Алгебра и анализ
Том 22 (2010), № 4

St. Petersburg Math. J.
Vol. 22 (2011), No. 4, Pages 573–614
S 1061-0022(2011)01159-8
Article electronically published on May 2, 2011

GRÖBNER–SHIRSHOV BASES OF THE LIE ALGEBRA D_n^+

A. N. KORYUKIN

Abstract. Over a field of characteristic 0, the reduced Gröbner–Shirshov bases (RGShB) are computed in the positive part D_n^+ of the simple finite-dimensional Lie algebra D_n for the canonical generators corresponding to simple roots, *under an arbitrary ordering of these generators* (i.e., an aritrary basis among the $n!$ bases is fixed and analyzed). In this setting, the RGShBs were previously computed by the author for the Lie algebras A_n^+, B_n^+, and C_n^+. For one ordering of the generators, the RGShBs of these algebras were calculated by Bokut and Klein (1996).

Introduction

In 1962 Shirshov [1], and in 1965 Buchberger [2] suggested essentially the same ideas for Lie polynomials (Shirshov) and commutative polynomials (Buchberger). Actually, they developed a universal way of computing a basis of an arbitrary algebra L of a certain variety. For this, the structure of the algebra L should be enriched somewhat: one needs to choose a set X of generators in L, to specify a basis of the free algebra of this variety that is determined by this set of generators, and to give a linear ordering of this basis.

After that, everything reduces to the properties of relations in the algebra L. Shirshov introduced the notion of a *composition* of (Lie) polynomials and the notion of a *set of polynomials closed* under composition. Later on, the sets closed under composition were called the Gröbner bases (GB) in the commutative case and the Gröbner–Shirshov bases (GShB) in the case of Lie algebras. Among other sets of Lie polynomials, the GsHBs are distinguished by the fact that, knowing a GShB, one can easily find a basis of the algebra L associated with it.

With the knowledge of a basis of the algebra, it is rather easy to solve some problems (for example, the word problem). Moreover, a GShB in fact contains much more information than a basis of the algebra. For this reason, the GShBs have gained recognition rather quickly, and a new area of study has been opened.

These new ideas found application immediately after the paper [1]. The first applications were given in the paper [1] itself; namely, the word problem was solved and a theorem was proved concerning the property of being free for Lie algebras with one defining relation. At the same time, in the paper [3], Shirshov used GShBs to disprove a conjecture on the structure of subalgebras of the free product of Lie algebras.

In [4], with the help of GShBs, Bokut′ proved that an arbitrary Lie algebra can be embedded in an algebraically closed algebra.

At present, the GBs and GShBs are actively used both in theoretic investigations and in computational techniques. One of the advantages of the new method is that it brings an algorithmic flavor in a search for answers to a large number of questions (for example, see the papers of Buchberger [5] and Gerdt [6] and also many other papers of

2010 *Mathematics Subject Classification.* Primary 17B22.
Key words and phrases. Gröbner–Shirshov, bases Lie algebras.

these authors). As a consequence, it became possible to apply computational techniques to many problems. For example, if a GShB of an arbitrary Lie algebra is recursive, then this algebra contains a linear recursive basis, and the question as to whether an arbitrary Lie polynomial belongs to an ideal of relations is solved algorithmically.

Thus, a GShB is an important object of study; an active investigation of this object began in the 1990s. Bokut′ and Klein found GShBs of the simple finite-dimensional Lie algebras A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2; see [7]–[10]. Moreover, together with Malcolmson, they described the GShBs for quantum universal enveloping algebras of type A_n [11] and proved that a set of Lie polynomials is a GShB of the corresponding Lie algebra if and only if it is a GShB in the associative sense [12].

The idea of a GShB, suggested in the cases of commutative algebras and Lie algebras, has received further development.

The recognition of some properties of algebras given with the help of GShBs was the subject of the paper [13] by Gateva-Ivanova and Latyshev. In the 1980s, Mikhalev extended the techniques of compositions to the case of superalgebras by proving the composition lemma for color superalgebras; see [14]–[16]. In [17], Bokut′, Kang, Lee, and Malcolmson found GShBs for the simple finite-dimensional Lie superalgebras of type A_n, B_n, C_n, or D_n. In [18, 19], Kang and Lee constructed the theory of GShBs for modules over Lie algebras and found GShBs (which were called Gröbner–Shirshov pairs in this case) for the irreducible sl_n-modules. Chibrikov [20] simplified significantly the notion of a GShB for modules: it became not a pair but a single set in a free module over a free algebra.

Moreover, the theory of GShBs was employed in the cases of conformal and vertex algebras. In the papers [21, 22], Roitman proved the existence of free conformal algebras (this fact is not a consequence of theorems in universal algebra, because conformal algebras do not form a variety). In [23], Bokut′, Fong, and Ke obtained the results of Roitman in a different way. Those authors applied ideas and techniques related to GShBs to associative conformal algebras [24].

Vasil′iev and Pavlov [25] listed all monomial orders (in the commutative case), and they plan to review all GBs of a finitely generated commutative algebra. The history of the theory of GShBs was presented in [26, 27].

In the papers [7, 8], for a certain ordering of generators, Bokut′ and Klein computed the reduced GShBs (BGShBs) for series of simple Lie algebras (the RGShBs are distinguished by the fact that they are uniquely determined by generators and the order on generators). For affine nontwisted Kac–Moody algebras, Poroshenko [28, 29] computed first a certain basis of a Lie algebra as a linear space (this basis was uniquely determined by the order on the Lie monomials that correspond to regular words, a so-called reduced basis), and then he found the RGShB uniquely determined by that basis. A conjecture was stated that, for the classical simple Lie algebras, the reduced basis, which is uniquely determined by the root system and the order on the set of monomials, can be described in terms of the root system. Later, this conjecture was confirmed for fields of characteristic 0.

So far, the RGShBs have been computed only for one ordering of the generators. Bokut′ proposed to compute RGShBs (for the order deg-lex) under an arbitrary ordering of generators. Thus, an entirely new problem arose: for classical simple Lie algebras, under an arbitrary ordering of generators, to describe the reduced bases and then the RGShBs uniquely determined by them in terms of the root system. In this wording, the generators of the Lie algebra are fixed, but the order on them is arbitrary, and the $n!$ RGShBs are analyzed simultaneously (n is the number of generators). This entirely new problem was solved by the author for the Lie algebras A_n^+, B_n^+, C_n^+ over a field of characteristic 0 in the papers [30]–[32], and for the Lie algebra D_n^+ in the present

paper. This is a new step in the development of the theory of GShBs. For this paper, the reduced bases of the Lie algebra D_n^+ were computed earlier in [33]. Before that, the reduced bases for the algebra D_n^+ had been found in [34, 8], but for only one ordering of generators.

§1. Reduced Gröbner bases (the commutative case)

First, we give a definition of a GB in the commutative case (which is much easier than the case of Lie algebras). Let A be a finitely generated commutative algebra given by a finite set of generators $X = \{x_1, \dots, x_n\}$ and a collection of relations. It can be regarded as the quotient algebra of polynomials $k[x_1, \dots, x_n]$ by a certain ideal I. The relations of the algebra A can be identified with the elements of I.

The set $\mathbb{Z}^n_{\geq 0}$ and, what is equivalent, the set of monomials $\{x^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0}\}$ are linearly ordered so that if $\alpha \prec \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma \prec \beta + \gamma$ (here the order relation is denoted by $\preceq$).

For an arbitrary polynomial f, the greatest monomial occurring in the representation of f as a linear combination of monomials is called the highest monomial for f, and its coefficient is the highest coefficient of f [35, p. 56, Definition 7].

A finite subset $G = \{g_1, \dots, g_s\}$ of an ideal I of the algebra of polynomials $k[x_1, \dots, x_n]$ is called a *Gröbner basis* or a *standard basis* of I if the ideal generated by the highest monomials of the polynomials in G coincides with the ideal generated by the highest monomials of the polynomials in I [35, p. 74, Definition 5].

In other words, a Gröbner basis (GB) of I is a finite set of relations (polynomials in I) such that the highest monomial of any relation is divisible by the highest monomial of a certain element in the GB. A GB of the ideal I may be also called a GB of the quotient algebra $k[x_1, \dots, x_n]/I$.

A GB depends on the ideal I and the monomial order. But, given an ideal I and a monomial order, many GBs may exist. This stimulates a search for conditions that determine a GB uniquely.

A *minimal* BG (see [35, p. 89, Definition 4]) is a Gröbner basis G such that 1) the highest coefficient of any polynomial of G is equal to 1; 2) for all $p \in G$, the highest monomial of p does not belong to the ideal generated by the highest monomials of the polynomials in $G - \{p\}$, or, for two distinct polynomials in G, the highest monomial of any of them does not divide the highest monomial of the other.

In this definition, there is a restriction only on the highest words of the polynomials in a GB, and a minimal GB is not determined uniquely by the ideal I and the monomial order.

A *reduced* GB (see [35, p. 90, Definition 5]) is a Gröbner basis G such that 1) the highest coefficient of any polynomial in G is equal to 1; 2) for all $p \in G$, no monomial of p is divisible by any highest monomial of the polynomials in $G - \{p\}$.

A reduced GB is a minimal GB. For any nonzero ideal I and any monomial order, a reduced GB exists and is determined uniquely (see [35, p. 90, Proposition 6]). In the present paper, we need some analysis of the notion of a reduced GB.

It is clear that the set of highest monomials of the elements in I is closed under multiplication by an arbitrary monomial. For this reason, all monomials can be split into two parts relative to the ideal I.

1) The highest monomials of the relations (elements of I). These monomials can be represented modulo I by linear combinations of lesser monomials.

2) The monomials that are not the highest monomials of relations. These monomials cannot be represented modulo I by linear combinations of lesser monomials. They form a basis of the quotient algebra $k[x_1, \dots, x_n]/I$.

The monomials that are not representable modulo I by a linear combination of lesser monomials are said to be *reduced*, and the other monomials are *nonreduced*. The set of reduced monomials forms a basis of the quotient algebra $k[x_1, \dots, x_n]/I$. This basis is said to be *reduced*. Any relation (an element of the ideal I) is a linear combination of relations of the form $e - \pi(e)$, where e is a nonreduced monomial and $\pi(e)$ is a unique linear combination of reduced monomials smaller than e.

The relations $e-\pi(e) = 0$ are said to be *canonical*. Associating the polynomial $e-\pi(e)$ with each nonreduced monomial e, we obtain a one-to-one correspondence between the set of nonreduced monomials and the set of canonical relations.

Now we can define a reduced GB of the ideal I as a set of canonical relations that generates the ideal I and is such that the highest monomials of the elements in this GB are not divisors of each other (we do not mention that a reduced GB consists of finitely many elements, because in a finitely generated commutative algebra this is true automatically). In other words, we can define a reduced Gröbner basis G as the set of all canonical relations $e - \pi(e)$ in which the highest monomial e is minimal among the nonreduced monomials under the order "the monomial e_1 is a divisor of the monomial e_2". From this definition, the property that the set of relations G generates the ideal I is obtained immediately.

A reduced Gröbner basis can also be defined as the smallest set of canonical relations that generates the quotient algebra $k[x_1, \dots, x_n]/I$ (I is an ideal). Indeed, first, if e is a nonreduced monomial such that any proper divisor of it (a monomial in the algebra of polynomials) is a reduced monomial, then the canonical relation $e - \pi(e)$ is an element of any Gröbner basis of canonical relations. Second, the set of all canonical relations $e - \pi(e)$, where e is a nonreduced monomial such that any proper divisor of it is a reduced monomial, forms a Gröbner basis of canonical relations and even a reduced Gröbner basis.

In the case of Lie algebras,

1) instead of an arbitrary monomial ordering, only the ordering deg-lex is used;

2) instead of (commutative) monomials, the Lie monomials corresponding to the Lyndon–Shirshov words (nonassociative regular words) are considered;

3) the ordering $\preceq$, "a monomial e_1 divides a monomial e_2", converts into the following ordering: "removing brackets in a nonassociative regular word e_1 results in a subword of the associative regular word obtained by removing the brackets in a nonassociative regular word e_2";

4) a reduced Gröbner basis converts into a reduced Gröbner–Shirshov basis, namely, the smallest set of canonical relations that generates the quotient Lie algebra, or the set of canonical relations $e - \pi(e)$ that correspond to minimal words e (under the ordering $\preceq$) among the nonreduced words.

§2. Reduced Gröbner–Shirshov bases of Lie algebras

In the sequel, X is an arbitrary linearly ordered finite set (of letters); $\mathrm{Lie}(X)$ is the Lie algebra over an arbitrary field k of characteristic 0 freely generated by the set X. The words are tacitly assumed to be associative and composed of elements of X. On the set of words, we consider the linear ordering lex, which differs from the lexicographic order only in the condition that a proper beginning of any word is greater than the word itself.

As in [36, 37], we say that a nonempty word u is *regular* if, for any two nonempty words v and w, from $u = vw$ it follows that $vw > wv$. The regular words were introduced independently by Lyndon in [38]. In [38, 36, 39], the authors also considered *regular nonassociative words*, which satisfy the following conditions: the letters are regular nonassociative words; removing the brackets in a regular nonassociative word results

in a regular word; if a nonassociative word u can be represented in the form $u = u_1 z$, then the nonassociative words u_1 and z are regular; if $u_1 = xy$, then $y \leq z$.

At present, the regular words (both associative and nonassociative) are commonly named the *Lyndon–Shirshov words*. Western authors call them the *Lyndon words* (see the survey [40]).

In [36], Shirsov showed that, in a regular associative word u, the brackets can be arranged uniquely so as to yield a regular nonassociative word, which will be denoted by (u). This arrangement of brackets is said to be *regular*.

We denote by $\mathrm{kReg}(X)$ the space with the basis consisting of all regular words in letters from X. We extend the regular arrangement of brackets to a linear mapping $(\cdot)$ of the space $\mathrm{kReg}(X)$ to the space with a basis of regular nonassociative words in letters from X. This mapping is bijective.

For an arbitrary regular word u, the image of the word (u) under the canonical mapping to the free algebra $\mathrm{Lie}(X)$ is denoted by $[u]$. Consider the linear mapping $[\cdot]$ of the space $\mathrm{kReg}(X)$ to the free algebra $\mathrm{Lie}(X)$ that takes each regular word u to the Lie monomial $[u]$. An element w of the space $\mathrm{kReg}(X)$ is called the support for (w) and $[w]$.

In [36, 39], it was shown that the mapping $[\cdot]$ is bijective and the set of Lie monomials $[u]$, where u is a regular associative word, is a basis of the algebra $\mathrm{Lie}(X)$.

Now we turn to bases and relations of an arbitrary finitely generated Lie algebra $L = \mathrm{Lie}(X)/J$ given by a finite set of generators X and an ideal of relations J. Relations that are different from zero will be called nontrivial.

On the set of words, we consider the linear ordering deg-lex: a word x is less than a word y if the length of x is less than that of y, or these words are of equal length but x is less than y in the lexicographic order.

A word $\bar{w}$ will be called the *highest* word of a nonzero element w of $\mathrm{kReg}(X)$ if for a nonzero element α of the field k, the word $w - \alpha\bar{w}$ is a linear combination of words that are less than u in the deg-lex order. For $S \subseteq \mathrm{kReg}(X)$ the set of highest words of the elements of S different from zero will be denoted by $\bar{S}$.

A regular word u is said to be *reduced* if $[u]$ modulo the ideal J is not a linear combination of elements $[v]$, where v is a regular word that is less than u. The elements of the form $[u] + J$, where u is a reduced word, constitute a basis of the algebra $\mathrm{Lie}(X)/J$. This basis is said to be *reduced*.

Remark 2.1. Shirshov showed that if $0 \neq s \in \mathrm{kReg}(X)$ and u is a regular word that contains the subword $\bar{s}$, then in the Lie ideal of the free Lie algebra generated by the element $[s]$, there is a nonzero element for which the highest word of its support coincides with u (see [36, Lemma 4]).

Remark 2.2. Remark 2.1 implies that if a regular word has a subword that is the highest word of the support of a nontrivial relation, then this word itself is the highest word of the support of a nontrivial relation.

Thus, if $S \subseteq \mathrm{kReg}(X) - \{0\}$ and $[S]$ is a set of nontrivial relations of the algebra L, then the supports of the elements of a reduced basis are $\bar{S}$-reduced, i.e., they contain no subwords from $\bar{S}$.

Question 2.3. What are the sets $[S]$ of nontrivial relations for which the converse is true, i.e., when are the $\bar{S}$-reduced words reduced?

A set of relations as in Question 2.3 enables one to find a reduced basis easily. Shirshov found partial operations (he called them *compositions*) with respect to which the sets as in Question 2.3 are closed; he showed in [36] that such sets are precisely the sets of nontrivial relations that are closed with respect to compositions. Subsequently, such sets of relations

were called the Gröbner–Shirshov bases (GShB) (see [7]). This definition suggests taking arbitrary sets of relations and closing them with respect to compositions, which was done in many papers. But if we consider an arbitrary ordering of the generators, then calculation of compositions seems to be impossible. For this reason, in the present paper we give the following definition of a GShB.

Definition 2.4. A GShB of a Lie algebra L is a set $[S]$ of nontrivial relations of L for which all $\bar{S}$-reduced words are reduced.

Now there is no reason for considering compositions, and our nearest goal is to further analyze the notion of a GShB. If is easy to show that any GShB $[S]$ is a set of generating relations.

In papers concerning GShBs, the Gröbner–Shirshov bases under consideration are composed exclusively of relations of the form $[u-\pi(u)]$, where u is a regular word and $\pi(u)$ is a linear combination of reduced words that are less than u in the deg-lex order. Such relations will be called *canonical* relations. It is clear that $\pi(u)$ is uniquely determined by u, and that any relation can be represented uniquely as a linear combination of canonical ones. We extend π to a linear mapping of the space $\mathrm{kReg}(X)$.

As was shown in [30, §2], the intersection of all GsHBs of canonical relations of the algebra $L = \mathrm{Lie}(X)/J$ is a GShB of this algebra. It is called the reduced GShB (RGShB); see [30, §2].

Remark 2.5. In [30, §2], it was also shown that the RGShB of an arbitrary finitely generated Lie algebra L is precisely the set of canonical relations $[u - \pi(u)]$ in which the highest word u is not reduced, but any proper regular subword of it is reduced.

This will be our working definition of the RGShB of an arbitrary finitely generated algebra. Remark 2.5 shows that to calculate the RGShB it remains to learn how to calculate regular words that are not reduced but any proper regular subword of them is reduced, and also to construct canonical relations with such highest words.

Now we consider the case where the ideal J is homogeneous with respect to the number of occurrences of each letter. Let $\mathbb{Z}X$ denote the free Abelian group generated by the set X. The operation in this group will be denoted additively. The algebra $\mathrm{Lie}(X)$ is graded by the group $\mathbb{Z}X$: as an element of the algebra $\mathrm{Lie}(X)$, any generator $x \in X$ is homogeneous and lies in the homogeneous component that corresponds to x regarded as an element of $\mathbb{Z}X$. This graduation induces a graduation on the algebra $L = \mathrm{Lie}(X)/J$.

The image of an arbitrary word u under the natural homomorphism of the semigroup of words to the group $\mathbb{Z}X$ will be denoted by $|u|$ and will be called the *composition* of the word u. Moreover, if the homogeneous component of the algebra L that corresponds to the element $|u|$ is not zero, we shall say that the composition of the word u is rooted. Now we recall some useful properties of reduced words.

Lemma 2.6. *Let $L = \mathrm{Lie}(X)/J$ be a finitely generated algebra such that the ideal of its relations is homogeneous with respect to the number of occurrences of each letter. Then*

1) *the composition of any reduced word is rooted*;

2) *each regular subword of a reduced word is reduced*;

3) *any regular subword of a reduced word* (*including the word itself*) *has rooted composition.*

Indeed, 1) follows immediately from the definitions; 2) is another form of Remark 2.2; 3) is a consequence of 1) and 2).

As in [30, 33], a regular word is called an RR-word if any regular subword of it (including the word itself) has rooted composition.

Remark 2.7. Under the assumptions of Lemma 2.6, the least word among the RR-words with one and the same composition is reduced.

In [33], it was shown that, for the algebra D_n^+, for any ordering of generators the RR-words are uniquely determined by their compositions, and the set of these words was calculated.

Now we consider the special case where for the algebra $L = \mathrm{Lie}(X)/J$ every RR-word is determined by its composition. A regular word is called an SR-*word* if its composition is not rooted but the compositions of all proper regular subwords of it are rooted.

Lemma 2.8. *Let* $L = \mathrm{Lie}(X)/J$ *be a finitely generated Lie algebra such that* (i) *its ideal of relations is homogeneous with respect to the number of occurrences of each letter*; (ii) *every* RR-*word is determined by its composition. Then*:

1) *the set of reduced words coincides with the set of* RR-*words*;

2) *the set of regular nonreduced words such that all proper regular subwords of them are reduced coincides with the set of* SR-*words*;

3) *the* RGShB *of* L *coincides with the set of elements of the free Lie algebra* $\mathrm{Lie}(X)$ *of the form* $[u]$*, where* u *is an arbitrary* SR-*word.*

Proof. 1) follows from 3) of Lemma 2.6 and from Remark 2.7; 2) is a consequence of Remark 2.5 and statement 1).

From Remark 2.5 and statement 1) we deduce that the RGShB of the algebra L is the set of relations of the form $[u - \pi(u)]$, where u is an SR-word and $\pi(u)$ is a uniquely determined linear combination of reduced words. Since u is an SR-word, its composition is not rooted, i.e., the homogeneous component of the algebra L that corresponds to the element $|u|$ consists only of 0. Therefore, $\pi(u) = 0$ and $[u - \pi(u)] = [u]$, as required. □

Our aim in the present paper is to calculate the RGShB of the Lie algebra D_n^+ for the generators corresponding to simple roots, under any ordering of these generators. For this, we construct the set of SR-words for the algebra D_n^+ for any ordering of generators.

§3. The algebra D_n^+, the root system, graphs of positive roots, and systems of occurrences

To describe the algebra D_n^+ and its generators that correspond to simple roots, we recall some well-known facts from the classical theory of finite-dimensional simple Lie algebras over a field of characteristic 0 (see [41, 42]).

The algebra D_n is a finite-dimensional simple decomposable Lie algebra. Let L be any finite-dimensional simple decomposable Lie algebra.

The term *decomposable* means that L contains a nilpotent subalgebra H (Cartan subalgebra) such that the Lie H-module L is the direct sum of H and the submodules $L_\alpha = \{x \in L \mid \exists n \geq 1\ \forall h \in H\ (\mathrm{ad}_h - \alpha(h) \cdot 1)^n \cdot x = 0\}$, where α ranges over the set of nonzero weights of the H-semiinvariants in L.

The nonzero weights of the H-semiinvariants are called *roots*. By definition, the roots are elements of the space H^* dual to H. The Cartan subalgebra is Abelian. The spaces L_α are one-dimensional; in particular, their elements are H-semiinvariants.

