FRACTIONAL MOMENTS OF AUTOMORPHIC $L$-FUNCTIONS

O. M. FOMENKO

Abstract. Upper and lower bounds for fractional moments of automorphic $L$-functions are found.

§1. Introduction

Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{(\text{Re}(s) > 1)}$$

be the Riemann zeta function. Let $k$ be a nonnegative real number, and let $T \geq 2$. Heath-Brown \[1\] investigated the behavior as $T \to \infty$ of the mean values

$$I_k(T) = \int_1^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} dt.$$ 

Consider the inequalities

(1.1) \hspace{1cm} I_k(T) \gg T(\log T)^{k^2}

and

(1.2) \hspace{1cm} I_k(T) \ll T(\log T)^{k^2}.

Heath-Brown proved the following result.

Theorem A (Heath-Brown \[1\]).

(i) If $k \geq 0$ is rational, then (1.1) holds.

(ii) If $k = 1/n > 0$, where $n$ is an integer, then (1.2) holds.

(iii) Under the Riemann hypothesis, (1.1) holds for all real $k \geq 0$.

(iv) Under the Riemann hypothesis, (1.2) holds for $0 \leq k \leq 2$.

Note that for $I_1(T)$ and $I_2(T)$, asymptotics are known; these classical facts were proved by Hardy and Littlewood and by Ingham, respectively.

Our purpose in the present paper is to transfer the result of Heath-Brown to some automorphic $L$-functions.

Consider the space $S_\kappa(\Gamma)$ of holomorphic cusp forms

$$f(z) = \sum_{n=1}^{\infty} a_f(n) \exp(2\pi i n z)$$

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of even weight \( \kappa \geq 12 \) with respect to the full modular group \( \Gamma = \text{SL}(2, \mathbb{Z}) \), \( z = x + iy \), \( y > 0 \); let \( S_\kappa(\Gamma)^+ \) be the set of all Hecke eigencuspforms with \( a_f(1) = 1 \) in this space. In the sequel it is always assumed that \( f \in S_\kappa(\Gamma)^+ \). Set

\[
\lambda_f(n) = a_f(n)/n^{\frac{\kappa-1}{2}};
\]

then \( \lambda_f(n) \) is an eigenvalue of the Hecke operator \( T_n \), \( n = 1, 2, 3, \ldots \). By the classical result by Deligne, for any prime \( p \) we have

\[
\lambda_f(p) = \alpha_p + \overline{\alpha}_p, \quad \alpha_p \overline{\alpha}_p = 1.
\]

We consider the following automorphic \( L \)-functions, defined initially on the half-plane \( \text{Re}(s) > 1 \):

1. \( L(s, f) = \prod_p (1 - \alpha_p p^{-s})^{-1}(1 - \overline{\alpha}_p p^{-s})^{-1} = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} \)
   (the Hecke \( L \)-function of \( f \));

2. \( L(s, \text{sym}^2 f) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \overline{\alpha}_p^2 p^{-s})^{-1} \)
   \(= \zeta(2s) \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s} = \sum_{n=1}^{\infty} c_n n^{-s} \)
   (the Shimura symmetric square \( L \)-function of \( f \));

3. \( L(s, f \times f) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1}(1 - p^{-s})^{-2}(1 - \overline{\alpha}_p^2 p^{-s})^{-1} \)
   \(= \zeta(2s) \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s} = \sum_{n=1}^{\infty} d_n n^{-s} \)
   (the \( L \)-function of the Rankin–Selberg convolution of \( f \) with itself).

The \( L \)-functions introduced above admit analytic continuation to the entire \( s \)-plane; the functions \( L(s, f) \) and \( L(s, \text{sym}^2 f) \) are entire, and \( L(s, f \times f) \) has a simple pole at \( s = 1 \) as a unique singularity. Each of these functions obeys a functional equation relating its values at \( s \) and \( 1 - s \). Below (up to §5) we need the functional equation for \( L(s, \text{sym}^2 f) \):

\[
\pi^{-3s/2} \Gamma \left( \frac{s}{2} + \frac{\kappa-1}{2} \right) \Gamma \left( \frac{s}{2} + \frac{\kappa-1}{2} \right) L(s, \text{sym}^2 f) = \pi^{-3(1-s)/2} \Gamma \left( \frac{1-s}{2} + \frac{\kappa-1}{2} \right) \Gamma \left( \frac{1-s}{2} + \frac{\kappa-1}{2} \right) L(1-s, \text{sym}^2 f).
\]

We introduce the mean value

\[
I_k(T, \text{sym}^2 f) = \int_1^{T} \left| L \left( \frac{1}{2} + it, \text{sym}^2 f \right) \right|^{2k} dt
\]

and, similarly, the mean values of \( L(s, f) \), \( L(s, f \times f) \), and the Dedekind zeta function \( \zeta_{K_3}(s) \) of a cubic field \( K_3 \). Below (up to §5) we deal only with the \( L \)-function \( L(s, \text{sym}^2 f) \).

**Theorem 1.**

(i) If \( k \geq 0 \) is rational, then

\[
I_k(T, \text{sym}^2 f) \gg T(\log T)^k.
\]

(ii) Under the Riemann hypothesis for \( L(s, \text{sym}^2 f) \), estimate (1.3) is true for all real \( k \geq 0 \).
(iii) Under the Riemann hypothesis for $L(s, \text{sym}^2 f)$, we have

$$I_k(T, \text{sym}^2 f) \ll T(\log T)^k$$

for $0 \leq k < \frac{2}{3}$.

