

SCHRÖDINGER OPERATOR ON THE AXIS WITH POTENTIALS DEPENDING ON TWO PARAMETERS

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Dedicated to Vasilii Mikhaïlovich Babich, a remarkable mathematician and personality

ABSTRACT. A Schrödinger operator on the axis is considered; its localized potential is the sum of a small potential and certain potentials with contracting supports, which can increase unboundedly when their supports are contracted. Sufficient conditions are presented for the absence (or existence) of eigenvalues for such an operator. In the case where eigenvalues exist, their asymptotic expansion is constructed.

INTRODUCTION

The study of one-dimensional Schrödinger operators with small potential has a long history. In [1, Chapter III, §22], it was proved that, in the case of a small rectangular potential well, the Schrödinger operator has a single eigenvalue, and the asymptotic expansion of this eigenvalue with respect to a small parameter was constructed.

The operator

$$(0.1) \quad -\frac{d^2}{dx^2} + \delta W(x), \quad 0 < \delta \ll 1,$$

where $W(x)$ is an integrable real function that decays sufficiently rapidly at infinity, was studied in [2, 3, 4, 5].

By the Birman–Schwinger method, it was proved that if

$$(0.2) \quad \langle W \rangle := \int_{-\infty}^{\infty} W(x) dx > 0,$$

then the operator (0.1) has no eigenvalues; if

$$(0.3) \quad \langle W \rangle \leq 0,$$

then the operator has a single eigenvalue λ_δ with the asymptotic expansions

$$(0.4) \quad \lambda_\delta = -\delta^2 \frac{\langle W \rangle^2}{4} (1 + \mathcal{O}(\delta)) \quad \text{if } \langle W \rangle < 0$$

and

$$(0.5) \quad \lambda_\delta = -\delta^4 \frac{1}{4} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^x W(t) dt \right)^2 dx \right) (1 + \mathcal{O}(\delta))$$

in the critical case where $\langle W \rangle = 0$.

2010 *Mathematics Subject Classification.* Primary 35J10, Secondary 35P20.

Key words and phrases. Schrödinger operator, perturbation.

Supported by RFBR-Volga region (grant no. 08-01-97016-r), a grant of the President of Russia for support of leading scientific schools (NSh-6249.2010.1) and by FPTs (02.740.110612). The second author was also supported by a grant of the President of Russia for support of young Doctors of Science (MD-453.2010.1).

Obviously, the operator

$$(0.6) \quad -\frac{d^2}{dx^2} + \mu^{-1}hW(x),$$

where $0 < \mu, h$ and $\mu^{-1}h \ll 1$, reduces to the operator (0.1) by the substitution $\delta = \mu^{-1}h$. Therefore, from (0.4) and (0.5) we see that if inequality (0.2) is fulfilled, then the operator (0.6) has no eigenvalues; if inequality (0.3) is fulfilled, then there exists a unique eigenvalue $\lambda_{\mu,h}$ with the asymptotic expansions

$$(0.7) \quad \lambda_{\mu,h} = - (h\mu^{-1})^2 \frac{\langle W \rangle^2}{4} (1 + \mathcal{O}(h\mu^{-1})) \quad \text{if } \langle W \rangle < 0$$

and

$$(0.8) \quad \lambda_{\mu,h} = -(h\mu^{-1})^4 \frac{1}{4} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^x W(t) dt \right)^2 dx \right)^2 (1 + \mathcal{O}(h\mu^{-1}))$$

if $\langle W \rangle = 0$.

Also, it is obvious that, by the change of variable $\xi = h^{-1}x$, the operator

$$(0.9) \quad -\frac{d^2}{dx^2} + \mu^{-1}W\left(\frac{x}{h}\right)$$

reduces to an operator of the form (0.1) for $\delta = h^2\mu^{-1}$. Therefore, for $h^2\mu^{-1} \ll 1$, formulas (0.4) and (0.5) show that if inequality (0.2) is fulfilled, then the operator (0.9) has no eigenvalues. If inequality (0.3) is fulfilled, then there exists a unique eigenvalue $\lambda_{\mu,h}$ with the asymptotic expansions

$$(0.10) \quad \lambda_{\mu,h} = - (h\mu^{-1})^2 \frac{\langle W \rangle^2}{4} (1 + \mathcal{O}(h^2\mu^{-1})) \quad \text{if } \langle W \rangle < 0$$

and

$$(0.11) \quad \lambda_{\mu,h} = -h^2 (h\mu^{-1})^4 \frac{1}{4} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^x W(t) dt \right)^2 dx \right)^2 (1 + \mathcal{O}(h^2\mu^{-1}))$$

if $\langle W \rangle = 0$.

