'_\Lambda$ such that $\{r_\Lambda(\xi_\alpha)\} \subset Y'$ is a net converging to any x_s, $1 \le s \le l$. In particular,

$$\text{(2.16)} \qquad f_3 := f_1 - f_2 \in H^\infty(X'_\Lambda).$$

Thus, f_3 admits a continuous extension $\widehat{f}_3$ to $\mathcal{M}_\Lambda$.

Now we prove that

($*$) *f_2 admits a continuous extension $\widehat{f}_2$ to $\mathcal{M}_\Lambda$.*

Indeed, by the definition of f and $r^*_\Lambda\rho_j$, the function f_1 defined by (2.10) has a continuous extension to $E(Y', \beta F_\Lambda)$; see [Br2, Proposition 2.1]. Thus, $f_2 := f_1 - f_3$ also admits a continuous extension to $E(Y', \beta F_\Lambda)$. (We keep the notation f_2 for this extension.) Now, if $\{\xi_\alpha\} \subset E(Y', \beta F_\Lambda)$ is a net converging to a point $\xi \in M_2$ (see (2.6)) such that $i_1^*(\xi) = x_s$ for some $1 \le s \le l$, then, by (2.15) (b), we get

$$\lim_\alpha f_2(\xi_\alpha) = 0.$$

Since $(i_1^*)^{-1}(x_s) = x_s$, this implies that the function $\widetilde{f}_2$ equal to 0 at each x_s and to f_2 on $E(cl(Y), \beta F_\Lambda)$ is continuous on M_2. Therefore, the function $\widehat{f}_2 := i_2^* \widetilde{f}_2$ is continuous on $\mathcal{M}_\Lambda$. Since the restriction of i_2^* to $E(Y, \beta F_\Lambda)$ is the identity map, $\widehat{f}_2$ is a continuous extension of f_2. This proves ($*$).

From (2.16) and ($*$) it follows that f_1 admits a continuous extension $\widehat{f}_1$ to $\mathcal{M}_\Lambda$. Now, the set $\widehat{N}_j$ (see (2.9)) is the union of $\widehat{r}_\Lambda^{-1}(N_j^*) \subset E(Y, \beta F_\Lambda)$ and $(i_2^* \circ i_1^*)^{-1}(O)$, where $O \subset\subset N_j$ is a neighborhood of x_j such that $\rho_j \equiv 1$ on O. The function f admits a continuous extension $\widetilde{f}$ on $\widehat{r}_\Lambda^{-1}(N_j^*)$, see [Br2, Proposition 2.1], and $f = f_1$ on $r_\Lambda^{-1}(O)$. Thus, the function $\widehat{f}$ defined by

$$\widehat{f} := \widehat{f}_1 \quad \text{on} \quad (i_2^* \circ i_1^*)^{-1}(O) \quad \text{and} \quad \widehat{f} := \widetilde{f} \quad \text{on} \quad \widehat{r}_\Lambda^{-1}(N_j^*)$$

is the required continuous extension of f to $\widehat{N}_j$.

Finally, in the case where $V = S_\Lambda$, we choose a C^∞-function ρ on R equal to 0 on $Y \setminus S$ and to 1 on $R \setminus Z$ and with $\text{supp}(d\rho) \subset\subset S$. Then, repeating the above arguments with ρ_j in place of ρ, we obtain the proof of the proposition in this case. We leave the details to the reader. $\square$

§3. Proof of Theorem 1.2

3.1. *Proof of Theorem* 2.2. Let $A = (a_{ij})$ be an $(n \times k)$-matrix, $k < n$, with entries in $H^\infty(X_\Lambda)$. Assume that the family of minors of order k of A satisfies the corona condition (1.1). By the corona theorem for $H^\infty(X_\Lambda)$, see Proposition 2.1, we can extend A continuously to $\mathcal{M}_\Lambda$ so that the family of minors of order k of the extended matrix $\widehat{A} = (\widehat{a}_{ij})$ satisfies (1.1) on $\mathcal{M}_\Lambda$ with the same δ as for A. Next, by [L, Theorem 3], to prove the theorem it suffices to find an $(n \times n)$-matrix $B = (b_{ij})$, $b_{ij} \in C(\mathcal{M}_\Lambda)$, such that $b_{ij} = \widehat{a}_{ij}$ for $1 \le j \le k$, $1 \le i \le n$, and $\det B = 1$.

