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ON THE COMPUTATION OF K-FUNCTIONALS

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Abstract. A new approach to the calculation of the sharp order of a K-functional is suggested. This approach employs the techniques of dyadic spaces.

§1. Introduction

The theory of local polynomial approximation makes it possible to construct dyadic analogs of certain spaces. The dyadic Hardy classes and the dyadic BMO are primary examples of this sort, which have found interesting applications in analysis.

In the author's paper [1], the dyadic Nikol′skiĭ–Besov spaces $B_p^{\lambda\theta}(F)$ were studied. We recall that the dyadic space $B_p^{\lambda\theta}(F)$ is defined in terms of local polynomial approximations linked with a dyadic family F; in essence, this space is a model of the classical space $B_p^{\lambda\theta}$ (the definition of $B_p^{\lambda\theta}(F)$ will be given below).

In [1] it was noted that there are at least two reasons for which the study of dyadic spaces is of interest. First, we deal with a new scale of spaces, and the functions belonging to them can be viewed as functions on a graph. Indeed, let us represent the dyadic family F as a tree. Its root vertex corresponds to the d-dimensional unit cube, and 2^d edges emanate from each vertex. Then the "weight" of each vertex is defined in terms of local approximation by polynomials on the cube corresponding to that vertex. This treatment of a space enables us to simplify the proofs, to make them more geometric, and to discard any restrictions on the integrability exponent p. It should be noted that the traditional approach to classical spaces, as developed by Nikol′skiĭ and Besov (see, e.g., [2]), is based on integral representations and is not applicable for $0 < p < 1$. So, dyadic spaces constitute a useful discretization of classical spaces and provide a useful language for the description of applied algorithms.

Second, dyadic B-spaces are closely related to classical B-spaces. Compared to its classical counterpart, the dyadic space $B_p^{\lambda\theta}(F)$ has a supplementary "parameter" F. It turns out that, for low smoothness, $B_p^{\lambda\theta}(F)$ does not depend on F and coincides with $B_p^{\lambda\theta}$ (see Theorem 3.3). Moreover, under a proper choice of dyadic families, $B_p^{\lambda\theta}$ is the intersection of 2^d dyadic spaces (see Corollary 3.1). Thanks to this, many results for B-spaces can be deduced from similar results for dyadic spaces. Our objective in this paper is to show how the techniques of dyadic spaces can be used in classical analysis; this will be done by considering the calculation of a K-functional as an example.

A different approach to the calculation of K-functionals was presented in [3]–[5].

The present paper can be regarded as a continuation of the author's paper [1]; it is organized as follows.

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In §2, we present definitions and state the main results. In §3, we relate the K-functionals for the couples of classical and of dyadic B-spaces and, up to the order of magnitude, calculate the K-functional for the couple $(B_p^{\lambda_1\theta_1}, B_p^{\lambda_2\theta_2})$. In §4 the techniques of local approximation are used to calculate the sharp order of magnitude of the K-functional for the couple $(L_p, \dot{W}_p^k)$, $0 < p \le \infty$. In §5, we suggest a new approach to the proof of a Bernstein-type inequality for splines with free nodes. Specifically, by interpolation techniques, we carry a similar inequality for dyadic spaces over to classical spaces. Finally, in §6 we calculate the sharp order of the K-functional for the couple (L_q, B_p^λ), $\lambda = d\big(\frac{1}{p} - \frac{1}{q}\big)$, $1 < p < q < \infty$, in terms of nonlinear spline-approximation.

§2. Definitions and the statements of main results

We remind the reader that the kth modulus of continuity of a function f in $L_p(Q_0)$, $0 < p \le \infty$, is defined by the formula

$$\omega_k(f,t)_p := \sup_{|h|\le t} \|\Delta_h^k f\|_{L_p(Q_{kh})},$$

where $Q_{kh} = \{x \in Q_0 \,:\, x + jh \in Q_0,\ j = 1, \dots, k\}$.

If the supremum is taken only over the vectors collinear to the axis Ox_i, we arrive at the definition of the partial continuity modulus $\omega_k^{(i)}(f,t)_p$. We denote

$$\bar{\omega}_k(f,t)_p := \sum_{i=1}^d \omega_k^{(i)}(f,t)_p.$$

Now, we define the Besov–Nikol′skiĭ spaces $B_p^{\lambda\theta}$, $0 < p \le \infty$. The classical space $B_p^{\lambda\theta}$, $p \ge 1$, is defined in terms of derivatives and divided differences or, equivalently, in terms of continuity moduli. But we also need these spaces for $0 < p < 1$ (rather than only for $p \ge 1$); there are two options for defining such spaces, which yield the same if $p \ge 1$.

The first definition (like the classical one) involves continuity moduli. The second definition, introduced by Peetre [6], is based on the Fourier transformation for distributions. The resulting space will be denoted by $B_p^{\lambda\theta}$ in the first case and by $\widehat{B}_p^{\lambda\theta}$ in the second. The spaces $\widehat{B}_p^{\lambda\theta}$, $0 < p < 1$, contain only integrable functions, which makes it possible to use integral representation methods. This is impossible for the scale $B_p^{\lambda\theta}$ (at least if $\lambda < d\big(\frac{1}{p} - 1\big)$, because then $B_p^{\lambda\theta}$ may contain nonintegrable functions). Peetre [6] showed that $\widehat{B}_p^{\lambda\theta}$ embeds in $B_p^{\lambda\theta}$, and the two spaces coincide if $\lambda > d\big(\frac{1}{p} - 1\big)$.

In the author's paper [7], principal properties of the scale $B_p^{\lambda\theta}$ with $0 < p < 1$ were studied. It turned out that the description (due to Nikol′skiĭ and Besov) of the functions in $B_p^{\lambda\theta}$, $p \ge 1$, in terms of entire functions can be extended to the case where $0 < p < 1$. This opened a possibility to modify the methods developed by Nikol′skiĭ and his followers for the scale $B_p^{\lambda\theta}$, $p \ge 1$, so as to make them applicable in this more general case. In what follows, we use the definition of $B_p^{\lambda\theta}$, $0 < p \le \infty$, in terms of continuity moduli.

Definition 2.1. A function $f \in L_p(Q_0)$, $0 < p \le \infty$, is said to belong to $B_p^{\lambda\theta}$ if the following quantity is finite:

$$|f|_{B_p^{\lambda\theta}} = \left(\sum_{n=0}^\infty \big(2^{n\lambda} w_k(f, 2^{-n})_p\big)^\theta \right)^{1/\theta};$$

here $0 < \lambda < k$, $0 < \theta \le \infty$.

As usual, the relevant quasinorm is introduced as follows:

$$\|f\|_{B_p^{\lambda\theta}} = |f|_{B_p^{\lambda\theta}} + \|f\|_{L_p}.$$

If $\theta = p$, we omit the parameter θ in the notation for $B_p^{\lambda\theta}$.

It turns out that, in this definition, the continuity modulus $\omega_k(f,t)_p$ can be replaced by the sum $\bar{\omega}_k(f,t)_p$ of the partial continuity moduli. See, e.g., [8] for $p \geq 1$; if $0 < p < 1$, this was proved in [9].

We pass to the definition of dyadic spaces. We remind the reader that the local polynomial approximation for a function $f \in L_p(Q_0)$, $0 < p \leq \infty$, is the function $Q \to E_k(f,Q)_p$, $Q \subset Q_0$, given by

$$E_k(f,Q)_p := \inf_{s \in P_k} \|f - s\|_{L_{p(Q)}}.$$

Here P_k is the space of polynomials of degree at most $k-1$ in each variable.

We denote by $P_k(\Pi)$ the space of piecewise polynomial functions (we mean polynomials in P_k) that are subordinate to a partition Π of Q_0; that is, $P_k(\Pi)$ consists of the functions of the form $\sum\limits_{Q \in \Pi} p_Q \chi_Q$, where $p_Q \in P_k$.

We observe that

$$\operatorname{dist}_{L_p}(f, P_k(\Pi)) = \left(\sum_{Q \in \Pi} E_k(f,Q)_p^p \right)^{1/p}.$$

Here and below, $L_p = L_p(Q_0)$ and $\|f\|_{L_p(Q)} = \|f\|_{p(Q)}$. If $Q = Q_0$, the symbol Q may be omitted.

Now, we pass to the definition itself of a dyadic family. Let F_n be the partition of the unit cube Q_0 in parallelepipeds that is formed by $(2^n - 1)d$ hyperplanes (there are $2^n - 1$ hyperplanes parallel to each particular coordinate hyperplane). Thus, F_0 consists of Q_0 only. We shall assume that the partition F_n consists of parallelepipeds whose edges have length[1] $\approx 2^{-n}$. In what follows, such parallelepipeds are called *quasicubes*. If F_{n+1} is a refinement of F_n (i.e., each quasicube in F_{n+1} is included in some quasicube in F_n), we call such partitions *dyadic*, and $F = \bigcup_{n=0}^{\infty} F_n$ a *dyadic family*. A partial case of dyadic partitions is presented by uniform partitions of Q_0 into cubes with edge length of precisely 2^{-n}. We use the notation D_n and $D = \bigcup_{n=0}^{\infty} D_n$ in this case.

