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GAUDIN HAMILTONIANS GENERATE THE BETHE ALGEBRA OF A TENSOR POWER OF THE VECTOR REPRESENTATION OF $\mathfrak{gl}_N$

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To Ludwig Dmitrievich Faddeev on his 75th birthday

Abstract. It is shown that the Gaudin Hamiltonians $H_1, \ldots, H_n$ generate the Bethe algebra of the n-fold tensor power of the vector representation of $\mathfrak{gl}_N$. Surprisingly, the formula for the generators of the Bethe algebra in terms of the Gaudin Hamiltonians does not depend on N. Moreover, this formula coincides with Wilson's formula for the stationary Baker–Akhiezer function on the adelic Grassmannian.

§1. Introduction

The Gaudin model describes a completely integrable quantum spin chain [G1, G2]. We consider the Gaudin model associated with the Lie algebra $\mathfrak{gl}_N$. Denote by $L_{\boldsymbol{\lambda}}$ the irreducible finite-dimensional $\mathfrak{gl}_N$-module with highest weight $\boldsymbol{\lambda}$. Consider a tensor product $\bigotimes_{a=1}^n L_{\boldsymbol{\lambda}^{(a)}}$ of such modules and two sequences of complex numbers: $K_1, \ldots, K_N$ and $z_1, \ldots, z_n$. Assume that the numbers $z_1, \ldots, z_n$ are distinct. The Hamiltonians of the Gaudin model are mutually commuting operators $H_1, \ldots, H_n$ acting on the space $\bigotimes_{a=1}^n L_{\boldsymbol{\lambda}^{(a)}}$,

$$H_a = \sum_{i=1}^N K_i\, e_{ii}^{(a)} + \sum_{i,j=1}^N \sum_{b\neq a} \frac{e_{ij}^{(a)} e_{ji}^{(b)}}{z_a - z_b}, \tag{1.1}$$

where the e_{ij} are the standard generators of $\mathfrak{gl}_N$, and $e_{ij}^{(a)}$ is the image of $1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(n-a)}$.

One of the main problems in the Gaudin model is to find eigenvalues and joint eigenvectors of the operators $H_1, \ldots, H_n$; see [B, RV, MTV1]. The Gaudin Hamiltonians appear also as the right-hand sides of the Knizhnik–Zamolodchikov equations; see [SV, RV, FFR, FMTV].

It was realized a long time ago that there are additional interesting operators commuting with the operators $H_1, \ldots, H_n$; see for example [KS, FFR]. Those operators are called the higher Gaudin Hamiltonians. To distinguish the operators $H_1, \ldots, H_n$, we will call them the classical Gaudin Hamiltonians. The algebra generated by all of the classical and higher Gaudin Hamiltonians is called the Bethe algebra. A useful formula for generators of the Bethe algebra was suggested in [T]; see also [MTV1, CT].

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In general, the Bethe algebra is larger than its subalgebra generated by the classical Gaudin Hamiltonians. Nevertheless, in this paper we show that if all factors of the tensor product $\bigotimes_{a=1}^{n} L_{\boldsymbol{\lambda}^{(a)}}$ are the standard vector representations of $\mathfrak{gl}_N$, then the classical Gaudin Hamiltonians generate the entire Bethe algebra. It is a surprising fact, because every tensor product of polynomial $\mathfrak{gl}_N$-modules is a submodule of a tensor power of the vector representation, and one may expect that the Bethe algebra of a tensor power of the vector representation is as general as the Bethe algebra of a tensor product of arbitrary representations. Another surprising fact is that our formula for the elements of the Bethe algebra in terms of the classical Gaudin Hamiltonians does not depend on N; see Theorem 3.2. The third surprise is that our formula is nothing else but Wilson's formula for the stationary Baker-Akhiezer function on the adelic Grassmannian [Wi].

Our theorem can be used to study the higher Gaudin Hamiltonians as functions of the classical Hamiltonians (or as limits of functions of the classical Gaudin Hamiltonians). Much more is known about the classical Hamiltonians than about the higher Hamiltonians.

Our proof of Theorem 3.2 is not elementary. We use the fact that the Bethe algebra is preserved under the $(\mathfrak{gl}_N, \mathfrak{gl}_n)$-duality and the completeness of the Bethe ansatz for a tensor product of vector representations and generic $K_1, \dots, K_N$, $z_1, \dots, z_n$.

§2. Bethe algebra

2.1. Lie algebras $\mathfrak{gl}_N$ and $\mathfrak{gl}_N[t]$. Let e_{ij}, $i, j = 1, \dots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{ij}, e_{sk}] = \delta_{js} e_{ik} - \delta_{ik} e_{sj}$. Let $\mathfrak{h} \subset \mathfrak{gl}_N$ be the Cartan subalgebra generated by e_{ii}, $i = 1, \dots, N$.