The proof is presented in §§2–4 and involves the approach of [1] (see also [2]). Similar results for the $L$-functions $L(s, f)$, $L(s, f \times f)$, and $\zeta_K^3(s)$ will be considered in §§5 and 6.

All constants implied by the $\ll$ (or $\gg$) notation may depend on $k$ (and on $f$, which is regarded as fixed) unless otherwise indicated. However, the constants are independent of the variable $\sigma$ introduced in §3. The symbols $a_1, a_2, a_3, \ldots$ will denote positive constants; $\varepsilon > 0$ is an arbitrarily small fixed number.

§2. Dirichlet polynomials

A multiplicative function $d_k(n)$ arises for real $k \geq 0$ from the relation

$$\zeta^k(s) = \prod_p (1 - p^{-s})^{-k} = \sum_{n=1}^\infty d_k(n)n^{-s} \quad (\text{Re}(s) > 1)$$

(for the details, see [1]). We have the formula

$$d_k(n) = \prod_{p^n || n} \frac{k(k+1) \cdots (k+1-1)}{\alpha!}.$$ 

We fix a branch of $\log L(s, \text{sym}^2 f)$:

$$\log L(s, \text{sym}^2 f) = \sum_p \sum_{j=1}^\infty \frac{(\alpha_p^{2j} + 1 + \alpha_p^{-2j})}{j p^{js}} \quad (\text{Re}(s) > 1)$$

and then use the formula

$$L^k(s, \text{sym}^2 f) = \exp\{k \log L(s, \text{sym}^2 f)\} \quad (\text{Re}(s) > 1)$$

to choose a branch of $L^k(s, \text{sym}^2 f)$.

Let the coefficients $\nu(n, k)$ be defined by the expansion

$$L^k(s, \text{sym}^2 f) = \sum_{n=1}^\infty \nu(n, k)n^{-s} \quad (\text{Re}(s) > 1).$$

For $\text{Re}(s) > 1$ we also have

$$L^k(s, \text{sym}^2 f) = \prod_p \sum_{m \geq 0, n \geq 0, l \geq 0} \frac{d_k(p^m)d_k(p^n)d_k(p^l)\alpha_p^{2m} \alpha_p^{-2n}}{p^{(m+n+l)s}},$$

which gives the inequality

$$|\nu(n, k)| \leq d_{3k}(n).$$

Consequently,

$$\nu(n, k) \ll n^\varepsilon.$$

The principal result of this section is as follows.

**Lemma 1.** For any fixed real $k \geq 0$, there exists $c'_k > 0$ such that

$$\left(\sigma - \frac{1}{2}\right)^{-k^2} \ll \sum_{n=1}^N \nu(n, k)^2 n^{-2\varepsilon} \ll \left(\sigma - \frac{1}{2}\right)^{-k^2}$$

uniformly for

$$\frac{1}{2} + \frac{c'_k}{\log N} \leq \sigma \leq 1.$$
Moreover,

\[(\log N)^k \ll \sum_{n=1}^{N} \nu(n, k)^2 n^{-1} \ll (\log N)^k.\]

We shall use the following result by Odoni.

**Theorem B** (Odoni [3]). Let \(f : \mathbb{N} \to \mathbb{R}\) be multiplicative, with \(f(n) \geq 0\) for all \(n \in \mathbb{N}\). Suppose that there exist constants \(a_1, a_2 > 1\) such that \(0 \leq f(p^r) \leq a_1 n^{a_2}\) for all \(n\), where \(p\) is a prime integer. Suppose also that there are constants \(c_0 > 0\) and \(0 < \gamma < 1\) such that

\[\sum_{p \leq x} f(p) = c_0 x (\log x)^{-1} + O(x (\log x)^{-1-\gamma})\]

as \(x \to \infty\). Then there is a constant \(A_0 > 0\) such that

\[\sum_{n \leq x} n^{-1} f(n) = A_0 (\log x)^{c_0} + O((\log x)^{c_0 - \gamma/2}).\]

We pass to the proof of Lemma 1. With every form \(f \in S_{\mu}(\Gamma)^+\), we associate a form \(F\) on \(GL(3)\) with the Fourier coefficients \((m, n)\) and the \(L\)-function

\[L(s, F) = \sum_{n=1}^{\infty} a(1, n)n^{-s}\]

such that

\[L(s, F) = L(s, \text{sym}^2 f).\]

The form \(F\) is called the Gelbart–Jacquet lift (the adjoint square lift) of \(f\); the existence of it was established in [4]. Let \(L(s, F \times F)\) be the \(L\)-function of the Rankin–Selberg convolution of the form \(F\) with itself. Recall that

\[L(s, F) = \prod_p (1 - \alpha p^{-s})^{-1}(1 - \bar{\alpha} p^{-s})^{-1}\]

(for simplicity, we set \(\alpha_p = \alpha\); then \(L(s, F \times F)\) has the Euler product

\[L(s, F \times F) = \prod_p (1 - \alpha^4 p^{-s})^{-1}(1 - \alpha^2 p^{-s})^{-1}(1 - p^{-s})^{-1}\]

\[\times (1 - \alpha^2 p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \bar{\alpha}^2 p^{-s})^{-1}(1 - p^{-s})^{-1}\]

\[\times (1 - \bar{\alpha}^2 p^{-s})^{-1}(1 - \bar{\alpha}^4 p^{-s})^{-1}.\]

The \(L\)-function \(L(s, F \times F)\) has a simple pole at \(s = 1\), and is analytic elsewhere, and obeys a functional equation relating its values at \(s\) and \(1 - s\) (see [5]).

