FUSION PROCEDURE FOR THE BRAUER ALGEBRA

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Dedicated to L. D. Faddeev on the occasion of his 75th birthday

Abstract. It is shown that all primitive idempotents for the Brauer algebra $B_n(\omega)$ can be found by evaluating a rational function in several variables that has the form of a product of $R$-matrix type factors. This provides an analog of the fusion procedure for $B_n(\omega)$.

§1. Introduction

It is well known that all primitive idempotents of the symmetric group $S_n$ can be obtained by taking certain limit values of the rational function

\[ \Phi(u_1, \ldots, u_n) = \prod_{1 \leq i < j \leq n} \left( 1 - \frac{s_{ij}}{u_i - u_j} \right), \]

where $s_{ij} \in S_n$ is the transposition of $i$ and $j$, $u_1, \ldots, u_n$ are complex variables, and the product is calculated in the group algebra $\mathbb{C}[S_n]$ in the lexicographical order on the pairs $(i, j)$. This construction, which is commonly referred to as the fusion procedure, goes back to Jucys [8] and Cherednik [5]. Detailed proofs were given by Nazarov [16]. A simple version of the fusion procedure was found in [13]; see also [14, Chapter 6] for applications to the Yangian representation theory and more references. In more detail, let $T$ be a standard tableau associated with a partition $\lambda$ of $n$, and let $c_k = j - i$ if the element $k$ occupies the cell of the tableau in row $i$ and column $j$. Then the consecutive evaluations

\[ \Phi(u_1, \ldots, u_n) \bigg|_{u_1 = c_1, u_2 = c_2, \ldots, u_n = c_n} \]

are well defined, and this value yields the corresponding primitive idempotent $E_\lambda^T$ multiplied by the product of the hooks of the diagram of $\lambda$.

In this paper we give a similar fusion procedure for the Brauer algebra $B_n(\omega)$. This algebra was introduced by Brauer in [4] and its structure and representation theory was studied by many authors; see, for instance, Wenzl [22], Nazarov [17], Leduc and Ram [11] and Rui [20]. We refer the reader to the survey paper by Barcelo and Ram [1] for the discussion of the Brauer algebra in the context of combinatorial representation theory and more references. The irreducible representations of $B_n(\omega)$ are indexed by all partitions of the nonnegative integers $n, n - 2, n - 4, \ldots$. If $\lambda$ is such a partition, then the up-down $\lambda$-tableaux $T$ parametrize basis vectors of the corresponding representation; see [2].
Consider the rational function
\[
\Psi(u_1, \ldots, u_n) = \prod_{1 \leq i < j \leq n} \left(1 - \frac{c_{ij}}{u_i + u_j}\right) \prod_{1 \leq i < j \leq n} \left(1 - \frac{s_{ij}}{u_i - u_j}\right)
\]
with the ordered products as in \((1.1)\); the elements \(c_{ij}, s_{ij} \in B_n(\omega)\) are defined in \((2)\) below. This function was first introduced by Nazarov \([18]\) \((3.14)\) in the context of representations of the classical Lie algebras and twisted Yangians.

Our main result is the following analog of the fusion procedure for the Brauer algebra: given an up-down \(\lambda\)-tableau \(T\), the consecutive evaluations
\[
(u_1 - c_1)^{p_1} \cdots (u_n - c_n)^{p_n} \Psi(u_1, \ldots, u_n)\big|_{u_1 = c_1, u_2 = c_2, \ldots, u_n = c_n}
\]
are well defined, and this value yields the corresponding primitive idempotent \(E_T^\lambda\) multiplied by a nonzero constant \(f(T)\), which is calculated in an explicit form. Here \(p_1, \ldots, p_n\) are certain integers depending on \(T\), which we call the \textit{exponents} of \(T\), and the \(c_i\) are the \textit{contents} of \(T\); see \((2)\) for precise definitions.