There exists a subset Π of the set of roots such that any root α can be uniquely represented in the form $\alpha = \sum a_\pi \cdot \pi$ (where the sum is taken over some elements π of Π; a_π is an integer different from zero), and the coefficients a_π have one and the same sign. Such a subset Π is called a *system of simple roots*. The roots that are sums of simple roots (as elements of the space H^*) are said to be *positive*.

The simple roots are linearly independent elements of the space H^*; in particular, the subgroup $\mathbb{Z}\Pi$ generated by simple roots in the additive group of H^* is the free Abelian

group generated by simple roots. Any root is an element of the group $\mathbb{Z}\Pi$. Thus, the algebra L is graded by the group $\mathbb{Z}\Pi$. In this graduation, any homogeneous component corresponding to a root is one-dimensional.

We denote by L^+ the subspace of L generated by all components L_α, where α is a positive root; L^+ is a subalgebra of L, and it is natural to call it the "positive part" of L. This subalgebra is generated by the H-semiinvariants the weights of which are simple roots. For each simple root π, we choose a nonzero H-semiinvariant of weight π. The set X of these elements generates the algebra L^+. The elements of X are precisely the generators that are of interest to us.

The algebra D_n^+ is the "positive part" of the algebra D_n, and n is the number of simple roots.

Remark 3.1. The relations of the algebra L are homogeneous with respect to the number of occurrences of each letter. Therefore, L is graded by the group $\mathbb{Z}X$. Moreover, the simple roots and the generators of the algebra L^+ are in one-to-one correspondence. Consequently, the graduations of the algebra L by the group $\mathbb{Z}\Pi$ and by the group $\mathbb{Z}X$ coincide. Hence, to know what elements of $\mathbb{Z}\Pi$ are positive roots is the same as to know what homogeneous components of the algebra L^+ are nonzero.

Remark 3.2. It should be noted that to calculate the RGShB of the algebra D_n^+, we only need to know the ordering of the generators and to know what elements of the group $\mathbb{Z}\Pi$ are positive roots (i.e., what homogeneous components of the algebra L^+ are nonzero).

Indeed, this information suffices for the calculation of the sets of RR-words and SR-words. As was shown in [33], the RR-words are determined by their compositions. Therefore, by Lemma 2.8, we can calculate the RGShB as well.

Remarks 3.1 and 3.2 show that there is no need to distinguish between letters and the simple roots corresponding to these letters. But if it is important to emphasize that we regard a letter q as a simple root, we shall write $|q|$ in place of q.

By a (*nonoriented*) *graph* we mean a pair $\langle X, R\rangle$, where X is a finite set (the set of vertices) and R is a symmetric antireflexive binary relation on X (the adjacency relation). Usually, the vertices of a graph are represented by points, and vertices x and y are joined by an edge if x and y are adjacent, i.e., $(x, y) \in R$. The vertices of an arbitrary graph that have at most one adjacent vertex are called *end vertices*.

A graph is *connected* if for any two vertices x and y there is a finite sequence of its vertices such that it begins with the vertex x, ends with the vertex y, and each subsequent vertex is adjacent to the preceding one. A connected graph is called an *interval* if at least one vertex of it has at most one adjacent vertex and each vertex has at most two adjacent vertices.

An arbitrary interval I either consists of only one vertex or has precisely two end vertices. If an interval I is a subgraph of a graph D without cycles, then I is determined by its end vertices. We denote $I = [x, y]$, where $\{x, y\}$ is the set of end vertices of the interval I; $[x, y) = [x, y] - \{y\}$; $(x, y) = [x, y) - \{x\}$. Clearly, $[x, y] = [y, x]$, $[x, y) = (y, x]$, $(x, y) = (y, x)$, and the sets $[x, y)$ and (x, y) are intervals.

For the root system D_n, we consider the graph the vertices of which are simple roots, and two vertices α and β are adjacent if the element $\alpha + \beta$ of the group $Z\Pi$ is again a root. Using the tables given at the end of the book [43], we can easily check the following:

(1) this graph is connected and has no cycles;

(2) any vertex of it has at most three adjacent vertices;

(3) there exists precisely one vertex z^* ($*$-center) that has three adjacent vertices;

(4) at least two vertices adjacent to the $*$-center are end vertices;

(5) this graph is determined by the number n of its vertices.

We denote this graph by $\bar{D}_n$.

The elements of the group $\mathbb{Z}X$ that can be represented as a sum of elements of X are called *positive*. We define a partial order in the group $\mathbb{Z}\Pi$: $\alpha \leq \beta$ if the element $\beta - \alpha$ is positive.

Remark 3.3. From [43] it can be seen that any positive root of the root system D_n is a positive element of the group $\mathbb{Z}\Pi$, obtained by summation of vertices of a connected subgraph of the graph $\bar{D}_n$, or the element $\sum D + [z^*, x]$, where D is a connected subgraph of the graph D_n containing the $*$-center and all three adjacent vertices; x is an arbitrary nonend vertex of D. Here $\sum D$ is the element of $\mathbb{Z}X$ obtained by summation of all vertices of D; $\sum D + [z^*, x]$ is the element $\sum D + \sum [z^*, x]$ (we sometimes omit the symbol $\sum$ to simplify the notation).

For any positive element $\alpha \in \mathbb{Z}\Pi$, we denote by $\bar{D}(\alpha)$ the subgraph of the graph $\bar{D}_n$ (the vertices of which are simple roots β such that $\beta \leq \alpha$) endowed with an additional structure: each vertex β is marked by the greatest positive integer d such that $d\beta \leq \alpha$ (this d is called the *multiplicity of* β).

From Remark 3.3 it is seen that any root α is a linear combination of simple roots with coefficients 0, 1, 2. We say that a vertex β of $\bar{D}(\alpha)$ is *single* if $2\beta \not\leq \alpha$, and β is a *double* vertex if $2\beta \leq \alpha$.

Also, the definitions show that, if we know the graph $\bar{D}(\alpha)$, we can find the root α itself (as an element of the group $\mathbb{Z}\Pi$).

We denote by 0 the set of end vertices of $\bar{D}_n$ that are adjacent to the $*$-center. The graph $\bar{D}_n$ contains at least four vertices. If $\bar{D}_n$ consists precisely of four vertices, then the set 0 has three elements. If $\bar{D}_n$ contains at least five vertices, then the set O has precisely two elements.

For any positive root α, we denote by $M^+(\alpha)$ the set of simple roots β such that $\alpha - \beta$ is a positive root.

For any positive root α such that $z^* \leq \alpha$, we denote by $O(\alpha)$ the set of end vertices in the graph $\bar{D}(\alpha)$ adjacent to the $*$-center.

If the graph $\bar{D}(\alpha)$ has double vertices, then we denote by $x_2^+(\alpha)$ a unique double vertex of $\bar{D}(\alpha)$ that belongs to $M^+(\alpha)$. It is a unique double vertex of $\bar{D}(\alpha)$ that has at most one adjacent double vertex and is different from the $*$-center if the graph has at least two double vertices.

If the graph $\bar{D}(\alpha)$ contains z^* and three vertices adjacent to it and has at least five vertices, then we denote by $x_1^+(\alpha)$ a unique end vertex of $\bar{D}(\alpha)$ that is not adjacent to z^*. Moreover, if $\bar{D}(\alpha)$ has double vertices, then we denote by $x_1^-(\alpha)$ a unique (single) vertex of this graph that is adjacent to $x_2^+(\alpha)$ and is such that $x_1^-(\alpha) \in (x_2^+(\alpha), x_1^+(\alpha)]$.

If the vertices of $\bar{D}_n$ are linearly ordered, then we shall talk about the *graph* D_n. The graph $\bar{D}(\alpha)$ with the linear order of the vertices induced from D_n will be called the *graph* $D(\alpha)$.

For an arbitrary ordered set M, we denote its greatest element by $\max(M)$. The greatest vertices of graphs and intervals will be denoted in the same way. We assume that the maximum of the empty set is always less than the maximum of a nonempty set.

Definition 3.4. A $*$-*system* (*of occurrences*) is a quadruple $\langle V, b, \preceq, p\rangle$, where

(1) V is a finite set (of occurrences of letters);

(2) $b : V \to \bar{D}_n$ is a mapping;

(3) $\preceq$ is a partial order on the set V such that any elements x, y of V are comparable if and only if $b(x) = b(y)$;

(4) $p \in V$ is a marked occurrence (of the letter $b(p)$) such that from $x \in V$ and $b(x) = b(p)$ it follows that $x = p$.

In an arbitrary $*$-system $\langle V, b, \preceq, p\rangle$, for an arbitrary set of occurrences M we denote by $\sum M$ the element of the group $\mathbb{Z}\Pi$ obtained by summation of the letters $b(x)$ over all $x \in M$. If $p \in M$, then the structure of a $*$-system is induced on the set M. We shall call it the subsystem of the $*$-system Ω and denote it by $\Omega|_M$. A $*$-system will be called the *$*$-system* (*of occurrences*) *of an element* $\alpha \in \mathbb{Z}\Pi$ if $\sum V = \alpha$; it will be denoted by $V^*(\alpha)$ or $\bar{V}(\alpha)$.

In an arbitrary $*$-system $\langle V, b, \preceq, p\rangle$, we can consider the occurrences of any letter x; these are elements of the set $b^{-1}(x)$. If this is a $*$-system of a root, then any letter x has at most two occurrences. They may be called the left and right ones. The left occurrence of the letter x is the smallest element of the set $b^{-1}(x)$ (under the order $\preceq$); the right occurrence is the greatest element. If the letter x has only one occurrence (i.e., if the set $b^{-1}(x)$ consists of one element), then the right and left occurrences coincide.

If the set of vertices has a linear order and in the $*$-system $V^*(\alpha)$ the marked occurrence is the occurrence of the leading vertex in the graph $D(\alpha)$, then we shall talk about the system $V(\alpha)$ of occurrences of the element α. Any system of occurrences can be viewed as a $*$-system.

An arbitrary word a can be regarded as a finite linearly ordered set V of occurrences of letters with the following order: an occurrence of x is less than an occurrence of y if the occurrence of x in the word a is placed to the left of the occurrence of y. In this situation, there are a finite set of letters and a finite set of occurrences of these letters. The notion of the "occurrence of a letter" can be regarded as a mapping b of the set of occurrences to the set of letters. We shall distinguish different occurrences of one and the same letter with the help of a partial order $\preceq$ on the set of occurrences under which occurrences are comparable if and only if they are occurrences of one and the same letter.

An arbitrary RR-word with one occurrence of its leading letter will be called an RR1-word. An RR2-word is an RR-word with two occurrences of its leading letter. Let a be an arbitrary RR1-word of composition α. If we mark a unique occurrence p of a certain letter, then the quadruple $\langle V, b, \preceq, p\rangle$ constructed above is a $*$-system of the root α. Moreover, if the marked occurrence p is the occurrence of the leading vertex in the graph $D(\alpha)$, then $\langle V, b, \preceq, p\rangle$ is the system of occurrences of the root α. We call it the system of occurrences of letters in the RR-word a. We may talk about subsystems of the $*$-system of occurrences of letters in an arbitrary RR-word.

We have introduced the graphs $\bar{D}_n$, $\bar{D}(\alpha)$ and $*$-systems, because in some cases we have no need to know the order on the set of vertices: only a certain marked occurrence is of importance.

In a $*$-system of an arbitrary root α, the set of all left occurrences will be called the *lower layer*, and the complement of this set is the *upper layer*.

An arbitrary $*$-system of a root α may be presented in the following way. We draw the graph $\bar{D}(\alpha)$ on a horizontal plane. On each of the single vertices of the graph $\bar{D}(\alpha)$, we place a checker piece, and on each double vertex we place a column of two pieces. The pieces on the bottom form the lower layer, and the remaining ones constitute the upper layer. The mapping "the occurrence of a letter" associates each piece with the vertex on which it is laid. We choose a column that consists of a single piece, and we mark this piece. If we mark the piece that is placed on the leading vertex of the graph $D(\alpha)$, we obtain the system of occurrences of the root α.

Definition 3.5. For an arbitrary $*$-system of occurrences of a root α, the following properties uniquely determine a binary relation R on the set of occurrences.

(1) if x, y are occurrences from the lower layer, then $(x, y) \in R$ is equivalent to the fact that $b(x) \in [b(p), b(y))$ and the vertices $b(x)$ and $b(y)$ are adjacent;

(2) if x is an occurrence from the lower layer and y is an occurrence from the upper layer, then $(x, y) \in R$ is equivalent to the fact that $b(y) \in [z^*, b(x))$ and the vertices $b(x)$ and $b(y)$ are adjacent;

(3) if x and y are occurrences from the upper layer, then $(x, y) \in R$ is equivalent to the fact that $b(x) \in [z^*, b(y))$ and the vertices $b(x)$ and $b(y)$ are adjacent.

Consider the oriented graph the vertices of which are occurrences; we join occurrences x and y by an arrow (oriented edge) if $(x, y) \in R$. This graph of the $*$-system Ω will be called the occurrence graph of Ω.

Remark 3.6. For an arbitrary root α, the occurrence graph $\bar{V}(\alpha)$ can be thought of in the following way. The mapping "occurrence of a letter", denoted by b, gives rise to a one-to-one correspondence between the occurrences of the lower layer and all vertices of the graph $\bar{D}(\alpha)$ and between the occurrences of the upper layer and the double vertices of the graph $\bar{D}(\alpha)$. In both layers we draw the corresponding edges and orient them; in the lower layer we choose the direction away from the marked occurrence, and in the upper layer the edges are directed away from the occurrence of the vertex z^*. For each occurrence x in the lower layer such that the vertex $b(x)$ is different from z^* and has an adjacent double vertex, we draw the oriented edge (x, y), where y is an occurrence from the upper layer such that $b(y) \in [z^*, b(x))$ and the vertices $b(x)$ and $b(y)$ are adjacent. We obtain the graph $\bar{V}(\alpha)$.

For the occurrence graph $\bar{V}(\alpha)$ of an arbitrary root α, we consider the family of all subsets U of the set of occurrences that satisfy the following condition.

T. If $x \in U$ and (x, y) is an edge of the graph $\bar{V}(\alpha)$, then $y \in U$.

Clearly, the empty set and the entire set V possess property T; the union and the intersection of subsets satisfying T also possess this property. Since the set of occurrences is finite, on the set of occurrences there exists a structure of topological space in which the closed sets are sets with property T. This topology is said to be *p-root*. The topological space arising on the set of occurrences is called the *p-space of occurrences* and is denoted in the same way as the occurrence graph. In the p-space of occurrences, we denote the closure of an arbitrary set M by $\overline{M}$ and the closure of an arbitrary occurrence x by $\bar{x}$.

Remark 3.7. In the p-space of occurrences of an arbitrary root, a subset U is open if and only if the following conditions are satisfied:

1) if the set U is nonempty, then $p \in U$ (here p is the marked occurrence);

2) if x and y are, respectively, the left and right occurrences of one and the same letter, then $y \in U$ implies $x \in U$;

3) if the set U is nonempty, then the element $\sum U$ is a root.

Remark 3.8. In the p-space of occurrences of a root α, the closed sets M are characterized by the following properties:

1) if $M \neq V$, then $p \notin M$, or the closure $\bar{p}$ of the marked occurrence p coincides with V (here V is the set of occurrences);

2) if x and y are, respectively, the left and the right occurrences of one and the same letter, then $x \in M$ implies $y \in M$;

3) if $M \neq V$, then the element $\sum(V - M)$ is a root.

Lemma 3.9. *Let Ω be the p-space of occurrences of an arbitrary root α, and let U be a nonempty open set. Then*:

1) *if $U' \subseteq U$, then the set U' is open in Ω if and only if U' is open in $\Omega|_U$*;

2) *if M is a set closed in Ω, then $U \cap M$ is closed in $\Omega|_U$*;

3) *if P is a set closed in $\Omega|_U$, then the set $P \cup (V - U)$ is closed in Ω (here V is the set of occurrences)*;

4) *for $x \in U$ the set $\bar{x} \cap U$ is the closure of the element x in $\Omega|_U$.*

Proof. Since U is a nonempty open set, the element $\sum U$ is a root. Moreover, being a nonempty open set, U contains the marked occurrence. Therefore, the restriction $\Omega|_U$ is a $*$-system (of the root $\sum U$). If Ω is a system of occurrences of the root α, then $\Omega|_U$ is also a system of occurrences. Assertion 1) is easily verified by definition.

2) If a set M is closed, then the set $V - M$ is open. Therefore, the set $(V - M) \cap U$ is open. For this reason, by property 1), the set $(V - M) \cap U$ is open in $\Omega|_U$. Therefore, its complement $U \cap M$ is closed in $\Omega|_U$.

3) The set $U - P$ is open in $\Omega|_U$. Hence, by property 1), the set $U - P$ is open in Ω. Consequently, its complement $P \cup (V - U)$ in V is closed in Ω.

4) Denote by $\bar{x}'$ the closure of the element x in $\Omega|_U$. By 2), the set $\bar{x} \cap U$ is closed in $\Omega|_U$. Moreover, $x \in \bar{x} \cap U$. Therefore, $\bar{x}' \subseteq \bar{x} \cap U$.

By 3), the set $\bar{x}' \cup (V - U)$ is closed. Moreover, $x \in \bar{x}' \cup (V - U)$. Therefore, $\bar{x} \subseteq \bar{x}' \cup (V - U)$. Thus, $\bar{x} \cap U \subseteq (\bar{x}' \cup (V - U)) \cap U = (\bar{x}' \cap U) \cup ((V - U) \cap U) = \bar{x}'$. The inclusions $\bar{x}' \subseteq \bar{x} \cap U$, $\bar{x} \cap U \subseteq \bar{x}'$ imply that $\bar{x}' = \bar{x} \cap U$, as required. □

Lemma 3.10. *If a is an arbitrary* RR1*-word and c is a beginning of the word a, then the set U of the occurrences in a that are occurrences in c is open.*

Proof. 1) Since the word a is regular, it begins with the occurrence of its leading letter. Therefore, its beginning c, if nonempty, also starts with the occurrence of the same letter. Consequently, property 1) of Remark 3.7 is fulfilled for the set U.

2) Since c is a beginning of the word a, property 2) is also fulfilled.

3) The beginning c of the word a, if nonempty, starts with a unique occurrence of its leading letter. Therefore, it is regular. Consequently, being a subword of the RR-word a, it has rooted composition, i.e., property 3) is also fulfilled. □

Corollary 3.11. *If x and y are occurrences from the system of occurrences of an arbitrary* RR1*-word a and $y \in \bar{x}$, then the occurrence y in a is not placed in a to the left of the occurrence x.*

Proof. Let c be the ending of the word a that begins with the occurrence x. By Lemma 3.10, the subset of occurrences M in the subword c is closed. Therefore, $y \in \bar{x} \subseteq M$, i.e., y is the occurrence in the ending c. □

Remark 3.12. For the p-space of occurrences of an arbitrary root, the property "a set $\{x\}$ is closed" is equivalent to the following:

(1) the occurrence x is right;

(2) if $V \neq \{p\}$, then the element $\sum V - |b(x)|$ is a root (here V is the set of occurrences and p is the marked occurrence);

(3) if $x = p$, then $V = \{p\}$.

Lemma 3.13. *In the space of occurrences of an arbitrary root, every nonempty closed set M has a closed one-element subset.*

Proof. The set $U = V - M$ is open (here V is the set of occurrences). Therefore, the element $\sum U$ is either zero or a positive root. From the structure of roots it is easily seen that there exists a simple root γ such that $\sum U \leq \sum V - \gamma$; the element $\sum V - \gamma$ is either zero or a positive root. Let x denote a right occurrence such that $|b(x)| = \gamma$. Then $x \notin U$, i.e., $\{x\} \subseteq M$. The set $V - \{x\}$ is open, i.e., the set $\{x\}$ is closed. □

Now using Lemma 3.9 and Remark 3.12, we can obtain the following inductive (with respect to the number of elements) characterization of closed sets.

Corollary 3.14. *In the p-space of occurrences Ω of an arbitrary root,*

1) *the empty set of occurrences is closed*;

2) *the closed singletons are described in Remark* 3.12;

3) *if $V \neq \{p\}$, $\{x\}$ is a closed singleton, and M is a set closed in the system of occurrences $\Omega|_{V-\{x\}}$, then the set $M \cup \{x\}$ is closed* (*here V is the set of occurrences*).

Remark 3.6 and the definition of closed sets imply the following statement.

Corollary 3.15. *Let $\Omega = \langle V, b, \preceq, p\rangle$ be the $*$-system of occurrences of an arbitrary root α, and let x be a right occurrence. Then*:

1) *if $x = p$, then the closure $\bar{x}$ of the occurrence x coincides with V*;

2) *if $x \neq p$, the graph $D(\alpha)$ has double vertices, and $b(x) \in O$ or x belongs to the upper layer, then $\bar{x} - \{x\}$ is the intersection of the upper layer and the set $b^{-1}(b(x), x_2^+(\alpha)]$*;

3) *if $x \neq p$, there are double vertices in the graph $D(\alpha)$, this graph contains at least five vertices, $b(p) \in O$, and $b(x) = x_1^-(\alpha)$, then $\bar{x}$ coincides with the set of right occurrences y such that $b(y) \in [x_2^+(\alpha), x_1^+(\alpha)]$*;

4) *if $b(x) \neq b(p)$, the vertex $b(x)$ and the vertex of the interval $(b(x), b(p)]$ adjacent to $b(x)$ are single in the graph $D(\alpha)$, then $\bar{x}$ coincides with the set of occurrences y such that $b(x) \in [b(y), b(p))$.*

Corollary 3.16. *Let $\langle V, b, \preceq, p\rangle$ be the $*$-system of occurrences of an arbitrary root α, and let x be a right occurrence. Then the element $\sum \bar{x}$ is a root and the vertex $b(x)$ has precisely one occurrence in the set $\bar{x}$. Therefore, the quadruple $\langle \bar{x}, b|_{\bar{x}}, \preceq|_{\bar{x}}, x\rangle$ is the $*$-system of occurrences of the root $\sum \bar{x}$, and it has the structures of a graph and of a topological space.*

Lemma 3.17. *Let $\Omega = \langle V, b, \preceq, p\rangle$ be the $*$-system of occurrences of an arbitrary root α, let $p' \in V$ be a right occurrence, and let $M \subseteq V$ be a closed set. Then the element $\sum \bar{p'}$ is a root, $\Omega' = \langle \bar{p'}, b|_{\bar{p'}}, \preceq|_{\bar{p'}}, p'\rangle$ is the $*$-system of occurrences of the root $\sum \bar{p'}$, and the set $M \cap \bar{p'}$ is closed in the space of occurrences Ω'.*

Proof. By Corollary 3.16, the element $\sum \bar{p'}$ is a root, the vertex $b(p')$ in the graph $\bar{D}(\sum \bar{p'})$ is single, and $\Omega' = \langle \bar{p'}, b|_{\bar{p'}}, \preceq|_{\bar{p'}}, p'\rangle$ is the $*$-system of occurrences of the root $\sum \bar{p'}$. Therefore, the space Ω' of occurrences is well defined.