Note that the matrix $\widehat{A}$ determines a trivial subbundle ξ of complex rank k in the trivial vector bundle $\theta^n := \mathcal{M}_\Lambda \times \mathbb{C}^n$ on $\mathcal{M}_\Lambda$. Let η be a subbundle of θ^n supplementary to ξ, i.e., $\xi \oplus \eta = \theta^n$. We shall prove that η is topologically trivial. Then a trivialization $s_1, s_2, \dots, s_{n-k} \in C(\mathcal{M}_\Lambda, \eta)$ (given by global continuous sections of η) will determine the required continuous extension B of the matrix $\widehat{A}$.

First, we prove that $\widehat{A}$ extends up to an invertible matrix on each $\widehat{N}_j$ and $\widehat{S}$; see (2.9).

Lemma 3.1. *Let $\widehat{V}$ be either one of $\widehat{N}_j$ or $\widehat{S}$. Then for $\widehat{A}|_{\widehat{V}}$ there is an $(n \times n)$-matrix $B_{\widehat{V}} = (b_{ij;\widehat{V}})$, $b_{ij;\widehat{V}} \in C(\widehat{V})$, such that $b_{ij;\widehat{V}} = \hat{a}_{ij}|_{\widehat{V}}$ for $1 \le j \le k$, $1 \le i \le n$, and $\det B_{\widehat{V}} = 1$. Moreover, $B_{\widehat{V}}|_V$ has entries in $H^\infty(V)$, where $V := \widehat{V} \cap X_\Lambda$.*

Proof. First, assume that $\widehat{V} = \widehat{N}_j$, so that $V = N^*_{j\Lambda}$ is an unbranched covering of $N^*_j \cong \mathbb{D}^*$; see (2.8). Then, by definition, $N^*_{j\Lambda} = \{N^*_{j\lambda}\}_{\lambda \in \Lambda}$, where each $N^*_{j\lambda} := N^*_{j\Lambda} \cap X_\lambda$ is an unbranched covering of N^*_j consisting of at most countably many connected components. Thus, each $N^*_{j\lambda}$ is biholomorphic to $\bigsqcup_{k \in K_\lambda} W_{j\lambda;k}$, $K_\lambda \subset \mathbb{N}$, where each $W_{j\lambda;k}$ is either $\mathbb{D}$ or $\mathbb{D}^*$. Now, $A|_{W_{j\lambda;k}}$ satisfies the conditions of Theorem 1.1 with the same δ as for A. Applying the main result of Tolokonnikov [T] for H^∞-matrices on $\mathbb{D}$, we see that there is a matrix $B_{j\lambda;k}$ with entries in $H^\infty(W_{j\lambda;k})$ that extends $A|_{W_{j\lambda;k}}$ in the sense of Theorem 1.1 and is such that $\det B_{j\lambda;k} = 1$ and

$$\sup_{j,\lambda,k} \|B_{j\lambda;k}\| \le C, \tag{3.1}$$

where C depends on δ, n, and the norm of A on X_Λ. (Here for a matrix $C = (c_{ij})$ with entries in $H^\infty(O)$ we set $\|C\| := \max_{i,j} \|c_{ij}\|_{H^\infty(O)}$.) In particular, (3.1) implies that the matrix $B_{j\Lambda}$ on $N^*_{j\Lambda}$ defined by

$$B_{j\Lambda}|_{W_{j\lambda;k}} = B_{j\lambda;k}, \quad 1 \le j \le l, \ \ k \in K_\lambda, \ \ \lambda \in \Lambda,$$

has entries in $H^\infty(N^*_{j\Lambda})$ and extends $A|_{N^*_{j\Lambda}}$, and $\det B_{j\Lambda} = 1$. By Proposition 2.5, $B_{j\Lambda}$ extends up to a continuous matrix $B_{\widehat{N}_j}$ on $\widehat{N}_j$. This matrix extends $\widehat{A}|_{\widehat{N}_j}$ and satisfies the required conditions of the lemma.