In the particular case where \(\lambda\) is a partition of \(n\), we thus reproduce some closely related results by Nazarov \([18]\); see, in particular, Propositions 3.2, 3.3 and formulas (3.20)–(3.23) there. In fact, he worked with wider classes of representations of the orthogonal and symplectic groups \(G_N\) parametrized by certain skew Young diagrams with \(n\) boxes. The natural action of \(G_N\) on the tensor power \((\mathbb{C}^N)^{\otimes n}\) commutes with the action of the Brauer algebra \(B_n(\omega)\) for a suitably specialized value of \(\omega\). Nazarov’s formulas for the idempotents provide remarkable analogs of the Young symmetrizers in an explicit form. Their images in \((\mathbb{C}^N)^{\otimes n}\) yield realizations of the representations of \(G_N\) associated with the skew Young diagrams. Note that the corresponding images of the factors in \((1.3)\) are the values of the Yang \(R\)-matrix and its transpose; cf. Remark \((3.3)\) below.

If \(\lambda\) is a partition of \(n\), then all exponents \(p_i\) are equal to zero, while the constant \(f(T)\) takes the same value as for \((1.2)\), thus making this case quite similar to that of the symmetric group. The existence of a special monomorphism \(\mathbb{C}[S_n] \to B_n(\omega)\) \((2)\) can be regarded as an ‘explanation’ of this analogy. If \(\lambda\) is a partition of \(n - 2f\) for some \(f \geq 1\), then the function \((1.3)\) can have zeros or poles of certain multiplicities at \(u_i = c_i\), so that in place of \((1.2)\) we need to take ‘regularized evaluations’ as in \((1.4)\).

The proof of our main theorem (Theorem 3.4) follows the approach of \([19]\) and is based on the construction of the primitive idempotents \(E_T^\lambda\) in terms of the Jucys–Murphy elements for the Brauer algebra. These elements were introduced independently by Nazarov \([17]\) and by Leduc and Ram \([11]\), where analogs of Young’s seminormal representations for the Brauer algebra were given. In a more general context of cellular algebras equipped with a family of Jucys–Murphy elements, the construction of the primitive idempotents and seminormal forms was given by Mathas \([12]\).

We expect that a result similar to Theorem 3.4 holds true for the Birman–Murakami–Wenzl algebras, which will be considered in our publications elsewhere; cf. \([9, 17]\).

We are pleased to dedicate this work to L. D. Faddeev on his 75th birthday. The fusion procedure originates in the work of his Leningrad School on the quantum inverse scattering method. A key role in this method is played by solutions of the Yang–Baxter equation \((R\)-matrices\). The fusion procedure is understood as a way to obtain new solutions of this equation out of old ones; see, e.g., \([10]\). As a result, in the quantum inverse scattering method \([21]\) one obtains a family of commuting transfer matrices that satisfy a number of functional equations (the so-called fusion relations). They are similar to the relations for the characters of the quantum linear groups (in the case of Hecke type \(R\)-matrices) and for the quantum \(SO\) and \(Sp\) groups (in the case of Birman–Murakami–Wenzl type \(R\)-matrices). The fundamental \(R\)-matrices of the Hecke and Birman–Murakami–Wenzl types were discussed in \([19]\).
§2. The Brauer algebra and its representations

Let \( n \) be a positive integer and \( \omega \) an indeterminate. An \( n \)-diagram \( d \) is a collection of \( 2n \) dots arranged into two rows with \( n \) dots in each row connected by \( n \) edges such that any dot belongs to only one edge. The product of two diagrams \( d_1 \) and \( d_2 \) is determined by placing \( d_1 \) above \( d_2 \) and identifying the vertices of the bottom row of \( d_1 \) with the corresponding vertices in the top row of \( d_2 \). Let \( s \) be the number of closed loops obtained in this placement. The product \( d_1 d_2 \) is given by \( \omega^s \) times the resulting diagram without loops. The Brauer algebra \( \mathcal{B}_n(\omega) \) is defined as the \( \mathbb{C}(\omega) \)-linear span of the \( n \)-diagrams with the multiplication defined above. The dimension of the algebra is \( 1 \cdot 3 \cdots (2n - 1) \).

The following presentation of \( \mathcal{B}_n(\omega) \) is well known; see, e.g., [3].