Consider the function

\[T(s) = \sum_{n=1}^{\infty} c_n^2 n^{-s}.\]

It is easily seen that

\[T(s) = L(s, F \times F) \cdot B(s),\]

where

\[B(s) = \prod_p X_p(p^{-s}),\]

\[X_p(t) = 1 + A_2 t^2 + A_3 t^3 + \cdots.\]

Using methods of the theory of distribution of primes, we show that there is a region

\[\sigma > 1 - C/ \log(|t| + 2), \quad C > 0,\]
that is free from zeros of $L(s, F \times F)$. This yields the asymptotic formula
\[
\sum_{p \leq x} c_p^2 \log p = x + O(x \exp(-D \sqrt{\log x})), \quad D > 0, \quad x \to \infty,
\]
which shows that
\begin{equation}
\sum_{p \leq x} c_p^2 = x \log^{-1} x + O(x \log^{-2} x).
\end{equation}

Since $\nu(p, k) = kc_p$, we have
\[
\sum_{p \leq x} \nu(p, k)^2 = k^2 x \log^{-1} x + O(k^2 x \log^{-2} x).
\]

By Theorem B,
\begin{equation}
\sum_{n \leq x} n^{-1} \nu(n, k)^2 = \frac{\exp(B_n k^2)}{\Gamma(k^2 + 1)} \cdot P \cdot (\log x)^{k^2} + O((\log x)^{k^2 - \gamma/2}),
\end{equation}
where $0 < \gamma < 1$ is an arbitrary but fixed number, and
\[
P = \prod_p \left\{ e^{-\nu(p, k)p^{-1}} \left( 1 + \nu(p, k)p^{-1} + \nu(p^2, k)p^{-2} + \cdots \right) \right\}.
\]
The constant of the leading term was calculated with the help of Wirsing’s result [6].

Now, the first part of Lemma 1 follows easily from (2.2) by summation by parts.

\section*{3. Convexity estimates}

The basis of the further considerations is the following result by Gabriel.

**Theorem C** (Gabriel [7]). Let $F(s)$ be regular in the infinite strip $\alpha < \Re(s) < \beta$, and continuous for $\alpha \leq \Re(s) \leq \beta$. Suppose $F(s) \to 0$ as $|\Im(s)| \to \infty$ uniformly for $\alpha \leq \Re(s) \leq \beta$. Then for $\alpha \leq \gamma \leq \beta$ and any $q > 0$ we have
\[
\int_{-\infty}^{\infty} |F(\gamma + it)|^q \, dt \leq \left\{ \int_{-\infty}^{\infty} |F(\alpha + it)|^q \, dt \right\}^{\frac{q - \gamma}{q - \alpha}} \left\{ \int_{-\infty}^{\infty} |F(\beta + it)|^q \, dt \right\}^{\frac{\gamma - \alpha}{q - \alpha}}.
\]

Consider some applications of Theorem C. Put
\[
w(t, T) = \int_{-T}^{2T} \exp\{-2k(t - \tau)^2\} \, d\tau,
\]
\[
J(\sigma, T) = \int_{-\infty}^{\infty} |L(\sigma + it, \text{sym}^2 f)|^{2k} w(t, T) \, dt.
\]

In Theorem C, we take
\[
F(s) = L(s, \text{sym}^2 f) \exp\{(s - i\tau)^2\}, \quad \gamma = \frac{1}{2}, \quad \alpha = 1 - \sigma, \quad \beta = \sigma, \quad \text{where} \quad \frac{1}{2} \leq \sigma \leq \frac{3}{4},
\]
\[
q = 2k > 0 \quad \text{and} \quad \tau \geq 2.
\]

By the functional equation and convexity,
\begin{equation}
L(\alpha + it, \text{sym}^2 f) \ll |L(\beta + it, \text{sym}^2 f)|(1 + |t|)^{3(\sigma - 1/2)},
\end{equation}
\begin{equation}
L(s, \text{sym}^2 f) \ll (|t| + 1)^{3/4 + \varepsilon} \quad (\Re(s) \geq \frac{1}{2}).
\end{equation}
Using (3.1), we obtain
\[ \int_{-\infty}^{\infty} |F(\alpha + it)|^{2k} \, dt \]
\[ \ll \left( \int_{-\infty}^{\tau/2} + \int_{\tau/2}^{3\tau/2} + \int_{3\tau/2}^{\infty} \right) |L(\sigma + it, \text{sym}^2 f)|^{2k} (1 + |t|)^{6k(\sigma-1/2)} \exp\{-2k(t - \tau)^2\} \, dt \]
\[ \ll \tau^{6k(\sigma-1/2)} \int_{-\infty}^{\infty} |L(\sigma + it, \text{sym}^2 f)|^{2k} \exp\{-2k(t - \tau)^2\} \, dt + \exp(-a_5 k \tau^2). \]

Lemma 3 and the above estimate yield
\[ \int_{-\infty}^{\infty} \left| L\left( \frac{1}{2} + it, \text{sym}^2 f \right) \right|^{2k} \exp\{-2k(t - \tau)^2\} \, dt \]
\[ \ll \tau^{3k(\sigma-1/2)} \int_{-\infty}^{\infty} |L(\sigma + it, \text{sym}^2 f)|^{2k} \exp\{-2k(t - \tau)^2\} \, dt + \exp(-a_4 k \tau^2). \]

Finally, integrating over \( T \leq \tau \leq 2T \), we get the following.