Now, let $\widehat{V} = \widehat{S}$ so that $V = S_\Lambda$ is an unbranched covering of S; see (2.8). In this case we argue as above, but instead of the result of [T] we use [Br2, Theorem 1.1] applied to the coverings $S_\lambda := S_\Lambda \cap X_\lambda$ of a bordered Riemann surface S. Then we obtain a matrix B_Λ on S_Λ with entries in $H^\infty(S_\Lambda)$ that extends $A|_{S_\Lambda}$ and satisfies $\det B_\Lambda = 1$. Applying Proposition 2.5 once again, we extend B_Λ continuously to $\widehat{S}$ so that the extended matrix $B_{\widehat{S}}$ satisfies the required conditions of the lemma. □

Let ξ_q be the quotient bundle of θ^n with respect to the subbundle ξ. By definition, ξ_q is isomorphic (in the category of continuous bundles on $\mathcal{M}_\Lambda$) to η. Thus it suffices to prove that ξ_q is topologically trivial.

Now, Lemma 3.1 implies straightforwardly that $\xi_q|_{\widehat{V}}$ is topologically trivial if $\widehat{V}$ is either one of $\widehat{N}_j$ or $\widehat{S}$. In particular, ξ_q is determined by a 1-cocycle defined on the open cover $\{\widehat{N}_1, \dots, \widehat{N}_l, \widehat{S}\}$ of $\mathcal{M}_\Lambda$ (see, e.g., [H] for the general theory of vector bundles). Since $\widehat{N}_i \cap \widehat{N}_j = \varnothing$ for $i \ne j$, this cocycle consists of continuous matrix-valued functions

$$C_i \in C(\widehat{N}_i \cap \widehat{S}, GL_{n-k}(\mathbb{C})), \quad 1 \le i \le l.$$

We set $\widetilde{N}_j := (i_1^*)^{-1}(N_j)$, $\widetilde{S} := (i_1^*)^{-1}(S)$; see (2.6). Then $\{\widetilde{N}_1, \dots, \widetilde{N}_l, \widetilde{S}\}$ is an open cover of M_2. Moreover, the map $i_2 : \mathcal{M}_\Lambda \to M_2$ is identical on each $\widehat{N}_i \cap \widehat{S}$; see Subsection 2.4. Therefore, each C_i can be regarded as a matrix-valued function on $\widetilde{N}_i \cap \widetilde{S}$. In particular, these functions determine a complex vector bundle $\widetilde{\xi}_q$ of rank $n-k$ on M_2 so that

$$i_2^* \widetilde{\xi}_q = \xi_q. \tag{3.2}$$

Lemma 3.2. *The bundle $\widetilde{\xi}_q$ is isomorphic to $\theta^{n-k-1}_{M_2} \oplus \theta$, where $\theta^{n-k-1}_{M_2} := M_2 \times \mathbb{C}^{n-k-1}$ is the trivial bundle and θ is a vector bundle of complex rank 1.*