From (0.7) and (0.10) it follows that if $\langle W \rangle < 0$, then the leading terms of the asymptotic expansions for the eigenvalues of the operators (0.6) and (0.9) coincide. In the critical case where $\langle W \rangle = 0$, from (0.8) and (0.11) it follows that the leading terms of the asymptotic expansions for the eigenvalues of these operators have distinct orders of smallness.

In the present paper, we study the eigenvalues of the operator

$$\mathcal{H}^{\mu,h} := -\frac{d^2}{dx^2} + \mu^{-1} \left(\sum_{j=1}^n V_j \left(\frac{x - x_j}{h} \right) + hW(x) \right), \quad 0 < h \ll 1,$$

under the assumption that there exists $\gamma > 0$ such that

$$(0.12) \quad \mu^{-1}h^{1/2} = o(h^\gamma).$$

Here, the x_j are arbitrary distinct numbers, and $V_1(x), \dots, V_n(x), W(x)$ are complex-valued functions of class $C_0^\infty(\mathbb{R})$ at least two of which are nonzero. Clearly, even for real functions, the operator $\mathcal{H}^{\mu,h}$ cannot be reduced to (0.1) by a change of variables.

The operator $\mathcal{H}_0 := -\frac{d^2}{dx^2}$ in $L_2(\mathbb{R})$ with domain $W_2^2(\mathbb{R})$ is selfadjoint, its discrete spectrum is empty, and the essential spectrum coincides with the nonnegative real axis. Since the functions V_j and W have compact support, the essential spectrum of the operator $\mathcal{H}^{\mu,h}$ regarded as an operator in $L_2(\mathbb{R})$ with domain $W_2^2(\mathbb{R})$ coincides with the nonnegative real axis (for the case of complex functions, see, e.g., [6]).

§1. STATEMENT OF MAIN RESULTS

Denote

$$\varkappa_{\mathbf{V},W} := \sum_{j=1}^n \langle V_j \rangle + \langle W \rangle.$$

In the present paper, we prove the following statements.

Theorem 1.1. *Under condition (0.12), if $\operatorname{Re} \varkappa_{\mathbf{V},W} > 0$, then the operator $\mathcal{H}^{\mu,h}$ has no eigenvalues converging to zero.*

If $\operatorname{Re} \varkappa_{\mathbf{V},W} < 0$, then the operator $\mathcal{H}^{\mu,h}$ has a unique eigenvalue $\lambda^{\mu,h}$ converging to zero; this eigenvalue is simple and

$$(1.1) \quad \lambda^{\mu,h} = -(\mu^{-1}h)^2 \frac{\varkappa_{\mathbf{V},W}^2}{4} (1 + \mathcal{O}(\mu^{-1}h)).$$

Formula (1.1) is a natural generalization of identities (0.7) and (0.10) to complex-valued potentials.

In the critical case, for simplicity of the statements and calculations below, we only deal with real-valued functions $V_1(x), \dots, V_n(x), W(x)$.

Theorem 1.2. *Let conditions (0.12) and $\langle W \rangle = \langle V_j \rangle = 0, j = 1, \dots, n$, be fulfilled. Then the operator $\mathcal{H}^{\mu,h}$ has a unique eigenvalue $\lambda^{\mu,h}$ converging to zero. This eigenvalue is simple and has the asymptotic expansion*

$$(1.2) \quad \lambda^{\mu,h} = -\frac{1}{4} (\mu^{-1}h)^4 \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^x W(t) dt \right)^2 dx \right)^2 (1 + \mathcal{O}(h + \mu^{-1}h)).$$

This theorem shows that, in the version $\varkappa_{\mathbf{V},W} = 0$ of the critical case in question, the potential $\mu^{-1}hW(x)$ is “principal”, and the leading term of the asymptotic expansion of the eigenvalue (1.2) coincides with the leading term of the asymptotic expansion of the eigenvalue (0.8) for the operator (0.6).