**Proposition 2.1.** The Brauer algebra \( \mathcal{B}_n(\omega) \) is isomorphic to the algebra with \( 2n - 2 \) generators \( s_1, \ldots, s_{n-1}, e_1, \ldots, e_{n-1} \) and the defining relations

\[
s_i^2 = 1, \quad e_i^2 = \omega e_i, \quad s_i e_i = e_i s_i = e_i, \quad i = 1, \ldots, n - 1, \\
s_i s_j = s_j s_i, \quad e_i e_j = e_j e_i, \quad s_i e_j = e_j s_i, \quad |i - j| > 1, \\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \\
s_i e_{i+1} e_i = s_{i+1} e_i, \quad e_{i+1} e_i s_{i+1} = e_i + 1 s_i, \quad i = 1, \ldots, n - 2.
\]

The generators \( s_i \) and \( e_i \) correspond to the following diagrams, respectively:

\[
\begin{array}{ccccccc}
& & & & \cdots & & \\
1 & 2 & \cdots & & i & i + 1 & n - 1 & n \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccc}
& & & & \cdots & & \\
1 & 2 & \cdots & & i & i + 1 & n - 1 & n \\
\end{array}
\]

The subalgebra of \( \mathcal{B}_n(\omega) \) generated over \( \mathbb{C} \) by \( s_1, \ldots, s_{n-1} \) is isomorphic to the group algebra \( \mathbb{C}[S_n] \) so that \( s_i \) can be identified with the transposition \((i, i + 1)\). Then for any \( 1 \leq i < j \leq n \) the transposition \( s_{ij} = (i, j) \) can be regarded as an element of \( \mathcal{B}_n(\omega) \). Moreover, \( e_{ij} \) will denote the element of \( \mathcal{B}_n(\omega) \) represented by the diagram in which the \( i \)th and \( j \)th dots in the top row, as well as the \( i \)th and \( j \)th dots in the bottom row are connected by an edge, while the remaining edges connect the \( k \)th dot in the top row with the \( k \)th dot in the bottom row for each \( k \neq i, j \). Equivalently, in terms of the presentation of \( \mathcal{B}_n(\omega) \) provided by Proposition 2.1

\[
s_{ij} = s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i \quad \text{and} \quad e_{ij} = s_{i,j-1} e_{j-1} s_{i,j-1}.
\]

The Brauer algebra \( \mathcal{B}_{n-1}(\omega) \) can be regarded as the subalgebra of \( \mathcal{B}_n(\omega) \) spanned by all diagrams in which the \( n \)th dots in the top and bottom rows are connected by an edge.

The Jucys-Murphy elements \( x_1, \ldots, x_n \) for the Brauer algebra \( \mathcal{B}_n(\omega) \) were introduced independently in [11] and [17]: they are given by the formulas

\[
x_r = \frac{\omega - 1}{2} + \sum_{k=1}^{r-1} (s_{kr} - e_{kr}), \quad r = 1, \ldots, n.
\]

The element \( x_n \) commutes with the subalgebra of \( \mathcal{B}_{n-1}(\omega) \). This implies that the elements \( x_1, \ldots, x_n \) of \( \mathcal{B}_n(\omega) \) commute pairwise. They can be used to construct a complete set of pairwise orthogonal primitive idempotents for the Brauer algebra following the approach of Jucys [9] and Murphy [15], see also [12] for a generalization to a wider class of cellular algebras. Namely, let \( \lambda \) be a partition of \( n - 2f \) for some \( f \in \{0, 1, \ldots, \lfloor n/2 \rfloor\} \). We shall identify partitions with their diagrams, so that if the parts of \( \lambda \) are \( \lambda_1, \lambda_2, \ldots \), then the corresponding diagram is a left-justified array of rows of unit boxes containing \( \lambda_1 \) boxes in the top row, \( \lambda_2 \) boxes in the second row, etc. The box in row \( i \) and column \( j \) is
of a diagram will be denoted as the pair \((i, j)\). An \(\text{up-down } \lambda\text{-tableau}\) is a sequence \(T = (\Lambda_1, \ldots, \Lambda_n)\) of diagrams such that for each \(r = 1, \ldots, n\) the diagram \(\Lambda_r\) is obtained from \(\Lambda_{r-1}\) by adding or removing one box, where \(\Lambda_0 = \emptyset\) is the empty diagram and \(\Lambda_n = \lambda\). To each up-down tableau \(T\) we attach the corresponding sequence of \(\textit{contents}\) \((c_1, \ldots, c_n)\), \(c_r = c_r(T)\), where

\[
c_r = \frac{\omega - 1}{2} + j - i \quad \text{or} \quad c_r = -\left(\frac{\omega - 1}{2} + j - i\right)
\]

