ON INTRINSIC ISOMETRIES TO EUCLIDEAN SPACE

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Abstract. Compact metric spaces that admit intrinsic isometries to the Euclidean $d$-space are considered. Roughly, the main result states that the class of such spaces coincides with the class of inverse limits of Euclidean $d$-polyhedra.

§1. Introduction

The intrinsic isometries are defined in §2; this is a variation of the notion of a path isometry, i.e., a map that preserves the lengths of curves. Any intrinsic isometry is a path isometry; the converse is not true in general.

The following statement is a reason for me to prefer intrinsic isometry.

Starting Proposition 1.1. Let $X$ be a compact metric space that admits an intrinsic isometry to the $d$-dimensional Euclidean space (further denoted by $\mathbb{E}^d$). Then $\dim X \leq d$, where $\dim$ denotes the Lebesgue covering dimension.

This statement is proved in §3. A similar statement for path isometry fails; see Example 4.2. Furthermore, also the Hausdorff dimension cannot be bounded. For example, the $\mathbb{R}$-tree admits an intrinsic isometry to $\mathbb{R}$ and it contains compact subsets of arbitrarily large Hausdorff dimension.

Here are some known results on length spaces that admit intrinsic isometry to $\mathbb{E}^d$.

Theorem 1.2. Let $\mathcal{R}$ be a $d$-dimensional Riemannian space and $f : \mathcal{R} \to \mathbb{E}^d$ a short map. Then, given $\varepsilon > 0$, there is an intrinsic isometry $\iota : \mathcal{R} \to \mathbb{E}^d$ such that

$$|f(x) - \iota(x)|_{\mathbb{E}^d} < \varepsilon$$

for any $x \in \mathcal{R}$.

In particular, any Riemannian $d$-space admits an intrinsic isometry to $\mathbb{E}^d$.

For path isometries, this theorem was proved in [7, 2.4.11], and the same proof works for intrinsic isometries. Applying this theorem, one can show that any limit of an increasing sequence of Riemannian metrics on a fixed $d$-dimensional manifold admits an intrinsic isometry to $\mathbb{E}^d$. (The proof is similar to the "if"-part of the main theorem.) In particular, any sub-Riemannian metric on a $d$-dimensional manifold admits an intrinsic isometry to $\mathbb{E}^d$.

Theorem 1.3. Let $\mathcal{P}$ be a Euclidean polyhedron and $f : \mathcal{P} \to \mathbb{E}^d$ a short map. Then, given $\varepsilon > 0$, there is a piecewise linear intrinsic isometry $\iota : \mathcal{P} \to \mathbb{E}^d$ such that

$$|f(x) - \iota(x)|_{\mathbb{E}^d} < \varepsilon$$

for any $x \in \mathcal{P}$.

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That is, a 1-Lipschitz map.
Corollary 1.4. Any \(d\)-dimensional Euclidean polyhedron admits a piecewise linear intrinsic isometry to \(\mathbb{E}^d\).

This corollary was proved in \[8\] for dimensions not exceeding 4, but a slight modification of the proof works in all dimensions; see \[10\]. The 2-dimensional case of this statement was proved in \[10\], but had been remaining unnoticed for many years and was reproved independently in \[2\].

A necessary and sufficient condition. Now we describe the main result of the paper.

A compact metric space \(X\) is called a pro-Euclidean space of rank at most \(d\) if it can be presented as an inverse limit \(X = \lim_{\leftarrow} P_n\) (see §2) of a sequence of Euclidean \(d\)-polyhedra \(P_n\).

Theorem 1.5 (Main Theorem). A compact metric space \(X\) admits an intrinsic isometry to \(\mathbb{E}^d\) if and only if \(X\) is a pro-Euclidean space of rank at most \(d\).

The proof is straightforward, and the statement is more interesting than the proof. It is rare when inverse limits help to solve a natural problem in metric geometry; the only other example I know is the characterization of the homogeneous locally compact metric spaces given in \[3\].

Note that the statement in Theorem 1.2 (in the compact case) is equivalent to the fact that any compact Riemannian \(d\)-space is a pro-Euclidean space of rank at most \(d\). The latter can be obtained directly from the following exercise. In particular, the main theorem provides an alternative proof to Theorem 1.2 in the compact case.

Exercise 1.6. Show that any compact Riemannian space admits a Lipschitz approximation by Euclidean polyhedra.