We denote by $V = \bigoplus_{i=1}^{N} \mathbb{C} v_i$ the standard N-dimensional vector representation of $\mathfrak{gl}_N$: $e_{ij} v_j = v_i$ and $e_{ij} v_k = 0$ for $j \neq k$.

Let M be a $\mathfrak{gl}_N$-module. A vector $v \in M$ is said to be *singular* if $e_{ij} v = 0$ for $1 \le i < j \le N$. We denote by M^{sing} the subspace of all singular vectors in M.

Let $\mathfrak{gl}_N[t] = \mathfrak{gl}_N \otimes \mathbb{C}[t]$ be the complex Lie algebra of $\mathfrak{gl}_N$-valued polynomials with the pointwise commutator. For $g \in \mathfrak{gl}_N$, we set $g(u) = \sum_{s=0}^{\infty} (g \otimes t^s) u^{-s-1}$.

We identify $\mathfrak{gl}_N$ with the subalgebra $\mathfrak{gl}_N \otimes 1$ of constant polynomials in $\mathfrak{gl}_N[t]$. Hence, any $\mathfrak{gl}_N[t]$-module has a canonical structure of a $\mathfrak{gl}_N$-module.

For each $a \in \mathbb{C}$, there exists an automorphism ρ_a of $\mathfrak{gl}_N[t]$, $\rho_a : g(u) \mapsto g(u-a)$. Given a $\mathfrak{gl}_N[t]$-module M, we denote by $M(a)$ the pullback of M through the automorphism ρ_a. As $\mathfrak{gl}_N$-modules, M and $M(a)$ are isomorphic by the identity map.

We have the evaluation homomorphism, $\mathfrak{gl}_N[t] \to \mathfrak{gl}_N$, $g(u) \mapsto g u^{-1}$. Its restriction to the subalgebra $\mathfrak{gl}_N \subset \mathfrak{gl}_N[t]$ is the identity map. For any $\mathfrak{gl}_N$-module M, we denote by the same letter the $\mathfrak{gl}_N[t]$-module obtained by pulling M back through the evaluation homomorphism.

2.2. Bethe algebra. Given an $N \times N$-matrix A with possibly noncommuting entries a_{ij}, we define its *row determinant* to be

$$\operatorname{rdet} A = \sum_{\sigma \in S_N} (-1)^{\sigma} \, a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{N\sigma(N)}.$$

Let $K_1, \dots, K_N$ be a sequence of complex numbers. Let ∂_u be the operator of differentiation in the variable u. Define the *universal differential operator* $\mathcal{D}^K$ by the formula

$$\mathcal{D}^K = \operatorname{rdet} \begin{pmatrix} \partial_u - K_1 - e_{11}(u) & -e_{21}(u) & \dots & -e_{N1}(u) \\ -e_{12}(u) & \partial_u - K_2 - e_{22}(u) & \dots & -e_{N2}(u) \\ \dots & \dots & \dots & \dots \\ -e_{1N}(u) & -e_{2N}(u) & \dots & \partial_u - K_N - e_{NN}(u) \end{pmatrix}.$$

This is a differential operator in u, whose coefficients are formal power series in u^{-1} with coefficients in $U(\mathfrak{gl}_N[t])$,

$$\mathcal{D}^K = \partial_u^N + \sum_{i=1}^N B_i^K(u)\,\partial_u^{N-i}, \quad B_i^K(u) = \sum_{j=0}^\infty B_{ij}^K u^{-j},$$

and $B_{ij}^K \in U(\mathfrak{gl}_N[t]),\ i=1,\dots,N,\ j\in\mathbb{Z}_{\ge 0}$. We have

$$\partial_u^N + \sum_{i=1}^N B_{i0}^K\,\partial_u^{N-i} = \prod_{i=1}^N (\partial_u - K_i). \tag{2.1}$$

The unital subalgebra of $U(\mathfrak{gl}_N[t])$ generated by B_{ij}^K, $i=1,\dots,N$, $j\in\mathbb{Z}_{>0}$, is called the *Bethe algebra* and is denoted by $\mathcal{B}^K$.

By [T, MTV1, CT], the algebra $\mathcal{B}^K$ is commutative, and $\mathcal{B}^K$ commutes with the subalgebra $U(\mathfrak{h})\subset U(\mathfrak{gl}_N[t])$. If all $K_1,\dots,K_N$ coincide, then $\mathcal{B}^K$ commutes with the subalgebra $U(\mathfrak{gl}_N)\subset U(\mathfrak{gl}_N[t])$.