if \(\Lambda_r\) is obtained by adding the box \((i, j)\) to \(\Lambda_{r-1}\) or by removing this box from \(\Lambda_{r-1}\), respectively. The primitive idempotents \(E_T = E_T^\lambda\) can now be defined by the following recurrence formula (we omit the superscripts indicating the diagrams, because they are determined by the up-down tableaux). Set \(\mu = \Lambda_{n-1}\) and consider the up-down \(\mu\)-tableau \(U = (\Lambda_1, \ldots, \Lambda_{n-1})\). Let \(\alpha\) be the box that is added to or removed from \(\mu\) to get \(\lambda\). Then

\[
E_T = E_U \frac{(x_n - a_1) \cdots (x_n - a_k)}{(c_n - a_1) \cdots (c_n - a_k)},
\]

where \(a_1, \ldots, a_k\) are the contents of all boxes excluding \(\alpha\), which can be removed from or added to \(\mu\) to get a diagram. When \(\lambda\) runs over all partitions of \(n, n-2, \ldots\) and \(T\) runs over all up-down \(\lambda\)-tableaux, the elements \(\{E_T\}\) yield a complete set of pairwise orthogonal primitive idempotents for \(B_n(\omega)\). They have the properties

\[
x_r E_T = E_T x_r = c_r(T) E_T, \quad r = 1, \ldots, n.
\]

Moreover, given an up-down tableau \(U = (\Lambda_1, \ldots, \Lambda_{n-1})\), we have the relation

\[
E_U = \sum_T E_T,
\]

summed over all up-down tableaux of the form \(T = (\Lambda_1, \ldots, \Lambda_{n-1}, \Lambda)\); we refer the reader to [11, 12] and [17] for more details. Relation (2.1) admits the following equivalent form:

\[
E_T = E_U \frac{u - c_n}{u - x_n} \bigg|_{u = c_n},
\]

where \(u\) is a complex variable. This relation is derived from (2.2) and (2.3) exactly as in the case of the symmetric group; see [13].

§3. The Fusion Procedure

Some combinatorial data extracted from the up-down tableaux will be convenient for the formulations below. Let \(U = (\Lambda_1, \ldots, \Lambda_{n-1})\) be a \(\mu\)-tableau; we define two infinite matrices \(m(U)\) and \(m'(U)\) whose rows and columns are labeled by positive integers and only a finite number of entries in each of the matrices is nonzero. The entry \(m_{ij}\) of the matrix \(m(U)\) (respectively, the entry \(m'_{ij}\) of the matrix \(m'(U)\)) equals the number of times the box \((i, j)\) was added (respectively, removed) in the sequence of diagrams \((\emptyset = \Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1})\). So, the difference \(m(U) - m'(U)\) is the matrix all of whose entries are zero except for the \(ij\)th matrix elements equal to 1 for which the corresponding boxes \((i, j)\) are contained in the diagram \(\mu\).

Example 3.1. For the up-down tableau

\[
U = \begin{pmatrix} \emptyset, & \emptyset, & \emptyset, & \emptyset, & \emptyset, & \emptyset, & \emptyset, & \emptyset, & \emptyset \end{pmatrix},
\]

the matrices are

\[
m(U) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad m'(U) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},
\]

respectively.
where the common zeros in both matrices have been omitted.

Furthermore, for each integer \( k \) we define the nonnegative integers \( d_k = d_k(U) \) and \( d'_k = d'_k(U) \) as the respective sums of the entries of the matrices \( m(U) \) and \( m'(U) \) on the \( k \)th diagonal: 

\[
d_k = \sum_{j-i=k} m_{ij}, \quad d'_k = \sum_{j-i=k} m'_{ij}.
\]

So, in Example 3.1, we have \( d_{-1} = d_0 = d_1 = 2 \), while \( d'_{-1} = d'_0 = d'_1 = 1 \) and the remaining values \( d_k \) and \( d'_k \) are zero.