For clarity, the remaining versions of the critical case where $\varkappa_{\mathbf{V},W} = 0$ are considered for two potentials.

Theorem 1.3. *Assume condition (0.12). Let*

$$\mathcal{H}^{\mu,h} = -\frac{d^2}{dx^2} + \mu^{-1} \left(V \left(\frac{x}{h} \right) + hW(x) \right),$$

and let $\langle V \rangle + \langle W \rangle = 0$, but $\langle W \rangle \langle V \rangle \neq 0$. Then the operator $\mathcal{H}^{\mu,h}$ has a unique eigenvalue $\lambda^{\mu,h}$ converging to zero. This eigenvalue is simple and has the asymptotic expansion

$$(1.3) \quad \lambda^{\mu,h} = -\frac{1}{4} (\mu^{-1}h)^4 \left(\int_{\operatorname{supp} W} \left(\int_p^x W(t) dt \right)^2 dx + q \langle W \rangle^2 \right)^2 \times (1 + \mathcal{O}(h + \mu^{-1}h)) \quad \text{if } 0 \notin \operatorname{supp} W,$$

where $q = \operatorname{dist}\{0, \operatorname{supp} W\}$, $p = \max_{x \in \operatorname{supp} W} |x|$, and

$$(1.4) \quad \lambda^{\mu,h} = -\frac{1}{4} (\mu^{-1}h)^4 \left(\int_{-\infty}^0 \left(\int_{-\infty}^x W(t) dt \right)^2 dx + \int_0^{\infty} \left(\int_x^{\infty} W(t) dt \right)^2 dx \right) \times (1 + \mathcal{O}(h + \mu^{-1}h)) \quad \text{if } 0 \in \operatorname{supp} W.$$

Comparing (1.3) and (1.4) with (0.8) and (0.11), we see that, in the case in question, the contribution of the narrow potential $\mu^{-1}V \left(\frac{x}{h} \right)$ to the leading term of the asymptotic expansion in question is as significant as that of the potential $\mu^{-1}hW(x)$.

Theorem 1.4. *Under condition (0.12), let*

$$\mathcal{H}^{\mu,h} = -\frac{d^2}{dx^2} + \mu^{-1} \left(V \left(\frac{x-x_1}{h} \right) - V \left(\frac{x-x_2}{h} \right) \right), \quad x_1 < x_2,$$

and suppose $\langle V \rangle \neq 0$. Then the operator $\mathcal{H}^{\mu,h}$ has a unique eigenvalue $\lambda^{\mu,h}$ converging to zero. This eigenvalue is simple and has the asymptotic expansion

$$(1.5) \quad \lambda^{\mu,h} = -\frac{1}{4} (\mu^{-1}h)^4 \left(\int_{x_1}^{x_2} \left(\int_{-\infty}^x V(t) dt \right)^2 dx \right)^2 (1 + \mathcal{O}(h + \mu^{-1}h)).$$

Formula (1.5) shows that, in this situation, the smallness order of the leading term of the eigenvalue differs from the case of a single narrow potential (see (0.11)) and is equal to that order in the case of a single small potential (see (0.8)).

§2. PRELIMINARY INFORMATION

In [7], the operator

$$(2.1) \quad \mathcal{H}_\varepsilon = -\frac{d^2}{dx^2} + \varepsilon \mathcal{L}_\varepsilon$$

was considered, where $0 < \varepsilon \ll 1$ is a small parameter and \mathcal{L}_ε is an arbitrary localized second order linear operator. Namely, there exists a bounded domain $Q \subset \mathbb{R}$ such that

$$(2.2) \quad \mathcal{L}_\varepsilon : W_{2,\text{loc}}^2(\mathbb{R}) \rightarrow L_2(\mathbb{R}; Q),$$