It suffices to consider the case where $M \subseteq \bar{p'}$ and to prove that if M is closed in Ω, then it is also closed in Ω'. The proof proceeds by induction on the number m of elements of the set $\bar{p'}$. The base of induction: for $m = 1$ the statement is obvious. The inductive step: consider $m \geq 2$ and assume that the statement is valid for $m - 1$. By Lemma 3.13, the set M has a closed one-element subset $\{x\}$. Then the set $U = V - \{x\}$ is open. Moreover, it is not empty ($\bar{p'} - \{x\} \subseteq U$ and the set $\bar{p'} - \{x\}$ is not empty because $m \geq 2$). Therefore, $p \in U$ and $\Omega|_U$ is the system of occurrences of the root $\sum U$. By statement 2) of Lemma 3.9, the set $U \cap M$ is closed in $\Omega|_U$.

Since the set $\{x\}$ is closed and the closed set $\bar{p'}$ contains at least two elements ($m \geq 2$), it follows that $x \neq p'$. Therefore, $p' \in V - \{x\} = U$. Consequently, by statement 4) of Lemma 3.9, the set $A = \bar{p'} \cap U$ is the closure of the occurrence p' in $\Omega|_U$.

By Corollary 3.16 (for the closure A in $\Omega|_U$ of the right occurrence p'), the element $\sum A$ is a root and $\Omega'|_A$ is the system of occurrences of the root $\sum A$.

By the inductive hypothesis, since the set $U \cap M$ closed in $\Omega|_U$, it is closed in $\Omega'|_A$.

Since the set $\{x\}$ is closed in Ω, the occurrence x in Ω is right by Remark 3.12. Moreover, the element $\sum A = \sum(\bar{p'} - \{x\})$ is a root and $x \neq p'$. Therefore, by the same remark, the set $\{x\}$ is closed in Ω'. Consequently, the set $A = \bar{p'} - \{x\}$ is open in Ω'. Since, moreover, the set $U \cap M$ is closed in $\Omega'|_A$, we see that the set $(U \cap M) \cup (\bar{p'} - A) = (U \cap M) \cup \{x\} = M$ is closed in Ω' by statement 3) of Lemma 3.9. $\square$

§4. RR-words of the Lie algebra D_n^+

In the sequel we tacitly assume that the set of letters is linearly ordered. We introduce the following notation:

Rt_1 is the set of roots whose graph has the property that its leading vertex p_1 is single;

$\mathrm{Rt}_{1,1}$ is the set of roots in Rt_1 whose graph involves double vertices and either the second-by-priority vertex p_2 is double or $p_2 \in O(\alpha)$;

$\mathrm{Rt}_{1,2}$ is the set of roots α in Rt_1 such that the graph of α has double vertices and contains at least five vertices, and $p_1 \in O(\alpha)$, $p_2 = x_1^-(\alpha)$;

$\mathrm{Rt}_{1,3}$ is the set of nonsimple roots α of Rt_1 whose graph has the property that the vertex p_2 and the vertex of the interval $[p_1, p_2)$ that is adjacent to it are single.

Lemma 4.1. *The set* $\{\mathrm{Rt}_{1,1}, \mathrm{Rt}_{1,2}, \mathrm{Rt}_{1,3}\}$ *is a partition of the set of all nonsimple roots in* Rt_1.

Proof. Clearly, $\mathrm{Rt}_{1,1}$, $\mathrm{Rt}_{1,2}$, and $\mathrm{Rt}_{1,3}$ are subsets of the set of nonsimple roots in Rt_1. Let p_1 and p_2 denote the leading and the second-by-priority vertices of the graph $D(\alpha)$, respectively, and let p' be the vertex of the interval $(p_2, p_1]$ adjacent to p_2.

If $\alpha \in \mathrm{Rt}_{1,2}$, then the vertex p_2 in the graph $D(\alpha)$ is single and $p_2 \notin O(\alpha)$, so that $\alpha \notin \mathrm{Rt}_{1,1}$. Consequently, the sets $\mathrm{Rt}_{1,1}$ and $\mathrm{Rt}_{1,2}$ do not intersect.

If $\alpha \in \mathrm{Rt}_{1,3}$, then the vertex p_2 in the graph $D(\alpha)$ is single and has a single adjacent vertex. Then $\alpha \notin \mathrm{Rt}_{1,1}$. Therefore, the sets $\mathrm{Rt}_{1,1}$ and $\mathrm{Rt}_{1,3}$ do not intersect.

If $\alpha \in \mathrm{Rt}_{1,2}$, then $p_1 \in O(\alpha)$, $p_2 = x_1^-(\alpha)$, and the vertex p' is double, which implies $\alpha \notin \mathrm{Rt}_{1,3}$. Consequently, the sets $\mathrm{Rt}_{1,2}$ and $\mathrm{Rt}_{1,3}$ do not intersect.

Assume that there exists a nonsimple root $\alpha \in \mathrm{Rt}_1$ such that $\alpha \notin \mathrm{Rt}_{1,1} \cup \mathrm{Rt}_{1,2} \cup \mathrm{Rt}_{1,3}$. Since $\alpha \notin \mathrm{Rt}_{1,1}$, the vertex p_2 in $D(\alpha)$ is single and $p_2 \notin O(\alpha)$. Since $\alpha \notin \mathrm{Rt}_{1,3}$, the vertex p_2' is double in $D(\alpha)$. Therefore, $p_2 = x_1^-(\alpha)$ and $p_1 \in O(\alpha)$, i.e., $\alpha \in \mathrm{Rt}_{1,2}$, a contradiction. Consequently, any nonsimple root in Rt_1 is an element of the set $\mathrm{Rt}_{1,1} \cup \mathrm{Rt}_{1,2} \cup \mathrm{Rt}_{1,3}$. □

Lemma 4.2. *Let u be an arbitrary* RR1*-word of length at least* 2*. Let p_2 be its second-by-priority letter, and let v be the ending of the word u that begins with the right occurrence of the letter p_2. Then*

1) *if $\alpha \in \mathrm{Rt}_{1,1}$, then $|v| = [p_2, x_2^+(\alpha)]$ (see* [33, Lemma 2.5]);

2) *if $\alpha \in \mathrm{Rt}_{1,2}$, then the graph $D(\alpha)$ contains at least five vertices, $p_2 = x_1^-(\alpha)$, and $|v| = [x_2^+(\alpha), x_1^+(\alpha)]$ (see* [33, Lemma 2.6]);

3) *if $\alpha \in \mathrm{Rt}_{1,3}$, then the graph $D(|v|)$ coincides with the connected component that contains the vertex p_2 of the graph obtained from $D(\alpha)$ by removing the edge $[p_2, p_2']$ (here p_2' is the vertex of the interval $[p_1, p_2)$ that is adjacent to p_2); see* [33, Lemma 2.4].

Corollary 4.3. *An arbitrary* RR1*-word is determined by its composition, i.e., it is reduced.*

For an arbitrary root α, we denote by $\mathrm{red}(\alpha)$ a unique reduced word of composition α. In the sequel, we shall refer to Lemma 4.2 when talking about the structure of RR1-words. The description in Corollary 3.15 of the closure of an arbitrary right occurrence enables us to state results on the structure of RR1-words without searching through all cases.

Corollary 4.4. *Let α be a positive root such that the leading vertex p_1 in its graph is single. Then an* RR1*-word u of composition α is unique. Moreover, if the root α is not simple, then the reduced word of composition $\sum \bar{p}'$ is an ending of the word u (here p' is the right occurrence of the second-by-priority letter of u, and $\bar{p}'$ is the closure of the occurrence p' in the space of occurrences of u).*

Lemma 4.5. *Let a be an* RR1*-word, let $\Omega = \langle V, f, \preceq, p\rangle$ be its system of occurrences, and let M be a set of occurrences closed in Ω and such that $M \neq V$. Then the word that is obtained from a by elimination of all occurrences belonging to M is an* RR1*-word.*

Proof. The proof proceeds by induction on the length n of a. The assertion for $n = 1$ is obvious. Let $n \geq 2$, and suppose the lemma is true for all $k < n$. Since $n \geq 2$, the word a has length at least 2. Therefore, being an RR-word, it has occurrences of at least two letters. Consequently, by Corollary 4.4, $a = \mathrm{red}(\sum U')\mathrm{red}(\bar{p'})$, where p' is the right occurrence of the second-by-priority letter of the word a, $U' = V - \bar{p'}$. We introduce the following notation: $U = V - M$; $b = \mathrm{red}(\sum U')$, $c = \mathrm{red}(\bar{p'})$; a', b', and c' are the words obtained from a, b, and c, respectively, by elimination of the occurrences belonging to M. Then $a = bc$ and $a' = b'c'$.

The set $U' = V - \bar{p'}$ is open. Moreover, it is not empty (because U' contains the occurrence of the leading letter of a). Therefore, the element $\sum U'$ is a root and $\Omega|_{U'}$ is a system of occurrences of the RR1-word b.

Since M is closed and U' is open, the set $U' \cap M$ is closed in $\Omega|_{U'}$ by statement 2) of Lemma 3.9. Moreover, the length of b is less than n. Therefore, by the inductive assumption, b' is an RR1-word.

If $\bar{p'} \subseteq M$, then $c' = 1$ and $a' = b'c' = b'$. Since b' is an RR1-word, it remains to consider the case where $\bar{p'} \nsubseteq M$, i.e., $p' \notin M$, or $p' \in U$.

By Lemma 3.17, the element $\sum \bar{p'}$ is a root, and the set $M \cap \bar{p'}$ is closed in the system of occurrences $\langle \bar{p'}, f|_{\bar{p'}}, \preceq|_{\bar{p'}}, p'\rangle$ of this root. Therefore, by the inductive assumption, c' is an RR1-word. Its set of occurrences is $\bar{p'} - M = \bar{p'} \cap U$. Thus, $c' = \mathrm{red}(\sum(\bar{p'} \cap U))$.

The set of occurrences in the word a' coincides with U. Now we show that the word a' is reduced, i.e., $a' = \mathrm{red}(\sum U)$.

The set $U = V - M$ is open. Moreover, it is not empty ($p' \in U$). Therefore, the element $\sum U$ is a root and $\Omega|_U$ is the system of occurrences of this root. Consequently, by statement 4) of Lemma 3.9, the closure of the element p' in the system $\Omega|_U$ coincides with $\bar{p'} \cap U$. Now Corollary 4.4 shows that the word $\mathrm{red}(\sum \bar{p'} \cap U) = c'$ is an ending of the word $\mathrm{red}(\sum U)$.

Thus, the words $a' = b'c'$ and $\mathrm{red}(\sum U)$ have one and the same composition and one and the same ending c', and the word b' (which can be obtained by elimination of the ending c') is reduced. Therefore, $a' = \mathrm{red}(\sum U)$. □

Corollary 4.6. *Let a be an* RR1*-word, let p_1 be its first letter, and let $q \neq p_1$ be a letter occurring in the word a and such that $|a| - |q|$ is a root. Then the word obtained from a by elimination of the right occurrence of q is an* RR1*-word.*

Lemma 4.7. *Let a be an* RR1*-word, let $q \in M^+(|a|)$, and let q be different from the first and the last letter of a. Then q is less than the letter that occurs in a immediately after the right occurrence of q.*

Proof. We denote by b the longest beginning of the word a that does not contain the right occurrence (in the word a) of the letter q. Since q is different from the first letter of the word a, the word b is not empty. Since q is different from the last letter of a, there exists a letter that occurs in a immediately after the right occurrence of q. We denote this letter by m. By the choice of m, we have $m \neq q$.

Suppose $q > m$. Then the word qm is regular, and moreover, it is a subword of the RR-word a. Therefore, the word qm has rooted composition, i.e., the vertices q and m are adjacent.

Since a is an RR1-word, it begins with a unique occurrence of the leading letter p_1. Moreover, since its beginning b is not empty, it follows that b also begins with a unique occurrence of the leading letter p_1 (in particular, $p_1 \leq |b|$) and $q < p_1$, $m < p_1$.

Therefore, the word bq is regular. Consequently, being a subword of the RR-word a, the word bq has rooted composition, i.e., $m \in M^+(|bqm|)$.

Since $m \neq q$ and, by the choice of the word b, the vertex q has the same multiplicity in the graphs $D(|a|)$ and $D(|bqm|)$. Moreover, $q \in M^+(|a|)$. Therefore, $q \in M^+(|bqm|)$.

For an arbitrary positive root β, the set $M^+(\beta)$ has adjacent vertices only in the case where the graph $D(\beta)$ contains precisely two vertices. Therefore, the graph $D(|bqm|)$ has two vertices. But this is impossible, because $p_1 \leq |b|$ and the vertices p_1, q, and m are distinct.

Thus, $q \leq m$ and $q \neq m$. Therefore, $q < m$, as required. □

With the help of Lemma 4.7, it is easy to obtain the following assertion.

Corollary 4.8. *For arbitrary* RR1*-words a and b with one and the same leading letter, if $|a| \leq |b|$, then $b \leq a$.*

Definition 4.9. For an arbitrary vertex p of the graph D_n, we introduce the following notation: O_p is the set of vertices x such that $\max[x, p] = p$; $\leq_p$ is a linear order on O_p such that $x <_p y$ and the greatest vertex of the intervals $[x, p]$ and $[y, p]$ that belongs to only one of them lies in the interval $[y, p]$.

Definition 4.10. Let $\alpha \in \mathrm{Rt}_1$, let $\Omega = \langle V, b, \preceq, p\rangle$ be the system of occurrences of the root α, and let V_1 be the set of all occurrences from the lower layer. Then an arbitrary set $M \subseteq V_1$ is said to be $\leq_1$*-closed* if for arbitrary occurrences x and y from the lower layer, the relations $b(x) \leq_{b(p)} b(y)$ and $y \in M$ imply $x \in M$. The intersection of $\leq_1$-closed sets is $\leq_1$-closed. For $M \subseteq V_1$, the intersection of all $\leq_1$-closed sets that contain M is called the $\leq_1$*-closure* of M.

Definition 4.11. For any system of occurrences $\langle V, b, \preceq, p\rangle$ of an arbitrary RR1-word v, we denote by $\leq_V$ the linear order on the set V under which $x \leq_V y$ (for $x, y \in V$) is equivalent to the fact that the occurrence x is not placed in the word b to the right of the occurrence y.

If we know the composition of an RR1-word, then, with the help of Lemma 4.2, we can determine what letter stands at the end of this word.

This is done in the following lemma.

Lemma 4.12. *Let $\alpha \in \mathrm{Rt}_1$, and let p_1 and q be the first and the last letter, respectively, in the word* $\mathrm{red}(\alpha)$. *Then*:

1. *If there are no double vertices in the graph $D(\alpha)$, then q is the greatest vertex under the order $\leq_{p_1}$ among the end vertices of this graph.*

2. *If the graph $D(\alpha)$ has double vertices and the following conditions* a)–c) *are fulfilled, then $q = x_1^+(\alpha)$*:

a) *in the graph $D(\alpha)$ the second-by-priority vertex p_2 is single and has an adjacent single vertex*;

b) $p_2 \in (p_1, x_1^+(\alpha)]$;

c) *if $p_1 \in O(\alpha)$ and $p_2 = x_1^-(\alpha)$, then $x_2^+(\alpha) < \max(p_2, x_1^+(\alpha)]$.*

3. *If the graph $D(\alpha)$ has double vertices and at least one of the conditions* a)–c) *is not satisfied, then $q = x_2^+(\alpha)$.*

Now we can describe the order $\leq_V$.

Lemma 4.13. *Let $\alpha \in \mathrm{Rt}_1$, let $\Omega = \langle V, b, \preceq, p\rangle$ be the system of occurrences of the root α, and let $x, y \in V$.*

1. *If x and y are occurrences from the lower layer, then $x \leq_V y \Leftrightarrow b(x) \leq_{b(p)} b(y)$.*

2. *Let U_x and U_y be the smallest sets open in the p-root topology that satisfy the condition $x \in U_x$, $y \in U_y$. If one of the sets U_x and U_y is a subset of the other, then $x \leq_V y \Leftrightarrow U_x \subseteq U_y$.*

3. *If the sets U_x and U_y are not comparable with respect to inclusion and one of the occurrences x and y belongs to the upper layer and the other to the lower layer, then $b(x) \neq b(y)$, the vertices $b(x)$ and $b(y)$ are not adjacent, the graph $D(\beta)$ has double vertices and adjacent single vertices ($\beta = \sum U_x \cup U_y$), $\{b(x), b(y)\} = M^+(\beta) = \{x_2^+(\beta), x_1^+(\beta)\}$, $b(p) \notin M^+(\beta)$. Moreover, the property that an occurrence $x' \in \{x, y\}$ from the upper layer is less under the order $\leq_V$ than an occurrence $y' \in \{x, y\}$ from the lower layer is equivalent to the fact that the second-by-priority letter p_2 of $b(U_x \cup U_y)$ lies in the interval $[x_1^-(\beta), b(y')] \cap (b(p), b(y')]$, and if $p_2 = x_1^-(\beta)$, then $b(x') < \max(x_1^-(\beta), b(y')]$.*

Proof. 1. The set of occurrences V_2 from the upper layer is closed in the p-root topology. Consequently, the set V_1 in the lower layer is open in the p-root topology. Therefore, by Lemma 4.5, the word d obtained from $\mathrm{red}(\alpha)$ by elimination of all occurrences belonging to V_2 is an RR1-word and $\Omega|_{V_1}$ is its system of occurrences. Applying statement 1 of Lemma 4.12 to the word d, we obtain assertion 1.

2. Assertion 2 follows from Lemma 3.10.

3. We apply statements 2, 3 of Lemma 4.12 to the shortest beginning of the word $\mathrm{red}(\alpha)$ that contains the occurrences x and y; this yields assertion 3. □

Now, for an arbitrary RR1-word, we describe the set of occurrences of the beginnings of that word that end with an arbitrary occurrence q.

Corollary 4.14. *Let a be an* RR1*-word, let $\Omega = \langle V, b, \preceq, p\rangle$ be the system of occurrences of the word a, and let $p_1 = b(p)$. Let $V_1 \subseteq V$ be the set of all occurrences from the lower layer, let $q \in V$, and let $M \subseteq V$ be the set of occurrences of the beginning of the word* $\mathrm{red}(a)$ *that ends with the occurrence q.*

1. *If $q \in V_1$ and condition A below is not satisfied for q, then $M = \bar{q}$, where $\bar{q}$ is the $\leq_1$-closure of the set $\{q\}$.*

A. *$V \neq V_1$ and the set V_1 has at least five elements; $q \neq p$, $b(q) \notin O \cup [z^*, p_1]$, and $\max[o, p_1) < p_2$ for any element $o \in O$ such that $o \neq p_1$ (here $p_2 = \max(p_1, b(q)]$); $z^* \neq p_2$. If the vertex p_2 is adjacent to z^*, then $z^* > \max(p_2, q]$.*

2. *If $q \in V_1$ and condition A is fulfilled for q, then the set M is open in the p-root topology and satisfies $\sum M = \sum \bar{q} + [z^*, x]$, where x is a double vertex of the interval $[z^*, p_2)$ that is most distant from z^* and below $p_1 \notin [z^*, x]$, and if x is adjacent to p_2, then $x > \max(p_2, b(q)]$.*

3. *If $q \notin V_1$ and condition B below is not satisfied for q, then $M = U_q \cup \overline{U_q \cap V_1}$, where $U_q \subseteq V$ is the smallest set among the open sets in the p-root topology for which $q \in U_q$; $\overline{U_q \cap V_1}$ is the $\leq_1$-closure of the set $U_q \cap V_1$.*

B. *$p_1 \in O$, the second-by-priority vertex p_2 from $b(\overline{U_q \cap V_1})$ is adjacent to $b(q)$, and $p_2 \notin [z^*, b(q)] \cup O$; moreover, p_2 has an adjacent single vertex in the graph $D(|a|)$ and this vertex is less than $b(q)$.*

4. *If $q \notin V_1$ and condition B is fulfilled for q, then $\sum M = [z^*, b(q)] + \sum O + [z^*, x]$, where x is a vertex of the graph $D(|a|)$ most distant from z^* and such that $x \notin O \cup [z^*, p_2]$, $b(q) > \max(p_2, x]$.*

Now, for an arbitrary RR1-word, we describe the set of occurrences of an arbitrary beginning of that word.

Corollary 4.15. *Let a be an* RR1*-word, let $\Omega = \langle V, b, \preceq, p\rangle$ be the system of occurrences of the word a, let V_1 be the set of all occurrences from the lower layer, and let $p_1 = b(p)$,*

$M \subseteq V$. Then M is the set of occurrences of a beginning of the word a if and only if the following conditions are satisfied.

1. *The set M is open in the p-root topology.*
2. *The set $M \cap V_1$ is $\leq_1$-closed.*
3. *Suppose the graph $D(|a|)$ has double vertices and contains at least five vertices and the set $M \cap V_1$ contains occurrences of the vertex z^* and of the three vertices adjacent to it. Let $M \cap V_1$ have at least five vertices, and let y be an occurrence from the lower layer such that the vertex $b(y)$ does not coincide with and is not adjacent to z^*. Moreover, let $b(p) \in O \cup [z^*, b(y))$ and let $b(y)$ be greater than all elements of the set $O \cup [z^*, b(y))$, except for p_1. Let x be an occurrence from the upper layer such that $b(x) \in [z^*, b(y))$, $p_1 \notin [z^*, b(x)]$, and the vertices $b(x)$ and $b(y)$ are not adjacent. Then $y \in M \Rightarrow x \in M$.*

   Moreover, let $p_1 \in O$, let y' be an occurrence from the lower layer such that $b(y) \in [z^*, b(y'))$ and $\max(b(y), b(y')] < b(y)$, and let x' be an occurrence from the upper layer such that $b(x')$ is a vertex of the interval $[z^*, b(y))$ adjacent to $b(y)$ and such that $b(x') < \max(b(y), b(y')]$. Then $y' \in M \Rightarrow x' \in M$.
4. *Suppose the graph $D(|a|)$ have double vertices and contains at least five vertices. Let $p_1 \in O$, let p_2 be a vertex of $D(|a|)$ such that $p_2 \notin O \cup \{z^*\}$ and p_2 is greater than all elements of the set $O \cup [z^*, p_2)$ except for p_1. Let y be an occurrence from the upper layer such that $b(y) \in [z^*, p_2)$ and the vertex $b(y)$ is adjacent to p_2, and let x be an occurrence from the lower layer such that $p_2 \in [z^*, b(x))$ and $b(y) > \max(p_2, b(x)]$. Then $y \in M \Rightarrow x \in M$.*

For the graph D_n, $n \geq 5$, we denote by o_1 the lesser vertex among the two end vertices adjacent to z^*.