Definition 2.2. The dyadic Nikol′skiĭ–Besov space $B_p^{\lambda\theta}(F)$ corresponding to a dyadic family F is the set of functions $f \in L_p(Q_0)$ for which the following quantity is finite:

$$|f|_{B_p^{\lambda\theta}(F)} := \left(\sum_{n=0}^{\infty} \left(2^{n\lambda} \operatorname{dist}_{L_p}(f, P_k(F_n)) \right)^{\theta} \right)^{1/\theta};$$

here $k > \lambda > 0$ and $0 < p, \theta \leq \infty$. The relevant quasinorm on $B_p^{\lambda\theta}(F)$ is defined by the formula

$$\|f\|_{B_p^{\lambda\theta}(F)} = |f|_{B_p^{\lambda\theta}(F)} + \|f\|_p.$$

Among all dyadic partitions, we isolate the so-called *special* partitions.

Let x be a vertex of Q_0. Consider the cube Q_0 obtained from Q_0 by a homothety with coefficient 3 and with center x. We split Q_x into 2nd congruent cubes and consider their intersections with Q_0. The resulting partition of Q_0 into parallelepipeds will be denoted by $F_n(x)$. We see that the $F_n(x)$ are dyadic partitions and $F(x) = \bigcup_{n=0}^{\infty} F_n(x)$ is a dyadic family. In what follows, the $F_n(x)$ are called *special dyadic partitions* and $F(x)$ is called a special dyadic family.

As usual, the Peetre K-functional of a couple (A,B) of spaces is defined by

$$K(f,t,A,B) = \inf_{f = f_1 + f_2} \{ \|f_1\|_A + t\|f_2\|_B \}.$$

[1]Throughout, $x \approx y$ means that there are positive constants c_1 and c_2 such that $c_1 x \leq y \leq c_2 x$.

Now, we formulate *the first result of the paper* about the relationship between the K-functionals for couples of classical and dyadic spaces.

Uniformly in f, we have

$$K(f,t,B_0,B_1) \approx \sum_x K(f,t,B_0(x),B_1(x));$$

here summation is over all vertices x of Q_0, and $B_j = B_p^{\lambda_j\theta_j}$, $B_j(x) = B_p^{\lambda_j\theta_j}(F(x))$, $j = 0,1$.

Thus, the calculation of the K-functional for a couple of B-spaces reduces to the calculation of K-functionals for dyadic spaces.

The knowledge of the K-functional for dyadic spaces will allow us to calculate the order of magnitude of the K-functional for classical spaces (see Corollary 3.2).

Now, we give the definition of the space $\dot{W}_p^k$. Let $\lambda = k-1+\frac{1}{p^*}$, where $p^* = \min(1,p)$.

Definition 2.3. The space $\dot{W}_p^k$ consists of all $f \in L_p(Q_0)$, $0 < p \le \infty$, such that the quantity

$$|f|_{\dot{W}_p^k} = \sup_n 2^{n\lambda}\bar{\omega}_k(f,2^{-n})_p$$

is finite.

If $1 < p < \infty$, this is a Sobolev space.

We are ready to formulate *the second result of the paper.*

Let $0 < p \le \infty$. Then

$$K(f,2^{-j\lambda},L_p,\dot{W}_p^k) \approx \bar{\omega}_k(f,2^{-j})_p.$$

The sharp order of magnitude for the K-functional of the couple $(L_p,\dot{W}_p^k)$ was found by Peetre in [6]. His method is not applicable if $0 < p < 1$, because then there are nonintegrable functions in L_p. The case of $0 < p < 1$ was treated by the author in [10]. In the present paper, a new approach is suggested, which is applicable whenever $0 < p \le \infty$ and involves the techniques of dyadic spaces.

The last result of the paper pertains to the computation of the K-functional for the couple (L_q,B_p^λ). In distinction to the previous cases, two different integrability exponents are now involved. This complicates the solution of the problem considerably. Since the answer will be in terms of nonlinear spline-approximation, we first give the definition of an approximating set.

Definition 2.4. A function s is said to belong to $P_k^n(F)$ if there exists a collection of dyadic quasicubes $\{Q_1,\dots,Q_n\}$ in F such that

$$s = \sum_{i=1}^n p_{Q_i}\chi_{Q_i};$$

here $p_{Q_i} \in P_k$.

It should be noted that there are no restrictions on the size of the quasicubes Q_i, and that they may fail to form a partition of Q_0. For instance, they may happen to be nested.

The functions belonging to $P_k^n(F)$ are piecewise polynomial. In order to obtain the approximating set, we add a smoothness condition. As usual, we denote by $C^{k-2}(Q_0)$ the space of functions defined on Q_0 and differentiable up to the order $k-2$. We put

$$S_k^n(F) := P_k^n(F) \cap C^{k-2}(Q_0).$$

Now, we formulate *the third result of the paper.*

If $f \in L_p(Q_0)$, $1 < p < q < \infty$, and $\lambda = d\left(\frac{1}{p} - \frac{1}{q}\right)$, then

$$K\left(f, 2^{-n\lambda}, L_q, B_p^\lambda\right) \approx 2^{-n\lambda}\left(\sum_{i=-1}^{n} \left(2^{i\lambda} e_i(f)_q\right)^p\right)^{\frac{1}{p}};$$

here $e_i(f)_q := \inf_{s \in S_k^{2^{id}}(D)} \|f - s\|_q$, $k > \lambda + 2$, and $e_{-1}(f)_q = \|f\|_q$.

The proof is based on a direct and an inverse approximation theorem. The direct theorem makes it possible to estimate the rate of approximation of a function in B_p^λ by the sets $S_k^n(D)$ (see Theorem 6.1 and the comments to it).

The inverse theorem says that a given rate of approximation implies that the function belongs to a certain particular space.

To prove the inverse theorem, we shall use the Bernstein inequality for dyadic spaces and interpolation techniques. The method is new and is simpler than those used earlier in the proof of that inequality (see [11]–[13]). It should be noted that the techniques of dyadic space enable us to obtain the result also for the couple (BMO, B_p^λ), $\lambda = \frac{d}{p}$, $p > 1$ (see Remark 6.1).

§3. Computation of the K-functional for a couple of B-spaces

First, we state a result very important for the subsequent presentation. It establishes the relationship between continuity moduli and local approximations.

Theorem 3.1. *If $f \in L_p(Q_0)$, $0 < p \le \infty$, then*

$$\bar{w}_k(f,t)_p \approx \sup_{\Pi_t} \left\{ \sum_{Q \in \Pi_t} E_k(f,Q)_p^p \right\}^{\frac{1}{p}}.$$

Here the supremum is taken over all families Π_t of congruent cubes with edge length of at most t and with mutually disjoint interiors.

For $p \ge 1$, Theorem 3.1 was proved by Brudnyĭ [14], and for $0 < p < 1$, it was proved by Storozhenko and Osval′d in [15] (see also the paper [16] by Nevskiĭ).

It turns out that the supremum in Theorem 3.1 can be taken over special dyadic partitions only (rather than over all families Π_t). To state the result, we denote

$$e_k\left(f, F_n(x)\right)_p := \operatorname{dist}_{L_p}\left(f, P_k\left(F_n(x)\right)\right).$$

Theorem 3.2. *Let $f \in L_p(Q_0)$, $0 < p \le \infty$. Then*

$$\bar{w}_k(f, 2^{-n})_p \approx \sum_x e_k\left(f, F_n(x)\right)_p;$$

here the summation is over all vertices x of Q_0.

Theorem 3.2 was proved by the author in [17]; it implies a statement about the relationship between classical and dyadic spaces.

Corollary 3.1. *Suppose $\lambda > 0$, $0 < p, \theta \le \infty$. If $f \in L_p(Q_0)$, then*

$$B_p^{\lambda\theta} = \bigcap_x B_p^{\lambda\theta}(F(x)),$$

where x runs over the vertices of Q_0.

We mention a case in which dyadic and classical spaces coincide.

Theorem 3.3. *If $0 < \lambda < \frac{1}{p}$, then*

$$B_p^{\lambda\theta} = B_p^{\lambda\theta}(F);$$

here either F is the special dyadic family $F(x)$, or $F = D$.

Theorem 3.3 was proved by the author in [18].

We recall that the dyadic Nikol′skiĭ–Besov space has been defined in terms of approximation by piecewise polynomial functions. The next result shows that the functions in $B_p^{\lambda\theta}$ can also be described in terms of approximation by smooth splines.

To state the result, we define the set of splines of degree at most $k-1$ in each variable and of defect 1 that are subordinate to a partition Π of Q_0. This set will be denoted by $S_k(\Pi)$:

$$S_k(\Pi) := P_k(\Pi) \cap C^{k-2}(Q_0).$$

Next, for a function f we denote by $s_n = s_n(f)$ the best approximation spline in $S_k(F_n)$, i.e.,

$$\|f - s_n\|_p = \operatorname{dist}_{L_p}(f, S_k(F_n)).$$

Theorem 3.4. *A function $f \in L_p(Q_0)$ belongs to $B_p^{\lambda\theta}$, $k > \lambda + 2$, $0 < p$, $\theta \le \infty$, if and only if*

a)

$$f = \sum_{n=0}^{\infty} (s_n - s_{n-1}) \quad (\textit{convergence in } L_p),$$

where $s_{-1} \equiv 0$;

b) *we have*

$$\left(\sum_{n=0}^{\infty} \left(2^{n\lambda} \|f - s_n\|_p\right)^{\theta} \right)^{\frac{1}{\theta}} < \infty.$$

Moreover,

$$|f|_{B_p^{\lambda\theta}} \approx \left(\sum_{n=0}^{\infty} \left(2^{n\lambda} \|f - s_n\|_p\right)^{\theta} \right)^{\frac{1}{\theta}}.$$

For $F = \bigcup_n U_{2^n(k-1)}$, where U_n is the uniform partition of Q_0 into cubes with edge length m^{-1}, Theorem 3.4 was proved by the author in [10]. Independently but much later, the statement was proved by DeVore and Popov in [19] in the case where $0 < \lambda < \min\left(k, k - 1 + \frac{1}{p}\right)$ and $F = D$.