where

$$L_2(\mathbb{R}; Q) := \{u : u \in L_2(\mathbb{R}), \text{supp } u \subset \bar{Q}\}$$

and

$$(2.3) \quad \|\mathcal{L}_\varepsilon u\|_{L_2(\mathbb{R})} \leq C_1 \|u\|_{W_2^2(Q)},$$

where the constant C_1 does not depend on ε . In [6], it was proved that the operator \mathcal{H}_ε in $L_2(\mathbb{R})$ with domain $W_2^2(\mathbb{R})$ is closed and its essential spectrum coincides with the nonnegative real semiaxis.

For small complex k , we define an operator $A(k) : L_2(\mathbb{R}; Q) \rightarrow W_{2,\text{loc}}^2(\mathbb{R})$ as follows:

$$A(k)g := \frac{1}{2k} \int_{-\infty}^{\infty} e^{-k|x-t|} g(t) dt.$$

The definition of $A(k)$ implies that if $\mathcal{R}(\lambda)$ is the resolvent of the operator $-\frac{d^2}{dx^2}$ in $L_2(\mathbb{R})$, then $A(k) = \mathcal{R}(-k^2)$ on functions in $L_2(\mathbb{R}; Q)$ for $\text{Re } k > 0$. We denote by I the identity operator and by

$$\tilde{A}(k) : L_2(\mathbb{R}; Q) \rightarrow W_2^2(Q) \quad \text{and} \quad T_\varepsilon(k) : L_2(\mathbb{R}; Q) \rightarrow L_2(\mathbb{R}; Q)$$

the operators acting as follows:

$$(2.4) \quad \tilde{A}(k)g := A(k)g - \frac{1}{2k} \langle g \rangle,$$

$$(2.5) \quad T_\varepsilon(k)g := \mathcal{L}_\varepsilon \tilde{A}(k)g.$$

In [7], the following results were obtained.

Proposition 1. *For small k , the equation*

$$(2.6) \quad 2k + \varepsilon \mathcal{F}_\varepsilon(k) = 0,$$

where

$$\mathcal{F}_\varepsilon(k) = \langle (I + \varepsilon T_\varepsilon(k))^{-1} \mathcal{L}_{\varepsilon(\mu,h)}[1] \rangle,$$

has a unique solution k_ε .

If $\operatorname{Re} k_\varepsilon < 0$, then the operator \mathcal{H}_ε has no eigenvalues converging to zero.
 If $\operatorname{Re} k_\varepsilon > 0$, then the operator \mathcal{H}_ε has a unique eigenvalue λ_ε converging to zero, and

$$(2.7) \quad \lambda_\varepsilon = -k_\varepsilon^2.$$

The definition of the function $\mathcal{F}_\varepsilon(k)$ shows that, for every $N \geq 2$, we have

$$(2.8) \quad \mathcal{F}_\varepsilon(k) = \langle \mathcal{L}_\varepsilon[1] \rangle + \sum_{j=1}^{N-1} (-1)^j \varepsilon^j \langle T_\varepsilon^j(k) \mathcal{L}_\varepsilon[1] \rangle + \varepsilon^N \mathcal{F}_{\varepsilon,N}(k),$$

where $\mathcal{F}_{\varepsilon,N}(k)$ is a function holomorphic in k and uniformly bounded with respect to ε .

§3. REDUCTION OF THE OPERATOR $\mathcal{H}^{\mu,h}$ TO THE OPERATOR \mathcal{H}_ε

Lemma 3.1. *Suppose $-\infty < a < x_0 < b < \infty$ and $c_0 > 0$. Then*

$$\int_{x_0 - c_0 h}^{x_0 + c_0 h} |u|^2 dx \leq C_2 h \|u\|_{W_2^2(a,b)}^2$$

for all functions u of class $W_2^2(a,b)$, where the constant C_2 is independent of h .