As a subalgebra of $U(\mathfrak{gl}_N[t])$, the algebra $\mathcal{B}^K$ acts on any $\mathfrak{gl}_N[t]$-module M. Since $\mathcal{B}^K$ commutes with $U(\mathfrak{h})$, it preserves the weight subspaces of M. If all $K_1,\dots,K_N$ coincide, then $\mathcal{B}^K$ preserves the subspace M^{sing} of singular vectors. If L is a $\mathcal{B}^K$-module, then the image of $\mathcal{B}^K$ in $\mathrm{End}(L)$ is called the *Bethe algebra of* L.

For our purposes it is convenient to consider another set of generators of the Bethe algebra $\mathcal{B}^K$, defined as follows. Let x be a new variable, and let

$$\begin{aligned}\Psi^K(u,x) &= \Big(x^N + \sum_{i=1}^N B_i^K(u)\,x^{N-i}\Big)\prod_{i=1}^N \frac{1}{x-K_i}\\ &= 1+\sum_{i=1}^\infty \Psi_i^K(u)\,x^{-i}.\end{aligned} \tag{2.2}$$

The series $\Psi_i^K(u),\ i\in\mathbb{Z}_{>0},$ are linear combinations of the series $B_i^K(u),\ i=1,\dots,N,$ and *vice versa*. Write

$$\Psi_i^K(u) = \sum_{j=1}^\infty \Psi_{ij}^K\,u^{-j}. \tag{2.3}$$

Then $\Psi_{ij}^K,\ i,j\in\mathbb{Z}_{>0}$, is a new set of generators of the Bethe algebra $\mathcal{B}^K$.

§3. Classical Gaudin Hamiltonians on $\bigotimes_{a=1}^n V(z_a)$

Recall that V is the vector representation of the Lie algebra $\mathfrak{gl}_N$. Consider the tensor product $\bigotimes_{a=1}^n V(z_a)$ of evaluation $\mathfrak{gl}_N[t]$-modules. The series $e_{ij}(u)$ acts on $\bigotimes_{a=1}^n V(z_a)$ as $\sum_{a=1}^n e_{ij}^{(a)}(u-z_a)^{-1}$, where $e_{ij}^{(a)}$ is the image of $1^{\otimes(a-1)}\otimes e_{ij}\otimes 1^{\otimes(n-a)}\in\big(U(\mathfrak{gl}_N)\big)^{\otimes n}$. We denote by $B_{ij},\Psi_{ij}\in\mathrm{End}(V^{\otimes n})$ the images of the elements $B_{ij}^K,\Psi_{ij}^K\in U(\mathfrak{gl}_N[t])$. Set

$$\begin{aligned}B_i(u) &= \sum_{j=0}^\infty B_{ij}u^{-j}, \quad \mathcal{D} = \partial_u^N + \sum_{i=1}^N B_i(u)\,\partial_u^{N-i},\\ \Psi_i(u) &= \sum_{j=1}^\infty \Psi_{ij}\,u^{-j}, \quad \Psi(u,x) = 1+\sum_{i=1}^\infty \Psi_i(u)\,x^{-i}.\end{aligned} \tag{3.1}$$

The series $B_i(u)$, $\Psi_i(u)$ sum up to rational functions of u with values in $\mathrm{End}(V^{\otimes n})$. Set in addition

$$\Psi_\dagger(x) = -\sum_{i=1}^\infty \Psi_{i1}x^{-i}.$$

Lemma 3.1. *We have*

$$\Psi_1(u) = -\sum_{a=1}^{n} \frac{1}{u - z_a}, \quad \Psi_2(u) = \sum_{a=1}^{n} \frac{1}{u - z_a}\Big(- H_a + \sum_{b\neq a} \frac{1}{z_a - z_b}\Big), \tag{3.2}$$

where

$$H_a = \sum_{i=1}^{N} K_i e_{ii}^{(a)} + \sum_{i,j=1}^{N} \sum_{b\neq a} \frac{e_{ij}^{(a)} e_{ji}^{(b)}}{z_a - z_b} \tag{3.3}$$

are the classical Gaudin Hamiltonians (1.1)*, and*

$$\Psi_\dagger(x) = \sum_{i=1}^{N} \sum_{a=1}^{n} \frac{e_{ii}^{(a)}}{x - K_i}.$$

Proof. The claim is straightforward. See also formula (8.5) and Appendix B in [MTV1]. □