In order to prove Theorem 3.4 for an arbitrary dyadic family F, it suffices to employ (as in [19]) the quasi-interpolation operator introduced in [20]. That operator makes it possible to replace approximation by piecewise polynomial function by spline approximation without changing the rate of the approximation.

Another method to "glue" piecewise polynomial functions together was involved in the proof of Theorem 3.4 given in [10]. That method was suggested by Brudnyĭ in [21] and it was extended by the author in [10] to the multidimensional case. It has turned out that this "gluing" algorithm can also be applied to special dyadic partitions.

Theorem 3.5. *If $f \in L_p(Q_0)$, $0 < p \le \infty$, then for every n there exists a smooth spline $\varphi_n = \varphi_n(f) \in S_k(U_{2^n(k-1)})$, $k \ge 2$, such that*

$$\|f - \varphi_n\|_p \le c \cdot \sum_x e_k(f, F_n(x))_p;$$

here summation is over all vertices x of Q_0.

Theorem 3.5 was proved by the author in [22].

We note that the spline φ_n in Theorem 3.5 not only approximates well the function $f \in L_p(Q_0)$, but also has many other properties. To state the result for $f \in L_p(Q_0)$, we denote by $g_{nx} = g_{nx}(f)$ a function in $P_k(F_n(x))$ such that

$$\|f - g_{nx}\|_p = e_k(f, F_n(x)).$$

Statement 3.1. *If $f \in L_p(Q_0)$, $p > 0$, $k > \gamma + 2$, $\gamma > 0$, and $0 < \theta \le \infty$, then*

$$\|\varphi_n - g_{nx}\|_p \le c \cdot \sum_y \|g_{ny} - g_{nx}\|_p. \tag{3.1}$$

Moreover,

$$\|\varphi_n\|_{B_p^{\gamma\theta}} \le c \cdot \left(\sum_y \|g_{ny}\|_{B_p^{\gamma\theta}(F(y))} + 2^{n\gamma} \left(\sum_y \|f - g_{ny}\|_p^\theta \right)^{\frac{1}{\theta}} \right) \tag{3.2}$$

and

$$\|f - \varphi_n\|_{B_p^{\gamma\theta}} \le c \cdot \sum_y \|f - g_{ny}\|_{B_p^{\gamma\theta}(F(y))}; \tag{3.3}$$

here summation is over all vertices of Q_0.

Proof. The first property was proved in [23]. To prove the second and the third properties, we need an auxiliary statement.

Lemma 3.1. *We have*

$$e_k(\varphi_n, F_i(x))_p \le c \cdot \left(\sum_y e_k(f, F_n(y))_p + e_k(g_{nx}, F_i(x))_p \right).$$

Proof. The triangle inequality yields

$$e_k(\varphi_n, F_i(x))_p \le c \cdot \left(\|\varphi_n - g_{nx}\|_p + \|g_{nx} - h_{ix}\|_p \right); \tag{3.4}$$

here $h_{ix} \in P_k(F_i(x))$.

Choosing h_{ix} so that

$$\|g_{nx} - h_{ix}\|_p = e_k(g_{nx}, F_i(x))_p,$$

we apply (3.1) to the first summand in (3.4) and take the definition of g_{nx} into account. □

Now, we prove inequality (3.2). Since $\varphi_n \in S_k(U_{2^n(k-1)})$ and the partitions $U_{2^n(k-1)}$ are dyadic, we see that $\varphi_n \in S_k(U_{2^i(k-1)})$ for $i \ge n$. Combined with Theorem 3.4, this implies that

$$|\varphi_n|_{B_p^{\gamma\theta}}^\theta \approx \sum_{i=0}^{n-1} \left(2^{i\gamma} \widetilde{e}_k(\varphi_n, U_{2^i(k-1)})_p \right)^\theta; \tag{3.5}$$

here $\widetilde{e}_k(\varphi_n, U_{2^i(k-1)})_p := \operatorname{dist}_{L_p}\left(\varphi_n, S_k(U_{2^i(k-1)})\right)$.

Now we pass to approximation by functions in $F_i(x)$, $i < n$. For this, we apply Theorem 3.5 to φ_n, obtaining

$$\widetilde{e}_k(\varphi_n, U_{2^i(k-1)})_p \le c \cdot \sum_x e_k(\varphi_n, F_i(x))_p. \tag{3.6}$$

We estimate the right-hand side of (3.6) by Lemma 3.1:

$$\widetilde{e}_k(\varphi_n, U_{2^i(k-1)})_p \le c \cdot \left(\sum_x \left(e_k(f, F_n(x))_p + e_k(g_{nx}, F_i(x))_p \right) \right),$$

and use this to continue (3.5):

$$|\varphi_n|_{B_p^{\gamma\theta}}^\theta \le c \cdot \left(\sum_x \sum_{i=0}^{n-1} \left(2^{i\gamma} e_k(g_{nx}, F_i(x))_p \right)^\theta + 2^{n\gamma\theta} \sum_x e_k(f, F_n(x))_p^\theta \right).$$

Observe that

$$\|\varphi_n\|_p \le c \cdot \sum_x \left(\|g_{nx}\|_p + e_k(f, F_n(x))_p \right)$$

by (3.1). The definition of the quasinorm in $B_p^{\gamma\theta}(F(x))$ and the last two inequalities yield (3.2).

We prove (3.3). Since $\varphi_n \in S_k(U_{2^n(k-1)})$ and $U_{2^i(k-1)} \subset U_{2^n(k-1)}$ for $i \geq n$, we obtain

$$|f-\varphi_n|^{\theta}_{B_p^{\gamma\theta}} \approx \sum_{i=0}^{n-1} \left(2^{i\gamma}\widetilde{e}_k(f-\varphi_n, U_{2^i(k-1)})_p\right)^{\theta} + \sum_{i=n}^{\infty} \left(2^{i\gamma}\widetilde{e}_k(f, U_{2^i(k-1)})_p\right)^{\theta}.$$

We denote the summands on the right by Σ_1 and Σ_2. An application of Theorem 3.5 to $f-\ \varphi_n$ yields

$$\widetilde{e}_k(f-\varphi_n, U_{2^i(k-1)})_p \leq c \cdot \sum_x e_k(f-\varphi_n, F_i(x))_p.$$

By analogy with Lemma 3.1, we use the triangle inequality to obtain

$$e_k(f-\varphi_n, F_i(x))_p \leq c \cdot e_k(f-g_{nx}, F_i(x))_p + \sum_y e_k(f, F_n(y))_p.$$

Taken together, these two inequalities enable us to estimate Σ_1:

$$\Sigma_1 \leq c \cdot \left(\sum_x \sum_{i=0}^{n-1} \left(2^{i\gamma} e_k(f-g_{nx}, F_i(x))_p\right)^{\theta} + 2^{n\gamma\theta} \sum_y \left(e_k(f, F_n(y))_p\right)^{\theta} \right).$$

We use Theorem 3.5 to estimate Σ_2:

$$\Sigma_2 \leq c \cdot \sum_x \sum_{i=n}^{\infty} \left(2^{i\gamma} e_k(f, F_i(x))_p\right)^{\theta}.$$

Now, $e_k(f, F_i(x))_p = e(f-g_{nx}, F_i(x))$ for $i \geq n$. Then, combining the estimates for Σ_1 and Σ_2, we arrive at

$$|f-\varphi_n|_{B_p^{\gamma\theta}} \leq c \cdot \left(\sum_x |f-g_{nx}|_{B_p^{\gamma\theta}(F(x))} + 2^{n\gamma} \left(\sum_y \left(e_k(f, F_n(y))_p\right)^{\theta} \right)^{\frac{1}{\theta}} \right).$$

We have used the definition of $B_p^{\lambda\theta}(F(x))$. Observe that the second summand is dominated by the first. By Theorem 3.5, this finishes the prof of (3.3). □

The properties of the spline φ_n mentioned in Statement 3.1 allow us to easily establish the relationship between the K-functionals for pairs of classical spaces and for pairs of dyadic spaces.

Theorem 3.6. *Suppose $\lambda > \mu \geq 0$. Then*

$$K(f, 2^{-n(\lambda-\mu)}, B_p^{\mu\theta_1}, B_p^{\lambda\theta_2}) \approx \sum_x K(f, 2^{-n(\lambda-\mu)}, B_p^{\mu\theta_1}(F(x)), B_p^{\lambda\theta_2}(F(x)));$$

here the summation is over all vertices x of Q_0. If $\mu = 0$, the spaces $B_p^{\mu\theta_1}$, $B_p^{\mu\theta_1}(F(x))$ should be replaced by L_p.

Proof. By Corollary 3.1, we have $\|f\|_{B_p^{\lambda\theta}(F(x))} \leq \|f\|_{B_p^{\lambda\theta}}$; then

$$K(f, 2^{-n(\lambda-\mu)}, B_1(x), B_2(x)) \leq K(f, 2^{-n(\lambda-\mu)}, B_1, B_2).$$

Here $B_1(x) := B_p^{\mu\theta_1}(F(x))$, $B_2(x) := B_p^{\lambda\theta_2}(F(x))$, and similarly $B_1 := B_p^{\mu\theta_1}$, $B_2 := B_p^{\lambda\theta_2}$.