A nonexample. Recall that a Minkowski space is a finite-dimensional real vector space with metric induced by a norm.

Proposition 1.7. Let \(\Omega\) be an open subset of a Minkowski \(d\)-space \(\mathbb{M}^d\). If \(\Omega\) admits an intrinsic isometry to \(\mathbb{E}^m\), then \(d \leq m\) and \(\mathbb{M}^d\) is isometric to \(\mathbb{E}^d\).

In particular, the condition in Theorem 1.1 on Lebesgue’s dimension is not sufficient.

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§2. Preliminaries

Standard definitions. Given a metric space \(X\) and two points \(x, x' \in X\), we denote by \(|xx'| = |xx'|_X\) the distance from \(x\) to \(x'\) in \(X\).

A length space is a metric space such that for any two points \(x, x'\) the distance \(|xx'|\) coincides with the infimum of the lengths of curves connecting \(x\) and \(x'\).

A map \(f : X \to Y\) between metric spaces \(X\) and \(Y\) is said to be short if for any \(x, x' \in X\) we have

\[ |f(x)f(x')|_Y \leq |xx'|_X.\]

A length space \(P\) is called a Euclidean \(d\)-polyhedron if there is a finite triangulation of \(P\) such that each simplex is isometric to a simplex in \(\mathbb{E}^d\).

\[2\]In fact, the same is true for a path isometry.
Intrinsic isometries and pullback metrics. Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and $f : \mathcal{X} \to \mathcal{Y}$ a continuous map. Given two points $x, x' \in \mathcal{X}$, a sequence of points $x = x_0, x_1, \ldots, x_n = x'$ is called an $\varepsilon$-chain from $x$ to $x'$ if $|x_{i-1}x_i| \leq \varepsilon$ for all $i > 0$. Set

$$\text{pull}_{f,\varepsilon}(x, x') = \inf \left\{ \sum_{i=1}^{n} |f(x_{i-1})f(x_i)|_{\mathcal{Y}} \right\},$$

where the infimum is taken along all $\varepsilon$-chains $(x_i)_{i=0}^{n}$ from $x$ to $x'$.

Clearly, $\text{pull}_{f,\varepsilon}$ is a premetric on $\mathcal{X}$, and $\text{pull}_{f,\varepsilon}(x, x')$ is monotone nonincreasing in $\varepsilon$. Thus, the (possibly infinite) limit

$$\text{pull}_f(x, x') = \lim_{\varepsilon \to 0} \text{pull}_{f,\varepsilon}(x, x')$$

is well defined. The premetric $\text{pull}_f$ is a pullback metric on $\mathcal{X}$, and $\text{pull}_f(x, x')$ is isometric to $[0, \infty]$.

A map $f : \mathcal{X} \to \mathcal{Y}$ between length spaces $\mathcal{X}$ and $\mathcal{Y}$ is an intrinsic isometry if

$$|xx'|_{\mathcal{X}} = \text{pull}_f(x, x')$$

for any $x, x' \in \mathcal{X}$.

Any intrinsic isometry is a short map. Moreover, it is easily seen that any intrinsic isometry preserves the lengths of curves. The converse fails; see §4.

Proposition 2.1. Let $\mathcal{X}$ be a compact (or even proper) metric space. Then existence of an intrinsic isometry $f : \mathcal{X} \to \mathcal{Y}$ implies that $\mathcal{X}$ is a length space.

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That is, it satisfies the triangle inequality, it is symmetric and nonnegative, and $\text{pull}_{f,\varepsilon}(x, x) = 0$, but it might happen that $\text{pull}_{f,\varepsilon}(x, x') = 0$ for $x \neq x'$.

That is, all closed bounded sets in $\mathcal{X}$ are compact.
The proof is left to the reader. Note that this is not true for general \( \mathcal{X} \). Consider two points connected by countably many unit intervals \( \mathbb{I}_n \) and one interval of length \( \frac{1}{2} \); we equip the resulting space with the natural intrinsic metric. Let us remove the interval of length \( \frac{1}{2} \) from our space. The metric on the remaining space \( \mathcal{X} \) is not intrinsic. Next, construct a map \( f : \mathcal{X} \to \mathbb{R} \) so that the restriction \( f_n = f|_{\mathbb{I}_n} \) is an intrinsic isometry, \( f_n(0) = 0 \), \( f_n(1) = \frac{1}{2} \), and the \( f_n(x) \) converge uniformly to \( \frac{x}{2} \). It is easily seen that \( f : \mathcal{X} \to \mathbb{R} \) is an intrinsic isometry.