Proof. Without loss of generality, we may assume that $[a, b] \subset (0, 1)$. We denote by $\dot{W}_2^1(0, 1)$ the completion of $C_0^\infty(0, 1)$ with respect to the norm $W_2^1(0, 1)$. In [8], it was proved that for all functions $v \in \dot{W}_2^1(0, 1)$ we have

$$(3.1) \quad \int_{x_0 - hc_0}^{x_0 + hc_0} |v(x)|^2 dx \leq C_3 h \|v\|_{W_2^1(0,1)}^2,$$

where the constant C_3 does not depend on h and v .

It is well known (see, e.g., [9]) that there exists a bounded linear extension operator $P : W_2^1(a, b) \rightarrow \dot{W}_2^1(0, 1)$ such that $u = Pu$ for $x \in (a, b)$ for every $u \in W_2^1(a, b)$. Therefore, by (3.1), we have

$$\begin{aligned} \int_{x_0 - hc_0}^{x_0 + hc_0} |u(x)|^2 dx &= \int_{x_0 - hc_0}^{x_0 + hc_0} |Pu(x)|^2 dx \leq C_3 h \|Pu\|_{W_2^1(0,1)}^2 \leq C_3 \|P\|^2 h \|u\|_{W_2^1(a,b)}^2 \\ &= C_2 h \|u\|_{W_2^1(a,b)}^2 \leq C_2 h \|u\|_{W_2^2(a,b)}^2 \end{aligned}$$

for every $u \in W_2^1(a, b)$. The lemma is proved. □

By \mathcal{V}_h , we denote the operator of multiplication by the function

$$\sum_{j=1}^n V_j \left(\frac{x - x_j}{h} \right) + hW(x).$$

Lemma 3.2. *Let Q be an arbitrary finite interval such that*

$$(3.2) \quad \operatorname{supp} W \cup \bigcup_{j=1}^n \{x_j\} \subset Q.$$

Then, for every $u \in W_{2,\text{loc}}^2(\mathbb{R})$ and all sufficiently small h , we have the estimate

$$(3.3) \quad \left\| h^{-\frac{1}{2}} \mathcal{V}_h u \right\|_{L_2(\mathbb{R})} \leq C_4 \|u\|_{W_2^2(Q)},$$

where C_4 is a constant independent of h .

Proof. Let $\mathcal{V}_h^{(j)}$ and \mathcal{W} denote the operators of multiplication by the functions $V_j\left(\frac{x-x_j}{h}\right)$ and $W(x)$, respectively. Obviously,

$$(3.4) \quad \|\mathcal{V}_h u\|_{L_2(\mathbb{R})} \leq \sum_{j=1}^n \|\mathcal{V}_h^{(j)} u\|_{L_2(\mathbb{R})} + h \|\mathcal{W}u\|_{L_2(\mathbb{R})}.$$

Let $c > 0$ be a number such that $\text{supp } V_j(t) \subset (-c, c)$ for every j . Then, by Lemma 3.1,

$$(3.5) \quad \begin{aligned} \|\mathcal{V}_h^{(j)} u\|_{L_2(\mathbb{R})}^2 &= \int_{x_j-hc}^{x_j+hc} \left| V_j\left(\frac{x-x_j}{h}\right) u(x) \right|^2 dx \\ &\leq \max_{t \in [-c, c]} |V_j(t)|^2 \int_{x_j-hc}^{x_j+hc} |u(x)|^2 dx \leq \tilde{C}_j h \|u\|_{W_2^2(Q)}^2. \end{aligned}$$

It is obvious that

$$(3.6) \quad \|\mathcal{W}u\|_{L_2(\mathbb{R})} = \|\mathcal{W}u\|_{L_2(Q)} \leq \tilde{C}_W \|u\|_{L_2(Q)} \leq \tilde{C}_W \|u\|_{W_2^2(Q)}.$$

Relations (3.4)–(3.6) lead to estimate (3.3). □

Lemma 3.3. *Suppose*

$$(3.7) \quad 0 < h \ll 1, \quad h = o(\mu^2),$$

and let Q satisfy condition (3.2),

$$(3.8) \quad \varepsilon(\mu, h) := \mu^{-1} h^{\frac{1}{2}}, \quad \mathcal{L}_{\varepsilon(\mu, h)} := h^{-\frac{1}{2}} \mathcal{V}_h.$$

Then

$$(3.9) \quad 0 < \varepsilon(\mu, h) \ll 1,$$

the operator $\mathcal{H}^{\mu, h}$ can be represented in the form (2.1), and relations (2.2) and (2.3) are valid.