**Lemma 2.** Suppose \( \frac{1}{2} \leq \sigma \leq \frac{3}{4}, \, k > 0, \text{ and } T \geq 2 \). Then
\[ J\left( \frac{1}{2}, T \right) \ll T^{3k(\sigma-1/2)} J(\sigma, T) + \exp(-a_5 k T^2). \]

Now we take \( F(s) = L(s, \text{sym}^2 f) \exp\{(s - i\tau)^2\}, \gamma = \sigma, \, \alpha = \frac{1}{2}, \, \beta = \frac{3}{2}, \, q = 2k > 0, \)
where \( \frac{1}{2} \leq \sigma \leq \frac{3}{4} \) and \( \tau \geq 2 \). Then
\[ \int_{-\infty}^{\infty} \left| F\left( \frac{1}{2} + it \right) \right|^{2k} \, dt \]
\[ \ll \int_{\tau/2}^{3\tau/2} \left| L\left( \frac{1}{2} + it, \text{sym}^2 f \right) \right|^{2k} \exp\{-2k(t - \tau)^2\} \, dt + \exp(-a_6 k \tau^2) \]
and
\[ \int_{-\infty}^{\infty} \left| F\left( \frac{3}{2} + it \right) \right|^{2k} \ll 1. \]

Theorem C implies that
\[ \int_{-\infty}^{\infty} |F(\sigma + it)|^{2k} \, dt \]
\[ \ll \left\{ \int_{-\infty}^{\infty} \left| L\left( \frac{1}{2} + it, \text{sym}^2 f \right) \right|^{2k} \exp\{-2k(t - \tau)^2\} \, dt \right\}^{3/2-\sigma} + \exp(-a_7 k \tau^2). \]

However,
\[ \int_{-\infty}^{\infty} |L(\sigma + it, \text{sym}^2 f)|^{2k} \exp\{-2k(t - \tau)^2\} \, dt \ll \int_{-\infty}^{\infty} |F(\sigma + it)|^{2k} \, dt + \exp(-a_8 k \tau^2). \]

Combining this with (3.2), integrating over \( T \leq \tau \leq 2T \), and using Hölder’s inequality, we obtain the following statement.

**Lemma 3.** Suppose \( \frac{1}{2} \leq \sigma \leq \frac{3}{4}, \, k > 0, \text{ and } T \geq 2 \). Then
\[ J(\sigma, T) \ll T^{\sigma-1/2} J\left( \frac{1}{2}, T \right)^{3/2-\sigma} + \exp(-a_9 k T^2). \]
We need some new notation. We write \( k = u/v \), where, in the case of Theorem 1 (i), \( u \) and \( v \) are positive coprime integers, in the case of (ii) \( v = 1 \) and \( u = k \) is any positive real number, and in the case of (iii) \( v = 1 \) and \( 0 < u = k < 2/3 \). We may assume that \( k > 0 \), because for \( k = 0 \), the theorem is trivial.

Now we define \( N = T^{1/2} \) in the cases of (i) and (ii), and \( N = T^{(10 + 3k)/12} \) in the case of (iii). We introduce the following quantities:

\[
S_N(s) = \sum_{n \leq N} \nu(n, k)n^{-s},
\]

\[
g(s) = L(s, \text{sym}^2 f)^n - S_N^2(s),
\]

\[
K(\sigma, T) = \int_{-\infty}^{\infty} |g(\sigma + it)|^{2/v} w(t, T) \, dt.
\]

In the cases of (ii) and (iii) the Riemann hypothesis for \( L(s, \text{sym}^2 f) \) (the extended Riemann hypothesis, ERH for short) is assumed to ensure analytic continuation of \( L(s, \text{sym}^2 f)^n \) into the half-plane \( \text{Re}(s) > \frac{1}{2} \).

For the proof of the following lemma we need the Montgomery–Vaughan mean value theorem.

**Theorem D** (Montgomery and Vaughan [8]). If \( \{b_n\} \) is a sequence of complex numbers such that \( \sum_{n \geq 1} n|b_n|^2 \) is convergent, then

\[
\int_0^T \left| \sum_{n=1}^{\infty} b_n n^{-it} \right|^2 dt = T \sum_{n=1}^{\infty} |b_n|^2 + O \left( \sum_{n=1}^{\infty} n|b_n|^2 \right),
\]

where the implied constant is absolute.

**Lemma 4.** Suppose \( \frac{1}{2} \leq \sigma \leq \frac{3}{4} \), \( k = u/v > 0 \), and \( T \geq 2 \). Then

\[
K(\sigma, T) \ll K \left( \frac{1}{2}, T \right)^{(5 - 4\sigma)/3} \left\{ (TN^{(\sigma - 3/2)/v})^{(4\sigma - 2)/3} + \exp \left\{ -a_{10}kT^2(2\sigma - 1) \right\} \right\},
\]

where, in the cases of (ii) and (iii), ERH is assumed.