**Proposition 2.2.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be metric spaces, let \( \mathcal{X} \) be compact, and let a continuous map \( f : \mathcal{X} \to \mathcal{Y} \) be such that

\[
\sup_{x,x' \in \mathcal{X}} \text{pull}_f(x,x') < \infty.
\]

Then, given \( \varepsilon > 0 \), there is \( \delta = \delta(f, \varepsilon) > 0 \) such that for any short map \( h : \mathcal{X} \to \mathcal{Y} \) such that

\[
|f(x)h(x)|_\mathcal{Y} < \delta \quad \text{for any} \quad x \in \mathcal{X}
\]

we have

\[
\text{pull}_f(x,x') < \text{pull}_h(x,x') + \varepsilon
\]

for any \( x, x' \in \mathcal{X} \).

The proof is a direct application of Lemma 2.3.

For a compact metric space \( \mathcal{X} \), we denote by \( \text{pack}_\varepsilon \mathcal{X} \) the maximal number of points in \( \mathcal{X} \) at a distance exceeding \( \varepsilon \) from each other. Clearly, \( \text{pack}_\varepsilon \mathcal{X} \) is finite for any \( \varepsilon > 0 \).

**Lemma 2.3.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be metric spaces, let \( \mathcal{X} \) be compact, and let \( f, h : \mathcal{X} \to \mathcal{Y} \) be two continuous maps.

Assume that \( |f(x)h(x)| < \delta \) for any \( x \in \mathcal{X} \). Then for any \( x, x' \in \mathcal{X} \) we have

\[
\text{pull}_{f,\varepsilon}(x,x') \leq \text{pull}_{h,\varepsilon}(x,x') + 4 \cdot \delta \cdot \text{pack}_\varepsilon \mathcal{X}.
\]

**Proof.** Suppose \( \text{pull}_{h,\varepsilon}(x,x') \leq \ell \), i.e., suppose there is an \( \varepsilon \)-chain \( \{x_i\}_{i=0}^n \) from \( x \) to \( x' \) such that

\[
(\ast) \quad \sum_{i=1}^n |h(x_{i-1})h(x_i)|_\mathcal{Y} < \ell.
\]

Since \( |h(x_i)f(x_i)| < \delta \), it follows that

\[
\text{pull}_{f,\varepsilon}(x,x') \leq \sum_{i=1}^n |f(x_{i-1})f(x_i)|_\mathcal{Y} < \sum_{i=1}^n |h(x_{i-1})h(x_i)|_\mathcal{Y} + 2 \cdot n \cdot \delta.
\]

Assume \( n \) is the smallest number for which there is an \( \varepsilon \)-chain satisfying (\ast). It suffices to show that

\[
n < 2 \cdot \text{pack}_\varepsilon \mathcal{X}.
\]

If \( n \geq 2 \cdot \text{pack}_\varepsilon \mathcal{X} \), there are indices \( i \) and \( j \) such that \( j - i > 1 \) and \( |x_ix_j| \leq \varepsilon \). We remove all elements \( x_k \) with \( i < k < j \) from the chain; i.e., we consider the new \( \varepsilon \)-chain

\[
x = x_0, \ldots, x_{i-1}, x_i, x_j, x_{j+1}, \ldots, x_n = x'.
\]

By the triangle inequality in \( \mathcal{Y} \), the new chain satisfies (\ast), so that \( n \) is not the smallest number, a contradiction.

**Proposition 2.4.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be metric spaces, let \( \mathcal{X} \) be compact, and let \( i : \mathcal{X} \to \mathcal{Y} \) be an intrinsic isometry.

Then, given \( \varepsilon > 0 \), there is \( \delta = \delta(i, \varepsilon) > 0 \) such that, for any connected set \( W \subset \mathcal{X} \),

\[
\text{diam} i(W) < \delta \quad \implies \quad \text{diam} W < \varepsilon.
\]
Proof. Suppose the contrary; then there is a sequence of connected subsets \( W_n \subset X \) such that \( \operatorname{diam}(W_n) \to 0 \) as \( n \to \infty \) but \( \operatorname{diam} W_n > \varepsilon \). Thus, there are two sequences of points \( x_n, x'_n \in W_n \) with \( |x_n x'_n| \geq \varepsilon \). We pass to a subsequence of \( n \) such that \( W_n \to W \) in the Hausdorff sense and \( x_n \to x, x'_n \to x' \). We obtain a closed connected subset \( W \subset X \) with two distinct points \( x \) and \( x' \) such that \( \iota(W) = p \) for some \( p \in Y \).