To formulate our main result, we introduce a diagonal matrix

$$Z = \operatorname{diag}(z_1, \dots, z_n) \tag{3.4}$$

and a matrix

$$Q = \begin{pmatrix} h_1 & \frac{1}{z_2 - z_1} & \frac{1}{z_3 - z_1} & \dots & \frac{1}{z_n - z_1} \\ \frac{1}{z_1 - z_2} & h_2 & \frac{1}{z_3 - z_2} & \dots & \frac{1}{z_n - z_2} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{z_1 - z_n} & \frac{1}{z_2 - z_n} & \frac{1}{z_3 - z_n} & \dots & h_n \end{pmatrix} \tag{3.5}$$

depending on new variables $h_1, \dots, h_n$. Set

$$\begin{aligned} \psi(u, x, z_1, \dots, z_n, h_1, \dots, h_n) &= \det\big(1 - (u - Z)^{-1}(x - Q)^{-1}\big), \\ \phi(x, z_1, \dots, z_n, h_1, \dots, h_n) &= \det(x - Q), \\ \psi_\dagger(x, z_1, \dots, z_n, h_1, \dots, h_n) &= \operatorname{tr}\big((x - Q)^{-1}\big). \end{aligned} \tag{3.6}$$

Theorem 3.2. *The Bethe algebra of* $\bigotimes_{a=1}^{n} V(z_a)$ *is generated by the classical Gaudin Hamiltonians* $H_1, \dots, H_n$*. More precisely,*

$$\Psi(u, x) = \psi(u, x, z_1, \dots, z_n, H_1, \dots, H_n).$$

In particular,

$$\psi_\dagger(x, z_1, \dots, z_n, H_1, \dots, H_n) = \sum_{i=1}^{N} \sum_{a=1}^{n} \frac{e_{ii}^{(a)}}{x - K_i}. \tag{3.7}$$

Remark. Since $\operatorname{tr}\big((x - Q)^{-1}\big) = \partial_x \log\big(\det(x - Q)\big)$, formula (3.7) can be written as

$$\phi(x, z_1, \dots, z_n, H_1, \dots, H_n) = \prod_{i=1}^{N} (x - K_i)^{\sum_{a=1}^{n} e_{ii}^{(a)}}.$$

Remark. The matrix $[Q, Z] + 1$ has rank one. For every distinct $z_1, \dots, z_n$ and every $h_1, \dots, h_n$, the pair (Q, Z) determines a point of the nth Calogero–Moser space, hence, a point of the adelic Grassmannian. The function $e^{ux}\psi(u, x, z_1, \dots, z_n, h_1, \dots, h_n)$ is the

stationary Baker–Akhiezer function of that point; see §3 in [Wi]. Theorem 3.2 says that the coefficients $\psi_{ij}(z_1,\dots,z_n,H_1,\dots,H_n)$ of the stationary Baker–Akhiezer function,

$$\begin{aligned} &e^{ux}\psi(u,x,z_1,\dots,z_n,H_1,\dots,H_n)\\ &\qquad = e^{ux}\biggl(1+\sum_{i,j=1}^{\infty}\psi_{ij}(z_1,\dots,z_n,H_1,\dots,H_n)\,u^{-j}x^{-i}\biggr), \end{aligned}$$

generate the Bethe algebra of $\bigotimes_{a=1}^n V(z_a)$. More remarks on this subject can be found in §5.

Corollary 3.3. *For distinct real $K_1,\dots,K_N$, and distinct real $z_1,\dots,z_n$, the joint spectrum of the classical Gaudin Hamiltonians $H_1,\dots,H_n$ acting on $\bigotimes_{a=1}^n V(z_a)$ is simple. That is, the classical Gaudin Hamiltonians have a joint eigenbasis, and for any two vectors of the eigenbasis at least one of the classical Gaudin Hamiltonians has different eigenvalues for those vectors.*

Proof. By [MTV5], for distinct real $K_1,\dots,K_N$, and distinct real $z_1,\dots,z_n$, the Bethe algebra of $\bigotimes_{a=1}^n V(z_a)$ has simple spectrum. Therefore, the classical Gaudin Hamiltonians have simple spectrum by Theorem 3.2. □

Corollary 3.4. *If $K_1,\dots,K_N$ coincide, and $z_1,\dots,z_n$ are distinct and real, then the joint spectrum of classical Gaudin Hamiltonians $H_1,\dots,H_n$ acting on $\bigl(\bigotimes_{a=1}^n V(z_a)\bigr)^{\mathrm{sing}}$ is simple.*