For the proof, in Theorem C we take

\[
F(s) = g(s) \exp \left\{ u(s - i\tau)^2 \right\},
\]

\[
\gamma = \sigma, \quad \alpha = \frac{1}{2}, \quad \beta = \frac{5}{4}, \quad \text{and} \quad q = \frac{2}{v}.
\]

Observe that

\[
\int_{-\infty}^{\infty} |F(\sigma + it)|^{2/v} dt \leq \left\{ \int_{-\infty}^{\infty} \left| F\left( \frac{1}{2} + it \right) \right|^{2/v} dt \right\}^{(5 - 4\sigma)/3} \times \left\{ \int_{-\infty}^{\infty} \left| F\left( \frac{5}{4} + it \right) \right|^{2/v} dt \right\}^{(4\sigma - 2)/3}.
\]

By (3.1), for \( \text{Re}(s) \geq \frac{1}{2} \) we have

\[
g(s) \ll (T + |t|)^{u + v}.
\]

Consequently,

\[
\int_{-\infty}^{\infty} \left| F\left( \frac{5}{4} + it \right) \right|^{2/v} dt \ll \int_{\tau/2}^{3\tau/2} \left| F\left( \frac{5}{4} + it \right) \right|^{2/v} dt + T^{2 + 2k} \exp(-a_{11}kT^2).
\]
Therefore, (3.3) and the above inequality imply that
\[
\int_{-\infty}^{\infty} |F(\sigma + it)|^{2/v} dt \\
\ll \left\{ \int_{-\infty}^{\infty} \left| F\left(\frac{1}{2} + it\right)\right|^{2/v} dt \right\}^{(5-4\sigma)/3} \left\{ \int_{\tau/2}^{3\tau/2} \left| F\left(\frac{5}{4} + it\right)\right|^{2/v} dt \right\}^{(4\sigma-2)/3} \\
+ \left\{ \int_{-\infty}^{\infty} \left| F\left(\frac{1}{2} + it\right)\right|^{2/v} dt \right\}^{(5-4\sigma)/3} \left\{ T^{2+2k} \exp(-a_{12}k\tau^2) \right\}^{(4\sigma-2)/3}.
\]

We integrate over \( T \leq \tau \leq 2T \) and apply Hölder’s inequality, obtaining
\[
K(\sigma,T) \ll K\left(\frac{1}{2},T\right)^{(5-4\sigma)/3} \\
\times \left\{ \int_{T}^{2T} \int_{\tau/2}^{3\tau/2} \left| g\left(\frac{5}{4} + it\right)\right|^{2/v} \exp\{-2k(t - \tau)^2\} dt d\tau \right\}^{(4\sigma-2)/3} \]
\[
+ K\left(\frac{1}{2},T\right)^{(5-4\sigma)/3} \exp\{-a_{13}kT^2(2\sigma - 1)\} \]
\[
\ll K\left(\frac{1}{2},T\right)^{(5-4\sigma)/3} \left\{ \int_{T/2}^{3T} \left| g\left(\frac{5}{4} + it\right)\right|^{2/v} dt \right\}^{(4\sigma-2)/3} \\
+ K\left(\frac{1}{2},T\right)^{(5-4\sigma)/3} \exp\{-a_{13}kT^2(2\sigma - 1)\}.
\]

First, we consider cases (ii) and (iii). We have
\[
g\left(\frac{5}{4} + it\right) = L^k\left(\frac{5}{4} + it, \text{sym}^2 f\right) - S_N\left(\frac{5}{4} + it\right) = \sum_{n>N} \frac{\nu(n,k)}{n^{5/4+it}}.
\]
By Theorem D,
\[
\int_{T/2}^{3T} \left| g\left(\frac{5}{4} + it\right)\right|^{2/v} dt \ll T \sum_{n=N}^{\infty} \nu(n,k)^2 n^{-5/2} + \sum_{n=N}^{\infty} \nu(n,k)^2 n^{-3/2} \ll TN^{\varepsilon-3/2}.
\]
In the case of (i),
\[
g(s) = \left\{ \sum_{n=1}^{\infty} \nu(n,u/v)n^{-s} \right\}^{v} - \left\{ \sum_{n=1}^{N} \nu(n,u/v)n^{-s} \right\}^{v} \\
= \sum_{n>N} a_n n^{-s} \quad (\text{Re}(s) > 1); \quad 0 \leq |a_n| \leq d_{3N}(n).
\]
We have
\[
\int_{T/2}^{3T} \left| g\left(\frac{5}{4} + it\right)\right|^{2/v} dt \ll TN^{(\varepsilon-3/2)/v}.
\]
Now, to prove Lemma 4 it remains to use (3.4).

\section{4. Proof of Theorem 1}

We define
\[
L(\sigma,T) = \int_{-\infty}^{\infty} |S_N(\sigma + it)|^2 \nu(t,T) dt
\]
for \( \frac{1}{2} \leq \sigma \leq \frac{3}{4} \). Note (see [1]) that
\[
\nu(t,T) \ll \exp\{- (T^2 + t^2)k/18\}
\]
for \( t \leq 0 \) and \( t \geq 3T \), whence

\[
L(\sigma, T) = \int_0^{3T} |S_N(\sigma + it)|^2 w(t, T) \, dt + O(1).
\]

Moreover, \( w(t, T) \ll 1 \) for all \( t \), and \( w(t, T) \gg 1 \) for \( 4T/3 \leq t \leq 5T/3 \). Theorem D yields

\[
\int_0^{3T} |S_N(\sigma + it)|^2 \, dt \ll T \sum_{n=1}^N \nu(n, k)^2 n^{-2\sigma}
\]

and

\[
\int_{4T/3}^{5T/3} |S_N(\sigma + it)|^2 \, dt \gg T \sum_{n=1}^N \nu(n, k)^2 n^{-2\sigma}.
\]