Since \( W \) is connected, for any \( \varepsilon > 0 \) there is an \( \varepsilon \)-chain \( (x_i)_{i=0}^n \) from \( x \) to \( x' \) such that \( \iota(x_i) = p \) for all \( i \). Thus, we have \( \operatorname{pull}_{\iota, \varepsilon}(x, x') = 0 \) for any \( \varepsilon > 0 \), i.e., \( \operatorname{pull}_{\iota}(x, x') = 0 \), a contradiction.

\[ \square \]

§3. Proofs

Proof of the Starting Proposition (1.1). Given \( \varepsilon > 0 \), choose \( \delta = \delta(\iota, \varepsilon) \) as in Proposition 2.1. Since \( \dim E^d = d \), there is a finite open covering \( \{U_i\}_{i=1}^n \) of \( \iota(X) \) with multiplicity at most \( d + 1 \) and such that \( \operatorname{diam} U_i < \delta \) for each \( i \).

Consider the covering \( \{V_\alpha\} \) of \( X \) by connected components of \( \iota^{-1}(U_i) \) for all \( i \). By Proposition 2.1, \( X \) is a length space. In particular, all sets \( V_\alpha \) are open. By Proposition 2.1, we have \( \operatorname{diam} V_\alpha < \varepsilon \). Clearly, the multiplicity of \( \{V_\alpha\} \) is at most \( d + 1 \). Thus, the claim follows.

Proof of the “if” part in the Main Theorem (1.5). Let \( X \) be a pro-Euclidean space of rank not exceeding \( d \). Assume \( (P_n)_{n=0}^\infty \) is a sequence of \( d \)-dimensional Euclidean polyhedra and that

\[ \varphi_{n,m} : P_m \to P_n \]

is an inverse system of short maps such that \( X = \varprojlim P_n \). Let \( \psi_n : X \to P_n \) be the projections.

By Theorem 1.3, given \( \varepsilon_{n+1} > 0 \) and a piecewise linear intrinsic isometry \( \iota_n : P_n \to \mathbb{E}^d \), there is a piecewise linear intrinsic isometry \( \iota_{n+1} : P_{n+1} \to \mathbb{E}^d \) such that

\[ |\iota_{n+1}(x)| \iota_n \circ \varphi_{n+1,n}(x) < \varepsilon_{n+1} \]

for any \( x \in P_n \). It remains to show that the sequence \( \varepsilon_n \) can be chosen in such a way that \( \iota_n \circ \psi_n \) converges to an intrinsic isometry \( \iota : X \to \mathbb{E}^d \).

We choose \( \varepsilon_{n+1} > 0 \) so that

\[ \varepsilon_{n+1} < \frac{1}{2} \min \{ \varepsilon_n, \delta(\iota_n, \frac{1}{n}) \} \]

where \( \delta(\iota_n, \frac{1}{n}) \) is as in Proposition 2.2. Since, clearly, \( \sum \varepsilon_i < \infty \), the following limit exists:

\[ \iota = \lim_{n \to \infty} \iota_n \circ \psi_n, \quad \iota : X \to \mathbb{E}^d. \]

Obviously, \( \iota \) is short. Next, for any \( x \in X \) we have

\[ |\iota(x) \iota_n \circ \psi_n(x)| < \sum_{i=n+1}^{\infty} \varepsilon_i < \delta(\iota_n, \frac{1}{n}). \]

Thus, by Proposition 2.2

\[ \operatorname{pull}_{\iota}(x, x') + \frac{1}{n} > \operatorname{pull}_{\iota_n \circ \psi_n}(x, x') \geq |\psi_n(x) \psi_n(x')|_{P_n}. \]

Since \( |\psi_n(x) \psi_n(x')|_{P_n} \to |xx'|_X \) as \( n \to \infty \), the map \( \iota : X \to \mathbb{E}^d \) is an intrinsic isometry.

\[ \square \]

Proof of the “only if” part in the Main Theorem (1.5). We shall give a construction of a polyhedron \( P \) associated with an intrinsic isometry \( \iota : X \to \mathbb{E}^d \) and a tiling of \( \mathbb{E}^d \) by coordinate \( a \)-cubes. (The space \( P \) will be glued out of \( a \)-cubes.) The construction will be done in such a way that if a tiling \( \tau' \) is a subdivision of a tiling \( \tau \), then for the corresponding polyhedra \( P' \) and \( P \) there will be a natural intrinsic isometry \( P' \to P \). Thus,
we shall construct the required inverse system of polyhedra out of nested subdivisions of $\mathbb{E}^d$.