We introduce the following notation: Rt_2 is the set of roots whose graph is such that its leading vertex is double; $\mathrm{Rt}_{2,1}$ is the set of roots from Rt_2 whose graph has only one double vertex; $\mathrm{Rt}_{2,2}$ and $\mathrm{Rt}_{2,3}$ are the sets of roots α from Rt_2 whose graph has at least two double vertices and $\max[o_1, p_1) < \max(p_1, x_1^-(\alpha)]$, $\max[o_1, p_1) > \max(p_1, x_1^-(\alpha)]$ (respectively). Note that $\{\mathrm{Rt}_{2,1}, \mathrm{Rt}_{2,2}, \mathrm{Rt}_{2,3}\}$ is a partition of the set Rt_2.

Lemma 4.16. *Let u be an* RR2*-word of composition α, let p_1 be the leading letter of u, and let u_2 be the ending of u that begins with the right occurrence of the letter p_1. Then one of the following statements is true*:

1) *$\alpha \in \mathrm{Rt}_{2,1}$, $p_1 = z^*$, and $|u_2| = [z^*, k]$, where k is an end vertex of the graph $D(\alpha)$ such that the interval $[z^*, k]$ contains the least vertex among those adjacent to z^* (see* [33, Lemma 3.8]);

2) *$\alpha \in \mathrm{Rt}_{2,2}$ and $|u_2| = [o_1, x_2^+(\alpha)]$, where o_1 is the lesser element among the two elements of the set $O(\alpha)$ (see* [33, Lemma 3.8]);

3) *$\alpha \in \mathrm{Rt}_{2,3}$ and $|u_2| = (m, x_1^+(\alpha)]$, where m is the vertex closest to p_1 among the vertices of the interval $[o_1, p_1)$ satisfying $m > \max(p_1, x_1^-(\alpha)]$ (see* [33, Lemma 3.9]).

Corollary 4.17. *An arbitrary* RR*-word is determined by its composition, i.e., it is reduced (see Corollary* 4.3 *and Lemma* 4.16).

Referring to Lemma 4.16, we shall speak about the structure of RR2-words. Now we state these results without searching through the cases.

Lemma 4.18. *Let α be a positive root whose graph is such that its leading vertex p_1 is double. Then the* RR2*-word u of composition α is unique. We introduce the following notation*: *u_2 is the ending of u that begins with the right occurrence of the letter p_1*; *m is the least vertex (in the order $\leq_{p_1}$) among the single vertices of the graph $D(\alpha)$ that have a double adjacent vertex*; *k is an end vertex of the graph $D(\alpha)$ such that $m \in (p_1, k]$*; *v is the beginning of the word $\mathrm{red}([z^*, x_2^+(\alpha)] + [m, k])$ that ends with the letter k. Then $|u_2|$ is the interval generated by the vertex $x_2^+(\alpha)$ and the interval $|v|$.*

§5. Regular words and SR-words

For an arbitrary word u of length at least 2, we introduce the following notation: $l(u)$ is the longest proper regular beginning of the word u; $r(u)$ is the longest proper regular ending of the word u; $\overline{l}(u)$ is the ending of the word u obtained by elimination of the beginning $l(u)$; $\overline{r}(u)$ is the beginning of the word u obtained by elimination of the ending $r(u)$.

Lemma 5.1. *A nonempty word is regular if and only if any proper ending of it is less than the word itself.*

Proof. Let v be a regular word, and let $v = ab$, where a and b are nonempty. Then $ba < ab = v$. Assume that $v \leq b$. Then $ba < ab \leq b$. Therefore, b is a beginning of the word ab, i.e., $ab = bc$ for some word c. Since $ba < ab = bc$, we have $a < c$.

Since $a \neq 1$, we have $c \neq 1$. Since the word v is regular, it follows that $cb < bc = ab$ and $c < a$, a contradiction. Thus, $b < v$.

Conversely, let v be a nonempty word any proper ending of which is less than v. Let $v = ab$, $a \neq 1 \neq b$. Then $b < ab$. Moreover, $ba \leq b$ (a nonempty beginning is not less than the word itself). Therefore, $ba \leq b < ab$, $ba < ab$. Since the representation $v = ab$ was chosen arbitrarily, the word v is regular. □

Corollary 5.2. *For any two regular words u and v, the word uv is regular if and only if $v < u$.*

Proof. Consider arbitrary regular words u and v. Since the regular words are nonempty, we have $u \neq 1 \neq v$. If the word uv is regular, then $v < uv$ by Lemma 5.1. Moreover, $uv < u$, because a proper beginning of an arbitrary word is greater than the word itself. Therefore, $v < uv < u$.

Conversely, let $v < u$. By Lemma 5.1, it suffices to prove that any proper ending b of the word uv is less than the word itself.

Assume that $uv \leq v$. Then $uv \leq v < u$. Therefore, u is a beginning of the word v, i.e., $v = uc$ for some c. Since $uc = v < u$, we have $c \neq 1$. Now, the inequality $uv \leq v$ yields $u^2c \leq uc$, $uc \leq c$. But the word $uc = v$ is regular. Consequently, by Lemma 5.1, $c < uc$, a contradiction. Thus, $v < uv$.

If b is an ending of the word v, then $b \leq v$ by Lemma 5.1. Therefore, $b \leq v < uv$.

Now we may assume that v is a proper ending of the word b, i.e., $u = ay$, $b = yv$, and $y \neq 1$. Since $b \neq uv$, we have $y \neq u$, i.e., $a \neq 1$. Therefore, y is a proper ending of the regular word u and, by Lemma 5.1, $y < u$. Thus, $b = yv < uv$. □

Lemma 5.3. *The union of two intersecting regular subwords of an arbitrary word is a regular subword.*

Proof. It suffices to show that if ab and bc are regular words and $b \neq 1$, then the word abc is regular. We may assume that $a \neq 1$ and $c \neq 1$. By Lemma 5.1, it suffices to prove that $d < abc$ for an arbitrary proper ending d of the word abc.

Since the word ab is regular and $a \neq 1 \neq b$, we have $b < ab$ by Lemma 5.1. Therefore, $bc < abc$. If $d \neq 1$ is an ending of the word bc, then, again by Lemma 5.1, $d \leq bc < abc$.

It remains to consider the case where bc is a proper ending of the word d, i.e., $d = ybc$, $a = xy$, $y \neq 1$. Since $d \neq abc$, we have $y \neq a$ and $x \neq 1$. Therefore, yb is a proper ending of the regular word ab and, by Lemma 5.1, $yb < ab$. Thus, $d = ybc < abc$. □

Corollary 5.4. *Any nonempty word u admits a unique representation as the product $u = u_1u_2 \cdots u_m$ of a nondecreasing sequence of regular words $u_1 \leq u_2 \leq \cdots \leq u_m$.*

Proof. Indeed, by Lemma 5.3, the longest regular subwords u_i of u do not intersect. Moreover, each occurrence of any letter of the word u lies in one of such longest regular subwords. Therefore, $u = u_1 \cdots u_m$. It is clear that the chain $u_1, \ldots, u_m$ is determined uniquely.

If $u_i > u_{i+1}$ for some $i = 1, \ldots, m-1$, then the word $u_i u_{i+1}$ is regular. But this is impossible, because u_i and u_{i+1} are longest proper subwords of u. Therefore, $u_i \leq u_{i+1}$ for all $i = 1, \ldots, m-1$. □

Corollary 5.5. *Let u be a regular word different from a letter, and let m be a positive integer such that the word $l(u)^m$ is a beginning of u. Then the ending of u obtained by removing the beginning $l(u)^m$ is regular. In particular, the word $\bar{l}(u)$ is regular.*

Proof. Suppose the contrary. Let c be the ending of the word u obtained by removing the beginning $l(u)^m$, and let $c = u_1 \cdots u_k$ be the decomposition as in Corollary 5.4. Then $k > 1$.

If $u_1 > l(u)$, then $l(u)u_1$ is a regular proper beginning of u, which is longer than $l(u)$; this contradicts the definition of $l(u)$. Therefore, $u_1 \leq l(u)$. Consequently, $u = l(u)^m u_1 \cdots u_k$, $l(u) \leq u_1 \leq \cdots \leq u_m$, $m + k > 1$, which contradicts Corollary 5.4 (the word u is regular). Therefore, the word c is regular. □

For arbitrary words u and v, we shall write $v \prec u$ if $u = wy_1u_1$ and $v = wy_2v_1$ for some (possibly empty) words w, u_1, v_1 and letters y_1, y_2 such that $y_1 > y_2$. Moreover, if the word v_1 is empty, we shall write $v \boxleft u$. Now we present a property of regular words that is most important in the present paper.

Corollary 5.6. *Let a be a regular word different from a letter. Then*:

1) *if m is the greatest positive integer such that $l(a)^m$ is a beginning of a, and c is an ending of a such that $a = l(a)^m c$, then the word c is regular and $c \boxleft l(a)$ (see* [31, Lemma 5.1]);

2) *two regular words b and c and a positive integer m such that $a = b^m c$ and b is not a beginning of c are uniquely determined by a.*

For an arbitrary regular word u different from a letter, we consider the representation $u = a^{i(u)}b$ as in Corollary 5.6. Here $i(u)$ is a positive integer and a, b are regular words such that $b \boxleft a$. Let $j(u)$ and $k(u)$ denote the number of occurrences of the leading letter of u in the word a (respectively, b). For arbitrary nonnegative integers i, j, k, we denote by $S(i, j, k)$ the set of SR-words u such that $i = i(u)$, $j = j(u)$, $k = k(u)$.

Lemma 5.7. *The set of* SR-*words splits into the sets $S(i, j, k)$, where* 1) $i, j \in \{1, 2\}$; 2) $0 \leq k \leq j$; 3) *for $i = 2$ and $j = 2$, only $k = 0$ is possible.*

Proof. 1) For an arbitrary SR-word u, we consider the representation $u = a^i b$, where a and b are regular words such that $b \boxleft a$. Suppose $i \geq 3$. Then, by Corollary 5.2, the word a^2b is regular. Moreover, a^2b is a proper subword of u. Therefore, the word a^2b has rooted composition. It follows that $k = 0$ (otherwise the word a^2b has at least three occurrences of its leading letter).

Next, $k = 0$ and $b \boxleft a$ imply that b is a letter. But then in the graph $D(|a^2b|)$ there is precisely one single vertex (b), which is impossible in the root system of D_n (the word a^2b has rooted composition), a contradiction. Thus, $i \in \{1, 2\}$.

Now, a and b are proper subwords of u. Moreover, since a and b are regular, they have rooted composition. Therefore, a has at most two occurrences of p_1.

Moreover, a has at least one occurrence of the leading letter p_1 of u (the word a begins with this letter). Consequently, $j \in \{1, 2\}$. Statement 1) is proved.

The word b does not end with the letter p_1 (because it is regular). This fact and $b \boxleft a$ imply statement 2).

3) For $i = 2$, the word ab is a proper subword of the SR-word $a^i b$. Moreover, $b < a$ because $b \boxleft a$. Therefore, the word ab is regular, and, being a regular proper subword of the SR-word $a^i b$, the word ab is an RR-word. Consequently, ab has rooted composition. Therefore, the number of occurrences $j + k$ of the letter p_1 in the word ab does not exceed 2. Hence, for $i = 2$ and $j = 2$, we have $k = 0$. □

Now it is easy to enumerate the triples (i, j, k) from Lemma 5.7: $(1, 1, 0)$, $(1, 1, 1)$, $(1, 2, 0)$, $(1, 2, 1)$, $(1, 2, 2)$, $(2, 1, 0)$, $(2, 1, 1)$, and $(2, 2, 0)$. This allows us to make the statement of Lemma 5.7 more specific.

Lemma 5.8. *The set of* SR-*words splits into the sets* $S(1, 1, 0)$, $S(1, 1, 1)$, $S(1, 2, 0)$, $S(1, 2, 1)$, $S(1, 2, 2)$, $S(2, 1, 0)$, $S(2, 1, 1)$, *and* $S(2, 2, 0)$.

Our purpose in this paper is to study each of the eight sets $S(i, j, k)$ separately.

Corollary 5.9. *For an arbitrary regular word x, any proper regular subword of it is a subword either of $l(x)$ or of $r(x)$ (see Lemma* 5.3).

Corollary 5.10. *An arbitrary word u is an* RR-*word if and only if it is regular, it has rooted composition, and $l(u)$ and $r(u)$ are* RR-*words (see Corollary* 5.9).

Corollary 5.11. *An arbitrary word u is an* SR-*word if and only if it is regular, its composition is not rooted, and $l(u)$ and $r(u)$ are* RR-*words (see Corollary* 5.9).

§6. The set $S(1, 1, 0)$

From Lemma 5.8 it is easily seen that $S(1, 1, 0)$ is precisely the set of SR-words with one occurrence of the leading letter. This fact and Corollary 5.11 imply the following statement.

Corollary 6.1. $S(1, 1, 0)$ *is precisely the set of words u such that*

1) *upon removing the last letter, we obtain an* RR1-*word*;
2) *the last letter of u is less than its first letter*;
3) *the element $|u|$ is not a root*;
4) *the word $r(u)$ is reduced*; *in particular, the ending of u that begins with the right occurrence of the second-by-priority letter of u is an* RR1-*word.*

By property 1) of Corollary 6.1, any word in $S(1, 1, 0)$ is uniquely represented in the form $\mathrm{red}(\alpha)q$, where α is a positive root and q is a letter. Next we analyze properties 3) and 4) in this corollary. For an arbitrary root α, let $M^-(\alpha)$ denote the set of simple roots β such that $\alpha + \beta$ is a root.

Remark 6.2. Let (α, q) be a pair in which α is a positive root and q is a letter. Then $q \in M^-(\alpha)$ (i.e., the element $\alpha + |q|$ is a root) if and only if either 1) q is not a vertex of the graph $D(\alpha)$ but has an adjacent vertex in it, or 2) q is a single vertex of the graph $D(\alpha)$ and the total multiplicity of the vertices adjacent to q is equal to 3.

Lemma 6.3. *Let v be an* RR1-*word, and let q be a letter that is less than the first letter p_1 of v. Moreover, suppose that the ending $v_2 q$ of the word vq that begins with the right occurrence of the second-by-priority letter p_2 of this word is reduced. Then the property "the word $r(vq)$ has two occurrences of its leading letter" is equivalent to the fact that*

1) $q < p_2$;
2) *the vertex p_2 in the graph $D(\alpha)$ is double (here $\alpha = |v|$)*;
3) *the graph $D(\alpha)$ has at least five vertices and $p_1 \notin O(\alpha)$*;
4) *the least of the vertices adjacent to p_2 is a vertex of the interval $[p_2, p_1]$, and it is a vertex of the graph $D(|v_2 q|)$.*

Proof. Since $q < p_1$, it follows that the word vq has occurrences of at least two letters and the second-by-priority letter p_2 of the word vq exists and is the first and the leading letter of the word $r(vq)$.

Assume that the word $r(vq)$ has two occurrences of its leading letter, i.e., the word $r(vq)$ has two occurrences of the letter p_2.

1. If $q \geq p_2$, then $q = p_2$ (because $q < p_1$). In this case $r(vq) = p_2$, which contradicts the fact that the word $r(vq)$ has two occurrences of the letter p_2.

2. Since the word $r(vq)$ has two occurrences of the letter p_2, property 1) implies property 2).

3. Assume that $p_1 \in O(\alpha)$. Introduce the following notation: v_2 is the ending of v that begins with the right occurrence of the letter p_2; v_1 is the longest subword of u that begins with the left occurrence of the letter p_2. Then the structure of RR1-words shows that $|v_2| = [p_2, x_2^+]$ and $|v_1| = I$, where $x_2^+ = x_2^+(\alpha)$ and I is an interval; if $p_2 = z^*$, then I coincides with the interval of all vertices of $D(\alpha)$ except for p_1, and if $p_2 \neq z^*$, then $I = [x, k]$, where k is an end vertex of $D(\alpha)$ such that $p_2 \in [z^*, k]$; x is the vertex of the interval $[o, p_2)$ adjacent to p_2; o is an end vertex of $D(\alpha)$ such that $o \notin \{p_1, k\}$.

In any case, the interval I contains all vertices of $[p_2, x_2^+]$, as well as all vertices of $D(\alpha)$ that have an adjacent vertex in $[p_2, x_2^+]$ (except for the vertex p_1).

The word v_2q is the ending of vq that begins with the right occurrence of the letter p_2. Thus, by assumption, the word v_2q is reduced. Therefore, the vertex q does not lie in the interval $[p_2, x_2^+]$ but has an adjacent vertex in it. The proved above shows that $|v_2q| \leq |v_1|$.

The word v_1 begins with the only occurrence of its leading letter. Therefore, it is a regular RR-word (as a regular proper subword of the RR-word v). Moreover, v_2q is also an RR-word and $|v_2q| \leq |v_1|$. Therefore, $v_1 \leq v_2q$ by Corollary 4.8. Thus, the word v_1v_2q is not regular.

But the regular word $r(vq)$ is the ending of vq that begins with the left occurrence of the letter p_2, i.e., $r(vq) = v_1v_2q$. This contradiction implies that $p_1 \notin O(\alpha)$. Moreover, the graph $D(\alpha)$ has double vertices and its vertex p_1 is single. Therefore, the graph $D(\alpha)$ has at least five vertices.

4. Since $p_1 \notin O(\alpha)$, the structure of RR1-words shows that $|v_2| = [p_2, x_2^+]$. The vertices of $D(|v_1|)$ are the vertices t of $D(\alpha)$ such that $p_2 \in [t, p_1)$; moreover, the vertex p_2 in $D(|v_1|)$ is single, but the other vertices of $D(|v_1|)$ have the same multiplicity in it as in the graph $D(\alpha)$. We denote by m' the vertex of the interval $[p_2, p_1]$ that is adjacent to p_2.

If m' is not a vertex of the graph $D(|v_2q|)$, then $|v_2q| \leq |v_1|$. In this case, $v_1 \leq v_2q$ by Corollary 4.8. Then the word v_1v_2q is not regular. But the word $v_1v_2q = r(vq)$ is regular. Thus, m' is a vertex of the graph $D(|v_2q|)$. Therefore, m' is the second letter of the word v_2q.

Denote by m the second letter of the word v_1. Then the word p_2m is a beginning of v_1. Being a regular beginning of the RR-word v_1, the word p_2m has rooted composition, i.e., the vertices p_2 and m are adjacent.

From the description of the graph $D(|v_1|)$ given above, it follows that $p_2 \in [m, p_1)$. Therefore, $m \notin [p_2, p_1)$. Also, $m' \in [p_2, p_1)$. Therefore, $m \neq m'$. Since the word $r(vq) = v_1v_2q$ is regular, we have $v_1 > v_2q$ by Corollary5.2. Since in addition m (m') is the second letter of the word v_1 (respectively, v_2q) and $m \neq m'$, we have $m > m'$.

The graph $D(|v_1|)$ contains any vertex adjacent to p_2 and different from m'. Therefore, the second letter of the word v_2 is the least of the vertices adjacent to p_2 and different from m'. This fact and $m > m'$ imply that m' is the least of the vertices adjacent to p_2.

Conversely, suppose all properties 1)–4) are fulfilled. Then the words v_1 and v_2 are

well defined and, as in the proof of property 4), we have $v_1 > v_2q$, i.e., the word v_1v_2q is regular. Thus, $r(vq) = v_1v_2q$. □

Lemma 6.4. *If under the assumptions of Lemma* 6.3*, properties* 1)–4) *are satisfied for the word vq, then either vq has rooted composition, or the composition of the word $r(vq)$ is not rooted.*

Proof. Indeed, since $q < p_2$, it follows that p_2 is the second-by-priority vertex in the graph $D(\alpha)$. Therefore, the word v_2 is not empty and begins with the letter p_2. Since the word v_2q is reduced, it has rooted composition, i.e., the element $|v_2| + |q|$ is a root. Therefore, the graph $D(|v_2| + |q|)$ is connected. Consequently, either q is a vertex of the graph $D(|v_2|)$, or it has an adjacent vertex in it. Therefore, either q is a vertex of the graph $D(|v|)$, or it has an adjacent vertex in it.

Therefore, if q is not a vertex of $D(|v|)$, then q has an adjacent vertex in this graph. In this case, the word vq has rooted composition.

It remains to consider the case where q is a vertex of $D(|v|)$. Since v is an RR1-word and the vertex p_2 in the graph $D(|v|)$ is double, the structure of RR1-words shows that $|v_2| = [p_2, x_2^+(|v|)]$. Moreover, q is either a vertex of the graph $D(|v_2|)$ or has an adjacent vertex in it. If q is vertex of $D([p_2, x_2^+(|v|)])$, then the composition of v_2q is not rooted, which is not true. Thus, q is not a vertex of the graph $D([p_2, x_2^+(|v|)])$ but has an adjacent vertex in it.

If $q \in [p_2, p_1]$, then $q = x_1^-(|v|)$ and the word vq has rooted composition. If $q \notin [p_2, p_1]$, then, using the structure of RR1-words, we compute the ending of v that begins with the left occurrence of the letter p_2. As a result, we conclude that if $p_2 \neq z^*$, then q has three occurrences in the word vq; if $p_2 = z^*$, then $q \in O$ and the letter q has two occurrences in the word vq. In both cases, the composition of the word $r(vq)$ is not rooted. □

Now we can make the description of the words in $S(1, 1, 0)$ given in Corollary 6.1 more precise.

Corollary 6.5. *$S(1, 1, 0)$ is precisely the set of words vq, where v is an* RR1*-word and q is a letter that is less than the first letter of v. Moreover,*

1) *the ending of vq that begins with the right occurrence of the second-by-priority letter of this word is an* RR1*-word*;

2) *the element $|v| + |q|$ is not a root*;

3) *properties* 1)–4) *in Lemma* 6.3 *cannot be satisfied simultaneously.*

Remark 6.6. An arbitrary word w is an RR1-word if and only if either it is a letter, or

1) the word $\overline{w}$ obtained from w by removing the last letter q is an RR1-word;

2) $q \in M^-(|\overline{w}|)$ and q is less than the leading letter of $\overline{w}$;

3) a reduced word with composition $|w|$ ends with the letter q.

Theorem 6.7. *The following statements* I *and* II *are valid.*

I. *$S(1, 1, 0)$ is precisely the set of words vq, where v is an* RR1*-word and q is a letter that is less than the first letter p_1 of v; moreover, one of the following alternative cases* 1 *and* 2 *occurs.*

1. *$|q| \not\leq \alpha$ (here $\alpha = |v|$). In this case, the graph $D(\alpha)$ contains no vertices adjacent to q, and q is greater than any vertex of $D(\alpha)$ except for p_1.*

2. *q is a vertex of $D(\alpha)$. In this case, the root α is not simple and one of the following alternative cases* 2.1 *or* 2.2 *occurs.*

2.1. *q coincides with the second-by-priority vertex p_2 of $D(\alpha)$. In this case, if the vertex q in $D(\alpha)$ is single, then in this graph the total multiplicity of the vertices adjacent to q is not equal to* 3.