Proof. By [MTV3], if $K_i=0$ for all $i=1,\dots,N$, and $z_1,\dots,z_n$ are real and distinct, then the Bethe algebra of $\bigl(\bigotimes_{a=1}^n V(z_a)\bigr)^{\mathrm{sing}}$ has simple spectrum. Therefore, the classical Gaudin Hamiltonians acting on $\bigl(\bigotimes_{a=1}^n V(z_a)\bigr)^{\mathrm{sing}}$ have simple spectrum by Theorem 3.2. The case of nonzero coinciding $K_1,\dots,K_N$ follows from the case of zero $K_1,\dots,K_N$, because $\sum_{i=1}^N e_{ii}^{(a)}=1$ for all $a=1,\dots,n$; see (3.3). □

§4. Proof of Theorem 3.2

4.1. Preliminary lemmas. For functions $f_1(x),\dots,f_m(x)$ of one variable, denote by

$$\mathrm{Wr}[f_1,\dots,f_m]=\det\begin{pmatrix} f_1 & f_1' & \dots & f_1^{(m-1)}\\ f_2 & f_2' & \dots & f_2^{(m-1)}\\ \dots & \dots & \dots & \dots\\ f_m & f_m' & \dots & f_m^{(m-1)}\end{pmatrix}$$

the Wronskian of $f_1(x),\dots,f_m(x)$. Set $\Delta=\prod_{1\le a<b\le n}(z_b-z_a)$,

$$P(u)=\prod_{a=1}^n(u-z_a)$$

and

$$P_a(u)=\prod_{b\ne a}\frac{u-z_b}{z_a-z_b},\quad a=1,\dots,n.$$

Let $f_a(x)=(x+\mu_a)e^{z_ax}$, $a=1,\dots,n$, where $\mu_1,\dots,\mu_n$ are new variables. Set

$$\begin{aligned} W(u,x)&=e^{-ux-\sum_{a=1}^n z_ax}\,\mathrm{Wr}\bigl[f_1(x),\dots,f_n(x),e^{ux}\bigr]\\ &=W_0(x)\biggl(u^n+\sum_{a=1}^n C_a(x)u^{n-a}\biggr). \end{aligned}$$

Clearly, $W_0(x)=e^{-\sum_{a=1}^n z_ax}\,\mathrm{Wr}\bigl[f_1(x),\dots,f_n(x)\bigr]$.

Lemma 4.1. *Let* $h_a = -\mu_a - \sum_{b\neq a} \frac{1}{z_a - z_b}$, $a = 1, \dots, n$. *Then*

$$W(u,x) = \Delta \cdot \det\bigl((u-Z)(x-Q)-1\bigr), \tag{4.1}$$

where the matrices Z *and* Q *are given by* (3.4) *and* (3.5). *In particular,*

$$W_0(x) = \Delta \cdot \det(x-Q). \tag{4.2}$$

Proof. First, we prove formula (4.2). Let S and T be $(n\times n)$-matrices with entries $S_{ab} = z_b^{a-1}$ and $T_{ab} = (a-1)z_b^{a-2}$, respectively. Clearly, $\det S = \Delta$. The entries of the matrix S^{-1} are determined by the relation $P_a(u) = \sum_{b=1}^n (S^{-1})_{ab}\, u^{b-1}$, so that the entries of $S^{-1}T$ are $(S^{-1}T)_{ab} = P_a'(z_b)$.

Let $M = \operatorname{diag}(\mu_1, \dots, \mu_n)$. Since $\partial_x^k f_a(x) = \bigl((x+\mu_a)z_a^k + kz_a^{k-1}\bigr)e^{z_a x}$, we have

$$W_0(x) = \det\bigl(S(x+M)+T\bigr) = \det S \cdot \det(x+M+S^{-1}T) = \Delta\cdot\det\bigl(x-Q\bigr).$$

To prove formula (4.1), set $z_{n+1} = u$. Let $\widehat{Q}$ be an $((n+1)\times(n+1))$-matrix with entries $\widehat{Q}_{ab} = (z_b - z_a)^{-1}$ for $a\neq b$, and

$$\widehat{Q}_{aa} = -\mu_a - \sum_{\substack{b=1\\ b\neq a}}^{n+1} \frac{1}{z_a - z_b},$$

where μ_{n+1} is a new variable. Set $f_{n+1}(x) = (x+\mu_{n+1})e^{z_{n+1}x}$. Similarly to (4.2), we have

$$e^{-\sum_{a=1}^{n+1} z_a x}\ \operatorname{Wr}\bigl[f_1(x), \dots, f_{n+1}(x)\bigr] = \Delta\cdot P(z_{n+1})\ \det(x-\widehat{Q}).$$