It remains to sum these inequalities over all 2^d special dyadic partitions.

We prove the reverse inequality. The definition of the K-functional shows that

$$K(f, 2^{-n(\lambda-\mu)}, B_1, B_2) \leq \|f-\varphi_n\|_{B_1} + 2^{-n(\lambda-\mu)}\|\varphi_n\|_{B_2}.$$

If $\mu = 0$, we estimate the first summand with the help of Theorem 3.4, and if $\mu > 0$, we estimate it with the help of (3.3). Also, we apply inequality (3.2) to the second summand. As a result, we obtain

$$K(f, 2^{-n(\lambda-\mu)}, B_1, B_2) \le c \cdot \Bigg(\sum_x \|f - g_{nx}\|_{B_1(x)} + 2^{n\mu} \Bigg(\sum_x \|f - g_{nx}\|_p^{\theta_2} \Bigg)^{\frac{1}{\theta_2}} + 2^{-n(\lambda-\mu)} \sum_x \|g_{nx}\|_{B_2(x)} \Bigg).$$

Since the second sum involves only 2^d summands, we can replace θ_2 by θ_1. After that, the first summand will dominate the second.

Next, in the course of the computation of K-functionals in [23], it was shown that the infimum is attained at the functions g_{nx}, i.e.,

$$K(f, 2^{-n(\lambda-\mu)}, B_1(x), B_2(x)) \approx \|f - g_{nx}\|_{B_1(x)} + 2^{-n(\lambda-\mu)} \|g_{nx}\|_{B_2(x)}.$$

This finishes the proof of Theorem 3.6. □

Corollary 3.2. *If $\lambda > \mu \ge 0$, then*

$$K(f, 2^{-n(\lambda-\mu)}, B_1, B_2) \approx \Bigg(\sum_{i=n}^{\infty} \big(2^{i\mu}\omega_i\big)^{\theta_1} \Bigg)^{\frac{1}{\theta_1}} + 2^{-n(\lambda-\mu)} \Bigg(\Bigg(\sum_{i=0}^{n} \big(2^{i\lambda}\omega_i\big)^{\theta_2} \Bigg)^{\frac{1}{\theta_2}} + \|f\|_p \Bigg);$$

here $\omega_i := \bar{\omega}_k(f, 2^{-i})_p$ and if $\mu = 0$, then $B_1 = L_p$ and the first sum on the right should be eliminated.

Proof. As in [23],

$$K\Big(f, 2^{-n(\lambda-\mu)}, B_1(x), B_2(x)\Big) \approx \Bigg(\sum_{i=n}^{\infty} \big(2^{i\mu} e_i(x)\big)^{\theta_1} \Bigg)^{\frac{1}{\theta_1}} + 2^{-n(\lambda-\mu)} \Bigg(\Bigg(\sum_{i=0}^{n} \big(2^{i\lambda} e_i(x)\big)^{\theta_2} \Bigg)^{\frac{1}{\theta_2}} + \|f\|_p \Bigg);$$

here $e_i(x) := e_k(f, F_i(x))_p$. Now Theorem 3.6 and Theorem 3.2 on the equivalence of a continuity modulus and the rate of piecewise polynomial approximation show that $\omega_i \approx \sum_x e_i(x)$. □

§4. The K-functional for the couple $(L_p, \dot{W}_p^k)$

In the preceding section we computed the K-functional of a couple of B-spaces. Now, we continue the computation of K-functionals.

Theorem 4.1. *If $0 < p \le \infty$, then*

$$K(f, 2^{-j\lambda}, L_p, \dot{W}_p^k) \approx \bar{\omega}_k(f, 2^{-j})_p.$$

Proof. We begin with two auxiliary statements. Suppose Q is a quasicube, $Q = J_1 \times \cdots \times J_d$. Then $|J_i| \approx |J_j|$. We split some of the segments J_i into two "nearly equal" parts (that is, the lengths of the resulting intervals should be comparable with that of J_i within a universal constant). This generates a splitting of Q into parallelepipeds; moreover, each edge of each parallelepiped is "nearly equal" in length to an edge of Q. This partition of Q into at most 2^d quasicubes will be denoted by $\widetilde{\Pi}(Q)$.

Now, let k^* be either k or $k+1$, and let φ be a spline of degree at most $k^* - 1$ and of smoothness at most $k^* - 2$ subordinate to $\widetilde{\Pi}(Q)$, that is, $\varphi \in S_{k^*}(\widetilde{\Pi}(Q))$. Though φ has derivatives only up to order $k^* - 2$, it has derivatives of arbitrary order almost everywhere.

Indeed, on each quasicube $Q_i \in \widetilde{\Pi}(Q)$, the function φ is a polynomial $p_i \in P_{k^*}$. We write $D^\beta \varphi(x) = D^\beta p_i(x)$ for $x \in Q_i$.

Next, put

$$R(S) := \big\{\beta = (\beta_1, \dots, \beta_d) \,:\, 0 \le \beta_i \le S,\ i = 1, \dots, d\big\}.$$

Using the above notation, we prove the following statement.

Lemma 4.1. *Suppose $\varphi \in S_{k^*}(\widetilde{\Pi}(Q))$. Then*

$$E_k(\varphi, Q)_{L_\infty} \le c \cdot \sum_{\beta \in \Gamma(k^*)} |Q|^{\frac{|\beta|}{d}} \|D^\beta \varphi\|_{L_\infty(Q)},$$

where $\Gamma(k^*) := R(k^* - 1) \setminus R(k^* - 2)$ and $|\beta| = \beta_1 + \dots + \beta_d$.

Proof. The definition of φ shows that

$$\varphi = \sum_{i=1}^{m} p_{Q_i} \chi_{Q_i},$$

where $m \le 2^d$.

By construction, all quasicubes Q_i are neighbors; that is, they have a common face. If the partition involves 2^d quasicubes, there is a unique common point; otherwise, the common points constitute an l-dimensional face with $1 \le l < d$. Let ξ_Q be an arbitrary inner point of Q that belongs to all Q_i.

We consider the Taylor series expansion of p_{Q_i} at ξ_Q:

$$p_{Q_i}(x) = \sum_{\beta \in R(k^*-1)} \frac{D^\beta p_{Q_i}(\xi_Q)}{\beta!} (x - \xi_Q)^\beta, \quad x \in Q_i.$$

Since $\varphi \in C^{k^*-2}(Q)$ and ξ_Q is an inner point of Q, for any cubes Q_i, Q_j in $\widetilde{\Pi}(Q)$ we have

$$D^\beta p_{Q_i}(\xi_Q) = D^\beta p_{Q_j}(\xi_Q) = D^\beta \varphi(\xi_Q)$$

for $\beta \in R(k^* - 2)$.

We introduce the polynomial p_Q defined by

$$p_Q(x) := \sum_{\beta \in R(k^*-2)} \frac{D^\beta \varphi(\xi_Q)}{\beta!} (x - \xi_Q)^\beta, \quad x \in Q.$$

Then

$$p_{Q_i}(x) - p_Q(x) = \sum_{\beta \in \Gamma(k^*)} \frac{D^\beta p_{Q_i}(\xi_Q)}{\beta!} (x - \xi_Q)^\beta, \quad x \in Q_i.$$

Since $\xi_Q \in Q_i$, for $x \in Q_i$ we have

$$\big|(x - \xi_Q)^\beta\big| \le c \cdot |Q_i|^{\frac{|\beta|}{d}}.$$

By construction, we have $|Q_i| \approx |Q|$; therefore,

$$\|p_{Q_i} - p_Q\|_{L_\infty(Q_i)} \le c \cdot \sum_{\beta \in \Gamma(k^*)} |Q|^{\frac{|\beta|}{d}} \|D^\beta p_{Q_i}\|_{L_\infty(Q_i)}.$$

We note that $p_Q \in P_{k^*-1}$ and k^* is either k or $k+1$. So, $p_Q \in P_k$. We have

$$E_k(\varphi, Q)_{L_\infty} \le \|\varphi - p_Q\|_{L_\infty(Q)} = \max_{i=1,\dots,m} \|p_{Q_i} - p_Q\|_{L_\infty(Q_i)}.$$

Using the preceding inequality, we continue the estimate:

$$E_k(\varphi, Q)_{L_\infty} \le c \cdot \sum_{\beta \in \Gamma(k^*)} |Q|^{\frac{|\beta|}{d}} \max_{i=1,\dots,m} \|D^\beta p_{Q_i}\|_{L_\infty(Q_i)}.$$

By definition,

$$\|D^\beta \varphi\|_{L_\infty(Q)} = \max_{i=1,\dots,m} \|D^\beta p_{Q_i}\|_{L_\infty(Q_i)}. \qquad \square$$

The next statement enables us to estimate the rate of approximation of a smooth spline s_j subordinate to a partition D_j by piecewise polynomial functions constructed in terms of finer special partitions $F_n(x)$, $j \le n$.

Let

$$k^* = \begin{cases} k, & 0 < p \le 1; \\ k+1, & p > 1. \end{cases}$$

Lemma 4.2. *If $0 < p \le \infty$ and $s_j \in S_{k^*}(D_j)$, then*

$$e_k(s_j, F_n(x))_p \le c \cdot 2^{(j-n)\lambda} e_k(s_j, F_j(x))_p,$$

where $j \le n$, $\lambda = k - 1 + \frac{1}{p^*}$, $p^* = \min(1, p)$.