2.2. *$q < p_2$. In this case,*

a) $|q| \not\leq |v_2|$, *but* q *has an adjacent vertex in the graph* $D(|v_2|)$ (v_2 *is the ending of the word* v *that begins with the right occurrence of the letter* p_2). *In particular,* $q \in M^-(|v_2|)$;

b) $q \notin M^-(\alpha)$;

c) *a reduced word with composition* $|v_2| + |q|$ *ends with the letter* q;

d) *if the vertex* p_2 *in the graph* $D(\alpha)$ *is double,* $p_1 \notin O(\alpha)$, *and the vertex of the interval* $[p_2, p_1]$ *adjacent to* p_2 *is the least of the vertices adjacent to* p_2, *then* $p_2 = x_2^+(\alpha)$ *and* $q \neq x_1^-(\alpha)$.

II. *Under the conditions of case* 2.2, *one of the following mutually excluding cases* 2.2.1, 2.2.2, *and* 2.2.3 *occurs.*

2.2.1. $\alpha \in \mathrm{Rt}_{1,1}$. *In this case, properties* a)–c) *in case* 2.2 *are equivalent to the following properties* 1)–3):

1) $q \notin [p_2, x_2^+(\alpha)]$, *but* q *has an adjacent vertex in the interval* $[p_2, x_2^+(\alpha)]$;

2) *it is impossible that the vertex* q *in the graph* $D(\alpha)$ *is single and simultaneously has an adjacent single vertex*;

3) q *is the greatest vertex* (*in the order* $\leq_{p_2}$) *among the end vertices of the graph* $[p_2, x_2^+(\alpha)] \cup \{q\}$.

2.2.2. $\alpha \in \mathrm{Rt}_{1,2}$. *In this case, the graph* $D(\alpha)$ *has at least five vertices and properties* a)–d) *of case* 2.2 *are equivalent to the fact that* $(q, x_1^+(\alpha)] = [x_2^+(\alpha), x_1^+(\alpha)]$, $\max\{q, x_2^+(\alpha)\} > \max(x_1^-(\alpha), x_1^+(\alpha)]$.

2.2.3. $\alpha \in \mathrm{Rt}_{1,3}$. *In this case, properties* a)–d) *of case* 2.2 *are equivalent to the following properties* 1)–3):

1) *the vertices* p_1 *and* p_2 *are not adjacent and* q *is the vertex of the interval* (p_2, p_1) *adjacent to* p_2;

2) *the vertex* q' *of the interval* $(q, p_1]$ *adjacent to* q *is single in the graph* $D(\alpha)$. *In* $D(\alpha)$, *there are precisely two vertices adjacent to* q: *these are* p_2 *and* q';

3) q *is greater than any vertex* x *of* $D(\alpha)$ *such that* $p_2 \in (x, q)$.

Proof. I. Consider an arbitrary word u in $S(1,1,0)$. We introduce the following notation: p_1 and q are the first and the last letters, respectively, of u; v is the word obtained from u by removing the last letter. Then $u = vq$; moreover, by Corollary 6.5, v is an RR1-word and $q < p_1$. Therefore, the element $\alpha = |v|$ is a root and p_1 is the leading vertex of the graph $D(\alpha)$.

1. Consider the case where $|q| \not\leq \alpha$. Then q has no adjacent vertices in $D(\alpha)$; otherwise the element $\alpha + |q|$ is a root.

Assume that in the word v there is a letter that is less than p_1 but greater than q. Then v has occurrences of at least two letters and its second-by-priority letter p_2 is greater than q. Denote by v_2 the ending of v that begins with the right occurrence of p_2. Then v_2q is a regular proper subword of the SR-word u, and thus it has rooted composition, i.e., the element $|v_2q|$ is a root. Therefore, the graph $D(|v_2q|)$ is connected. Since, in addition, $0 \neq |v_2| \leq |v|$, the graph $D(|vq|)$ is also connected. But since q does not have adjacent vertices in the graph $D(|v|)$, this is possible only if $v = q$. But the word v is not a letter. Thus, the vertex q is greater than any vertex of $D(\alpha)$ except for p_1.

Conversely, consider a pair (v, q) (where v is an RR1-word and q is a letter that is less than the first letter p_1 of v) for which the assertions of case 1 are true. With the help of Corollary 6.5, we show that $vq \in S(1,1,0)$.

Since q is greater than any vertex of $D(\alpha)$ except for p_1 (here $\alpha = |v|$), we have $r(vq) = q$. Therefore, property 1) of Corollary 6.5 is fulfilled.

Since $D(\alpha)$ contains no vertices adjacent to q, the graph $D(\alpha + |q|)$ is not connected. Therefore, property 2) of Corollary 6.5 is fulfilled.

Since q is greater than any vertex of $D(\alpha)$ except for p_1, property 1) of Lemma 6.3 fails. Therefore, property 3) of Corollary 6.5 holds. By Corollary 6.5, $vq \in S(1,1,0)$.

2. Let q be a vertex of $D(\alpha)$. Then the root α is not simple; otherwise $v = q$ and $vq = q^2$, but the word vq is regular.

Denote by p_2 the second-by-priority vertex of $D(\alpha)$. Then $q \leq p_2$, because q is a vertex of $D(\alpha)$ and $q < p_1$. Therefore, either $q = p_2$, or $q < p_2$.

2.1. Consider the case where $q = p_2$. Since vq is an SR-word, its composition $|v| + |q|$ is not rooted, i.e., $q \notin M^-(\alpha)$. From Remark 6.2 we see that if the vertex q in $D(\alpha)$ is single, then in this graph the total multiplicity of the vertices adjacent to q is not equal to 3.

Conversely, consider a pair (v, q) (where v is an RR1-word and q is a letter that is less than the first letter p_1 of v) for which the assertions of case 2.1 are valid. With the help of Corollary 6.5, we show that $vq \in S(1, 1, 0)$. Since $q = p_2$, we have $r(vq) = q$. Therefore, property 1) of Corollary 6.5 is fulfilled.

Thus, q is a vertex of $D(\alpha)$, and if $2|q| \not\leq \alpha$, then, in the graph $D(\alpha)$, the total multiplicity of the vertices adjacent to q is not equal to 3. Therefore, by Remark 6.2, we obtain $q \notin M^-(\alpha)$, i.e., property 2) of Corollary 6.5 holds. Since $q = p_2$, property 1) of Lemma 6.3 is not fulfilled. Therefore, property 3) of Corollary 6.5 also holds. By Corollary 6.5, $vq \in S(1, 1, 0)$.

2.2. Consider the case where $q < p_2$. Denote by v_2 the ending of the word v that begins with the right occurrence of the letter p_2. Then v_2q is a regular proper subword of the SR-word vq, and so it is an RR-word. Moreover, it has a unique occurrence of its leading letter p_2. Therefore, v_2q is an RR1-word. In particular, it has rooted composition, i.e., the element $|v_2| + |q|$ is a root, or $q \in M^-(|v_2|)$. Therefore, the graph $D(|v_2q|)$ is connected.

a) Assume that $|q| \leq |v_2|$. Since $q \in M^-(|v_2|)$, it follows that the vertex q in the graph $D(|v_2|)$ is single and the total multiplicity in this graph of the vertices adjacent to q is equal to 3, i.e., either in the graph $D(|v_2|)$ there are no double vertices, this graph contains z^* and the three vertices adjacent to z^*, and $q = z^*$, or the vertex q in $D(|v_2|)$ has an adjacent double vertex and an adjacent single vertex. In both cases, the graph $D(|v_2|)$ is not an interval.

Only for $\alpha \in \mathrm{Rt}_{1,3}$ the graph $D(|v_2|)$ is not an interval. Therefore $\alpha \in \mathrm{Rt}_{1,3}$. By Lemma 4.2, for any vertex of $D(|v_2|)$ (different from p_2) the total multiplicity of the vertices adjacent to it is the same as in the graph $D(\alpha)$. Since $q < p_2$ and, by assumption, q is a vertex of $D(|v_2|)$, we conclude that in the graph $D(\alpha)$, as in the graph $D(|v_2|)$, the total multiplicity of the vertices adjacent to q is equal to 3, i.e., $q \in M^-(\alpha)$, which is impossible, because the composition of the word vq as an SR-word is not rooted.

Thus, $|q| \not\leq |v_2|$. Since $q \in M^-(|v_2|)$, q has an adjacent vertex in the graph $D(|v_2|)$. Therefore, property a) is fulfilled.

b) $q \notin M^-(\alpha)$, because the composition of the SR-word vq is not rooted.

c) The word v_2q, being a proper subword of the SR-word vq, is reduced. It has the composition $|v_2| + |q|$. Hence, we obtain property c).

d) Since $vq \in S(1, 1, 0)$, it follows that, by Corollary 6.5, properties 1)–4) of Lemma 6.3 cannot be fulfilled simultaneously. Since $q < p_2$, property 1) of Lemma 6.3 is fulfilled. Therefore, properties 2)–4) of Lemma 6.3 cannot hold simultaneously, i.e., if the vertex p_2 in the graph $D(\alpha)$ is double, $p_1 \notin O(\alpha)$, and the least of the vertices adjacent to p_2 is the vertex p_2' of the interval $[p_2, p_1]$, then $|p_2'| \not\leq |v_2q|$.

Consider the following case: the vertex p_2 in the graph $D(\alpha)$ is double, $p_1 \notin O(\alpha)$, and the least of the vertices adjacent to p_2 is the vertex p_2' of the interval $[p_2, p_1]$. In this case, $|v_2| = [p_2, x_2^+(\alpha)]$, $x_1^-(\alpha) \in (p_2, p_1]$, $p_2' \in (p_2, x_1^-(\alpha)]$.

If $p_2 \neq x_2^+(\alpha)$, then $p_1 \notin O(\alpha)$ implies that $p_2' \leq [p_2, x_2^+(\alpha)] \leq |v_2q|$, which contradicts

$|p_2'| \not\leq |v_2q|$. Therefore, $p_2 = x_2^+(\alpha)$. Then $p_2' = x_1^-(\alpha)$. Since $|p_2'| \not\leq |v_2q|$, we have $q \neq x_1^-(\alpha)$, i.e., property d) is fulfilled.

Conversely, let v be an RR1-word different from a letter, let q be a vertex of the graph $D(\alpha)$ (here $\alpha = |v|$), let q be less than the second-by-priority letter p_2 of v, and suppose that properties a)–d) of case 2.2 occur. We use Corollary 6.5 to show that $vq \in S(1,1,0)$.

Since the ending v_2 of v that begins with the right occurrence of the letter p_2 is an RR1-word, property 1) of Remark 6.6 is fulfilled for the word v_2q. From property a) and the inequality $q < p_2$, we obtain property 2) of Remark 6.6 for v_2q. Therefore, by Remark 6.6, v_2q is an RR1-word, i.e., property 1) of Corollary 6.5 is fulfilled for the word vq. Property b) implies property 2) of Corollary 6.5.

If the vertex p_2 in the graph $D(\alpha)$ is double, $p_1 \notin O(\alpha)$, and the least of the vertices adjacent to p_2 is the vertex p_2' of the interval $[p_2, p_1]$, then property d) implies that $|p_2'| = |x_1^-(\alpha)| \not\leq |p_2| + |q| = |v_2| + |q| = |v_2q|$, i.e., properties 2)–4) of Lemma 6.3 cannot be fulfilled simultaneously, and thus, property 3) in Corollary 6.5 occurs. Therefore, by Corollary 6.5, $vq \in S(1,1,0)$.

II. Since the root α is not simple in case 2.2, Lemma 4.1 shows that $\alpha \in \mathrm{Rt}_{1,1} \cup \alpha \in \mathrm{Rt}_{1,2} \cup \alpha \in \mathrm{Rt}_{1,3}$.

2.2.1. In the case where $\alpha \in \mathrm{Rt}_{1,1}$, from Lemma 4.2 we have $|v_2| = [p_2, x_2^+(\alpha)]$, whence we conclude that property a) of case 2.2 is equivalent to property 1).

We show that if property 1) occurs, then properties b) and 2) are equivalent. If $2|q| \leq \alpha$, then properties b) and 2) are fulfilled. Moreover, $|q| \leq \alpha$. Therefore, it suffices to consider the case where q is a single vertex of $D(\alpha)$.

Property 1) implies that the vertex q of $D(\alpha)$ has an adjacent double vertex in $D(\alpha)$. Therefore, in $D(\alpha)$, the total multiplicity of the vertices adjacent to q is either 2 or 3. Since the vertex q in the graph $D(\alpha)$ is single, the condition $q \notin M^-(\alpha)$ is equivalent to the fact that the total multiplicity of the vertices adjacent to q is not equal to 3. This implies that if property 1) holds, then properties b) and 2) are equivalent.

Since $\alpha \in \mathrm{Rt}_{1,1}$, property 1) implies that the graph $D(|v_2| + |q|)$ contains no double vertices. This fact and Lemma 4.2 imply that if property 1) holds, then properties c) and 3) are equivalent.

2.2.2. Let $\alpha \in \mathrm{Rt}_{1,2}$. Then the graph $D(\alpha)$ has at least five vertices, $p_2 = x_1^-(\alpha)$, and Lemma 4.2 implies that $|v_2| = [x_2^+(\alpha), x_1^+(\alpha)]$.

Assume that conditions a)–d) of case 2.2 are satisfied. Then from condition a) we obtain $(q, x_1^+(\alpha)] = [x_2^+(\alpha), x_1^+(\alpha)]$. Moreover, $p_2 = x_1^-(\alpha)$ is the greatest vertex of the interval $|v_2| + |q| = [q, x_1^+(\alpha)]$. This fact and property c) imply that $\max\{q, x_2^+(\alpha)\} > \max(x_1^-(\alpha), x_1^+(\alpha)]$.

Conversely, let $(q, x_1^+(\alpha)] = [x_2^+(\alpha), x_1^+(\alpha)]$ and $\max\{q, x_2^+(\alpha)\} > \max(x_1^-(\alpha), x_1^+(\alpha)]$. Then, from the relation $(q, x_1^+(\alpha)] = [x_2^+(\alpha), x_1^+(\alpha)]$ we obtain properties a) and b); the inequality $\max\{q, x_2^+(\alpha)\} > \max(x_1^-(\alpha), x_1^+(\alpha)]$ implies property c). Property d) is fulfilled, because the vertex p_2 in the graph $D(\alpha)$ is not double.

2.2.3. Consider the case where $\alpha \in \mathrm{Rt}_{1,3}$. Assume that conditions a)–d) of case 2.2 are satisfied. Since q is a vertex of $D(\alpha)$ but not of $D(|v_2|)$, and it has an adjacent vertex in $D(|v_2|)$, the relation $\alpha \in \mathrm{Rt}_{1,3}$ and Lemma 4.2 show that q coincides with the vertex of the interval $(p_2, p_1]$ adjacent to p_2. Since $q < p_1$, property 1) holds.

Since $q \in (p_1, p_2)$, the vertex q has exactly two adjacent vertices in $[p_1, p_2]$. Moreover, $[p_1, p_2] \leq \alpha$. Therefore, q has at least two adjacent vertices in the graph $D(\alpha)$ and the total multiplicity of the vertices adjacent to q is at least 2.

The vertex q is single in the graph $D(\alpha)$, because q coincides with the vertex of the interval $[p_2, p_1]$ that is adjacent to p_2, and $\alpha \in \mathrm{Rt}_{1,3}$. Therefore, in the graph $D(\alpha)$ the total multiplicity of the vertices adjacent to q does not exceed 3. Consequently,

from property b) it follows that, in the graph $D(\alpha)$, the total multiplicity of the vertices adjacent to q is equal to 2. Since $q \in (p_2, p_1)$ and $[p_2, p_1] \leq \alpha$, property 2) holds.

Since $\alpha \in \mathrm{Rt}_{1,3}$ and q coincides with the vertex of the interval $[p_2, p_1]$ adjacent to p_2, Lemma 4.2 shows that the vertices of the graph $D(|v_2|)$ are precisely p_2 and the vertices x of $D(\alpha)$ that satisfy the condition $p_2 \in (x, q)$. Moreover, in $D(|v_2 q|)$, q is an end vertex, and the leading vertex (p_2) of this graph is single and adjacent to q. This fact and property c) imply property 3).

Conversely, assume that conditions 1)–3) are satisfied. Since $\alpha \in \mathrm{Rt}_{1,3}$, Lemma 4.2 and property 1) show that the vertices of the graph $D(|v_2|)$ are precisely p_2 and the vertices x of the graph $D(\alpha)$ that satisfy the condition $p_2 \in (x, q)$. Moreover, the vertex q is adjacent to p_2. This implies property a).

Since $\alpha \in \mathrm{Rt}_{1,3}$, we have $2|p_2| \not\leq \alpha$. Therefore, property 2) implies that the total multiplicity of the vertices in $D(\alpha)$ that are adjacent to q is equal to 2. Since $\alpha \in \mathrm{Rt}_{1,3}$, property 1) shows that the vertex q is single. This yields property b). Since the vertices of the graph $D(|v_2|)$ are precisely p_2 and the vertices x of the graph $D(\alpha)$ that satisfy $p_2 \in (x, q)$, property 3) implies property c). Property d) is fulfilled because $2|p_2| \not\leq \alpha$. $\square$

§7. The set $S(1, 2, 0)$

Remark 7.1. By definition, $S(1, 2, 0)$ is precisely the set of SR-words $u = vw$, where v and w are regular words, $w \boxleq v$, and the word v has two occurrences of the leading letter of the word u, while w has none of them.

Corollary 7.2. *$S(1, 2, 0)$ is precisely the set of* SR*-words vq, where v is an* RR2*-word and q is a letter that is less than the leading letter p_1 of v.*

Proof. Indeed, consider an arbitrary word u in $S(1, 2, 0)$ and its representation $u = vw$ as in Remark 7.1. Since v has the occurrence of the leading letter p_1 of the word u, it follows that v is not empty and p_1 is its leading letter. Since the word v is regular, it begins with its leading letter p_1. Since w has no occurrences of p_1 and $w \boxleq v$, we see that w is a letter such that $q < p_1$.

Since w is not empty, we conclude that v is a regular proper subword of the SR-word vw, so that it is reduced; moreover, it has two occurrences of its leading letter p_1.

Conversely, if v is a reduced word with two occurrences of its leading letter p_1, q is a letter such that $q < p_1$, and vq is an SR-word, then Remark 7.1 shows that $vq \in S(1, 2, 0)$. $\square$

Corollary 7.3. *$S(1, 2, 0)$ is precisely the set of words vq for which*

1) *v is an* RR2*-word*;

2) *q is a letter that is less than the leading letter p_1 of v*;

3) *$q \notin M^-(|v|)$*;

4) *the word $v_2 q$ is reduced* (*here v_2 is the ending of v that begins with the right occurrence of p_1*).

Proof. Consider an arbitrary word u in $S(1, 2, 0)$. We denote by q the last letter of u and by v the beginning of u obtained by removing the last letter. Then $u = vq$ and, by Corollary 7.2, properties 1) and 2) hold.

Since the composition of vq, as an SR-word, is not rooted, property 3) also holds.

The word $v_2 q$ begins with the only occurrence of its leading letter p_1, so that it is regular. Being a regular proper subword of the SR-word vq, the word $v_2 q$ is reduced, i.e., property 4) holds.

Conversely, let conditions 1)–4) be satisfied for a word $u = vq$. Since the word v is regular (property 1)), condition 2) shows that the word vq is regular. Property 3) implies

that the composition of vq is not rooted. From properties 1) and 2) we conclude that $l(u) = l(vq) = v$ is an RR-word. Property 4) implies that $r(u) = v_2q$ is an RR-word; together with Corollary 5.11, this proves that u is an SR-word. Now, Corollary 7.2 yields $u \in S(1,2,0)$. □

Lemma 7.4. *The set $S(1,2,0)$ consists precisely of the words $u = vq$ for which $v = \mathrm{red}(\alpha)$ and*

1) *α is a root whose graph is such that its leading vertex p_1 is double*;

2) *q is a vertex of the graph $D(\alpha)$ such that $q < p_1$*;

3) *$|q| \not\leq |v_2|$, but q has an adjacent vertex in the graph $D(|v_2|)$ (here v_2 is the ending of v that begins with the right occurrence of the letter p_1)*;

4) *if the vertex q in the graph $D(\alpha)$ is single, then it has no adjacent single vertices*;

5) *q is greater (under the order $\leq_{p_1}$) than any other vertex of $D(|v_2|)$.*

Proof. Consider $u \in S(1,2,0)$. We denote by q the last letter of u and by v the beginning of u obtained by removing the last letter. Then $u = vq$, and property 1) of Corollary 7.3 yields 1).

2. By property 4) of Corollary 7.3, the element $|v_2| + |q|$ is a root. Therefore, q has an adjacent vertex in $D(|v_2|)$. Consequently, q has an adjacent vertex in the graph $D(|v|)$ as well. If $|q| \not\leq |v|$, then $q \in M^-(|v|)$, which contradicts property 3) of Corollary 7.3. Therefore, $|q| \leq |v|$. This fact and property 2) of Corollary 7.3 imply statement 2).

3. By property 4) of Corollary 7.3, $q \in M^-(|v_2|)$. Moreover, by Lemma 4.16, the graph $D(|v_2|)$ is an interval. This yields 3).

4. Since q is a vertex of $D(\alpha)$ (see 2)), using 3) and considering the cases of Lemma 4.16 item by item, we see that q has an adjacent double vertex in $D(\alpha)$.

Assume that the vertex q in the graph $D(\alpha)$ is single. Since q has an adjacent double vertex, it follows that the total multiplicity of the vertices adjacent to q is either 2 or 3. Furthermore, $q \notin M^-(\alpha)$. Therefore, the total multiplicity of the vertices adjacent to q is equal to 2. This proves statement 4).

5. Since the graph $D(|v_2|)$ is an interval, from 3) we deduce that the graph $D(|v_2q|)$ contains no double vertices. This fact and property 4) of Corollary 7.3 imply 5).

Conversely, consider a pair (α, q) and the word $v = \mathrm{red}(\alpha)$ for which statements 1)–5) are true; we use Corollary 7.3 to show that $vq \in S(1,2,0)$. From 1) and 2) we obtain properties 1) and 2) of Corollary 7.3, respectively.

Since q is a vertex of $D(\alpha)$ (see 2)), using 3) and considering the cases of Lemma 4.16, we conclude that q has a double adjacent vertex in $D(\alpha)$.

Assume that the vertex q in $D(\alpha)$ is single. Since q has a double adjacent vertex, the total multiplicity of the vertices adjacent to q is equal either to 2 or to 3. Using 4) we deduce that the total multiplicity of the vertices adjacent to q is equal to 2. Therefore, property 3) of Corollary 7.3 holds.