Finally, for each integer \( k \) we introduce the parameters \( g_k = g_k(U) \) and \( g'_k = g'_k(U) \) by

\[
g_k = \delta_{k0} + d_{k-1} + d_{k+1} - 2d_k, \quad g'_k = d'_{k-1} + d'_{k+1} - 2d'_k.
\]

Now the exponents \( p_1, \ldots, p_n \) of an up-down \( \lambda \)-tableau \( T = (\Lambda_1, \ldots, \Lambda_n) \) are defined inductively, so that \( p_k \) depends only on the first \( r \) diagrams \( (\Lambda_1, \ldots, \Lambda_r) \) of \( T \). Hence, it suffices to define \( p_n \). Taking \( U = (\Lambda_1, \ldots, \Lambda_{n-1}) \), we set

\[
p_n = 1 - g_{k_n}(U) \quad \text{or} \quad p_n = 1 - g'_{k_n}(U),
\]

respectively, if \( \Lambda_n \) is obtained from \( \Lambda_{n-1} \) by adding a box on the diagonal \( k_n \) or by removing a box on the diagonal \( k_n \).

**Example 3.2.** The exponents for the up-down tableau

\[
T = (\quad \square \quad \quad \square \quad \quad \square \quad \quad \square \quad \quad )
\]

are \( p_1 = p_2 = p_3 = 0, p_4 = p_5 = 1, \) and \( p_6 = 2 \).

The constants \( f(T) \) mentioned in the Introduction are defined inductively by the formula

\[
f(T) = f(U) \varphi(U, T),
\]

where \( U = (\Lambda_1, \ldots, \Lambda_{n-1}) \) and \( T = (\Lambda_1, \ldots, \Lambda_n) \). Here

\[
\varphi(U, T) = \prod_{k \neq k_n} (k_n - k)^{g_k} \prod_{k} (k_n + k + \omega - 1)^{g'_k}
\]

or

\[
\varphi(U, T) = \prod_{k \neq k_n} (-k_n + k)^{g'_k} \prod_{k} (-k_n - k - \omega + 1)^{g_k}
\]

if \( \Lambda_n \) is obtained from \( \Lambda_{n-1} \) by adding or removing a box on the diagonal \( k_n \), respectively, where the products are taken over all integers \( k \), while \( g_k = g_k(U) \) and \( g'_k = g'_k(U) \). Note that only a finite number of the parameters \( g_k \) and \( g'_k \) are nonzero, so that each product in the above formulas contains only a finite number of factors not equal to 1.

**Proposition 3.3.** If \( T = (\Lambda_1, \ldots, \Lambda_n) \) is an up-down \( \lambda \)-tableau and \( \lambda \) is a partition of \( n \), then all exponents \( p_1, \ldots, p_n \) of \( T \) are equal to zero, and \( f(T) \) equals the product of the hooks of \( \lambda \).

**Proof.** Set \( U = (\Lambda_1, \ldots, \Lambda_{n-1}) \) and \( \mu = \Lambda_{n-1} \). The nonzero entries of the matrix \( m(U) \) are equal to 1; these are the \( ij \)th matrix elements such that the corresponding boxes \((i, j)\) are contained in the diagram \( \mu \). Furthermore, all entries of the matrix \( m'(U) \) are zero. Hence, the parameters \( g'_k(U) \) are all zero, while the nonzero values of \( g_k(U) \) are equal to \( \pm 1 \). The value 1 (respectively, \(-1\)) corresponds to those diagonals \( k \) where a box can
be well defined. The corresponding value coincides with \( f(T) \) is also verified easily. \( \square \)

Consider the rational function \( \Psi(u_1, \ldots, u_n) \) with values in the Brauer algebra \( \mathcal{B}_n(\omega) \), defined by (3.3). Now we can prove our main theorem.

**Theorem 3.4.** For any up-down tableau \( T = (\Lambda_1, \ldots, \Lambda_n) \), the consecutive evaluations

\[
(u_1 - c_1)^{p_1} \ldots (u_n - c_n)^{p_n} \Psi(u_1, \ldots, u_n)\big|_{u_1=c_1} \bigm|_{u_2=c_2} \ldots \bigm|_{u_n=c_n}
\]

are well defined. The corresponding value coincides with \( f(T) E_T \).