Proof. We choose a piecewise polynomial function $g_j(x) \in P_k(F_j(x))$ in such a way that

$$e_k(s_j, F_j(x))_p = \|s_j - g_j(x)\|_p.$$

Since $g_j(x) \in P_k(F_j(x))$ and $F_n(x)$ is a refinement of $F_j(x)$ ($j \le n$), we can view $g_j(x)$ as an element of $P_k(F_n(x))$. We have

$$e_k(s_j, F_n(x))_p = e_k(s_j - g_j(x), F_n(x))_p.$$

Put $\varphi_j = s_j - g_j(x)$. Recall that s_j is a spline subordinate to the partition D_j, and $g_j(x)$ is a piecewise polynomial function subordinate to the special dyadic partition $F_j(x)$. Since D_j is a refinement of $F_j(x)$, we may regard φ_j as a piecewise polynomial function subordinate to the partition D_j.

So, we must estimate the rate of approximation of φ_j by piecewise polynomial functions subordinate to a finer partition $F_n(x)$ ($j \le n$).

By the definition of the best approximation,

$$e_k(\varphi_j, F_n(x))_p = \Bigg(\sum_{Q \in F_n(x)} E_k(\varphi_j, Q)_p^p \Bigg)^{\frac{1}{p}}.$$

We split the set of all quasicubes $Q \in F_n(x)$ into two subsets. To the first subset (denoted by G_1) we attribute all quasicubes that lie in the interior of some cube $Y \in D_j$. To the second subset (denoted by G_2) we attribute the quasicubes that intersect a face of some cube Y. For $i = 1, 2$, we put

$$A_i(p) := \sum_{Q \in G_i} E_k(\varphi_j, Q)_p^p.$$

To estimate $A_1(p)$, we consider two cases: $p > 1$ and $0 < p \le 1$. If $0 < p \le 1$, then $s_j \in S_k(D_j)$; therefore $\varphi_j \in P_k(D_j)$. Consequently, $E_k(\varphi_j, Q)_p = 0$ for $Q \in G_1$. Thus, $A_1(p) = 0$ if $0 < p \le 1$.

Let $p > 1$. Then $s_j \in S_{k+1}(D_j)$ and $\varphi_j \in P_{k+1}(D_j)$; therefore, $E_k(\varphi_j, Q)_p \ne 0$. But since Q lies in the interior of some $Y \in D_j$, we see that, on Q, φ_j is a polynomial belonging to P_{k+1}. We show that

$$E_k(\varphi_j, Q)_p \le c \cdot \sum_{\alpha \in \Gamma(k+1)} |Q|^{\frac{|\alpha|}{d}} \|D^\alpha \varphi_j\|_{p(Q)}. \tag{4.1}$$

Clearly, $E_k(\varphi_j, Q)_p \le |Q|^{\frac{1}{p}} E_k(\varphi_j, Q)_{L_\infty}$. Since φ_j on Q is a polynomial in P_{k+1}, the Taylor formula shows that

$$E_k(\varphi_j, Q)_{L_\infty} \le c \cdot \sum_{\alpha \in \Gamma(k+1)} \frac{\|D^\alpha \varphi_j\|_{C(Q)}}{\alpha!} |Q|^{\frac{|\alpha|}{d}}.$$

It remains to use again the fact that φ_j is a polynomial on Q and to recall that, on a finite-dimensional space, all norms are equivalent, whence

$$\|D^\alpha \varphi_j\|_{C(Q)} \le c \cdot |Q|^{-\frac{1}{p}} \|D^\alpha \varphi_j\|_{p(Q)}.$$

Taken together, the last two inequalities yield (4.1).

Now we estimate $A_1(p)$ for $p > 1$. We have

$$A_1(p) \le c \cdot \sum_{Q \in G_1} \sum_{\alpha \in \Gamma(k+1)} |Q|^{\frac{|\alpha| p}{d}} \|D^\alpha \varphi_j\|^p_{p(Q)}.$$

Selecting the quasicubes $Q \in F_n(x)$ that are included in one and the same quasicube $Y \in D_j$, we split G_1 into mutually nonintersecting sets:

$$G_1 = \bigcup_Y \{Q \in F_n(x) : Q \subset Y\}.$$

Then

$$A_1(p) \le c \cdot \sum_{Y \in D_j} \sum_{Q \subset Y} \sum_{\alpha \in \Gamma(k+1)} |Q|^{\frac{|\alpha| p}{d}} \|D^\alpha \varphi_j\|^p_{p(Q)}.$$

The quasicubes Q do not intersect and $|Q| \approx 2^{-nd}$, so

$$A_1(p) \le c \cdot \sum_{\alpha \in \Gamma(k+1)} \sum_{Y \in D_j} |Q|^{\frac{|\alpha| p}{d}} \|D^\alpha \varphi_j\|^p_{p(Y)}.$$

Since φ_j is a polynomial on Y, by a Markov-type inequality we obtain

$$\|D^\alpha \varphi_j\|_{p(Y)} \le c \cdot |Y|^{-\frac{|\alpha|}{d}} \|\varphi_j\|_{p(Y)}.$$

Thus,

$$A_1(p) \le c \cdot \sum_{\alpha \in \Gamma(k+1)} \sum_{Y \in D_j} \left(\frac{|Q|}{|Y|}\right)^{\frac{|\alpha| p}{d}} \|\varphi_j\|^p_{p(Y)}.$$

Now, $|\alpha| \ge k$ for $\alpha \in \Gamma(k+1)$. Since $|Q| \le |Y|$, we see that

$$\left(\frac{|Q|}{|Y|}\right)^{\frac{|\alpha| p}{d}} \le \left(\frac{|Q|}{|Y|}\right)^{\frac{kp}{d}} \approx 2^{(j-n)kp}.$$

Finally, for $p > 1$ we arrive at

$$A_1(p) \le c \cdot 2^{(j-n)kp} \sum_{Y \in D_j} \|\varphi_j\|^p_{p(Y)} = c \cdot 2^{(j-n)kp} \|\varphi_j\|^p_p.$$

We pass to the estimation of $A_2(p)$. Recall that

$$A_2(p) = \sum_{Q \in G_2} E_k(\varphi_j, Q)^p_p,$$

where the summation is over the quasicubes that intersect faces of cubes in D_j.

A quasicube $Q \in G_2$ can intersect at most 2^d cubes D_j. We use a property of quasicubes taken from different partitions that was proved in [18]. Since $Q \in F_n(x)$ and $Y \in D_j$, we see that Q and Y are "in engagement", that is,

$$|Q \cap Y| \approx \min(|Q|, |Y|) = |Q|.$$

It follows that the faces of the cubes $Y \in D_j$ split $Q \in F_n(x)$ into quasicubes comparable in size with Q. We obtain a "nearly" uniform partition $\widetilde{\Pi}(Q)$ as in Lemma 4.1. Viewing a spline $s_j \in S_{k^*}(D_j)$ as a function on a quasicube $Q \in G_2$, we may write $s_j \mid_Q \in S_{k^*}(\widetilde{\Pi}(Q))$. Next, the piecewise polynomial function $g_j(x)$ is subordinate to the partition $F_j(x)$, and $F_n(x) \subset F_j(x)$. Therefore, on smaller quasicubes $Q \in F_n(x)$, the function $g_j(x)$ coincides with a polynomial in P_k. Since $P_k \subset P_{k^*}$ and $P_{k^*} \subset S_{k^*}(\widetilde{\Pi}(Q))$, we see that $g_j(x) \in S_{k^*}(\widetilde{\Pi}(Q))$. Recall that $\varphi_j = s_j - g_j(x)$; consequently, $\varphi_j \mid_Q \in S_{k^*}(\widetilde{\Pi}(Q))$.

Applying Lemma 4.1, we obtain

$$E_k(\varphi_j, Q)_p \le c \cdot |Q|^{\frac{1}{p}} \sum_{\beta \in \Gamma(k^*)} |Q|^{\frac{|\beta|}{d}} \|D^\beta \varphi_j\|_{L_\infty(Q)}.$$

Denote by $Y(Q)$ the set of all cubes $Y \in D_j$ that intersect a fixed quasicube Q. Then $Q \subset \bigcup_{Y \in Y(Q)} Y$, $|Y(Q)| \le 2^d$, and

$$\|D^\beta \varphi_j\|_{L_\infty(Q)} \le \sum_{Y \in Y(Q)} \|D^\beta \varphi_j\|_{L_\infty(Y)}.$$

We apply the last two inequalities to estimate $A_2(p)$. We have

$$A_2(p) \le c \cdot \sum_{Q \in G_2} |Q| \Bigg(\sum_{\beta \in \Gamma(k^*)} |Q|^{\frac{|\beta|}{d}} \sum_{Y \in Y(Q)} \|D^\beta \varphi_j\|_{L_\infty(Y)} \Bigg)^p.$$

Since the sets $\Gamma(k^*)$ and $Y(Q)$ are finite, we can continue the estimate by placing the exponent p under the summation sign. Next, since φ_j is a polynomial on Y, the Markov inequality together with an inequality for different metrics implies that

$$\|D^\beta \varphi_j\|_{L_\infty(Y)} \le c \cdot |Y|^{-\frac{1}{p} - \frac{|\beta|}{d}} \|\varphi_j\|_{p(Y)}.$$