We set $a_n = \frac{1}{2^n}$ and $r_n = \frac{1}{2^n} \cdot a_n$. Fixing $n$ for a while, consider the tiling of $\mathbb{E}^d$ by coordinate $a_n$-cubes. We want to construct a Euclidean polyhedron $P_n$ associated with this tiling.

The image $i(X)$ is covered by a finite number of such $a_n$-cubes, say $\{\Box_n^i\}$. For each $\Box_n^i$, consider all connected components $\{W_n^i\}$ of

$$B_{r_n}(i^{-1}(\Box_n^i)) \subset X,$$

where $B_r(S)$ denotes the $r$-neighborhood of a set $S$.

By Proposition 2.1, $X$ is a length space. In particular, each set $W_n^i$ is open and contains a ball of radius $r_n$. Thus, for fixed $i$, the collection of open sets $\{W_n^i\}$ is finite. Therefore, the set of all $\{W_n^i\}$ for all $\{\Box_n^i\}$ forms a finite open cover of $X$. For each $W_n^i$, we make an isometric copy $\Box_n^i$ of $\Box_n^i$, and fix an isometry $i_n^i : \Box_n^i \to \Box_n^i$. The Euclidean polyhedron $P_n$ is glued from the $\Box_n^i$ by the following rule: we glue $\Box_n^i$ to $\Box_n^{i_2j_2}$ along $(i_n^{i_2j_2})^{-1} \circ i_n^{i_1j_1}$ if and only if $W_n^{i_1j_1} \cap W_n^{i_2j_2} \neq \emptyset$. (The map $(i_n^{i_2j_2})^{-1} \circ i_n^{i_1j_1}$ sends one of the faces of $\Box_n^{i_1j_1}$ isometrically to a face of $\Box_n^{i_2j_2}$.)

The resulting polyhedron $P_n$ admits a natural piecewise intrinsic isometry $i_n : P_n \to \mathbb{E}^d$, defined by $i_n(x) = i_n^i(x)$ for $x \in \Box_n^i$. Next, there is a uniquely determined intrinsic isometry $\varphi_{m,n} : P_m \to P_n$ for $m \geq n$ that satisfies $i_m = i_n \circ \varphi_{m,n}$ and

$$\varphi_{m,n}(\Box_{n}^i) \subset \Box_{n}^i \subset P_n \Rightarrow W_{m}^{i'j'} \subset W_{n}^{i'j'} \subset X.$$
for all large \( n \). Thus, the \( x_i \) form an \( \varepsilon \)-chain from \( x \) to \( x' \), and (\(*\)) implies that

\[
\text{pull}_{\varepsilon}^{i\varepsilon}(x, x') \leq \sum_{i=1}^{m} |i(x_{i-1}) i(x_i)| < \sum_{i=1}^{m} |i_n \circ \gamma_n(t_{i-1}) \circ \gamma_n(t_i) + 4 \cdot a_n \cdot \left[ \frac{\ell}{8} \right] ,
\]

which contradicts (\(*\)) if \( n \) is sufficiently large. \( \Box \)

Remark 1. In the inverse system \((\varphi_{m,n}, \mathcal{P}_n)\) constructed above, the images of \( \varphi_{m,n} \) form a \( \sqrt{d} \cdot a_n \)-net in \( \mathcal{P}_n \). It follows that the space \( X \) is isometric to the Gromov–Hausdorff limit of \( \mathcal{P}_n \) (see also \( \S 2 \)).

Proof of Proposition 1.7. The inequality \( d \leq m \) follows from the Starting Proposition \( \text{Proposition 1.1} \). In the proof of the second part, we use the following two statements.

1) Assume \( \varepsilon : \Omega \subset \mathcal{M}^d \rightarrow \mathbb{E}^m \) is an intrinsic isometry; then it is a Lipschitz map for a Euclidean structure on \( \Omega \). Thus, by Rademacher’s theorem (see [12, 3.1.6]), the differential \( d_p \varepsilon \) is well defined for almost all \( p \in \Omega \).