Proof. Inequality (3.9) follows from (3.7) and (3.8). By (3.8), we have

$$\mathcal{H}^{\mu, h} = -\frac{d^2}{dx^2} + \mu^{-1} \mathcal{V}_h = -\frac{d^2}{dx^2} + \mu^{-1} h^{\frac{1}{2}} h^{-\frac{1}{2}} \mathcal{V}_h = -\frac{d^2}{dx^2} + \varepsilon(\mu, h) \mathcal{L}_{\varepsilon(\mu, h)}.$$

Consequently, the operator $\mathcal{H}^{\mu, h}$ can be represented in the form (2.1). From (3.8) and (3.3), it follows that

$$\|\mathcal{L}_{\varepsilon(\mu, h)} u\|_{L_2(\mathbb{R})} = \|h^{-\frac{1}{2}} \mathcal{V}_h u\|_{L_2(\mathbb{R})} \leq C_5 \|u\|_{W_2^2(Q)}.$$

In other words, inequality (2.3) is valid for the operator $\mathcal{L}_{\varepsilon(\mu, h)}$. Now, obviously, the definition of $\mathcal{L}_{\varepsilon(\mu, h)}$ implies relation (2.2). □

In the sequel, we assume that Q is an arbitrary finite interval satisfying (3.2).

§4. PROOF OF THEOREM 1.1

Using (2.5), (2.8), and Lemma 3.3, we get

$$(4.1) \quad \begin{aligned} \mathcal{F}_{\varepsilon(\mu, h)}(k) &= h^{-\frac{1}{2}} \left(\langle \mathcal{V}_h[1] \rangle + \sum_{j=1}^{N-1} (-1)^j \mu^{-j} \langle (\mathcal{V}_h \tilde{A}(k))^j \mathcal{V}_h[1] \rangle \right) \\ &\quad + (\mu^{-1} h^{\frac{1}{2}})^N \mathcal{F}_{\varepsilon(\mu, h), N}(k). \end{aligned}$$

We prove that the following estimate uniform in k is valid for small k :

$$(4.2) \quad \left| \langle (\mathcal{V}_h \tilde{A}(k))^j \mathcal{V}_h[1] \rangle \right| = \mathcal{O}(h^{j+1}), \quad j \geq 1.$$

The definition (2.4) of $\tilde{A}(k)$ shows that

$$\tilde{A}(k)g = \frac{1}{2k} \int_{-\infty}^{\infty} (e^{-k|x-y|} - 1)g(y) dy$$

for $g \in L_2(\mathbb{R}; Q)$. Expanding the function $e^{-k|x-y|}$ in a series in k , we obtain

$$(4.3) \quad \tilde{A}(k)g = \frac{1}{2} \sum_{i=1}^{\infty} (-1)^i \frac{k^{i-1}}{i!} \int_{-\infty}^{\infty} |x-y|^i g(y) dy, \quad x \in Q.$$

Consequently,

$$(4.4) \quad (\mathcal{V}_h \tilde{A}(k)) \mathcal{V}_h[1] = \frac{1}{2} h \mathcal{V}_h F_1,$$

where

$$(4.5) \quad \begin{aligned} F_1(x, h, k) &= \sum_{i=1}^{\infty} (-1)^i \frac{k^{i-1}}{i!} f_{1,i}(x, h), \\ f_{1,i}(x, h) &= \int_{-\infty}^{\infty} \left(\sum_{j=1}^n |x-x_j - hy|^i V_j(y) + |x-y|^i W(y) \right) dy. \end{aligned}$$