**Proof.** The proof of the theorem will follow from a sequence of lemmas.

**Lemma 3.5.** The function \( \Psi(u_1, \ldots, u_n) \) can be written in the equivalent form

\[
\Psi(u_1, \ldots, u_n) = \prod_{r=2, \ldots, n} \left( 1 - \frac{e_{r-1,r}}{u_{r-1} + u_r} \right) \ldots \left( 1 - \frac{e_{1,r}}{u_1 + u_r} \right) \times \left( 1 - \frac{s_{1,r}}{u_1 - u_r} \right) \ldots \left( 1 - \frac{s_{r-1,r}}{u_{r-1} - u_r} \right),
\]

where the factors are ordered in accordance with the increasing values of \( r \).

**Proof.** This follows by using the following easily verified identities for the rational functions in \( u \) and \( v \) with values in \( \mathcal{B}_n(\omega) \): if \( i < j < r \), then

\[
(1 - \frac{e_{ir}}{u})(1 - \frac{e_{jr}}{v})(1 - \frac{s_{ij}}{u-v}) = (1 - \frac{s_{ij}}{u})(1 - \frac{e_{jr}}{v})(1 - \frac{e_{ir}}{u}).
\]

If the indices \( i, j, k, l \) are distinct, then the elements \( e_{ij} \) and \( e_{kl} \) of \( \mathcal{B}_n(\omega) \) commute. Therefore, we can represent the first product occurring in (3.3) as

\[
\prod_{1 \leq i < j \leq n} \left( 1 - \frac{e_{ij}}{u_i + u_j} \right) = \prod_{1 \leq i < j \leq n-1} \left( 1 - \frac{e_{ij}}{u_i + u_j} \right) \left( 1 - \frac{e_{1,n}}{u_1 + u_n} \right) \ldots \left( 1 - \frac{e_{n-1,n}}{u_{n-1} + u_n} \right).
\]

Now, using identities (3.5) repeatedly, we get

\[
(1 - \frac{e_{1,n}}{u_1 + u_n}) \ldots (1 - \frac{e_{n-1,n}}{u_{n-1} + u_n}) \prod_{1 \leq i < j \leq n-1} \left( 1 - \frac{s_{ij}}{u_i - u_j} \right) = \prod_{1 \leq i < j \leq n-1} \left( 1 - \frac{s_{ij}}{u_i - u_j} \right) \left( 1 - \frac{e_{n-1,n}}{u_{n-1} + u_n} \right) \ldots \left( 1 - \frac{e_{1,n}}{u_1 + u_n} \right).
\]

Hence, the function (3.3) can be written as

\[
\Psi(u_1, \ldots, u_n) = \Psi(u_1, \ldots, u_{n-1}) \left( 1 - \frac{e_{n-1,n}}{u_{n-1} + u_n} \right) \ldots \left( 1 - \frac{e_{1,n}}{u_1 + u_n} \right) \times \left( 1 - \frac{s_{1,n}}{u_1 - u_n} \right) \ldots \left( 1 - \frac{s_{n-1,n}}{u_{n-1} - u_n} \right),
\]

and the decomposition (3.4) follows by induction on \( n \). \( \square \)

Lemma 3.5 allows us to use the induction on \( n \) to prove the theorem. By the inductive hypothesis, setting \( u = u_n \) we get

\[
(u_1 - c_1)^{p_1} \ldots (u_n - c_n)^{p_n} \Psi(u_1, \ldots, u_n)\big|_{u_1=c_1} \bigm|_{u_2=c_2} \ldots \bigm|_{u_n-1=c_{n-1}} = f(U) E_T (u - c_n)^{p_n} \left( 1 - \frac{e_{n-1,n}}{c_{n-1} + u} \right) \ldots \left( 1 - \frac{e_{1,n}}{c_1 + u} \right) \times \left( 1 - \frac{s_{1,n}}{c_1 - u} \right) \ldots \left( 1 - \frac{s_{n-1,n}}{c_{n-1} - u} \right),
\]

where \( U \) is the up-down tableau \( (\Lambda_1, \ldots, \Lambda_{n-1}) \). The next lemma will allow us to simplify this expression.
Lemma 3.6. We have
\[
E_U \left( 1 - \frac{e_{n-1,n}}{c_{n-1} + u} \right) \cdots \left( 1 - \frac{e_{1,n}}{c_1 + u} \right) \left( 1 - \frac{s_{1,n}}{c_1 - u} \right) \cdots \left( 1 - \frac{s_{n-1,n}}{c_{n-1} - u} \right)
\]
(3.8)
\[= \frac{u - c_n}{u - c_{n-1}} \frac{u - c_{n-1}}{u - c_{n-2}} \cdots \left( 1 - \frac{1}{u - c_1} \right) E_U \frac{u - c_n}{u - x_n} \]