2) For any curve \( \gamma(t) \) with natural parameter in a metric space, for almost all values of the parameter \( t_0 \) we have

\[
|\gamma(t_0) \gamma(t_0 + \varepsilon)| = \varepsilon + o(\varepsilon);
\]

see [4, 2.7.5].

Let \( \|\ast\| \) denote the norm that induces the metric on \( \mathcal{M}^d \). Fixing \( u \) with \( \|u\| = 1 \), consider the pencil of lines of the form \( p + u \cdot t \) in \( \Omega \). The two statements above imply that \( |d_p \varepsilon(v)| = \|v\| \). Hence, we obtain the parallelogram identity

\[
2 \cdot (\|v\|^2 + \|w\|^2) = \|v + w\|^2 + \|v - w\|^2
\]

for any two vectors \( v \) and \( w \). Thus, the norm \( \|\ast\| \) is Euclidean. \( \Box \)

\( \S 4. \) About path isometries

In this section we relate the notion of an intrinsic isometry defined in [2] to the more common (but less natural) notion of a path and weak path isometry.

Definition 4.1. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two length spaces. A map \( \varepsilon : \mathcal{X} \rightarrow \mathcal{Y} \) is called

1) a path isometry if for any path \( \gamma : [0, 1] \rightarrow \mathcal{X} \) we have

\[
\text{length} \gamma = \text{length} \varepsilon \circ \gamma ;
\]

2) a weak path isometry if for any rectifiable path \( \gamma : [0, 1] \rightarrow \mathcal{X} \) we have

\[
\text{length} \gamma = \text{length} \varepsilon \circ \gamma .
\]

As was noted in [2] any intrinsic isometry is a path isometry (and therefore, a weak path isometry). Later we shall show that the converse fails. Similar counterexamples for weak path isometries are much simpler: one can take a left-invariant sub-Riemannian metric \( d \) on the Heisenberg group \( \mathcal{H} \); then factoring by the center gives a weak path isometry \((\mathcal{H}, d) \rightarrow \mathbb{E}^2 \) (which is not a path isometry and, thus, not an intrinsic isometry).

Example 4.2. There is a length space \( \mathcal{X} \) and a path isometry \( f : \mathcal{X} \rightarrow \mathbb{R} \) such that \( f^{-1}(0) \) is a nontrivial connected subset.

Moreover, in such an example the Lebesgue covering dimension of \( f^{-1}(0) \) can be made arbitrarily large.
In particular, an analog of Proposition 1.1 fails for path isometries.

The following construction was suggested by D. Burago; it is based on two ideas: 1) the construction in [5, 3.1], and 2) the construction of a pseudoarc given in [9] (see also the survey [11] and the references therein). In fact, in the first part of the construction, $f^{-1}(0)$ will be homeomorphic to a pseudoarc, and in the second part $f^{-1}(0)$ will be homeomorphic to a product of pseudoarcs.

**Proof.** The space $X$ will be a completion $\Gamma$ of a certain metric graph $\tilde{\Gamma}$.

First, we describe the construction of $f$ modulo the construction of $\Gamma$. Set $\Gamma = \Gamma \setminus \Gamma$. Consider $f: \tilde{\Gamma} \to \mathbb{R}$, where $f(x)$ is the distance from $x$ to $\Gamma$. Then $f$ is a path isometry on $\Gamma$ and $f(\Gamma) = 0$. To finish that proof, we need to construct $\Gamma$ in such a way that

(i) $\Gamma$ is connected and contains more than one point;
(ii) $f$ is a path isometry on the entire $\tilde{\Gamma}$ (not only on $\Gamma$).

**Construction of $\Gamma$.** For two real intervals $I$ and $J$, a continuous map $h$ of $I$ onto $J$ is said to be $\varepsilon$-crooked if for any $t_1 < t_2$ in $I$ there are $t_1 < t'_2 < t'_1 < t_2$ such that $|h(t'_i) - h(t_i)| \leq \varepsilon$ for $i \in \{1, 2\}$. The existence of an $\varepsilon$-crooked map for any given $I$ and $\varepsilon > 0$ can be proved easily by induction on $n = \left\lceil \frac{1}{\varepsilon} \cdot \text{length} J \right\rceil$.