By Lemma 4.16, the graph $D(|v_2|)$ is an interval. This fact and statement 3) imply that the graph $D(|v_2q|)$ has no double vertices. Moreover, 1) and 2) show that v_2 is an RR1-word. Next, statement 5) and Remark 6.6 imply property 4) of Corollary 7.3. Therefore, by Corollary 7.3, $vq \in S(1,2,0)$. □

Theorem 7.5. *$S(1,2,0)$ is precisely the set of words vq for which $v = \mathrm{red}(\alpha)$, $\alpha \in \mathrm{Rt}_2$, and q is a letter that is less than the leading letter p_1 of the word v. Moreover, one of the following statements* 1–3 *is true.*

1. *$\alpha \in \mathrm{Rt}_{2,1}$. In this case,*

a) *$q \in O(\alpha)$ and q is not the least among the vertices adjacent to z^**;

b) *$q > \max(z^*, k]$, where k is an end vertex of the graph $D(\alpha)$ such that the interval $(z^*, k]$ contains the least of the vertices adjacent to z^*.*

2. $\alpha \in \mathrm{Rt}_{2,2}$. *In this case, either* $q = x_1^-(\alpha)$ *and the graph* $D(\alpha)$ *contains no adjacent single vertices, or* $q = o_2 > \max(p_1, x_1^-(\alpha)]$.

3. $\alpha \in \mathrm{Rt}_{2,3}$. *In this case, either* $o_1 = m$ *and* $q \in O(\alpha)$, *or* $o_1 \neq m = q$. *Here* m *is the vertex of the interval* $[o_1, p_1)$ *that is closest to* p_1 *and satisfies* $m > \max(p_1, x_1^-(\alpha)]$.

Proof. It suffices to compute the composition of the word v_2 with the help of Lemma 4.16. We have $|v_2| = [z^*, k]$ for $\alpha \in \mathrm{Rt}_{2,1}$, $|v_2| = [o_1, x_2^+(\alpha)]$ for $\alpha \in \mathrm{Rt}_{2,2}$, and $|v_2| = (m, x_1^+(\alpha)]$ for $\alpha \in \mathrm{Rt}_{2,3}$. Then we apply Lemma 7.4. □

§8. The sets $S(2,1,0)$ and $S(2,2,0)$

Denote $S(2,0) = S(2,1,0) \cup S(2,2,0)$.

Remark 8.1. By definition, the words from $S(2,0)$ are precisely the SR-words $u = v^2 w$, where v and w are regular words, $w \sqsubset v$, and w has no occurrences of the leading word of u.

Lemma 8.2. *The words in* $S(2,0)$ *are precisely the* SR-*words* $u = v^2 q$, *where* v *is a regular word and* q *is a letter, and* $q < p_1$ (p_1 *is the first letter of* v).

Proof. For $u \in S(2,0)$, we consider the representation $u = v^2 w$ as in Remark 8.1. Since w has no occurrences of the leading letter $p_1(u)$ of u, we conclude that $p_1(u)$ coincides with the leading letter $p_1(v)$ of v. Since v is regular, it begins with its leading letter, i.e., $p_1 = p_1(v)$. Since $w \sqsubset v$ and w has no occurrences of the letter $p_1(u)$, it follows that w is a letter and $w < p_1(u) = p_1(v) = p_1$.

Conversely, let v be a regular word, let q be a letter, let $q < p_1$, where p_1 is the first letter of v, and let $v^2 q$ be an SR-word. Then q is a regular word that has no occurrences of the leading letter of the word $v^2 q$ and $q \sqsubset v$. Therefore, by Remark 8.1, $v^2 q \in S(2,0)$. □

Lemma 8.3. *The words in* $S(2,0)$ *are precisely the words* $u = v^2 q$, *where* v *is a regular word and* q *is a letter*; *moreover,* vq *is an* RR-*word.*

Proof. For $u \in S(2,0)$, we consider the representation $u = v^2 q$ as in Lemma 8.2. Since the word v is regular, it is not empty. Then vq is a proper regular subword of the SR-word $v^2 q$; thus, it is an RR-word.

Conversely, let v be a regular word and q a letter; moreover, let vq be an RR-word. Since vq is an RR-word, it is regular. Since v and vq are regular words and $v > vq$, the word $v^2 q$ is regular by Corollary 5.2,

Since q is a letter and the word v is not empty (it is regular), the composition of the word $v^2 q$ is not rooted (the graph $D(|v^2 q|)$ has double vertices and at most one single vertex).

The word v is regular and q is a letter. Therefore, $l(v^2 q) = va$, where a is a beginning of the word v. If $a \neq 1$, then $va \leq v$, $v \leq a$ (a nonempty beginning of any word is not less than the word itself), and thus $va \leq a$, which contradicts Lemma 5.1. Consequently, $a = 1$ and $l(v^2 q) = va = v$. Being a regular subword of the RR-word vq, v is an RR-word. Therefore, $l(v^2 q)$ is an RR-word.

The word vq is regular and $v \neq 1$. Therefore, $r(v^2 q) = wvq$, where w is an ending of v such that $w \neq v$. Assume that $w \neq 1$. Then $vq < wvq$ (Lemma 5.1), $wvq < w$ (a proper beginning of a word is greater than the word itself), and $w < v$ (Lemma 5.1). Therefore, $vq < w < v$, so that v is a beginning of w. But w is an ending of v and $w \neq v$. Therefore, $w = 1$ and $r(v^2 q) = vq$. Consequently, $r(v^2 q)$ is an RR-word.

Combining the proved above with Corollary 5.11, we see that $v^2 q$ is an SR-word. Moreover, q is less than the first letter of v (the word vq is regular). Thus, by Lemma 8.2, $v^2 q \in S(2,0)$. □

A regular word v of length at least two is said to be *prolonged* if the word $\bar{v}$ obtained from v by removing the last letter is regular.

Corollary 8.4. *The relation $u = \bar{v}v$ determines a one-to-one correspondence between the words u in $S(2,0)$ and the prolonged reduced words v. Here, $\bar{v}$ is the word obtained from v by elimination of the last letter.*

Theorem 8.5. 1. *The relation $u = \bar{v}v$ determines a one-to-one correspondence between the words u in $S(2,1,0)$ and the reduced words v of length at least 2. Here, $\bar{v}$ is the word obtained from v by elimination of the last letter.*

2. *The relation $u = \mathrm{red}(\alpha - |q|)\mathrm{red}(\alpha)$ determines a one-to-one correspondence between the words u in $S(2,1,0)$ and the nonsimple roots α. Here, q is the last letter of the word* $\mathrm{red}(\alpha)$; *this letter can be computed starting with the root α and using Corollary* 4.14.

Proof. For $\alpha \in \mathrm{Rt}_1$ the word v begins with the only occurrence of its leading letter. Therefore, any nonempty beginning of it is regular. Being a regular beginning of the RR-word v, this beginning is an RR-word. Consequently, any RR1-word different from a letter is prolonged. Now we apply Corollary 8.4. □

Lemma 8.6. *Let $v = \mathrm{red}(\alpha)$ be an* RR2*-word, and let v_2 be the ending of v that begins with the right occurrence of the leading letter p_1 of v. Then the word v is not prolonged if and only if the least (under the order $\leq_{p_1}$) of the single vertices of $D(\alpha)$ that have a double adjacent vertex is the greatest (under the order $\leq_{p_1}$) among the end vertices of the interval $D(|v_2|)$.*

Proof. Since, by Lemma 4.18, the graph $D(|v_2|)$ contains only one of the single vertices of $D(\alpha)$ that have a double adjacent vertex, we see that $D(|v_2|)$ is an interval.

Consider a word v_1 such that $v = v_1v_2$. By [33, Lemma 3.1], we have $v_2 \prec v_1$, i.e., $v_1 = ax_1b$, $v_2 = ax_2c$, and $x_2 < x_1$ for some words a, b, c and letters x_1, x_2. By [33, Lemma 3.7], x_2 is the least (under the order $\leq_{p_1}$) among the single vertices of the graph $D(\alpha)$ that have a double adjacent vertex. Since the word a is not empty (it begins with the leading letter of v), it follows that the word v is not prolonged if and only if $c = 1$ or, equivalently, the reduced word of composition $|v_2|$ ends with the letter x_2, which is true if and only if the vertex x_2 in the interval $D(|v_2|)$ is an end vertex and is the greatest (under the order $\leq_{p_1}$) among the end vertices of this interval. □

Examining the cases of Lemma 4.2 and using Lemma 8.6, we obtain the following statement.

Corollary 8.7. *Let $v = \mathrm{red}(\alpha)$ be an* RR2*-word, and let q be its last letter.*

1. *For $\alpha \in \mathrm{Rt}_{2,1}$, q is an end vertex such that the interval $[z^*, q]$ contains the least of the vertices adjacent to z^*. In this case, the word v is prolonged if and only if in the graph $D(\alpha)$, the least of the vertices adjacent to z^* has a single adjacent vertex.*

2. *For $\alpha \in \mathrm{Rt}_{2,2}$, the word v is prolonged if and only if the leading vertex p_1 of the graph $D(\alpha)$ is different from $x_2^+(\alpha)$ and $\max[o_1, p_1) < \max(p_1, x_2^+(\alpha)]$. For $\alpha \in \mathrm{Rt}_{2,2}$, if the word v is prolonged, then $q = x_2^+(\alpha)$.*

3. *For $\alpha \in \mathrm{Rt}_{2,3}$, the word v is prolonged if and only if the graph $D(\alpha)$ has adjacent single vertices. For $\alpha \in \mathrm{Rt}_{2,3}$, if the word v is prolonged, then $q = x_1^+(\alpha)$.*

Theorem 8.8. 1. *The relation $u = \bar{v}v$ determines a one-to-one correspondence between the words u in $S(2,2,0)$ and the prolonged reduced words v described in Corollary* 8.7 *($\bar{v}$ is the word obtained from v by removing the last letter).*

2. *The relation $u = \mathrm{red}(\alpha - |q|)\mathrm{red}(\alpha)$ determines a one-to-one correspondence between the words u in $S(2,2,0)$ and the compositions α of the prolonged reduced words described*

in Corollary 8.7. *Here q is the last letter of the word* $\mathrm{red}(\alpha)$, *which is calculated starting with the root α in accordance with Corollary* 8.7.

§9. The set $S(1,1,1)$

In the sequel, P is the set of pairs (v,w) of RR1-words with one and the same leading letter and such that $w \boxleq v$.

Remark 9.1. The relation $u = vw$ determines a one-to-one correspondence between the elements u in the set $S(1,1,1)$ and the pairs $(v,w) \in P$ satisfying the condition that $|v| + |w|$ is not a root.

Lemma 9.2. *For $(v,w) \in P$, the element $|v| + |q|$ is a root (here q is the last letter of the word w).*

Proof. The RR-word w has rooted composition. Therefore, the graph $D(|w|)$ is connected, so that the vertex q has an adjacent vertex in the graph $D(|\bar{w}|)$ (here $\bar{w}$ is the word obtained from w by removing the last letter). Moreover, $|\bar{w}| \le |v|$. Consequently, q has an adjacent vertex in the graph $D(|v|)$.

If q is not a vertex of $D(|v|)$, then $|v| + |q|$ is a root. Thus, we may assume that q is a vertex of $D(|v|)$.

Suppose q is a double vertex of $D(|v|)$. Since the word $w = \bar{w}q$ has rooted composition, q is not a double vertex of $D(|\bar{w}|)$. Moreover, $|\bar{w}| \le |v|$. Therefore, $|w| \le |v|$. By Lemma 4.8, we obtain $w \ge v$, which contradicts the relation $w \boxleq v$.

Thus, q is a single vertex of the graph $D(|v|)$. If q has no occurrences in the word $\bar{w}$, then $|w| \le |v|$ and, by Lemma 4.8, we obtain $w \ge v$, which contradicts the relation $w \boxleq v$.

Thus, q is a single vertex of the graph $D(|\bar{w}|)$. Moreover, the element $|w| = |\bar{w}| + |q|$ is a root. Therefore, either $q = z^*$ and q has three adjacent vertices in $D(|\bar{w}|)$, or the vertex q of $D(|\bar{w}|)$ has an adjacent single vertex y and an adjacent double vertex. Moreover, $|\bar{w}| \le |v|$ and q is a single vertex of $D(|v|)$. Therefore, in the second case the vertex y in $D(|v|)$ is also single (because the double vertices of $D(|v|)$ form a connected subgraph). Consequently, in any case $|v| + |q|$ is a root. □

Corollary 9.3. *For an arbitrary word $u = vw$ in $S(1,1,1)$ (here $(v,w) \in P$), its last letter q lies in $M^+(|v| + |q|)$.*

Lemma 9.4. *For $(v,w) \in P$, let q be the last letter of w. Then the number of occurrences of q in v is smaller precisely by* 1 *than that in w.*

Proof. Since $|\bar{w}| \le |v|$, $|w| = |\bar{w}| + |q|$, we have $|v| \ge |w| - |q|$. Therefore, $d_q(v) \ge d_q(w) - 1$, where $d_q(v)$ and $d_q(w)$ are the numbers of occurrences of q in v and w, respectively.

Assume that the assertion of the lemma does not hold, i.e., $d_q(v) \ne d_q(w) - 1$. Then $d_q(v) \ge d_q(w)$. Since $d_q(w) \ge 1$, we have $d_q(v) \ge 1$, i.e., the word v has at least one occurrence of q. Denote by t the beginning of v that ends with the last occurrence of q.

Since $w \boxleq v$ and $d_q(v) \ge d_q(w)$, the word t has length greater than that of w. This fact and $w \boxleq v$ imply that $w \boxleq t$. Consequently, $w < t$.

Therefore, the word t is regular (it begins with a unique occurrence of its leading letter). Moreover, t is a subword of the RR-word v. Consequently, t is an RR-word. In particular, it has rooted composition. Moreover, $|w| \le |t|$. Now, by Lemma 4.8, we obtain $w \ge t$, which contradicts the inequality $w < t$ proved above. □

Lemma 9.5. *For $(v,w) \in P$, the pair (α, q), where q is the last letter of w and $\alpha = |v| + |q|$, possesses the following properties.*

1) *α is a positive root whose graph is such that its leading vertex p_1 is single and the set $M^+(\alpha)$ has at least two elements different from p_1;*

2) *q is an element of the set $M^+(\alpha)$ that is different from p_1 and from the last letter of the reduced word of composition α.*

Proof. By Lemma 9.2, α is a root. Since $(v, w) \in P$, the word w is regular and has length at least two. Therefore, w begins with its leading letter p_1, and its last letter q is less than p_1. Consequently, the vertex p_1 in the graph $D(\alpha)$ is leading and single. Since $\alpha = |v| + |q|$ and the word v has rooted composition, we obtain $q \in M^+(\alpha)$. Since $q < p_1$, we have $q \neq p_1$.

We denote by a the reduced word of composition α and by q' the last letter of a. It remains to show that $q' \neq q$. Suppose the contrary: $q' = q$. Then the word a ends with q. Moreover, a begins with the only occurrence of its leading letter. Therefore, the word $\bar{a}$ obtained from a by removing the last letter is regular. Being a regular proper subword of the RR-word a, it is an RR-word. But the RR-words are determined by their compositions. Therefore, $\bar{a} = v$. Consequently, $a = vq$. Also, $w \boxleft v$, whence $w \boxleft a$. Thus, $w < a$. Since $|w| \leq |a|$, Lemma 4.8 yields $w \geq a$, which contradicts the inequality $w < a$ proved above. □

We use the notation Base for the set of all pairs (α, q) satisfying properties 1) and 2) of Lemma 9.5.

Remark 9.6. For an arbitrary root β, any single vertex q of the graph $D(\beta)$ such that $q \in M^+(\beta)$ is an end vertex in $D(\beta)$.

Lemma 9.7. *An* RR1*-word cannot have the subword q^2, where q is a letter.*

Proof. Any RR1-word $a = bq^2c$ begins with its leading letter p_1 and does not have other occurrences of this letter. Therefore, $q < p_1$ and b begins with the letter p_1. Then $|bq^2|$ is a root, q is a double vertex of the graph $D(|bq^2|)$, and $q \in M^+(|bq^2|)$. The double vertex q of $D(|bq^2|)$ is a single and not an end vertex in the graph $D(|bq|)$.

The nonempty beginnings b and bq of the RR1-word a have rooted composition. Therefore, $q \in M^+(|bq|)$. What we have proved above contradicts Remark 9.6. □

Lemma 9.8. *Let $(\alpha, q) \in$* Base*, let p_1 be the leading vertex of the graph $D(\alpha)$, and let v and w be* RR1*-words with common leading letter p_1. Let $|v| = \alpha - |q|$. Then $w \boxleft v$ is equivalent to the fact that w is the beginning of the reduced word a of composition α that ends with the right occurrence of the letter q.*

Proof. Recall that $(\alpha, q) \in$ Base if and only if the vertex p_1 in the graph $D(\alpha)$ is single, $q < p_1$, $\alpha - |q|$ is a root, and the word $a = \mathrm{red}(\alpha)$ does not end with q.

Since $\alpha - |q|$ is a root, q has an occurrence in a. Denote by b and c two words such that $a = bqc$, $|q| \not\leq |c|$.

Since a is an RR-word, it is regular. Since the leading vertex p_1 of the graph $D(\alpha)$ is single, the word a has only one occurrence of p_1. For this reason, the regular word a begins with the only occurrence of its leading letter p_1. Moreover, $a = bqc$ and $q < p_1$. Therefore, the word b begins with the only occurrence of its leading letter p_1. In particular, b is nonempty and $p_1 \leq |b|$.

Since a does not end with the letter q, the word c is not empty. Denote by m the first letter of c. Since $q \in M^+(|a|)$, by Lemma 4.7 we have $q < m$.

Since a is an RR1-word and $q \in M^+(|a|)$, Corollary 4.6 shows that the word bc obtained from a by removing the right occurrence of the letter q is an RR1-word. Since it has the same composition as the RR-word v, we have $v = bc$. The relation $v = bc$ implies that the word bm is a beginning of v.

Let $w = bq$. Since bm is a beginning of v and $q < m$, we have $w = bq \boxleft v$.

Conversely, let $w \boxleft v$, and let $\bar{w}$ be a word such that $w = \bar{w}q$. From $\bar{w}q \boxleft v$ it follows that $\bar{w}$ is a beginning of v.

Assume that w is shorter than the word bq. Since $\bar{w}$ is a beginning of the word $v = bc$, we conclude that $\bar{w}$ is a beginning of b different from b. Therefore, $|w| < |bq|$. Hence $|w| < |bq|$. Since w and bq are RR1-words with one and the same leading letter p_1, Lemma 4.8 shows that $bq < w$.

Since $w \boxleft v$, the word w is not longer than the word b, and b is a beginning of v, we have $w \boxleft b$. Therefore, $w \prec b$ and $w \prec bq$. Hence $w < bq$, which contradicts the inequality $bq < w$.

Thus, $w = \bar{w}q$ is not shorter than the word bq, i.e., $\bar{w}$ is not shorter than b. Since $\bar{w}$ is a beginning of the word $v = bc$, we have $\bar{w} = bd$, where d is a beginning of c. Therefore, bqd is a beginning of the RR1-word $a = bqc$. In particular, bqd is an RR-word. The word $w = bdq$ is also an RR-word and has the same composition as bqd. Therefore, $bdq = bqd$, $dq = qd$, and $d = q^k$ for some nonnegative integer k. If $k \geq 1$, then the RR1-word bdq ends with two occurrences of the same letter, which contradicts Lemma 9.7. Therefore, $k = 0$, $d = 1$, and $w = bdq = bq$. □

Corollary 9.9. *From Lemmas* 9.5 and 9.8 *we conclude that, associating the pair* (α, q), *where* q *is the last letter of the word* w *and* $\alpha = |v| + |q|$, *with a pair* $(v, w) \in P$, *we obtain a one-to-one correspondence between the sets* P *and* Base. *Moreover, the preimage of a pair* $(\alpha, q) \in \mathrm{Base}$ *is* (v, w), *where* $v = \mathrm{red}(\alpha - |q|)$ *and* w *is the beginning of the word* $\mathrm{red}(\alpha)$ *that ends with the right occurrence of the letter* q.

Corollary 9.10. *From Remark* 9.1 *and Corollary* 9.9 *it follows that the set* $S(1, 1, 1)$ *consists precisely of the words* vw, *where* $v = \mathrm{red}(\alpha - |q|)$, w *is the beginning of the word* $\mathrm{red}(\alpha)$ *that ends with the right occurrence of the letter* q, $(\alpha, q) \in \mathrm{Base}$, *and the element* $|v| + |w|$ *is not a root.*

Lemma 9.11. *Suppose* $(v, w) \in P$ *and* $|v| + |w|$ *is a root. Then*:

1) *the graphs* $D(|v| + |w|)$ *and* $D(\alpha)$ *have one and the same vertices* (*here* $\alpha = |v| + |q|$ *and* q *is the last letter of* w);

2) *the leading vertex* p_1 *of the graph* $D(|v| + |w|)$ *is double in it*;

3) q *is an end vertex in the graph* $D(|v| + |w|)$;

4) *the graph* $D(|v| + |w|)$ *contains the* $*$*-center and the three vertices adjacent to it*;

5) *if the vertex* q *is not adjacent to the* $*$*-center, then all nonend vertices of* $D(|v|+|w|)$ *are double in this graph.*

Proof. 1) Since $|v| = \alpha - |q|$, we have $\alpha - |q| \leq |v| \leq \alpha$. Moreover, $|q| \leq |w| \leq \alpha$. Therefore, $\alpha \leq |v| + |w| \leq 2\alpha$, and the graphs $D(|v| + |w|)$ and $D(\alpha)$ have one and the same vertices.

2) Since $|p_1| \leq |v|$, $|p_1| \leq |w|$, and $|v| + |w|$ is a root, the vertex p_1 in the graph $D(|v| + |w|)$ is double.

3) By Lemma 9.4, the letter q has an odd number of occurrences in any word of composition $|v| + |w|$. Moreover, $|v| + |w|$ is a root. Therefore, q has precisely one occurrence in an arbitrary word of composition $|v| + |w|$. Therefore, by Lemma 9.4, q does not have occurrences in an arbitrary word of composition $|v|$. Moreover, $\alpha = |v|+|q|$. Consequently, q is a single vertex of the graph $D(\alpha)$. But $q \in M^+(\alpha)$ ($(\alpha, q) \in \mathrm{Base}$). Therefore, q is an end vertex of the graph $D(\alpha)$. Now statement 1) implies that q is an end vertex of the graph $D(|v| + |w|)$.

4) The graph $D(|v| + |w|)$ has double vertices (for example, p_1, because $p_1 \leq |v|$ and $p_1 \leq |w|$). Therefore, the graph $D(|v| + |w|)$ contains the $*$-center and the three vertices adjacent to it.

5) Consider the case where the vertex q is not adjacent to the $*$-center. We have $z^* \le |v| + |w|$ (statement 4)). Now statement 3) implies that the interval $[z^*, q)$ consists precisely of all nonend vertices of the graph $D(|v| + |w|)$.

Moreover, from statement 2) it follows that p_1 is not an end vertex in the graph $D(|v| + |w|)$. Therefore, $p_1 \in [z^*, q)$. In particular, the interval $[z^*, q)$ is not empty. Consequently, there is a vertex y of the graph $D(|v| + |w|)$ such that $[z^*, y] = [z^*, q)$.