Hence, Lemma 1 shows that

\[
L(\sigma, T) \ll \int_0^{3T} |S_N(\sigma + it)|^2 \, dt + O(1).
\]

Lemma 5. In the cases of (i) and (ii) we have the estimate

\[
J\left(\frac{1}{2}, T\right) \gg T(\log T)^{k^2},
\]

where in case (ii), ERH is assumed. In the case of (iii), under ERH we have the estimate

\[
J\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2}.
\]

We start the proof of the first part of the lemma with the trivial inequality

\[
|S_N(\sigma)|^{2/\nu} \ll |L(s, \text{sym}^2 f)|^{2k} + |g(s)|^{2/\nu},
\]

whence it follows that

\[
L(\sigma, T) \ll J(\sigma, T) + K(\sigma, T).
\]

Similarly,

\[
J(\sigma, T) \ll K(\sigma, T) + L(\sigma, T)
\]

and

\[
K\left(\frac{1}{2}, T\right) \ll L\left(\frac{1}{2}, T\right) + J\left(\frac{1}{2}, T\right).
\]

First, let \( K\left(\frac{1}{2}, T\right) \leq T \). Then, by (4.2) and (4.3) with \( \sigma = \frac{1}{2} \), we have

\[
T(\log T)^{k^2} \ll L\left(\frac{1}{2}, T\right) \ll J\left(\frac{1}{2}, T\right) + K\left(\frac{1}{2}, T\right) \ll J\left(\frac{1}{2}, T\right) + T,
\]

and the assertion is proved.

Now, let \( K\left(\frac{1}{2}, T\right) > T \). By Lemma 4,

\[
K(\sigma, T) \ll K\left(\frac{1}{2}, T\right) N^{-2\lambda(\sigma-1/2)/\nu} + T(5^{4\sigma}/3)^{3} \exp\{-a_{10}kT^2(2\sigma - 1)\}
\]

\[
\ll K\left(\frac{1}{2}, T\right) N^{-2\lambda(\sigma-1/2)/\nu},
\]

\[
\ll K\left(\frac{1}{2}, T\right) N^{-2\lambda(\sigma-1/2)/\nu},
\]

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where we have set
\[ \lambda = 1 - \frac{2\varepsilon}{3}. \]
Recall that \( N = T^{1/2} \). Relations (4.3), (4.6), and (4.5) imply that
\[
L(\sigma, T) \ll J(\sigma, T) + K(\sigma, T) \ll J(\sigma, T) + K\left(\frac{1}{2}, T\right) T^{-\lambda \sigma^{1/2}/v}.
\]
\[
\ll J(\sigma, T) + \left\{ L\left(\frac{1}{2}, T\right) + J\left(\frac{1}{2}, T\right) \right\} T^{-\lambda \sigma^{1/2}/v}.
\]
Thus, either
\[
L(\sigma, T) \ll L\left(\frac{1}{2}, T\right) T^{-\lambda \sigma^{1/2}/v} \tag{4.7}
\]
or
\[
L(\sigma, T) \ll J(\sigma, T) + J\left(\frac{1}{2}, T\right) T^{-\lambda \sigma^{1/2}/v}. \tag{4.8}
\]
We write \( \sigma - \frac{1}{2} = c(\log T)^{-1} \). Then (4.7) and (4.1), (4.2) yield
\[
T \left( \sigma - \frac{1}{2} \right)^{-k^2} = c^{-k^2} T(\log T)^{k^2} \ll L(\sigma, T)
\]
\[
\ll \left\{ L\left(\frac{1}{2}, T\right) T^{-\lambda \sigma^{1/2}/v} \ll T(\log T)^{k^2} \exp(-\lambda c/v),
\]
whence
\[ \exp(\lambda c/v) \leq C_1 c^{k^2} \]
with \( C_1 = C_1(k) \). Since this fails if \( c = c(k) \) is sufficiently large, the alternative (4.8) holds true. Inequality (4.8) and Lemma 3 imply that
\[
L(\sigma, T) \ll T^{-1/2} J\left(\frac{1}{2}, T\right)^{3/2 - \sigma} + J\left(\frac{1}{2}, T\right) T^{-\lambda \sigma^{1/2}/v}. \]
With this value of \( c \) in \( \sigma - \frac{1}{2} = c(\log T)^{-1} \), the above inequality and (4.1) give
\[
T(\log T)^{k^2} c^{-k^2} \ll L(\sigma, T)
\]
\[
\ll J\left(\frac{1}{2}, T\right) \left\{ e^{c} J\left(\frac{1}{2}, T\right)^{-c(\log T)^{-1}} + \exp(-\lambda c/v) \right\} \ll J\left(\frac{1}{2}, T\right),
\]
whence
\[ T(\log T)^{k^2} \ll J\left(\frac{1}{2}, T\right), \]
which proves the first part of the lemma.

We pass to the proof of the second part of the lemma. If \( K\left(\frac{1}{2}, T\right) \leq T \), then, by (4.4) with \( \sigma = \frac{1}{2} \) and (4.2),
\[
J\left(\frac{1}{2}, T\right) \ll L\left(\frac{1}{2}, T\right) + K\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2} + T \ll T(\log T)^{k^2},
\]
and the assertion is proved.