Proof. Note that the Jucys–Murphy element \(x_n\) commutes with \(E_U\), and the inverses of the expressions occurring in the product are found by
\[
\left( 1 - \frac{s_{r,n}}{c_r - u} \right)^{-1} \left( 1 - \frac{1}{u - c_r} \right) = \left( 1 + \frac{s_{r,n}}{c_r - u} \right)
\]
and
\[
\left( 1 - \frac{e_{r,n}}{c_r + u} \right)^{-1} = \left( 1 + \frac{e_{r,n}}{c_r + u - \omega} \right),
\]
where we have used the relations \(s_{r,n}^2 = 1\) and \(e_{r,n}^2 = \omega e_{r,n}\). Hence, relation (3.8) is equivalent to
\[
E_U \left( 1 + \frac{s_{n-1,n}}{c_{n-1} - u} \right) \cdots \left( 1 + \frac{s_{1,n}}{c_1 - u} \right) \left( 1 + \frac{e_{1,n}}{c_1 + u - \omega} \right) \cdots \left( 1 + \frac{e_{n-1,n}}{c_{n-1} + u - \omega} \right) = E_U \frac{u - x_n}{u - c_1}.
\]
(3.9)

We embed the Brauer algebra \(B_n(\omega)\) in \(B_m(\omega)\) for some \(m \geq n\) and use induction on \(n\) to verify the more general identity
\[
E_U \left( 1 + \frac{s_{n-1,m}}{c_{n-1} - u} \right) \cdots \left( 1 + \frac{s_{1,m}}{c_1 - u} \right) \left( 1 + \frac{e_{1,m}}{c_1 + u - \omega} \right) \cdots \left( 1 + \frac{e_{n-1,m}}{c_{n-1} + u - \omega} \right) = E_U \frac{u - x_n}{u - c_1},
\]
(3.10)
where
\[
x_n^{(m)} = \frac{\omega - 1}{2} + \sum_{k=1}^{n-1} (s_{km} - e_{km}).
\]

By (2.3), \(E_U = E_W E_U\), where \(W\) is the up-down tableau \((\Lambda_1, \ldots, \Lambda_{n-2})\). Hence, using the inductive hypothesis, we can write the left-hand side of (3.10) as
\[
E_U \left( 1 + \frac{s_{n-1,m}}{c_{n-1} - u} \right) E_W \left( 1 + \frac{e_{n-1,m}}{c_{n-1} + u - \omega} \right) = \frac{1}{u - c_1} E_U \left( u - x_n^{(m)} + \frac{s_{n-1,m} (u - x_n^{(m)})}{c_{n-1} - u} + \frac{(u - x_n^{(m)}) e_{n-1,m}}{c_{n-1} + u - \omega} \right.
\]
\[+ \left. \frac{s_{n-1,m} (u - x_n^{(m)}) e_{n-1,m}}{c_{n-1} - u} \right) \cdot \left( \frac{u - x_n^{(m)}}{c_{n-1} + u - \omega} \right)
\]
Now we use the following relations in \(B_m(\omega)\), which are valid for \(1 \leq r < n - 1\):
\[
s_{n-1,m} s_{r,m} = s_{r,n-1} s_{n-1,m}, \quad s_{n-1,m} e_{r,m} = e_{r,n-1} s_{n-1,m}
\]
and
\[
s_{r,m} e_{n-1,m} = e_{r,n-1} e_{n-1,m}, \quad e_{r,m} e_{n-1,m} = s_{r,n-1} e_{n-1,m}.
\]
They imply that
\[
s_{n-1,m} x_n^{(m)} = x_{n-1} s_{n-1,m}
\]
and
\[
x_n^{(m)} e_{n-1,m} = (\omega - 1 - x_{n-1}) e_{n-1,m}.
\]
Together with the relation \( E_U x_{n-1} = c_{n-1} E_U \) implied by (2.2), this allows us to bring the left-hand side of (3.10) to the form

\[
\frac{1}{u - c_1} E_U \left( u - x_{n-1}^{(m)} - s_{n-1,m} + c_{n-1,m} \right) = E_U \frac{u - x_n^{(m)}}{u - c_1},
\]
as required.