It is easy to see that

$$\operatorname{Wr}\bigl[f_1(x), \dots, f_n(x), e^{ux}\bigr] = \lim_{\mu_{n+1}\to\infty}\bigl(\mu_{n+1}^{-1}\operatorname{Wr}\bigl[f_1(x), \dots, f_{n+1}(x)\bigr]\bigr)$$

and

$$\lim_{\mu_{n+1}\to\infty}\bigl(\mu_{n+1}^{-1}\det(x-\widehat{Q})\bigr) = \det\bigl(x-Q-(u-Z)^{-1}\bigr).$$

Then

$$W(u,x) = \Delta\cdot P(u)\ \det\bigl(x-Q-(u-Z)^{-1}\bigr) = \Delta\cdot\det\bigl((u-Z)(x-Q)-1\bigr).$$

The lemma is proved. □

The complex vector space spanned by the functions $f_1, \dots, f_n$ is the kernel of the monic differential operator

$$D = \partial_x^n + \sum_{a=1}^n C_a(x)\partial_x^{n-a}. \tag{4.3}$$

The function $\psi(u,x)$, defined by (3.6), has the following expansion as $u\to\infty$, $x\to\infty$:

$$\psi(u,x) = 1 + \sum_{i=1}^\infty\sum_{j=1}^\infty \psi_{ij}u^{-j}x^{-i}. \tag{4.4}$$

Here we suppressed the arguments $z_1, \dots, z_n$, $h_1, \dots, h_n$. Set

$$\psi_i(u) = \sum_{j=1}^\infty \psi_{ij}u^{-j}, \quad i\in\mathbb{Z}_{>0}.$$

Lemma 4.2. *We have*

$$\psi_1(u) = -\sum_{a=1}^{N} \frac{1}{u - z_a}, \quad \psi_2(u) = \sum_{a=1}^{N} \frac{1}{u - z_a} \Big(- h_a + \sum_{b \neq a} \frac{1}{z_a - z_b} \Big), \tag{4.5}$$

and

$$\sum_{i=1}^{\infty} \psi_{i1}\, x^{-i} = -\operatorname{tr}\big((x - Q)^{-1}\big). \tag{4.6}$$

Proof. The proof is straightforward using formulae (3.5), (3.6). □

4.2. Proof of Theorem 3.2. Denote $\mathcal{D}_{\mathrm{reg}} = P(u)\,\mathcal{D}$. By Theorem 3.1 in [MTV2], we have

$$\mathcal{D}_{\mathrm{reg}} = \sum_{i=0}^{N} \sum_{a=0}^{n} A_{ia} u^a \partial^i, \quad A_{ia} \in \mathrm{End}(V^{\otimes n}), \tag{4.7}$$

and

$$\sum_{a=0}^{n} A_{Na} u^a = P(u), \quad \sum_{i=0}^{N} A_{in} \partial^i = R(\partial_u), \quad R(x) = \prod_{i=1}^{N} (x - K_i).$$

Let $v \in \bigotimes_{a=1}^{n} V(z_a)$ be an eigenvector of the Bethe algebra, $A_{ia} v = \alpha_{ia} v$, $\alpha_{ia} \in \mathbb{C}$, for all (i, a). Consider a scalar differential operator

$$D_v = \sum_{i=0}^{N} \sum_{a=0}^{n} \alpha_{ia}\, x^i \partial_x^a.$$

Notice that we changed $u \mapsto \partial_x$, $\partial_u \mapsto x$ compared with (4.7). By Theorem 3.1 in [MTV2] and Theorem 12.1.1 in [MTV4], the kernel of D_v is generated by the functions $(x + \mu_a) e^{z_a x}$, $a = 1, \ldots, n$, with suitable $\mu_a \in \mathbb{C}$. Let

$$h_a = -\mu_a - \sum_{b \neq a} \frac{1}{z_a - z_b}, \quad a = 1, \ldots, n.$$

Lemma 4.3. *We have $H_a v = h_a v$ for all $a = 1, \ldots, n$.*

Proof. We have $D_v = R(x)\, D$, where D is given by (4.3). Then Lemma 4.1 and formulae (2.2), (2.3), (3.1) show that the eigenvalues of the operators Ψ_{ij} are the numbers ψ_{ij} given by (4.4): $\Psi_{ij} v = \psi_{ij} v$. The claim follows from comparing formulae (3.2) and (4.5). □