![Graph of an $\varepsilon$-crooked map.](image)

We fix a sequence of real intervals $J_n$ with short $\frac{1}{2^n}$-crooked maps $h_n: J_n \to J_{n-1}$. The topological inverse limit $J_n = \lim \leftarrow J_n$ is a compact connected set that has no non-trivial paths.

We can think of $J_n$ as of a (linear) metric graph with length of each edge at most $\frac{1}{2^n}$. We construct a graph $\Gamma$ from the disjoint union $\bigsqcup_n J_n$ by joining each vertex $v$ of $J_n$ to a vertex of $J_{n-1}$ that is closest to $h_n(v)$ by an edge of length $\frac{1}{2^n}$. Then $\Gamma$ is homeomorphic to $J_\infty$; this yields $\Gamma$.

Let $\Gamma_n$ denote the finite subgraph of $\Gamma$ formed by all vertices in $J_1$, $J_2$, $\ldots$, $J_n$. Note that there is a short map $\Gamma_n \to \Gamma_{n-1}$ that is identical on $\Gamma_{n-1}$. It follows that, for any path $\alpha: [0, 1] \to \Gamma$, the total length of $\alpha \setminus \tilde{\Gamma}$ is at least $|\alpha(0)\alpha(1)|\tilde{\Gamma}$, and (ii) follows.

**Second part.** We construct a graph $\Gamma^{(m)}$ so as to make $\Gamma^{(m)}$ homeomorphic to the product of $m$ copies of $\tilde{\Gamma}$.

We do this in the case where $m = 2$; the other cases are similar. The set of vertices of $\Gamma^{(2)}$ is the disjoint union $\bigsqcup_n (\text{Vert } J_n \times \text{Vert } J_n)$, where Vert $J_n$ denotes the set of vertices.

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5 That is, a locally finite graph with intrinsic metric such that each edge is isometric to a real interval.
of \( J_n \). We connect two vertices \((x, y) \in \text{Vert} J_n \times \text{Vert} J_n\) and \((x', y') \in \text{Vert} J_k \times \text{Vert} J_k\) if and only if the pairs \((x, x')\) and \((y, y')\) were connected in \( \Gamma \); the length of this edge will be the maximum of the lengths of the edges \(xx'\) and \(yy'\) (we assume that a vertex is connected to itself by an edge of length 0).

Clearly, there is a homeomorphism \( \hat{\Gamma}^{(2)} \to \hat{\Gamma} \times \hat{\Gamma} \). Note that there are two short coordinate projections \( \varsigma_1, \varsigma_2 : \hat{\Gamma}^{(2)} \to \hat{\Gamma} \). Thus, for any path \( \alpha : [0, 1] \to \bar{\Gamma} \), the total length of \( \alpha \) is at least \( \max_i |\varsigma_i \circ \alpha(0)| \varsigma_i \circ \alpha(1)| \). This ensures that \( \hat{\Gamma}^{(2)} \) is bi-Lipschitz equivalent to the product \( \hat{\Gamma} \times \hat{\Gamma} \).

\[ \square \]

§5. Comments and open questions

A length space \( M \) is called a Minkowski \( d \)-polyhedron if there is a finite triangulation of \( M \) such that each simplex is isometric to a simplex in a Minkowski space. Accordingly, a compact metric space \( X \) is called a pro-Minkowski space of rank not exceeding \( d \) if it can be presented as an inverse limit of Minkowski \( d \)-polyhedra.

**Question 5.1.** Is it true that any length space with Lebesgue’s covering dimension \( d \) is a pro-Minkowski space of rank \( d \)?

Or even more specifically:

**Question 5.2.** Is it true that any metric space homeomorphic to a disk is a pro-Minkowski space of rank 2?

One can reformulate this philosophically: *Is there any essential difference between the Finsler metric and the general metric on an \( n \)-manifold?* This question was asked by D. Burago; it served as the original motivation for this paper (see also [5, Theorem 1]).

If we remove the restriction on dimension, then the answer to the above question is YES. Namely, the following exercise can be solved by using the Kuratowski embedding \( x \mapsto \text{dist}_x \).

**Exercise 5.3.** Show that any compact length space is an inverse limit of Minkowski polyhedra \( \mathcal{M}_n \) with \( \dim \mathcal{M}_n \to \infty \).

**Question 5.4.** Is it true that any path isometry from a closed Euclidean ball to Euclidean space is an intrinsic isometry?

**References**


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