Since $|v| = \alpha - |q|$, $q \ne y$, and $|y| \le \alpha$, we have $|y| \le |v|$.

Since $p_1 \le |w|$, $q \le |w|$, and the graph $D(|w|)$ is connected ($|w|$ is a root), we have $[p_1, q] \le |w|$. Moreover, $y \in [p_1, q]$. Therefore, $|y| \le |w|$.

The inequalities $|y| \le |v|$ and $|y| \le |w|$ imply that $2|y| \le |v| + |w|$, i.e., the vertex y is double in the graph $D(|v| + |w|)$. Therefore, all the vertices of the interval $[z^*, y]$ are double in the graph $D(|v| + |w|)$, i.e., all nonend vertices of $D(|v| + |w|)$ are double in this graph. □

Lemma 9.12. *For* $(\alpha, q) \in \mathrm{Base}$*, the property that* $|v| + |w|$ *is a root is equivalent to the following properties* 1)–4)*. Here* $v = \mathrm{red}(\alpha) - |q|$ *and* w *is the beginning of the word* $\mathrm{red}(\alpha)$ *that ends with the right occurrence of the letter* q*.*

1) *The graph* $D(\alpha)$ *contains the* $*$*-center and the three vertices adjacent to it.*

2) *The leading vertex* p_1 *of the graph* $D(\alpha)$ *is single and is not an end vertex.*

3) q *is the least (under the order* $\le_{p_1}$*) among the end vertices of the graph* $D(\alpha)$*.*

4) *If the vertex* q *is not adjacent to the* $*$*-center, then an arbitrary vertex* x *of* $D(\alpha)$ *is double in it if and only if* $x \in [z^*, p_1)$*,* $\max[x, p_1) > \max(p_1, q]$*.*

Proof. Assume that $(\alpha, q) \in \mathrm{Base}$ and the element $|v| + |w|$ is a root. We prove properties 1)–4).

1) By statements 1) and 4) of Lemma 9.11, the graph $D(\alpha)$ has $*$-center.

2) Since $p_1 \le |v|$, $p_1 \le |w|$, and $|v| + |w|$ is a root, it follows that the vertex p_1 is double in the graph $D(|v| + |w|)$ and is single in the graphs $D(|v|)$ and $D(|w|)$.

Since $(\alpha, q) \in \mathrm{Base}$, we have $q \ne p_1$. Moreover, $|v| = \alpha - |q|$. Therefore, the vertex p_1 has one and the same multiplicity in the graphs $D(v)$ and $D(\alpha)$. Moreover, the vertex p_1 is single in $D(|v|)$. Hence it is also single in $D(\alpha)$.

From statements 1) and 2) of Lemma 9.11 we deduce that p_1 is not an end vertex of $D(\alpha)$.

3) By statement 3) of Lemma 9.11, q is an end vertex of $D(\alpha)$.

Consider an arbitrary end vertex x of $D(\alpha)$ different from q. Since $|v| = \alpha - |q|$, we have $|x| \le |v|$.

Assume that $x \le_{p_1} q$. Lemma 4.13 implies that $|x| \le |w|$. But in this case $2|x| \le |v| + |w|$, i.e., x is a double end vertex in the graph $D(|v| + |w|)$, which is impossible. Hence, we obtain property 3).

4) Consider the case where the vertex q is not adjacent to the $*$-center. Let x be an arbitrary vertex in $[z^*, p_1)$.

Since the vertex q in the graph $D(\alpha)$ is an end vertex and is not adjacent to z^*, but the vertex p_1 of $D(\alpha)$ is not an end vertex, we have $p_1 \in [z^*, q)$. For this reason, $p_1 \in (x, q)$, and $\max(p_1, q] < \max[x, p_1)$ if and only if $q <_{p_1} x$ or, equivalently, $|x| \not\le |w|$, by Lemma 4.13. Since x is not an end vertex in $D(\alpha)$, we have $2|x| \le |v| + |w|$ by Lemma 9.12. Therefore, $|x| \not\le |w|$ if and only if $2|x| \le |v|$. Since $p_1 \in (x, q)$, we have $x \ne q$. Consequently, $2|x| \le |v|$, i.e., $2|x| \le \alpha$.

Now, for an arbitrary double vertex x of $D(\alpha)$ we have $2|x| \le \alpha$, and property 2) implies that $x \in [z^*, p_1)$. This and the said above yield the inequality $\max(p_1, q] < \max[x, p_1) \Leftrightarrow 2|x| \le \alpha$. Conversely, if $x \in [z^*, p_1)$ and $\max(p_1, q] > \max[p_1, q]$, then we have $2|x| \le \alpha$. Property 4) is proved.

Conversely, suppose properties 1)–4) are fulfilled for a pair $(\alpha, q) \in$ Base. We show that the element $|v| + |w|$ is a root. Since $(\alpha, q) \in$ Base, the element $\alpha - |q|$ is a root.

First, we consider the case where $q \in O(\alpha)$. Since $\alpha - |q|$ is a root, the graph $D(\alpha)$ has no double vertices. By property 1), the graph $D(\alpha)$ has at least four vertices.

Consider the subcase where the graph $D(\alpha)$ has precisely four vertices. From property 2) we have $p_1 = z^*$. Now property 3) implies that α is the sum of the vertex z^* and the vertices adjacent to it; $|v| = \alpha - |q|$ is the sum of the vertex z^* and the vertices adjacent to it except for q; $|w| = |q| + |z^*|$; and $|v| + |w|$ is a root the graph of which has precisely four vertices and has a double vertex.

Now we consider the subcase where the graph $D(\alpha)$ has at least five vertices. Let o be the end vertex of $D(\alpha)$ that is adjacent to z^* and is different from q, and let x be an end vertex of $D(\alpha)$ that is not adjacent to z^*. Then $p_1 \in [z^*, x)$ (by property 2), the vertex p_1 in the graph $D(\alpha)$ is not an end vertex). Property 3) yields $q < o$ and $\max[q, p_1) < \max(p_1, x]$.

Since the graph $D(\alpha)$ has no double vertices, $\{q, o, x\}$ is the set of its end vertices, and $x \notin O(\alpha)$, we have $\alpha = |q| + |o| + [z^*, x]$. Therefore, $|v| = \alpha - |q| = |o| + [z^*, x] = [o, x]$. Now, by Corollary 4.14, $|w| = [q, m)$, where m is the vertex of the interval $(p_1, x]$ that is closest to p_1 and satisfies $\max[q, p_1) < \max(p_1, m]$ (the inequality $\max[q, p_1) < \max(p_1, x]$ shows that such a vertex m exists).

By property 3), $\max[q, p_1) > \max(p_1, x]$. Therefore, $x \neq m$. Thus, $|v| + |w| = [o, x] + [q, m)$ is a root (because $O(\alpha) = \{o, q\}$, $x \neq m$).

Now we consider the case where $q \notin O(\alpha)$. Since, by property 3), the vertex q in the graph $D(\alpha)$ is an end vertex, q is not adjacent to the $*$-center. Also, by property 2), p_1 is not an end vertex in $D(\alpha)$. Therefore, $p_1 \in [z^*, q)$. Property 3) implies that $\max(p_1, q] < \max(p_1, k]$ for $k \in O$.

Denote by m the vertex of the interval $[z^*, p_1)$ such that m is most distant from p_1 and satisfies the inequality $\max[m, p_1) < \max(p_1, q]$.

Since q is an end vertex in $D(\alpha)$ and $q \notin O(\alpha)$, the vertices of the graph $D(\alpha)$ are precisely the vertices of the graph $O \cup [z^*, q]$. Property 4) implies that the double vertices of $D(\alpha)$ are vertices of the interval $[z^*, m)$. Therefore, $\alpha = \sum O + [z^*, q] + [z^*, m)$. Moreover, $|v| = \alpha - |q|$ and, by Corollary 4.14, $|w| = [m, q]$. Thus, $|v| + |w| = \alpha - |q| + [m, q] = \sum O + [z^*, q] + [z^*, q)$ is a root. □

Theorem 9.13. *The words in $S(1, 1, 1)$ are in one-to-one correspondence with the pairs (α, q) from* Base *for which at least one of the conditions* 1)–4) *of Lemma* 9.12 *is not fulfilled. The correspondence is given by the relation $u = vw$, where v is the reduced word of composition $\alpha - |q|$ and w is the beginning of the reduced word of composition α that ends with the right occurrence of the letter q.*

§10. The set $S(1, 2, 1)$

Remark 10.1. By definition, $S(1, 2, 1)$ is precisely the set of SR-words u such that in the representation $u = v^k w$, where $w \boxleft v$, we have $k = 1$, the word v has precisely two occurrences of the leading letter of u, and w has only one occurrence of this letter.

We denote by $\bar{P}_{2,1}$ the set of pairs (v, w) of RR-words with common leading letter p_1 such that the word v has two occurrences of p_1, the word w has one occurrence of p_1, and $w \boxleft v$.

We use Remark 10.1 and Corollary 5.11 to deduce the following statement.

Corollary 10.2. *$S(1, 2, 1)$ is precisely the set of words vw, where $(v, w) \in \bar{P}_{2,1}$ and $r(vw)$ is an* RR-*word.*

Proof. For $u \in S(1,2,1)$, we consider the representation $u = vw$ from Remark 10.1. Since the words v and w are regular, we have $v \neq 1 \neq w$. Therefore, v and w are regular proper SR-words of vw, so that they are RR-words. Therefore, $(v,w) \in \bar{P}_{2,1}$.

By definition, the word $r(vw)$ is a regular proper subword of vw. Since vw is an SR-word, it follows that $r(vw)$ is an RR-word.

Conversely, for an arbitrary pair $(v,w) \in \bar{P}_{2,1}$ such that $r(vw)$ is an RR-word we show that $vw \in S(1,2,1)$. Since $w \boxleq v$, we have $w < v$ and thus, by Corollary 5.2, the word vw is regular. Since the word vw has three occurrences of its leading letter, we conclude that its composition is not rooted.

Since v and w are RR-words, the word v is regular and $w \neq 1$. Therefore, $v \neq vw$ and v is a beginning of the word $l(vw)$. Denote by $\bar{w}$ the beginning of the word w obtained from w by removing the last letter. Since $w \neq 1$, it follows that $l(vw)$ is a beginning of the word $v\bar{w}$. Since v is a beginning of the word $l(vw)$, we have $l(vw) = va$, where a is a beginning of $\bar{w}$. Since $w \boxleq v$, $\bar{w}$ is a beginning of v. Since a is a beginning of $\bar{w}$, it follows that a is a beginning of v.

Suppose $a \neq 1$. The word w begins with the only occurrence of its leading letter p_1. Therefore, the nonempty beginning a of the word w begins with the only occurrence of the letter p_1. Consequently, the word a is regular. Moreover, $v \leq a$ (a is a nonempty beginning of the word v). Thus, the words v, a, and $va = l(vw)$ are regular and $v \leq a$, which contradicts Corollary 5.2. Therefore, $a = 1$ and $l(vw) = va = v$ is an RR-word.

Since $r(vw)$ is an RR-word, from the above argument and Corollary 5.11 we conclude that vw is an SR-word. Since $(v,w) \in \bar{P}_{2,1}$, Remark 10.1 shows that $vw \in S(1,2,1)$. □

For an arbitrary RR2-word v, we denote by v_1 and v_2 regular words with a common leading letter that satisfy the condition $v = v_1v_2$. Denote by Q the set of pairs (α, q), where $\alpha \in \mathrm{Rt}_2$ and $q \in M^-(|v_1|)$ is a letter that is less than the leading vertex p_1 of the graph $D(\alpha)$; let v_1 be the longest beginning of the word $v = \mathrm{red}(\alpha)$ that has only one occurrence of the letter p_1. For any root γ and any vertex q of the graph $D(\gamma)$, we denote by $w_q(\gamma)$ the beginning of the word $\mathrm{red}(\gamma + |q|)$ that ends with the right occurrence of the letter q.

Lemma 10.3. *Associating the pair $(|v|, q)$, where q is the last letter of the word w, with a pair $(v,w) \in \bar{P}_{1,2}$, we obtain an injective mapping of $\bar{P}_{1,2}$ onto Q. Under this mapping, the preimage of a pair (α, q) is the pair $(v, w_q(|v_1| + |q|))$, where $v = \mathrm{red}(\alpha)$ and v_1 is the longest beginning of the word v that has only one occurrence of the leading letter.*

Proof. Let $(v,w) \in \bar{P}_{1,2}$, and let p_1 and q be the first and last letters of w. The words v and w begin with p_1, and $w \boxleq v$. Therefore, the word w has length at least two. Moreover, w is regular. Therefore, $q < p_1$. Since $w \boxleq v$, it follows that either $w = v_1q$, or $w \boxleq v_1$. In both cases, $q \in M^-(|v_1|)$ (in the second case, by Lemma 9.2). Therefore, $(v,q) \in Q$, i.e., there exists a mapping $f : \bar{P}_{1,2} \to Q$ that takes a pair $(v,w) \in \bar{P}_{1,2}$ to the pair $(|v|, q)$, where q is the last letter of w.

Consider $(\alpha, q) \in Q$, $v = \mathrm{red}(\alpha)$. If the word $\mathrm{red}(|v_1| + |q|)$ ends with q, then we have $w_q(|v_1|+|q|) = v_1q$. If the word $\mathrm{red}(|v_1|+|q|)$ does not end with q, then $w_q(|v_1|+|q|) \boxleq v_1$ by Corollary 9.9. In both cases, $(v, w_q(|v_1| + |q|)) \in \bar{P}_{1,2}$ and $f(v, w_q(|v_1| + |q|)) = (v,q)$. Therefore, the mapping f is surjective.

Let $(v,w) \in \bar{P}_{1,2}$, and let q be the last letter of w. Since $w \boxleq v$, we have either $w = v_1q$ or $w \boxleq v_1$. If $w = v_1q$, then the word $\mathrm{red}(|v_1| + |q|)$ ends with the letter q, i.e., $w_q(|v_1|+|q|) = v_1q = w$. If $w \neq v_1q$, then $w \boxleq v_1$ and, by Corollary 9.9, $w = w_q(|v_1|+|q|)$. Therefore, the mapping f is injective. □

Theorem 10.4. I. $S(1,2,1) = \{\mathrm{red}(\alpha)w_q(|v_1| + |q|) : (\alpha, q) \in Q\}$.

II. *One of the following statements* 1–3 *is true.*

1. *$\alpha \in \mathrm{Rt}_{2,1}$, q is the least of the vertices adjacent to z^*, and $w_q(|v_1|+|q|) = z^*q$.*

2. *$\alpha \in \mathrm{Rt}_{2,2}$, and one of the following alternative statements* 2.1 *or* 2.2 *is fulfilled.*

2.1. *$q = o_1$. In this case, $|w_q(|v_1|+|q|)| = [o_1, x)$, where x is the vertex of the interval $(p_1, x_1^-(\alpha)]$ closest to p_1 and such that the second-by-priority vertex of the interval $[o_1, x)$ lies in the interval $[o_1, p_1)$.*

2.2. *The graph $D(\alpha)$ contains not all vertices of the graph D_n, and q is a unique vertex not lying in the graph $D(\alpha)$ but having an adjacent vertex in it. In this case, one of the following alternative statements* 2.2.1 *or* 2.2.2 *is fulfilled.*

2.2.1. *$o_2 > \max(p_1, q]$ (here $o_2 \in O$, $o_2 > o_1$) and $|w_q(|v_1|+|q|)| = [z^*, q]$.*

2.2.2. *$o_2 < \max(p_1, q]$, $|w_q(|v_1|+|q|)| = [o_2, q]$, and $w_q(|v_1|+|q|) = v_1q$.*

3. *$\alpha \in \mathrm{Rt}_{2,3}$. In this case, one of the alternative statements* 3.1 *or* 3.2 *is fulfilled.*

3.1. *$q = x_1^-(\alpha)$ and $|w_q(|v_1|+|q|)| = [m, x_1^-(\alpha)]$, where m is the vertex of the interval $[z^*, p_1]$ most distant from p_1 and such that the second-by-priority vertex of the interval $[m, x_1^-(\alpha)]$ lies in the interval $(p_1, x_1^-(\alpha)]$.*

3.2. *$q = m$, $|w_q(|v_1|+|q|)| = [o_1, m] + [o_2, x_2^+(\alpha)]$, and $w_q(|v_1|+|q|) = v_1q$.*

Proof. I. For $u \in S(1,2,1)$, by Corollary 10.2, we have $u = vw$, where $(v,w) \in \bar{P}_{2,1}$. By Lemma 10.3, $(|v|, q) \in Q$ and $w = w_q(|v_1|+|q|)$, where q is the last letter of w. Therefore, $S(1,2,1) \subseteq \{\mathrm{red}(\alpha)w_q(|v_1|+|q|) : (\alpha, q) \in Q\}$. From 10.2, it is clear that to prove the reverse inclusion it suffices to check the relation $r(vw) = w$ for $v = \mathrm{red}(|v|)$, $w = w_q(|v_1|+|q|)$, $(|v|, q) \in Q$. This relation follows from the inequality $v_2 \le w_q(|v_1|+|q|)$ for $v = \mathrm{red}(|v|)$, $(|v|, q) \in Q$. Below, we prove this inequality for all leaves of the tree of alternatives in statement II.

II. The set $\{\mathrm{Rt}_{2,1}, \mathrm{Rt}_{2,2}, \mathrm{Rt}_{2,3}\}$ is a partition of the set Rt_2.

1. For $\alpha \in \mathrm{Rt}_{2,1}$, from Lemma 4.2 we obtain $|v_2| = [z^*, k]$, where k is an end vertex of the graph $D(\alpha)$ such that the interval $[z^*, k]$ contains the least vertex m_1 among the vertices adjacent to z^*. Since $|v| = |v_1|+|v_2|$, it follows that $|v_1|$ is the largest interval that consists of vertices of $D(\alpha)$ and satisfies the condition $|m_1| \not\le |v_1|$. Therefore, $M^-(|v_1|) = \{m_1\}$. Thus, $q = m_1$, $|v_1|+|q|$ is a graph that contains z^* and the three vertices adjacent to it and has no double vertices. Now Corollary 4.14 shows that $w_q(|v_1|+|q|) = z^*q$ is a beginning of the word v_2. Therefore, $v_2 \le w_q(|v_1|+|q|)$.

2. For $\alpha \in \mathrm{Rt}_{2,2}$, from Lemma 4.2 we obtain $|v_2| = [o_1, x_2^+(\alpha)]$. Consequently, $|v_1| = [o_2, x_1^+(\alpha)]$ and $M^-(|v_1|) = \{o_1, x_0(\alpha)\}$, where $x_0(\alpha)$ is a unique vertex that does not lie in the graph $D(\alpha)$ but has an adjacent vertex in it.

Since $\alpha \in \mathrm{Rt}_{2,2}$, we have $\max[o_1, p_1) < \max(p_1, x_1^-(\alpha)]$. Therefore, there exists a vertex x of the interval $(p_1, x_1^-(\alpha)]$ closest to p_1 and such that $\max[o_1, p_1) < x$. Denote $a = \mathrm{red}([z^*, x))$.

Since $[o_1, x) \le |v_2| = [o_1, x_2^+(\alpha)]$, Corollary 4.14 shows that ao_1 is a beginning of the word v_2, by the choice of the vertex x.

2.1. For $q = o_1$, we see that $|v_1|+|q| = |o_1| + [o_2, x_1^+(\alpha)]$ is a graph without double vertices, which consists of all vertices of the graph $D(\alpha)$. Since $\max[o_1, p_1) < \max(p_1, x_1^-(\alpha)]$, from Corollary 4.14, using the choice of the vertex x, we conclude that $|w_q(|v_1|+|q|)| = [o_1, x)$. Therefore, $|w_q(|v_1|+|q|)| = [o_1, x) \le [o_1, x_1^-(\alpha)) = [o_1, x_2^+(\alpha)] = |v_2|$, i.e., $|w_q(|v_1|+|q|)| \le |v_2|$ and, by Corollary4.8, $v_2 \le w_q(|v_1|+|q|)$.

2.2. For $q = x_0(\alpha)$, we have $|v_1|+|q| = [o_2, q]$. Since $\max[z^*, p_1) \le \max[o_1, p_1) < \max(p_1, x_1^-(\alpha)] \le \max(p_1, q]$, we have $[z^*, q] \le |w_q(|v_1|+|q|)| \le |v_1|+|q| = [o_2, q]$.

2.2.1. For $o_2 > \max(p_1, q]$, we have $|w_q(|v_1|+|q|)| = [z^*, q]$. Since $[z^*, x] \le |w_q(|v_1|+|q|)| = [z^*, q]$, Corollary 4.14 and the choice of the vertex x show that ax is a beginning of the word $w_q(|v_1|+|q|)$. Since ao_1 is a beginning of the word v_2 and $o_1 \le \max[o_1, p_1) < x$, we have $v_2 \le ao_1 < w_q(|v_1|+|q|)$.

2.2.2. For $o_2 < \max(p_1, q]$, we have $|w_q(|v_1| + |q|)| = [o_2, q]$. Since $o_1 < o_2$, the choice of the vertex x shows that any element of the set $\{o_2, x\}$ is greater than the second-by-priority vertex of the interval $[z^*, x)$. Therefore, by Corollary 4.14, ay is a beginning of the word $w_q(|v_1|+|q|) = \mathrm{red}([o_2, q])$, where y is the least of the elements of the set $\{o_2, x\}$. Since $o_1 < y$ and ao_1 is a beginning of the word v_2, we have $v_2 \le ao_1 < w_q(|v_1| + |q|)$.

3. For $\alpha \in \mathrm{Rt}_{2,3}$, from Lemma 4.2 we obtain $|v_2| = [m, x_1^+(\alpha)]$, where m is the vertex of the interval $[z^*, p_1]$ that is most distant from p_1 and satisfies the condition that the second-by-priority vertex of the interval $[m, x_1^-(\alpha)]$ lies in the interval $(p_1, x_1^-(\alpha)]$. Then $|v_1| = [o_1, m) + [o_2, z^*, x_2^+(\alpha)]$. Therefore, $M^-(\alpha) = \{x_1^-(\alpha), m\}$.

3.1. $q = x_1^-(\alpha)$. Then $|v_1| + |q| = [o_1, m) + [o_2, z^*, x_1^-(\alpha)]$, and, by the choice of the vertex m and Corollary 4.14, we obtain $|w_q(|v_1| + |q|)| = [m, x_1^-(\alpha)]$. Therefore, $|w_q(|v_1|+|q|)| = [m, x_1^-(\alpha)] \le [m, x_1^+(\alpha)] = |v_2|$, and, by Corollary 4.8, $v_2 \le w_q(|v_1|+|q|)$.

3.2. $q \ne x_1^-(\alpha)$. Then $m \ne p_1$, $q = m$, and the vertex q in the graph $D(|v_1| + |q|)$ is double. Therefore, $|v_1| + |q| = [o_1, m] + [o_2, x_2^+(\alpha)]$.