By Theorem 10.5.1 in [MTV4], if $K_1, \ldots, K_N$ and $z_1, \ldots, z_n$ are generic, then the Bethe algebra of $\bigotimes_{a=1}^{n} V(z_a)$ has an eigenbasis. Hence, by Lemmas 4.1 and 4.3, for such $K_1, \ldots, K_N$, $z_1, \ldots, z_n$ we have

$$\Psi(u, x) = \psi(u, x, z_1, \ldots, z_n, H_1, \ldots, H_n).$$

Since both sides of this identity are meromorphic functions of $K_1, \ldots, K_N$ and $z_1, \ldots, z_n$, the identity holds true for all $K_1, \ldots, K_N$, $z_1, \ldots, z_n$. The theorem is proved. □

§5. Bethe algebra and functions on the Calogero–Moser space

5.1. Calogero–Moser space $\mathcal{C}_n$. Let $\mathcal{M}_n$ be the space of complex matrices of size $n\times n$. The group GL_n acts on $\mathcal{M}_n\oplus\mathcal{M}_n$ by conjugation, $g:(X,Y)\mapsto(gXg^{-1},gYg^{-1})$. Denote $\widehat{\mathcal{F}}_n=\mathbb{C}[\mathcal{M}_n\oplus\mathcal{M}_n]^{GL_n}$.

Let $\mathcal{C}_n\subset\mathcal{M}_n\oplus\mathcal{M}_n$ be the subset of pairs (X,Y) with the matrix $[X,Y]+1$ having rank one. The set $\mathcal{C}_n$ is GL_n-invariant. Let $\mathcal{I}_n\subset\widehat{\mathcal{F}}_n$ be the ideal of functions vanishing on $\mathcal{C}_n$. By definition, the algebra $\mathcal{F}_n=\widehat{\mathcal{F}}_n/\mathcal{I}_n$ is the algebra of functions on the nth Calogero–Moser space; see [Wi].

Consider a function

$$\phi(u,x,X,Y)=\det\bigl(1-(u-Y)^{-1}(x-X)^{-1}\bigr), \tag{5.1}$$

depending on matrices X,Y and variables u,x. It has an expansion as $u\to\infty$, $x\to\infty$:

$$\phi(u,x,X,Y)=1+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\phi_{ij}(X,Y)u^{-j}x^{-i} \tag{5.2}$$

with $\phi_{ij}\in\widehat{\mathcal{F}}_n$ for any (i,j).

Lemma 5.1 ([MTV6])**.** *The algebra $\mathcal{F}_n$ is generated by the images of ϕ_{ij}, $i,j\in\mathbb{Z}_{>0}$.*

5.2. Bethe algebra and functions on $\mathcal{C}_n$. In this section we treat $K_1,\dots,K_N$ and $z_1,\dots,z_n$ as variables. Set

$$\mathcal{E}_{N,n}=\mathrm{End}(V^{\otimes n})\otimes\mathbb{C}[K_1,\dots,K_N,z_1,\dots,z_n].$$

We identify the algebras $\mathrm{End}(V^{\otimes n})$ and $\mathbb{C}[K_1,\dots,K_N,z_1,\dots,z_n]$ with the respective subalgebras $\mathrm{End}(V^{\otimes n})\otimes 1$ and $1\otimes\mathbb{C}[K_1,\dots,K_N,z_1,\dots,z_n]$ of $\mathcal{E}_{N,n}$.

The operators B_{ij} and Ψ_{ij}, defined in §3, depend on $K_1,\dots,K_N$, $z_1,\dots,z_n$ polynomially, so we view them as elements of $\mathcal{E}_{N,n}$. Denote by $\mathcal{B}_{N,n}$ the unital subalgebra of $\mathcal{E}_{N,n}$ generated by B_{ij}, $i=1,\dots,N$, $j\in\mathbb{Z}_{\ge 0}$.

Lemma 5.2. *The algebra $\mathcal{B}_{N,n}$ is generated by Ψ_{ij}, $i=1,\dots,N$, $j\in\mathbb{Z}_{>0}$, and symmetric polynomials in $K_1,\dots,K_N$.*

Proof. By formula (2.1), we have

$$x^N+\sum_{i=1}^{N}B_{i0}\,x^{N-i}=\prod_{i=1}^{N}(x-K_i),$$

so that symmetric polynomials in $K_1,\dots,K_N$ belong to $\mathcal{B}_{N,n}$. Formula (2.2) yields

$$\Bigl(x^N+\sum_{i=1}^{N}\sum_{j=0}^{\infty}B_{ij}\,u^{-j}x^{N-i}\Bigr)\prod_{i=1}^{N}\frac{1}{x-K_i}=1+\sum_{i=1}^{\infty}\sum_{i=1}^{\infty}\Psi_{ij}\,u^{-j}x^{-i}.$$

Therefore, the elements Ψ_{ij} are linear combinations of the elements B_{ij} with coefficients being symmetric polynomials in $K_1,\dots,K_N$, and *vice versa*. That proves the claim. □