Now, let \( K\left(\frac{1}{2}, T\right) > T \). Recall that \( N = T^{(10+3k)/12} \).

Using (4.4), (4.6), and (4.5) we get
\[
J(\sigma, T) \ll L(\sigma, T) + K(\sigma, T)
\]
\[
\ll L(\sigma, T) + \left\{ L\left(\frac{1}{2}, T\right) + J\left(\frac{1}{2}, T\right) \right\} T^{-\lambda (\sigma-1/2)(10+3k)/6}.
\]
Thus, either

\begin{equation}
J(\sigma, T) \ll J\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma - 1/2)(10 + 3k)/6}
\end{equation}

or

\begin{equation}
J(\sigma, T) \ll L(\sigma, T) + L\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma - 1/2)(10 + 3k)/6}.
\end{equation}

Take \( \sigma - 1/2 = c(\log T)^{-1} \). With the help of (4.9) and Lemma 2, we obtain

\begin{equation}
J\left(\frac{1}{2}, T\right) \ll J\left(\frac{1}{2}, T\right) T^{3k(\sigma - 1/2) - \lambda(\sigma - 1/2)(10 + 3k)/6}
= J\left(\frac{1}{2}, T\right) \exp\{c(3k - \lambda k/2 - 5\lambda/3)\}.
\end{equation}

Since \( \lambda = 1 - \frac{2\varepsilon}{3} \) may be chosen arbitrarily close to 1, for \( 0 < k < \frac{2}{3} \) and sufficiently large \( c \), we get a contradiction. Now, combining (4.10) with (4.1), (4.2), and Lemma 2, we prove the second part of Lemma 5.

At this point, it is easy to prove Theorem 1. The properties of \( w(t, T) \) imply the inequality

\begin{equation}
J\left(\frac{1}{2}, T\right) \ll I_k(3T, \text{sym}^2 f) + \exp(-a_{14}KT^2),
\end{equation}

which gives (1.3) with the help of Lemma 5. Next, the properties of \( w(t, T) \) and Lemma 5 give

\begin{equation}
I_k(5T/3, \text{sym}^2 f) - I_k(4T/3, \text{sym}^2 f) \ll J\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2}.
\end{equation}

Replacing \( T \) by \( (4/5)^n T \) and summing over \( n \), we deduce (1.4), completing the proof of Theorem 1.

We finish with a remark. Jutila [2] observed that if \( k = 1/n, n = 1, 2, 3, \ldots \), then the implied constants in Theorem A can be taken to be independent of \( k \). Jutila indicated the necessary changes in Heath-Brown’s proof; as a corollary, he obtained information on large deviations of \( \log |\zeta(\frac{1}{2} + it)| \). To get a result uniform for \( k = 1/n \), certain similar changes can also be made in the proof of Theorem 1.

\section{5 The automorphic \( L \)-functions \( L(s, f) \) and \( L(s, f \times f) \)}

In this section we state analogs of Theorem 1 for \( L(s, f) \) and \( L(s, f \times f) \).

**Theorem 2.** (i) If \( k \geq 0 \) is rational, then

\begin{equation}
I_k(T, f) \gg T(\log T)^{k^2},
\end{equation}

where

\begin{equation}
I_k(T, f) = \int_1^T \left| L\left(\frac{1}{2} + it, f\right)\right|^{2k} dt.
\end{equation}

(ii) Under the Riemann hypothesis for \( L(s, f) \), estimate (5.1) is valid for all real \( k \geq 0 \).

(iii) Under the Riemann hypothesis for \( L(s, f) \), we have

\begin{equation}
I_k(T, f) \ll T(\log T)^{k^2}
\end{equation}

for \( 0 \leq k < 1 \).
Theorem 3.

(i) If $k \geq 0$ is rational, then

$$I_k(T, f \times f) \gg T (\log T)^{2k^2},$$

where

$$I_k(T, f \times f) = \int_1^T \left| L \left( \frac{1}{2} + it, f \times f \right) \right|^{2k} dt.$$

(ii) Under the Riemann hypothesis for $L(s, f \times f)$, estimate (5.2) is valid for all real $k \geq 0$.

(iii) Under the Riemann hypothesis for $L(s, f \times f)$, we have

$$I_k(T, f \times f) \ll T (\log T)^{2k^2}$$

for $0 \leq k < \frac{1}{2}$.

We do not present the proofs because they are much similar to that of Theorem 1.

In connection with Theorems 2 and 3, we make several remarks. We begin with Theorem 2.

1. Good [9] proved that

$$I_1(T, f) = B_1 T \log T + B_2 T + O(T^{2/3 + \epsilon}).$$

2. Part of the results of Theorem 2 was announced earlier by Laurinčikas [10].

3. As an analog of the asymptotics (2.1) we mention the formula

$$\sum_{p \leq x} \lambda_1^2(p) = x \log^{-1} x + O(x \log^{-2} x).$$

This was proved by Moreno [11].

Now we proceed to Theorem 3.

1. An analog of the asymptotics (2.1) is

$$\sum_{p \leq x} d_p^2 = 2x \log^{-1} x + O(x \log^{-2} x).$$

For the proof, we can use the identity

$$\lambda_1^2(p) = (\alpha_p^2 + 2 + \alpha_p^2)^2 = (\alpha_p^2 + 1 + \alpha_p^2)^2 + 1 + 2(\alpha_p^2 + 1 + \alpha_p^2),$$

which is equivalent to

$$d_p^2 = c_p^2 + 1 + 2c_p.$$

It can be shown that

$$\sum_{p \leq x} c_p \ll x \log^{-2} x.$$

Therefore, using (2.1), we arrive at (5.3).