Since $\alpha \in \mathrm{Rt}_{2,3}$, we have $\max(p_1, x_2^+(\alpha)] \le \max(p_1, x_1^-(\alpha)] < \max(p_1, o_1]$. Moreover, $p_1 \in (m, x_2^+(\alpha)]$. Therefore, the word $\mathrm{red}([o_1, m] + [o_2, x_2^+(\alpha)])$ ends with the letter m. Consequently, this word coincides with $w_q(|v_1|+|q|)$, and $|w_q(|v_1|+|q|)| = |v_1|+|m|$. Since v_1 is an RR1-word, we have $w_q(|v_1|) = v_1 m$. By [33, Lemma 3.1], $v_2 \prec v_1$. Therefore, $v_2 \prec v_1 m = w_q(|v_1| + |q|)$, $v_2 \le w_q(|v_1| + |q|)$. □

§11. The sets $S(1, 2, 2)$, $S(2, 1, 1)$ and the main results

By definition, $S(1, 2, 2)$ is the set of all SR-words u such that

1) the word $l(u)$ has precisely two occurrences of the leading letter p_1 of u;

2) the word $(l(u))^2$ is not a beginning of u;

3) the word $\bar{l}(u)$ has precisely two occurrences of the letter p_1.

By Corollary 5.6, $S(1, 2, 2)$ is the set of all SR-words of the form $u = vw$, where v and w are regular words having two occurrences of the leading letter p_1 of u and such that $w \boxleq v$.

Since v and w are regular proper subwords of the SR-word u, it follows that v and w are RR-words. They have two occurrences of their leading letter p_1 each. Let v_1, v_2, w_1, and w_2 be RR1-words such that $v = v_1 v_2$ and $w = w_1 w_2$.

Lemma 11.1. *For any word $u = vw$ in $S(1, 2, 2)$, where v and w are regular words each of which has two occurrences of the leading letter p_1 for u and $w \boxleq v$, the following conditions are fulfilled*:

1) $v_1 = w_1$ *and* $w_2 \boxleq v_2$;

2) *the roots* $|v_2|$ *and* $|w_2|$ *are not comparable.*

Proof. 1. Since $w \boxleq v$ and each of the words v and w has two occurrences of p_1, condition 1) is fulfilled.

2. By condition 1), $w_2 \boxleq v_2$. Therefore, the length of v_2 is not less than that of w_2. Assume that the assertion does not hold. Then $|w_2| \le |v_2|$. Since $w_2 \boxleq v_2$, we can apply Lemma 9.2 to conclude that $|v_2| + |q|$ is a root (q is the last letter of w_2). Moreover, from Lemma 4.16 it follows that the graph $D(|v_2|)$ is an interval. Therefore, $|q| \not\le |v_2|$, so that $|q| \le |w_2| \not\le |v_2|$, a contradiction. □

Lemma 11.2. *The set $S(1, 2, 2)$ is empty.*

Proof. Let $u \in S(1, 2, 2)$; let v and w be regular words each of which has two occurrences of the leading letter p_1 of u, and let $u = vw$, $w \boxleq v$. By Lemma 11.1, we have $v_1 = w_1$. Denote $\alpha_1 = |v_1|$. Let $\min_v$, $\min_w$ be the least (under the order $\le_{p_1}$) of the single vertices of the graph $D(|v|)$ (respectively, the graph $D(|w|)$) that have a double adjacent vertex.

The structure of RR2-words shows that $\min_v$ is a unique vertex of D_n that does not lie in $D(|v_1|)$ but has an adjacent vertex in it. Similarly, $\min_w$ is a unique vertex of D_n that does not lie in $D(|w_1|)$ but has an adjacent vertex in it. Moreover, $v_1 = w_1$. Therefore, $\min_v = \min_w$; we denote $\min = \min_v$.

If $\min \in O$, then $|v_2| = [\min, x_2^+(|v|)]$ and $|w_2| = [\min, x_2^+(|w|)]$; thus, the roots $|v_2|$ and $|w_2|$ are comparable, which contradicts Lemma 11.1. Therefore, $\min \notin O$.

The graph $D(|v_1|)$ contains all single vertices of the graph $D(|v|)$ that have an adjacent double vertex except for min. Therefore, the graph $D(|v_1| + |\min|)$ contains all single vertices of $D(|v|)$ that have an adjacent double vertex. Since $\min \notin O$, the double vertices of $D(|v|)$ are not end vertices of the graph $O \cup \{\min\}$. Similarly, the double vertices of $D(|w|)$ are not end vertices of the graph $O \cup \{\min\}$. Consequently, $[z^*, x_2^+(|v|)] = [z^*, x_2^+(|w|)]$, $x_2^+(|v|) = x_2^+(|w|)$. We denote $x_2^+ = x_2^+(|v|)$.

Let I be the set of all vertices of the interval $[z^*, x_2^+] \cup \{\min\}$ that do not exceed the vertex min in the order $\leq_{p_1}$. The set I is an interval and $[p_1, \min] \subseteq I$. Since min is an end vertex of the interval $[z^*, x_2^+] \cup \{\min\}$ and $\min \in I \subseteq [z^*, x_2^+] \cup \{\min\}$, it follows that min is an end vertex of the interval I. Since $\min \neq p_1$, the interval I contains at least two vertices. Let m denote the end vertex of I different from min.

Since $\min \notin O$, the structure of RR2-words shows that $|v_2|$ is an interval in which m is an end vertex, and $|\min| \leq |v_2|$. Similarly, $|w_2|$ is an interval in which m is an end vertex, and $|\min| \leq |w_2|$. Since $m \neq \min$, it follows that the roots $|v_2|$ and $|w_2|$ are comparable, which contradicts Lemma 11.1. □

The set $S(2,1,1)$

By definition, $S(2,1,1)$ is the set of SR-words u such that

1) the word $l(u)$ has precisely one occurrence of the leading letter p_1 of u;

2) the word $(l(u))^2$ is a beginning of u, but $(l(u))^3$ is not;

3) the ending of the word u obtained by removing the beginning $(l(u))^2$ from u has precisely one occurrence of the letter p_1.

Remark 11.3. By Corollary 5.6, $S(2,1,1)$ is the set of SR-words $u = a^2b$, where a and b are regular words with a common leading letter that have a unique occurrence of this letter and are such that $b \boxleft a$.

Lemma 11.4. 1. *For $u \in S(2,1,1)$, $r(u)$ is not a prolonged* RR2*-word and $\overline{r}(u) = \overline{r}(r(u))$.*

2. *If an* RR2*-word v is not prolonged, then $\overline{r}(v)v \in S(2,1,1)$ and $r(\overline{r}(v)v) = v$.*

Proof. 1. Consider an arbitrary word u in $S(2,1,1)$. By Remark 11.3, there exist uniquely determined regular words a and b with a common leading letter p_1 that have only one occurrence of this letter and satisfy $u = a^2b$, $b \boxleft a$.

Since $b \boxleft a$, we have $b < a$, so that the word ab is regular. Moreover, the word a^2b has precisely three occurrences of the letter p_1 and its ending ab begins with the letter p_1 and has precisely two occurrences of this letter. Therefore, $r(u) = ab$.

Being a regular proper subword of the SR-word u, ab is an RR-word. Since ab has exactly two occurrences of the leading letter p_1, we conclude that ab is an RR2-word. Moreover, $b \boxleft a$. Therefore, the RR2-word $r(u) = ab$ is not prolonged. Moreover, $\overline{r}(r(u)) = a = \overline{r}(u)$.

2. Consider an arbitrary nonprolonged RR2-word v. Denote $v_1 = \overline{r}(v)$. Since v_1 is a beginning of v and $v \neq v_1$, we have $v < v_1$. Therefore, the word $u = v_1v$ is regular. Since the word u has three occurrences of the leading letter p_1, its composition is not rooted.

By [33, Lemma 3.1], we have $v_2 \prec v_1$, where $v_2 = r(v)$. Therefore, $v_1 = zx_1c$ and $v_2 = zx_2d$, where z, c, and d are words, and x_1 and x_2 are letters such that $x_2 < x_1$.

Since the regular word v is not prolonged, we have $d = 1$ and $v_2 = zx_2$. Then $v_1^2 z$ is the beginning of the word $v_1 v$ obtained from $v_1 v$ by removing the last letter. Therefore, $l(u)$ is a beginning of the word $v_1^2 z$. Moreover, z is a beginning of the word v_1. Therefore, $l(v_1 v) = v_1$ is an RR-word. Also, $r(u) = v$ is an RR-word. By Corollary 5.11, the above implies that $v_1 v$ is an SR-word. Moreover, $u = v_1^2 v_2$ and $v_2 = zx_2 \boxleq zx_1 c = v_1$. Consequently, $u \in S(2,1,1)$ by Remark 11.3. Since the words $\bar{r}(v)$ and v are regular and have common leading letter, and the word $\bar{r}(v)$ has exactly one occurrence of this letter, we have $r(\bar{r}(v)v) = v$. □

Lemma 11.5. *An arbitrary* RR2*-word a is not prolonged if and only if*

1) *the least vertex m^- in the order $\leq_{p_1}$ among the single vertices of the graph $D(|a|)$ that have a double adjacent vertex has no single adjacent vertices (p_1 is the first letter of the word a);*

2) *if $m^- \in O$, then the second-by-priority vertex of the interval $[m^-, x_2^+(|a|)]$ lies in the interval $[m^-, p_1)$.*

Proof. Let a be an arbitrary nonprolonged word; let a_1 and a_2 be words with one occurrence of the letter p_1 and such that $a = a_1 a_2$.

Since a is an RR2-word, from [33, Lemma 3.1] it follows that $a_1 = dx_1 b$ and $a_2 = dx_2 c$, where d, b, and c are words, x_1 and x_2 are letters, and $x_2 < x_1$. By [33, Lemma 3.7], $x_2 = m^-$. Since the regular word a is not prolonged, we have $c = 1$ and $a_2 = dx_2 = dm^-$.

1. Assume that property 1) is not fulfilled. Let y be a single vertex of the graph $D(|a|)$ adjacent to m^-. Suppose $|y| \leq |a_1|$. Then $m^- \in [p_1, y] \leq |a_1|$, which is incorrect. Thus, $|y| \not\leq |a_1|$, i.e., $|y| \leq |a_2| = |dm^- c|$.

All vertices of the graph $D(|d|)$ are double in the graph $D(|a|)$, and the vertex y is single in $D(|a|)$. Therefore, $|y| \not\leq |d|$. Moreover, $y \neq m^-$. Consequently, $|y| \leq |c|$, so that the word c is nonempty. We denote by $\bar{c}$ and $\bar{a_2}$ the words obtained from c (respectively, from a_2) by removing the last letter.

Then $\bar{a_2} = dx_2\bar{c} < dx_1 b = a_1$. Therefore, the word $a_1\bar{a_2}$ is regular. Since the word $\bar{a_2}$ is nonempty, we have $a_1\bar{a_2} < a_1$; thus, the word $a_1^2\bar{a_2}$ is regular. Being a proper regular subword of the SR-word $a_1^2 a_2$, $a_1^2\bar{a_2}$ is an RR-word, and thus, it has rooted composition, which is impossible because $3|p_1| \leq |a_1^2\bar{a_2}|$. This contradiction shows that property 1) is fulfilled.

2. If $m^- \in O$, then the structure of RR2-words for the word a shows that $|a_2| = [m^-, x_2^+(|a|)]$. Since $a_2 = dm^-$, it follows that an RR1-word of composition $[m^-, x_2^+(|a|)]$ ends with the letter m^-, which implies property 2).

Conversely, let a be an RR2-word satisfying 1) and 2). Since a is an RR2-word, in accordance with [33, Lemma 3.1] we have $a_1 = dx_1 b$ and $a_2 = dx_2 c$, where d, b, and c are words, x_1 and x_2 are letters, and $x_2 < x_1$. By [33, Lemma 3.7], we have $x_2 = m^-$. From properties 1) and 2) it follows that the word a_2 ends with the letter m^-. Therefore, $c = 1$, $a_2 = dm^-$, and the word a is not prolonged. □

Lemma 11.6. *Associating the word $r(u)$ with every word u in $S(2,1,1)$, we obtain an injective mapping onto the set of nonprolonged* RR2*-words. The inverse mapping takes any nonprolonged* RR2*-word v to the word $\bar{r}(v)v$ (see Lemma* 11.4*).*

Now we state the main results of the paper.

Theorem 11.7. *The set of* SR*-words of the Lie algebra D_n^+ is partitioned into the sets $S(1,1,0)$, $S(1,2,0)$, $S(2,1,0)$, $S(2,2,0)$, $S(1,1,1)$, $S(1,2,1)$, and $S(2,1,1)$ described in statements* 6.7*,* 7.5*,* 8.5*,* 8.8*,* 9.13*,* 10.4*, and* 11.6*.*

Theorem 11.8. *The reduced Gröbner–Shirshov basis of the Lie algebra D_n^+ is precisely the set of relations of the form $[u] = 0$, where u is an* SR*-word and $[u]$ is the element of the free Lie algebra obtained from u by the regular arrangement of brackets.*

References

[1] A. I. Shirshov, *Some algorithm problems for Lie algebras*, Sibirsk. Mat. Zh. **3** (1962), no. 2, 292–296. (Russian) MR0183753 (32:1231)

[2] B. Buchberger, *An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal*, Ph. D Thesis, Univ. Innsbruck, Austria, 1965. (German)

[3] A. I. Shirshov, *On a hypothesis of the theory of Lie algebras*, Sibirsk. Mat. Zh. **3** (1962), no. 2, 297–301. (Russian) MR0182684 (32:167)

[4] L. A. Bokut′, *Embedding Lie algebras into algebraically closed Lie algebras*, Algebra i Logika Sem. **1** (1962), no. 2, 47–53. (Russian) MR0152557 (27:2535)

[5] B. Buchberger and F. Winkler (eds.), *Gröbner bases and applications*, London Math. Soc. Lecture Notes Ser., vol. 251, Cambridge Univ. Press, Cambridge, 1998. MR1699811 (2000e:13003)

[6] V. P. Gerdt and W. Lassner, *Isomorphism verification for complex and real Lie algebras by Gröbner basis technique*, Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics (Acireale, 1992) (N. H. Ibragimov et al., eds.), Kluwer Acad. Publ., Dordrecht, 1993, pp. 245–254. MR1259560 (95b:17001)

[7] L. A. Bokut and A. A. Klein, *Serre relations and Gröbner–Shirshov bases for simple Lie algebras.* I, Internat. J. Algebra Comput. **6** (1996), no. 4, 389–400. MR1414346 (97k:17005)

[8] ———, *Serre relations and Gröbner–Shirshov bases for simple Lie algebras.* II, Internat. J. Algebra Comput. **6** (1996), no. 4, 401–412. MR1414346 (97k:17005)

[9] ———, *Gröbner–Shirshov bases for exceptional Lie algebras.* I, Ring Theory (Miskolc, 1996), J. Pure Appl. Algebra **133** (1998), 51–57. MR1653694 (99i:17010)

[10] ———, *Gröbner–Shirshov bases for exceptional Lie algebras E_6, E_7, E_8*, Algebras and Combinatorics (Hong Kong, 1997), Springer, Singapore, 1999, pp. 37–46. MR1733166 (2001c:17018)

[11] L. A. Bokut and P. Malcolmson, *Gröbner–Shirshov bases for quantum enveloping algebras*, Israel J. Math. **96** (1996), 97–113. MR1432728 (97k:17013)

[12] ———, *Gröbner–Shirshov bases for relations of Lie algebra and its enveloping algebra*, Algebras and Combinatorics (Hong Kong, 1997), Springer, Singapore, 1999, pp. 47–54. MR1733167 (2001c:17020)

[13] T. Gateva-Ivanova and V. Latyshev, *On recognisable properties of associative algebras. Commutational aspects of commutative algebra*, J. Symbolic Comput. **6** (1988), no. 2–3, 371–388. MR0988424 (90f:68090)

[14] A. A. Mikhalev, *A composition lemma and the word problem for color Lie superalgebras*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1989**, no. 5, 88–91; English transl., Moscow Univ. Math. Bull. **44** (1989), no. 5, 87–90; *Superalgebras of free Lie p-superalgebras*, Mat. Zametki **43** (1988), no. 2, 178–191; English transl., Math. Notes **43** (1988), no. 1–2, 99–106. MR1029790 (90m:17005); MR0939518 (89f:17025)

[15] ———, *Shirshov composition techniques in Lie superalgebras* (*noncommutative Gröbner bases*), Tr. Semin. I. G. Petrovskogo no. 18 (1995), 277–289; English transl., J. Math. Sci. **80** (1996), no. 5, 2153–2160. MR1425053 (97k:17045)

[16] ———, *The composition lemma for color Lie superalgebras and for Lie p-superalgebras*, Proc. of the Internat. Conf. on Algebra, Pt. 2 (Novosibirsk, 1989), Contemp. Math., vol. 131, Pt. 2, Amer. Math. Soc., Providence, RI, 1992, pp. 91–104. MR1175824 (93h:17010)

[17] L. A. Bokut, S.-J. Kang, K. H. Lee, and P. Malcolmson, *Gröbner–Shirshov bases for Lie superalgebras and their universal enveloping algebras*, J. Algebra **217** (1999), 461–495. MR1700511 (2000e:17006)

[18] S.-J. Kang and K.-H. Lee, *Gröbner–Shirshov bases for representation theory*, J. Korean Math. Soc. **37** (2000), 55–72. MR1749085 (2001a:16030)

[19] ———, *Gröbner–Shirshov bases for irreducible sl_{n+1}-modules*, J. Algebra **232** (2000), no. 1, 1–20. MR1783910 (2001h:17015)

[20] E. S. Chibrikov, *On free conformal Lie algebras*, Vestnik Novosibirsk. Gos. Univ. Ser. Mat. Mekh. Inf. **4** (2004), no. 1, 65–83. (Russian)

[21] M. Roitman, *On free conformal and vertex algebras*, J. Algebra **217** (1999), 496–527. MR1700512 (2001i:17036)

[22] ———, *Universal enveloping conformal algebras*, Selecta Math. (to appear).

[23] L. A. Bokut′, Y. Fong, and W.-F. Ke, *Free associative conformal algebras*, Proc. of the 2nd Tainan–Moscow Algebra and Combinatorics Workshop (Tainan, 1997), Springer-Verlag, Hong Kong, 2000, pp. 13–25.

[24] ———, *Composition–diamond lemma for associative conformal algebras*, J. Algebra **272** (2004), no. 2, 739–774. MR2028080 (2005f:16050)

[25] N. N. Vasil′ev and D. Pavlov, *Enumeration of finite monomial orders and Young tableaux*, Internat. Conf. Polynom. Comput. Algebra (2008), pp. 71–76.

[26] L. A. Bokut′, W.-F. Ke, P. S. Kolesnikov, and Y. Fong, *Gröbner and Gröbner–Shirshov bases in algebra, and conformal algebras*, Fundam. Prikl. Mat. **6** (2000), no. 3, 669–706. (Russian) MR1801321 (2001j:17050)

[27] L. A. Bokut′ and P. S. Kolesnikov, *Gröbner–Shirshov bases: from inception to the present time*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **272** (2000), 26–67; English transl., J. Math. Sci. (N.Y.) **116** (2003), no. 1, 2894–2916. MR1811792 (2002a:16030)

[28] E. N. Poroshenko, *Gröbner–Shirshov bases for the affine nontwisted Kac–Moody algebras*, Thesis, Novosibirsk, 2002. (Russian)

[29] ———, *Groebner–Shirshov bases for the Kac–Moody algebras of the type* $A_n^{(1)}$, Comm. Algebra **30** (2002), 2617–2637. MR1908229 (2003e:17041)

[30] A. N. Koryukin, *Gröbner–Shirshov bases for the Lie algebra* A_n, Algebra Logika **44** (2005), no. 2, 131–147; English transl., Algebra Logic **44** (2005), no. 2, 73–81. MR2170693 (2006g:17015)

[31] ———, *Gröbner–Shirshov bases of the Lie algebra* B_n^+, Algebra i Analiz **20** (2008), no. 1, 93–137; English transl., St. Petersburg Math. J. **20** (2009), no. 1, 65–94. MR2411970 (2009b:17021)

[32] ———, *Gröbner–Shirshov bases for the Lie algebra* C_n^+. I, Nauchn. Vestnik Novosibirsk. Tekhn. Univ. **2006**, no. 4(25), 139–154; II, ibid., **2007**, no. 1(26), 89–99; III, ibid., **2007**, no. 2(27), 107–117; IV, ibid., **2007**, no. 3(28), 111–120; V, ibid., **2007**, no. 4(29), 63–76; VI, ibid., **2008**, no. 1(30), 121–130. (Russian)

[33] A. N. Koryukin and K. P. Shum, *Reduced bases of the Lie algebra* D_n^+, Sib. Zh. Ind. Mat. **9** (2006), no. 4, 90–104. (Russian) MR2304801 (2007k:17009)

[34] P. Lalonde and A. Ram, *Standard Lyndon bases of Lie algebras and enveloping algebras*, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1821–1830. MR1273505 (95h:17013)

[35] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, Springer-Verlag, New York, 1997, 536 pp. MR1417938 (97h:13024)

[36] A. I. Shirshov, *On free Lie rings*, Mat. Sb. (N.S.) **45** (1958), no. 2, 113–122. (Russian) MR0099356 (20:5796)

[37] ———, *Selected works of A. I. Shirshov* (L. A. Bokut′, V. Latyshev, I. Shestakov, and E. Zelmanov, eds.), Birkhäuser, Basel, 2009, 242 pp. MR2547481

[38] R. Lyndon, *On Burnside's problem*, Trans. Amer. Math. Soc. **77** (1954), 202–215. MR0064049 (16:218b)

[39] K. T. Chen, R. H. Fox, and R. C. Lyndon, *Free differential calculus.* IV. *The quotient groups of the lower central series*, Ann. of Math. (2) **68** (1958), 81–95. MR0102539 (21:1330)

[40] L. A. Bokut and Yuqun Chen, *Gröbner–Shirshov bases for Lie algebras: after A. I. Shirshov*, Southeast Asian Bull. Math. **31** (2007), no. 6, 1057–1076. MR2386984 (2008m:17009)

[41] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Grad. Texts in Math., vol. 9, Springer-Verlag, New York–Berlin, 1972. MR0323842 (48:2197)

[42] N. Jacobson, *Lie algebras*, Intersci. Tracts in Pure Appl. Math., vol. 10, Intersci. Publ., New York–London, 1962. MR0143793 (26:1345)

[43] N. Bourbaki, *Éléments de mathématique.* Fasc. 34. *Groupes et algèbre de Lie*, Actualités Sci. Indust., No. 1337, Hermann, Paris, 1968. MR0240238 (39:1590)

Sobolev Institute of Mathematics, Siberian Branch, Russian Academy of Sciences, Academician Koptyug Avenue 4, Novosibirsk 630090, Russia

E-mail address: koryukin@ngs.ru

Received 18/MAR/2009

Translated by N. B. LEBEDINSKAYA