So,

$$A_2(p) \le c \cdot \sum_{Q \in G_2} \frac{|Q|}{|Y|} \sum_{\beta \in \Gamma(k^*)} \left(\frac{|Q|}{|Y|} \right)^{\frac{|\beta|}{d} p} \sum_{Y \in Y(Q)} \|\varphi_j\|_{p(Y)}^p.$$

We use the relations $|Q| \approx 2^{-nd}$ and $|Y| \approx 2^{-jd}$ for $j \le n$, and the fact that $\beta \in \Gamma(k^*)$ to obtain

$$\sum_{\beta \in \Gamma(k^*)} \left(\frac{|Q|}{|Y|} \right)^{\frac{|\beta|}{d} p} \le c \cdot 2^{(j-n)(k^*-1)p}.$$

We continue the estimate for $A_2(p)$:

$$A_2(p) \le c \cdot 2^{(j-n)(d+(k^*-1)p)} \sum_{Q \in G_2} \sum_{Y \in Y(Q)} \|\varphi_j\|_{p(Y)}^p.$$

We want to change the order of summation. For this, we must estimate the number of quasicubes Q that intersect faces of a fixed cube $Y \in D_j$. Let $R_n(Y)$ denote the set of such quasicubes. Then

$$\sum_{Q \in G_2} \sum_{Y \in Y(Q)} \|\varphi_j\|_{p(Y)}^p = \sum_{Y \in D_j} |R_n(Y)| \, \|\varphi_j\|_{p(Y)}^p.$$

We show that

$$|R_n(Y)| \le c \cdot 2^{(d-1)(n-j)}.$$

For that, it suffices to estimate the number of quasicubes that intersect a $(d-1)$-dimensional face. Since any edge of the quasicube Y is roughly 2^{-j} and any edge of Q is roughly 2^{-n}, there are at most $c \cdot 2^{(d-1)(n-j)}$ such cubes.

Thus,

$$A_2(p) \le c \cdot 2^{(j-n)(1+(k^*-1)p)} \|\varphi_j\|_p^p.$$

Let $0 < p \le 1$. Then $1 + (k^* - 1)p = 1 + (k-1)p = \lambda p$. If $p > 1$, then $1 + (k^* - 1)p = 1 + kp > kp = \lambda p$. Since $j \le n$, we have

$$A_2(p) \le c \cdot 2^{(j-n)\lambda p} \|\varphi_j\|_p^p$$

for any p.

We recall that a similar estimate was obtained for $A_1(d)$ for $p > 1$, and $A_1(p) = 0$ for $0 < p \le 1$. It follows that

$$e_k^p(s_j, F_n(x))_p = A_1(p) + A_2(p) \le c \cdot 2^{(j-n)\lambda p} \|\varphi_j\|_p^p.$$

It remains to use the fact that

$$\|\varphi_j\|_p = \|s_j - g_j(x)\|_p = e_k(s_j, F_j(x))_p.$$

This finishes the proof of Lemma 4.2. □

Now, we finish the proof of Theorem 4.1. Let $f = f_0 + f_1$. Then

$$\bar{\omega}_k(f_0 + f_1, 2^{-j})_p \le c \cdot \left(\bar{\omega}_k(f_0, 2^{-j})_p + \bar{\omega}_k(f_1, 2^{-j})_p\right)$$

by properties of the continuity modulus. Clearly,

$$\bar{\omega}_k(f_0, 2^{-j})_p \le c \cdot \|f_0\|_p$$

and

$$\bar{\omega}_k(f_1, 2^{-j})_p \le 2^{-j\lambda} \sup_n 2^{n\lambda} \bar{\omega}_k(f_1, 2^{-n})_p.$$

Next,

$$\bar{\omega}_k(f, 2^{-j})_p \le c \cdot \left(\|f_0\|_p + 2^{-j\lambda} \sup_n 2^{n\lambda} \bar{\omega}_k(f_1, 2^{-n})_p\right).$$

We recall that

$$\|f_1\|_{\dot{W}_p^k} := \sup_n 2^{n\lambda} \bar{\omega}_k(f_1, 2^{-n})_p,$$

where $\lambda = k - 1 + \frac{1}{p^*}$.

Taking the infimum over all representations $f = f_0 + f_1$, we obtain

$$\bar{\omega}_k(f, 2^{-j})_p \le c \cdot K\left(f, 2^{-j\lambda}, L_p, \dot{W}_p^k\right).$$

We prove the reverse inequality. The definition of the K-functional shows that

$$K\left(f, 2^{-j\lambda}, L_p, \dot{W}_p^k\right) \le \|f - s_j\|_p + 2^{-j\lambda} \|s_j\|_{\dot{W}_p^k}.$$

We choose $s_j \in S_{k^*}(D_j)$ in such a way that

$$\|f - s_j\|_p \le c \cdot \bar{\omega}_{k^*}(f, 2^{-j})_p.$$

The existence of such a spline was proved, e.g., in [20].

Now we estimate the quantity $\bar{\omega}_k(s_j, 2^{-n})_p$. By Theorem 3.2,

$$\bar{\omega}_k(s_j, 2^{-n})_p \approx \sum_x e_k(s_j, F_n(x))_p.$$

If $j \le n$, we can apply Lemma 4.2, obtaining

$$\bar{\omega}_k(s_j, 2^{-n})_p \le c \cdot 2^{(j-n)\lambda} \sum_x e_k(s_j, F_j(x))_p.$$

Using Theorem 3.2 once again, we arrive at

$$\bar{\omega}_k(s_j, 2^{-n})_p \le c \cdot 2^{(j-n)\lambda} \bar{\omega}_k(s_j, 2^{-j})_p.$$

Then

$$\sup_{n\geq j} 2^{n\lambda}\bar{\omega}_k(s_j, 2^{-n})_p \leq c\cdot 2^{j\lambda}\bar{\omega}_k(s_j, 2^{-j})_p$$

and

$$\|s_j\|_{\dot{W}_p^k} \leq c\cdot \sup_{n\leq j} 2^{n\lambda}\bar{\omega}_k(s_j, 2^{-n})_p.$$

Collecting the estimates, we see that

$$K\left(f, 2^{-j\lambda}, L_p, \dot{W}_p^k\right) \leq c\cdot \left(\|f-s_j\|_p + 2^{-j\lambda}\sup_{n\leq j} 2^{n\lambda}\bar{\omega}_k(s_j, 2^{-n})_p\right).$$

Clearly,

$$\bar{\omega}_k(s_j, 2^{-n})_p \leq c\cdot (\bar{\omega}_k(f, 2^{-n})_p + \|f-s_j\|_p).$$

It remains to refer to the choice of s_j, from which it follows that

$$\|f-s_j\|_p \leq c\cdot \bar{\omega}_{k^*}(f, 2^{-j})_p \leq c\cdot \bar{\omega}_k(f, 2^{-j})_p.$$

Here we have used the fact that $k^* \geq k$. Now,

$$K\left(f, 2^{-j\lambda}, L_p, \dot{W}_p^k\right) \leq c\cdot 2^{-j\lambda}\sup_{n\leq j} 2^{n\lambda}\bar{\omega}_k(f, 2^{-n})_p.$$

We employ the following property of the continuity modulus:

$$\bar{\omega}_k(f, N\cdot t)_p \leq c\cdot N^{k-1+\frac{1}{p^*}}\bar{\omega}_k(f, t)_p,$$

where N is a natural number.

For $n \leq j$ we have

$$\bar{\omega}_k(f, 2^{-n})_p \leq c\cdot 2^{(j-n)\lambda}\bar{\omega}_k(f, 2^{-j})_p.$$

Continuing the estimate of the K-functional, we obtain

$$K\left(f, 2^{-j\lambda}, L_p, \dot{W}_p^k\right) \leq c\cdot \bar{\omega}_k(f, 2^{-j})_p.$$

This finishes the proof of Theorem 4.1. □

§5. Bernstein inequality

In [1], a Bernstein-type inequality was proved for dyadic spaces, specifically, the inequality

$$|s_n|_{B_p^\lambda(F)} \leq c\cdot n^{\frac{\lambda}{d}}\|s_n\|_q,$$

where $s_n \in P_k^n(F)$, $\frac{\lambda}{d} = \frac{1}{p} - \frac{1}{q}$, $0 < p < q \leq \infty$.

Our purpose in this section is to prove a similar inequality for classical spaces.

Theorem 5.1. *Suppose that the numbers λ, p, q, and d satisfy $0 < \lambda - [\lambda] < \frac{1}{p}$, $\frac{\lambda}{d} = \frac{1}{p} - \frac{1}{q}$, $1 \leq p < q \leq \infty$. Then for $s_n \in S_k^n(D)$ and $k > \lambda + 2$ we have*

$$\|s_n\|_{B_p^\lambda} \leq c\cdot n^{\frac{\lambda}{d}}\|s_n\|_q.$$

Proof. We use a result proved in [24], which says that, if $0 < \lambda - [\lambda] < \frac{1}{p}$ and $p \geq 1$, then any function $f \in B_p^{\lambda\theta}$ satisfies the inequality

$$|f|_{B_p^{\lambda\theta}} \leq c\cdot |f|_{B_p^{\lambda\theta}(D)}, \tag{5.1}$$

where c is independent of f.

We have $s_n = \sum_{i=1}^n p_{Q_i}\chi_{Q_i}$. Choose a cube of the smallest size among the Q_i; suppose it belongs to the partition D_N. Since the family D is dyadic, we may assume that the spline s_n is subordinate to the uniform partition D_N, i.e., $s_n \in S_k(D_N)$.