Let Z,Q be the matrices given by (3.4), (3.5). For any $f\in\widehat{\mathcal{F}}_n$, define a function $\bar f$ of the variables $z_1,\dots,z_n$, $h_1,\dots,h_n$by the formula

$$\bar f(z_1,\dots,z_n,h_1,\dots,h_n)=f(Q,Z).$$

Lemma 5.3. *The function $\bar f$ depends only on the image of f in $\mathcal{F}_n$.*

Proof. Since the matrix $[Q,Z]+1$ has rank one, the pair (Q,Z) belongs to $\mathcal{C}_n$. □

Theorem 5.4. *For any $f\in\widehat{\mathcal{F}}_n$, we have $\bar f(z_1,\dots,z_n,H_1,\dots,H_n)\in\mathcal{B}_{N,n}$. In particular, $f(z_1,\dots,z_n,H_1,\dots,H_n)$ is a polynomial in $z_1,\dots,z_n$.*

Proof. By Lemmas 5.3 and 5.1, it suffices to prove the claim for the functions $\phi_{ij}(X,Y)$. Since $\bar{\phi}_{ij} = \psi_{ij}$ by (5.1), (5.2), (3.6), and (4.4), and $\psi_{ij}(z_1,\dots,z_n,H_1,\dots,H_n) = \Psi_{ij}$ by Theorem 3.2, the statement follows from Lemma 5.2. □

Example. Let $N = n = 2$. Then $Z = \mathrm{diag}(z_1, z_2)$,

$$Q = \begin{pmatrix} h_1 & (z_2 - z_1)^{-1} \\ (z_1 - z_2)^{-1} & h_2 \end{pmatrix},$$

$$H_1 = K_1 e_{11}^{(1)} + K_2 e_{22}^{(1)} + \frac{\Omega}{z_1 - z_2},$$

$$H_2 = K_1\, e_{11}^{(2)} + K_2 e_{22}^{(2)} + \frac{\Omega}{z_2 - z_1},$$

$$\Omega = e_{11}^{(1)} e_{11}^{(2)} + e_{12}^{(1)} e_{21}^{(2)} + e_{21}^{(1)} e_{12}^{(2)} + e_{22}^{(1)} e_{22}^{(2)}.$$

Let $f(X,Y) = \mathrm{tr}(X^2)$. Then $\bar{f}(z_1,z_2,H_1,H_2) = H_1^2 + H_2^2 - 2(z_1 - z_2)^{-2}$ is a polynomial in z_1, z_2.

Remark. It is known that $\widehat{\mathcal{F}}_n$ is spanned by the functions

$$\mathrm{tr}(X^{m_1} Y^{m_2} X^{m_3} Y^{m_4} \cdots),$$

where $m_1, m_2, \dots$ are nonnegative integers; see [W].

Theorems 3.2 and 5.4 show that the assignment $\gamma : f \mapsto \bar{f}(z_1,\dots,z_n,H_1,\dots,H_n)$ defines an algebra homomorphism $\widehat{\mathcal{F}}_n \to \mathcal{B}_{N,n}$ that sends ϕ_{ij} to Ψ_{ij}. By Lemma 5.3, this homomorphism factors through $\mathcal{F}_n$. By Lemma 5.2, the images of $\widehat{\mathcal{F}}_n$ tensored with the algebra of symmetric polynomials in $K_1,\dots,K_N$ generate $\mathcal{B}_{N,n}$.

We showed in [MTV6] that for $n = N$, the homomorphism γ induces an isomorphism of $\mathcal{F}_N$ with the quotient of $\mathcal{B}_{N,N}$ by the relations

$$\Psi_{i1} = -\sum_{j=1}^{N} K_j^{i-1}, \quad i \in \mathbb{Z}_{>0}.$$

In other words, let

$$(V^{\otimes N})_{\mathbf{1}} = \left\{ v \in V^{\otimes N} \;\middle|\; \sum_{a=1}^{N} e_{ii}^{(a)} v = v,\ i = 1,\dots,N \right\}.$$

Each element of $\mathcal{B}_{N,N}$ induces an element of

$$\mathrm{End}\big((V^{\otimes N})_{\mathbf{1}}\big) \otimes \mathbb{C}[K_1,\dots,K_N,z_1,\dots,z_N].$$

Then $\mathcal{F}_N$ is isomorphic to the image of $\mathcal{B}_{N,N}$ in

$$\mathrm{End}\big((V^{\otimes N})_{\mathbf{1}}\big) \otimes \mathbb{C}[K_1,\dots,K_N,z_1,\dots,z_N].$$

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