Lemma 3.6 shows that, in order to complete the proof of the theorem, we need to show that the rational function

\[
f(U) (u - c_1) \prod_{r=1}^{n-1} \left( 1 - \frac{1}{(u - c_r)^2} \right) (u - c_n)^{p_n-1} \cdot E_U \frac{u - c_n}{u - x_n},
\]
is regular at \( u = c_n \) and that its value equals \( f(T) E_T \). Using the parameters (3.1), we can write this expression as

\[
f(U) \prod_k \left( u - \frac{\omega - 1}{2} - k \right)^{g_k} \prod_k \left( u + \frac{\omega - 1}{2} + k \right)^{g_k} (u - c_n)^{p_n-1} \cdot E_U \frac{u - c_n}{u - x_n},
\]
where \( k \) runs over the set of integers. If the diagram \( \Lambda_n \) is obtained from \( \Lambda_{n-1} \) by adding or removing a box on the diagonal \( k_n \), then the value of the content \( c_n \) is given by the respective formulas

\[
c_n = \frac{\omega - 1}{2} + k_n \quad \text{or} \quad c_n = -\left( \frac{\omega - 1}{2} + k_n \right).
\]
The definition of the exponents (3.2) and the constants \( f(T) \) in (3.3) together with (2.4) imply the desired statement.

The following corollary is immediate from Proposition 3.3 and Theorem 3.4; cf. [13, 18].

**Corollary 3.7.** If \( T = (\Lambda_1, \ldots, \Lambda_n) \) is an up-down \( \lambda \)-tableau and \( \lambda \) is a partition of \( n \), then the consecutive evaluations

\[
\Psi(u_1, \ldots, u_n) \bigg|_{u_1 = c_1} \bigg|_{u_2 = c_2} \cdots \bigg|_{u_n = c_n}
\]
are well defined. The corresponding value coincides with \( H(\lambda) E_T \), where \( H(\lambda) \) is the product of the hooks of \( \lambda \).

**Remark 3.8.** In two particular cases where \( \lambda \) is a row- or column-diagram with \( n \) boxes, one can write alternative multiplicative expressions associated with the respective tableaux. Namely, the primitive idempotent corresponding to the only up-down \( (n) \)-tableau is proportional to

\[
\prod_{1 \leq i < j \leq n} \left( 1 + \frac{s_{ij}}{j-i} - \frac{e_{ij}}{j-i+\omega/2-1} \right),
\]
while the primitive idempotent corresponding to the up-down \( (1^n) \)-tableau is proportional to

\[
\prod_{1 \leq i < j \leq n} \left( 1 - \frac{s_{ij}}{j-i} \right),
\]
with both products taken in the lexicographical order on the pairs \( (i,j) \). These formulas are easily verified by using the well-known fact that the rational function

\[
R_{ij}(u) = 1 - \frac{s_{ij}}{u} + \frac{e_{ij}}{u - \omega/2 + 1}
\]
is a solution of the Yang–Baxter equation

\[
R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u);
\]
see [23]. These multiplicative formulas for the idempotents do not seem to have natural analogs for general up-down tableaux. Note, however, that the following alternative rational function in the case of $B_3(\omega)$ can be used instead of $\Psi(u_1, u_2, u_3)$ in the formulation of the fusion procedure:

$$\tilde{\Psi}(u_1, u_2, u_3) = \left(1 - (u_1 - u_2) s_1 + \frac{u_1 - u_2 - 1}{u_1 + u_2} e_1\right) \left(1 - (u_1 - u_3) s_2 + \frac{u_1 - u_3 - 2}{u_2 + u_3} e_2\right) \times \left(1 - (u_1 - u_2) s_1 + \frac{u_1 - u_2 - 1}{u_1 + u_2} e_1\right).$$

ACKNOWLEDGMENTS

We are grateful to Maxim Nazarov and Oleg Ogievetsky for valuable discussions. We acknowledge the support of the Australian Research Council. The first author would like to thank the School of Mathematics and Statistics of the University of Sydney for warm hospitality during his visit.

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Received 15/JAN/2010

Originally published in English