Theorem 3.4 now shows that $s_n \in B_p^\lambda$.

We write the following chain of inequalities:

$$\|s_n\|_{B_p^\lambda} \le c \cdot \|s_n\|_{B_p^\lambda(D)} \le c \cdot n^{\frac{\lambda}{d}} \|s_n\|_q.$$

The first inequality is a consequence of (5.1). The second is true because $s_n \in S_k^n(D)$; therefore, $s_n \in P_k^n(D)$ and the Bernstein inequality for dyadic spaces applies. Theorem 5.1 is proved. □

In order to formulate the next result, we recall the definition of the approximation space $E_{\lambda\theta}(A, X)$ introduced in [25] (see also [26]). Let $A := \{A_n\}_{n=0}^\infty$ be a sequence of subsets of a Banach space X that satisfy $\alpha A_n \subset A_n$ for $\alpha \in R$ and $A_n + A_m \subset A_{n+m}$. A function $f \in X$ is said to belong to $E_{\lambda\theta}(A, X)$ if the following quantity is finite:

$$\|f\|_{E_{\lambda\theta}(A,X)} = \|f\|_X + \left(\sum_{n=1}^\infty \left(n^{\frac{\lambda}{d}} e_n(f)_X \right)^\theta \frac{1}{n} \right)^{\frac{1}{\theta}}, \tag{5.2}$$

where $e_i(f)_X = \operatorname{dist}(f, A_i)_X$.

Theorem 5.2. *Suppose $\lambda = d\left(\frac{1}{p} - \frac{1}{q}\right)$ and $k > \lambda + 2$. Then*

$$B_p^\lambda \supset E_{\lambda p}(A, L_q),$$

where $A = \{S_k^n(D)\}_{n=1}^\infty$. Here $0 < p < q < \infty$ if $d = 1$; $1 < p < q < \infty$ if $d \ne 1$; and $1 < p < \infty$ if $d \ne 1$, $q = \infty$.

Proof. Given μ, d, and q, we define p_μ by $\frac{1}{p_\mu} = \frac{\mu}{d} + \frac{1}{q}$.

A set $G(q, d)$ is said to be *admissible* if $\mu - [\mu] < \frac{1}{p_\mu}$ for $\mu \in G(q, d)$, and $p_\mu > 1$ if $d > 1$. Put $\alpha_q = \frac{1}{q}\min(1, q-1)$ and $\beta_{d,q} = \max\left(0, d - 1 - \left[\frac{d}{q}\right]\right)$. We show that the set

$$G(q,d) = \begin{cases} (0, +\infty) & \text{if } q \ne \infty, \ d = 1, \\ (0, \alpha_q) \cup \left(\beta_{d,q}, d - \frac{d}{q}\right) & \text{if } q \ne \infty, \ d \ne 1, \\ \left(1, 1 + \frac{1}{d}\right) \cup (d-1, d) & \text{if } q = \infty, \ d \ne 1, \end{cases}$$

is admissible.

If $d = 1$ and $q \ne \infty$, then $\mu = \frac{1}{p_\mu} - \frac{1}{q} < \frac{1}{p_\mu}$.

Now, let $d \ne 1$, $q \ne \infty$, and $\mu \in (0, \alpha_q)$. Then $[\mu] \ge 0$ and $\mu - [\mu] < \alpha_q \le \frac{1}{q} < \frac{1}{p_\mu}$. Furthermore, since $\mu < \alpha_q \le \frac{q-1}{q} < d\left(1 - \frac{1}{q}\right)$, we have $\frac{1}{p_\mu} = \frac{\mu}{d} + \frac{1}{q} < 1$ and $p_\mu > 1$.

Now, let $\mu \in \left(\beta_{d,q}, d - \frac{d}{q}\right)$. Then $[\mu] \ge \beta_{d,q}$ and $\mu - [\mu] \le \mu - \left(d - 1 - \left[\frac{d}{q}\right]\right)$. Note that $\mu - \left(d - 1 - \left[\frac{d}{q}\right]\right) = (d-1)\left(\frac{\mu}{d} - 1\right) + \frac{\mu}{d} + \left[\frac{d}{q}\right]$. Since $\mu < d - \frac{d}{q}$, we have

$$\mu - [\mu] < \frac{\mu}{d} - \frac{d}{q} + \frac{1}{q} + \left[\frac{d}{q}\right] \le \frac{1}{p_\mu}.$$

The inequality $p_\mu > 1$ follows from the estimate $\mu < d - \frac{d}{q}$.

Next, let $q = \infty$, $d \ne 1$, and $\mu \in \left(1, 1 + \frac{1}{d}\right)$. Then $[\mu] = 1$ and $\mu - [\mu] < \frac{1}{d} < \frac{\mu}{d} = \frac{1}{p_\mu}$. Since $p_\mu > 1$, it follows that $\mu < d$.

Finally, let $\mu \in (d-1, d)$. Then $[\mu] = d - 1$ and

$$\mu - [\mu] = (d-1)\left(\frac{\mu}{d} - 1\right) + \frac{\mu}{d} < \frac{\mu}{d} = \frac{1}{p_\mu}.$$

Also, $p_\mu = \frac{d}{\mu} > 1$.

Thus, the set $G(q, d)$ is admissible.

Assume now that $q < \infty$. Since $p > 1$, we have $\lambda \in (0, d - \frac{d}{q})$. Given q, d, and λ, we choose nonintegers λ_i, $i = 0, 1$, such that $\lambda_i \in G(q, d)$ and $\lambda_0 < \lambda < \lambda_1$. The structure of $G(q, d)$ makes this possible. Indeed, if $\lambda \in G(q, d)$, we can take $\lambda_0 = \lambda - \varepsilon$ and $\lambda_1 = \lambda + \varepsilon$.

If $\lambda \notin G(q,d)$, we take $\lambda_0 \in (0,\alpha_q)$ and $\lambda_1 \in \left(\beta_{d,q}, d-\frac{d}{q}\right)$. Having chosen λ_i, $i=0,1$, we choose p_{λ_i} so as to ensure the relation $\frac{1}{p_{\lambda_i}} = \frac{\lambda_i}{d} + \frac{1}{q}$. Then the parameters $(d,q,\lambda_i,p_{\lambda_i})$ satisfy the assumptions of Theorem 5.1, and we can write the Bernstein inequality

$$\|s_n\|_{B^{\lambda_i}_{p_i}} \le c_i \cdot n^{\frac{\lambda_i}{d}} \|s_n\|_q. \tag{5.3}$$

We remind the reader that, by the Peetre–Sparr theorem (see [27]), the Bernstein inequality

$$\|a\|_{Y_i} \le \gamma_i n^{\alpha_i} \|a\|_X \quad (a \in A_n)$$

implies that

$$(Y_0, Y_1)_{\theta s} \supset E_{\alpha_\theta s}(A,X).$$

Here $\alpha_\theta = (1-\theta)\alpha_0 + \theta\alpha_1$, $0<\theta<1$.

By (5.3), this result implies that

$$\left(B^{\lambda_0}_{p_0}, B^{\lambda_1}_{p_1}\right)_{\theta s} \supset E_{\lambda_\theta s}(A, L_q). \tag{5.4}$$

We choose θ in such a way that $\lambda_\theta = (1-\theta)\lambda_0 + \theta\lambda_1 = \lambda$ and put $s = p_\theta$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}$. It remains to interpolate between the spaces on the left-hand side of (5.4). For this, we use a result of [19]:

$$\left(B^{\lambda_0}_{p_0}, B^{\lambda_1}_{p_1}\right)_{\theta p_\theta} = B^{\lambda_\theta}_{p_\theta} = B^{\lambda}_{p}.$$

This proves Theorem 5.2 for $q \ne \infty$.

Now let $q=\infty$. First, we consider the case where $1<p<d$, i.e., $\lambda = \frac{d}{p} \in (1,d)$. We choose nonintegers λ_0, λ_1 satisfying $1<\lambda_0<\lambda<\lambda_1<d$ and $\lambda_0,\lambda_1 \in G(\infty,d)$. Put $p_i = \frac{d}{\lambda_i}$, $i=0,1$. Much as in the case of $q<\infty$, it can be shown that

$$B^{\lambda}_{p} \supset E_{\lambda p}(A, L_\infty). \tag{5.5}$$

Next, consider the case where $p \ge d$. Then $\lambda = \frac{d}{p} \le 1$. We choose λ_1 in such a way that $\lambda \le 1 < \lambda_1 < d$. Since $1<\lambda_1<d$, we see that $p_1 = \frac{d}{\lambda_1} \in (1,d)$. We can apply (5.5) for the couple (λ_1, p_1), obtaining

$$\left(L_\infty, B^{\lambda_1}_{p_1}\right)_{\frac{\lambda}{\lambda_1}p} \supset \left(L_\infty, E_{\lambda_1,p_1}(A,L_\infty)\right)_{\frac{\lambda}{\lambda_1}p}. \tag{5.6}$$

After interpolation on the right-hand side, we obtain $E_{\lambda,p}(A,L_\infty)$. We want to replace L_∞ on the left by BMO. Recall that

$$|f|_{BMO} := \sup_{Q\subset Q_0} \frac{E_1(f,Q)_L}{|Q|}$$

(see [28]). Therefore, $BMO \supset L_\infty(Q_0)$, and by (5.6) we have

$$\left(BMO, B^{\lambda_1}_{p_1}\right)_{\frac{\lambda}{\lambda_1}p} \supset E_{\lambda,p}(A,L_\infty).$$

Now, the Peetre–Svensson theorem (see [29]) allows us to calculate the space on the left:

$$\left(BMO, B^{\lambda_1}_{p_1}\right)_{\frac{\lambda}{\lambda_1}p} = B^{\lambda}_{p}.$$

We arrive at

$$B^{\lambda}_{p} \supset E_{\lambda,p}(A,L_\infty),$$

which proves Theorem 5.2 also in the case of $q=\infty$. □

The next result shows that Theorem 5.2 makes redundant a restriction imposed on λ in Theorem 5.1.