Proof. Since dim $M_2 = 2$, see Corollary 2.4, the Freudenthal expansion theorem [F] shows that M_2 can be presented as the inverse limit of a family of compact polyhedra $\{Q_j\}_{j\in J}$ with $\dim Q_j \le 2$. Let $\pi_j : M_2 \to Q_j$ be the continuous projections in the inverse limit construction. Then, using a well-known theorem about continuous maps of inverse limits of compact spaces (see, e.g., [ES]) and the fact that all complex vector bundles of rank d on M_2 can be obtained as pullbacks of the universal bundle $EU(d)$ on the classifying space $BU(d)$ of the unitary group $U(d) \subset GL_d(\mathbb{C})$ under some continuous maps $M_2 \to BU(d)$ (see, e.g., [H]), we conclude that for each complex vector bundle E on M_2 of rank d there is $j_0 \in J$ and a complex vector bundle E_{j_0} of rank d on Q_{j_0} such that the pullback $\pi_{j_0}^* E_{j_0}$ is isomorphic to E. Assume that $d := n-k \ge 2$. Consider the bundle E_{j_0} on Q_{j_0}. Since Q_{j_0} is a CW-complex of dimension at most 2, the only obstruction to the existence of $d-1$ orthonormal sections for E_{j_0} lies in the group $H^4(Q_{j_0}, \mathbb{Z})$ (see, e.g., [H]). Since this group is trivial, E_{j_0} always has such sections. Thus, E_{j_0} is isomorphic to $\theta_{j_0}^{n-k-1} \oplus \theta_{j_0}$, where $\theta_{j_0}^{n-k-1} := Q_{j_0} \times \mathbb{C}^{n-k-1}$ and θ_{j_0} is a vector bundle on Q_{j_0} of complex rank 1. Since $\pi_{j_0}^* E_{j_0}$ is isomorphic to E, this implies the required statement. □

This lemma and (3.2) imply that $\xi_q \cong \eta$ is isomorphic to $\theta^{n-k-1} \oplus i_2^*\theta$, where $\theta^{n-k-1} = \mathcal{M}_\Lambda \times \mathbb{C}^{n-k-1}$ is the trivial bundle. Now, for the first Chern classes (which are additive with respect to the operation of the direct sum of bundles) we have the following identity:

$$0 = c_1(\theta^n) = c_1(\xi) + c_1(\eta) = c_1(\theta^{n-k-1} \oplus i_2^*\theta) = c_1(i_2^*\theta). \tag{3.3}$$

Here we have used the fact that the Chern classes of trivial bundles are zero.

Relation (3.3) shows that the first Chern class of the complex rank 1 vector bundle $i_2^*\theta$ is zero. Thus, this bundle is topologically trivial (see, e.g., [H]). Combining this fact with the above isomorphism for η, we get

$$\eta \cong \theta^{n-k} := \mathcal{M}_\Lambda \times \mathbb{C}^{n-k}.$$

This completes the proof of Theorem 2.2. □

3.2. *Proof of Theorem* 1.1. Let $\Lambda_{Y;M,\delta}$ be defined as the set of all possible couples (A, X), where X is a connected covering of Y and A is an $(n \times k)$-matrix on X satisfying the conditions of Theorem 1.1 with a fixed δ in the corona condition (1.1) for the family of minors of order k and such that $\|A\| \le M$. For $\Lambda := \Lambda_{Y;M,\delta}$ we consider the $(n \times k)$-matrix $\mathcal{A}$ with the entries in $H^\infty(X_\Lambda)$ defined as follows:

$$\mathcal{A}|_{X_\lambda} := A, \quad \lambda = (A, X) \in \Lambda, \quad X_\lambda := X. \tag{3.4}$$

Then, clearly, $\mathcal{A}$ satisfies the conditions of Theorem 2.2 on X_Λ. By that theorem, there is an $(n \times n)$-matrix $\widetilde{\mathcal{A}}$ with entries in $H^\infty(X_\Lambda)$ and with det $\widetilde{\mathcal{A}} = 1$ that extends $\mathcal{A}$. For $\lambda = (A, X) \in \Lambda$ we set

$$\widetilde{A} := \widetilde{\mathcal{A}}|_X.$$

Then $\widetilde{A}$ extends A, det $\widetilde{A} = 1$, and

$$\|\widetilde{A}\| \le C(\|A\|, \delta, M, Y).$$

The proof of Theorem 1.1 is complete. □

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Department of Mathematics and Statistics, University of Calgary, Calgary, Canada
E-mail address: albru@math.ucalgary.ca

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