2. We define the coefficients $\nu_1(n, k)$ by the expansion (the choice of a branch is similar to that of $L^k(s, \text{sym}^2 f)$)

$$L^k(s, f \times f) = \sum_{n=1}^\infty \nu_1(n, k)n^{-s} \quad (\text{Re}(s) > 1).$$

An analog of Lemma 1 looks like this:

for fixed real $k \geq 0$, there exists $c_k^2 > 0$ such that

$$\left( \sigma - \frac{1}{2} \right)^{-2k^2} \ll \sum_{n=1}^N \nu_1(n, k)^2 n^{-2\sigma} \ll \left( \sigma - \frac{1}{2} \right)^{-2k^2}.$$
uniformly for
\[ \frac{1}{2} + \frac{c''_k}{\log N} \leq \sigma \leq 1. \]
Moreover, we have
\[ (\log N)^{2k^2} \ll \sum_{n=1}^{N} \nu_1(n,k)^2 n^{-1} \ll (\log N)^{2k^2}. \]

3. In order to avoid the singularity of \( \zeta(s) \) at \( s = 1 \), Heath-Brown applied yet another convexity theorem of Gabriel [7]. We can employ that theorem in the case of the \( L \)-function \( L(s, f \times f) \), which is holomorphic in the \( s \)-plane except for a simple pole at \( s = 1 \).

4. Since
\[ L(s, f \times f) = \zeta(s)L(s, \text{sym}^2 f), \]
the Riemann hypothesis for \( L(s, f \times f) \) is the conjunction of the Riemann hypotheses for \( \zeta(s) \) and \( L(s, \text{sym}^2 f) \).

5. In [12], it was proved that
\[ I_1(T, f \times f) \ll T^{11/6+\varepsilon}, \]
which can be deduced from the estimate
\[ I_1(T, \text{sym}^2 f) \ll T^{3/2+\varepsilon}. \]

§6. The Dedekind zeta function \( \zeta_{K_3}(s) \)

Let \( K_3 \) be the cubic field over \( \mathbb{Q} \) given by adjoining a root of the polynomial \( x^3 + ax^2 + bx + c \) of discriminant \( D < 0 \) with the Galois group \( S_3 \). Consider \( \zeta_{K_3}(s) \), the Dedekind zeta function of the field \( K_3 \):
\[ \zeta_{K_3}(s) = \sum_{m=1}^{\infty} M(m)m^{-s} \quad (\text{Re}(s) > 1), \]
where \( M(m) \) is the number of integral ideals in the field \( K_3 \) of norm \( m \). It is known (see [12] [13]) that
\[ \zeta_{K_3}(s) = \zeta(s)L(s, \varphi), \]
where \( \varphi(z) \) is a holomorphic Hecke eigencuspsform of weight 1 with respect to the congruence subgroup \( \Gamma_0(|D|) \),
\[ \varphi(z) = \sum_{m=1}^{\infty} a(m) \exp(2\pi imz). \]
We have
\[ M(m) = \sum_{d|m} a(d); \]
in particular, \( M(p) = 1 + a(p) \).

**Theorem 4.**

(i) If \( k \geq 0 \) is rational, then
\[ I_k(T, K_3) \gg T(\log T)^{2k^2}, \]
where
\[ I_k(T, K_3) = \int_{1}^{T} \left| \zeta_{K_3}\left(\frac{1}{2} + it\right) \right|^{2k} dt. \]
(ii) Under the Riemann hypothesis for \( \zeta_{K_3}(s) \), estimate (6.2) is valid for all real \( k \geq 0 \).
(iii) Under the Riemann hypothesis for \( \zeta_{K_3}(s) \), we have
\[
I_k(T, K_3) \ll T(\log T)^{2k/3}
\]
for \( 0 \leq k < \frac{2}{3} \).

The proof is similar to that of Theorems 1–3. Several remarks are in order.

1. An analog of the asymptotics (2.1) is
\[
\sum_{p \leq x} M(p)^2 = 2x \log^{-1} x + O(x \log^{-2} x).
\]
This can be proved with the help of the identity
\[
M(p)^2 = 1 + 2a(p) + a^2(p)
\]
and some well-known facts.

2. The results of Hardy and Littlewood
\[
I_1(T) \sim T \log T
\]
and Good [9]
\[
I_1(T, \varphi) \sim B_1 T \log T,
\]
combined with the Cauchy inequality and (6.1), show that
\[
I_{1/2}(T, K_3) \ll T \log T.
\]

3. In [14] it was proved that
\[
I_1(T, K_3) \ll T^{5/4+\varepsilon}.
\]

4. The results of Heath-Brown [15]
\[
I_6(T) \ll T^{2+\varepsilon}
\]
and Jutila [16]
\[
I_3(T, \varphi) \ll T^{2+\varepsilon},
\]
combined with the Hölder inequality and (6.1) yield
\[
I_2(T, K_3) \ll T^{2+\varepsilon}.
\]

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St. Petersburg Branch, Steklov Mathematical Institute, 27 Fontanka, St. Petersburg 191023, Russia
E-mail address: fomenko@pdmi.ras.ru

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