Corollary 5.1. *Suppose* $\lambda = d\left(\frac{1}{p} - \frac{1}{q}\right)$. *Then for* $s_n \in S_k^n(D)$ *and* $k > \lambda + 2$ *we have*

$$\|s_n\|_{B_p^\lambda} \le c \cdot n^{\frac{\lambda}{d}} \|s_n\|_q.$$

Here $0 < p < q < \infty$ *if* $d = 1$; $1 < p < q < \infty$ *if* $d \ne 1$; *and* $1 < p < \infty$ *if* $d \ne 1$, $q = \infty$.

Proof. By Theorem 5.2, we have

$$|f|_{B_p^\lambda} \le c \cdot \left(\sum_{i=0}^{\infty} \left(2^{i\lambda} \operatorname{dist}_{L_q}\left(f, S_k^{2^{id}}(D)\right)\right)^p \right)^{\frac{1}{p}}.$$

Since $S_k^n(D) \subset S_k^m(D)$ for $n \le m$, the sum on the right can be replaced with

$$\left(\sum_{i=1}^{\infty} \left(i^{\frac{\lambda}{d}} \operatorname{dist}_{L_q}\left(f, S_k^i(D)\right)\right)^p \frac{1}{i} \right)^{\frac{1}{p}}.$$

We take $f = s_n \in S_k^n(D)$. Then

$$\|s_n\|_{B_p^\lambda} \le c \cdot \left(\sum_{i=1}^{n-1} \left(i^{\frac{\lambda}{d}} \operatorname{dist}_{L_q}\left(s_n, S_k^i(D)\right)\right)^p \frac{1}{i} \right)^{\frac{1}{p}} + \|s_n\|_p.$$

It remains to observe that $\operatorname{dist}_{L_q}(s_n, S_k^i(D)) \le \|s_n\|_q$, $\|s_n\|_p \le c \cdot \|s_n\|_q$ for $p < q$, and

$$\left(\sum_{i=1}^{n-1} i^{\frac{\lambda p}{d} - 1} \right)^{\frac{1}{p}} \le c \cdot n^{\frac{\lambda}{d}}. \qquad \square$$

Remark 5.1. The Bernstein inequality for dyadic Besov spaces that was proved in [1] admits a refinement in the case of $q = \infty$. Namely, it can be shown that

$$|s_n|_{B_p^\lambda(F)} \le c \cdot n^{\frac{\lambda}{d}} |s_n|_{BMO_p^k(F)}, \tag{5.7}$$

where $\lambda = \frac{d}{p}$, $0 < p < \infty$, $s_n \in P_k^n(F)$, $k > \lambda$.

Here the dyadic space $BMO_p^k(F)$ is defined as the set of all $f \in L_p(Q_0)$, $0 < p \le \infty$, for which the following quantity is finite:

$$|f|_{BMO_p^k(F)} := \sup_{Q \subset Q_0, Q \in F} \frac{E_k(f, Q)_p}{|Q|^{\frac{1}{p}}}.$$

In [30] it was shown that

$$|f|_{BMO} \approx \sup_{Q \subset Q_0} \frac{E_k(f, Q)_p}{|Q|^{\frac{1}{p}}},$$

$p \ge 1$. Therefore, $BMO \subset BMO_p^k(F)$ for $p \ge 1$, so that $BMO_p^k(F)$ can be replaced by BMO in (5.7). From the resulting inequality much as in Theorem 5.2, it can be proved that

$$B_p^\lambda \supset E_{\lambda,p}(A, BMO); \tag{5.8}$$

here $\lambda = \frac{d}{p}$, $1 < p < \infty$, $d > 1$.

§6. The K-functional for the couple (L_q, B_p^λ)

We give a statement about the rate of approximation in L_q for functions belonging to B_p^λ.

Theorem 6.1. *Suppose that $\lambda = d\big(\frac{1}{p} - \frac{1}{q}\big)$, $0 < p < q < \infty$, and let $0 < p \le 1$ if $q = \infty$. If $f \in B_p^\lambda$, then there exists a function $s_n \in S_k^{cn}(D)$, $c = c(k,d)$, such that*

$$\|f - s_n\|_q \le c(k,d,p,q) \cdot n^{-\frac{\lambda}{d}} |f|_{B_p^\lambda};$$

here $k > \lambda + 2$.

A result about approximation by splines with free nodes for a "limit" exponent was first obtained by Brudnyĭ [31] in the one-dimensional case. The multidimensional case was announced in [32] (see [33] for a detailed proof). Independently, the same result was obtained in [11] for $0 < p < q < \infty$ and in [12] for $q = \infty$.

Using interpolation (much as in [33]), we can improve Theorem 6.1 as follows.

Theorem 6.2. *If $\lambda = d\big(\frac{1}{p} - \frac{1}{q}\big)$, then*

$$B_p^\lambda \subset E_{\lambda,p}(A, X_q),$$

where

$$X_q = \begin{cases} L_q & \text{if } \ 0 < q < \infty; \\ L_\infty & \text{if } \ q = \infty, \ \ p < 1; \\ BMO & \text{if } \ q = \infty, \ \ p \ge 1, \end{cases}$$

$A = \{S_k^n\}_{n=1}^\infty$, $k > \lambda + 2$.

Together with Theorem 5.2 (see also Remark 5.1), Theorem 6.2 makes it possible to describe the space B_p^λ in terms of nonlinear approximation in L_q. We use this to calculate the K-functional for the couple (L_q, B_p^λ).

Theorem 6.3. *Suppose that $\lambda = d\big(\frac{1}{p} - \frac{1}{q}\big)$, and $1 \le p < q < \infty$ for $d \ne 1$ and $0 < p < q < \infty$ for $d = 1$. Then*

$$K(f, 2^{-n\lambda}, L_q, B_p^\lambda) \approx 2^{-n\lambda} \cdot \left(\sum_{i=-1}^{n} \big(2^{i\lambda} e_i(f)_q\big)^p \right)^{\frac{1}{p}};$$

here $e_i(f)_q := \operatorname{dist}(f, S_k^{2^{id}})_{L_q}$, $e_{-1}(f)_q = \|f\|_q$, $k > \lambda + 2$.

Proof. Put

$$M(f) := 2^{-n\lambda} \cdot \left(\sum_{i=-1}^{n} \big(2^{i\lambda} e_i(f)_q\big)^p \right)^{\frac{1}{p}}.$$

Since $e_i(f)_q \le \|f\|_q$, we have $M(f_0) \le c \cdot \|f_0\|_q$. On the other hand, Theorem 6.2 shows that $M(f_1) \le c \cdot 2^{-n\lambda} \|f_1\|_{B_p^\lambda}$. Thus, for $f = f_0 + f_1$ we obtain

$$M(f) \le c \cdot (M(f_0) + M(f_1)) \le c \cdot \Big(\|f_0\|_q + 2^{-n\lambda} \|f_1\|_{B_p^\lambda} \Big).$$

Taking the infimum over all representations of f in the form $f = f_0 + f_1$, we arrive at

$$M(f) \le c \cdot K(f, 2^{-n\lambda}, L_q, B_p^\lambda).$$

We prove the reverse inequality. By the definition of the K-functional, we have

$$K(f, 2^{-n\lambda}, L_q, B_p^\lambda) \le \|f - s_n\|_q + 2^{-n\lambda} \|s_n\|_{B_p^\lambda}. \tag{6.1}$$

Let $s_n \in S_k^{2^{nd}}$ satisfy $\|f - s_n\|_q = e_n(f)_q$. Theorem 5.2 implies the estimate

$$\|s_n\|_{B_p^\lambda} \le c \cdot \left(\sum_{i=-1}^{n-1} \left(2^{i\lambda} e_i(s_n)_q\right)^p \right)^{\frac{1}{p}}. \tag{6.2}$$

We have used the fact that $s_n \in S_k^{2^{id}}$ for $i \ge n$. We can continue (6.2) by taking into account the fact that $e_i(f - s_n)_q \le e_n(f)_q$. Then $e_i(s_n)_q \le c \cdot e_i(f)_q$ for $i \le n$. Together with (6.2), this yields

$$\|s_n\|_{B_p^\lambda} \le c \cdot 2^{n\lambda} M(f). \tag{6.3}$$

Taken together, (6.3) and (6.1) imply $K(f, 2^{-n\lambda}, L_q, B_p^\lambda) \le c \cdot M(f)$. We have used the inequality $\|f - s_n\|_q \le M(f)$. □

Remark 6.1. Employing Remark 5.1 (inequality (5.8)) in place of Theorem 5.2, we can estimate the order of the K-functional for the couple (BMO, B_p^λ), $\lambda = \frac{d}{p}$. For this, in Theorem 6.3 we replace L_q by BMO and take $p \in (1, \infty)$, $d > 1$.

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