From the definition of $f_{1,i}$, it follows that for sufficiently small h we have

$$(4.6) \quad |f_{1,i}(x, h)| \leq C_5 M^i, \quad x \in \bar{Q},$$

where $M = 2R$ and R is a positive number such that the supports of the functions $V_j(x)$ and $W(x)$ and the points x_j lie in $(-R, R)$. Consequently, the series (4.5) converges uniformly for $x \in Q$ and small k and h , and

$$(4.7) \quad |\mathcal{V}_h F_1(x, h, k)| \leq C_6.$$

Therefore, by (4.4) and (4.5),

$$\left\langle (\mathcal{V}_h \tilde{A}(k)) \mathcal{V}_h[1] \right\rangle = \frac{1}{2} h \sum_{i=1}^{\infty} (-1)^i \frac{k^{i-1}}{i!} \langle \mathcal{V}_h f_{1,i}(x, h) \rangle.$$

Hence, by (4.6), we obtain

$$\begin{aligned} \left| \left\langle (\mathcal{V}_h \tilde{A}(k)) \mathcal{V}_h[1] \right\rangle \right| &\leq \frac{1}{2} h \sum_{i=1}^{\infty} \frac{|k|^{i-1}}{i!} \left| \langle \mathcal{V}_h f_{1,i}(x, h) \rangle \right| \\ &\leq \frac{1}{2} h \sum_{i=1}^{\infty} \frac{|k|^{i-1}}{i!} \langle |\mathcal{V}_h[1]| |f_{1,i}(x, h)| \rangle \leq h C_6 \langle |\mathcal{V}_h[1]| \rangle. \end{aligned}$$

Next,

$$\begin{aligned} \langle |\mathcal{V}_h[1]| \rangle &= \int_{-\infty}^{\infty} \left| \sum_{j=1}^n V_j \left(\frac{y-x_j}{h} \right) + hW(y) \right| dy \\ &\leq h \sum_{j=1}^n \int_{-\infty}^{\infty} |V_j(y)| dy + h \int_{-\infty}^{\infty} |W(y)| dy \leq C_7 h. \end{aligned}$$

The last two inequalities combine to yield

$$(4.8) \quad \left| \left\langle (\mathcal{V}_h \tilde{A}(k)) \mathcal{V}_h[1] \right\rangle \right| \leq C_7 h^2.$$

We have proved (4.2) for $j = 1$.

For $j = 2$, we use the representation

$$(\mathcal{V}_h \tilde{A}(k))^2 \mathcal{V}_h[1] = \frac{1}{2} h \mathcal{V}_h \tilde{A}(k) \mathcal{V}_h F_1 = \frac{1}{4} h^2 \mathcal{V}_h F_2,$$

where

$$F_2(x, h, k) = \sum_{i=1}^{\infty} (-1)^i \frac{k^{i-1}}{i!} f_{2,i}(x, h, k),$$

$$f_{2,i}(x, h, k) = \int_{-\infty}^{\infty} \left(\sum_{j=1}^n |x - x_j - hy|^i V_j(y) F_1(x_j + hy, h, k) + |x - y|^i W(y) F_1(y, h, k) \right) dy.$$

Taking (4.7) into account and repeating the above estimates, we see that (4.2) is true for $j = 2$.

The further proof of (4.2) proceeds by an easy induction.

Using (4.1), (3.8), (4.2), (0.12), and the fact that N is arbitrary, we get

$$(4.9) \quad \varepsilon(\mu, h) \mathcal{F}_{\varepsilon(\mu, h)}(k) = \mu^{-1} \langle \mathcal{V}_h[1] \rangle + \mu^{-2} h^2 \mathcal{G}(k, h, \mu),$$

where $\mathcal{G}(k, h, \mu)$ is a function holomorphic in k and uniformly bounded in μ and h .

By the definition of \mathcal{V}_h ,

$$(4.10) \quad \begin{aligned} \langle \mathcal{V}_h[1] \rangle &= \int_{-\infty}^{\infty} \left(\sum_{j=1}^n V_j \left(\frac{x - x_j}{h} \right) + hW(x) \right) dx \\ &= h \sum_{j=1}^n \int_{-\infty}^{\infty} V_j(t) dt + h \int_{-\infty}^{\infty} W(x) dx = h \varkappa_{\mathbf{V}, W}. \end{aligned}$$

Relations (4.9) and (4.10) show that equation (2.6) takes the form

$$k = -\frac{h}{2\mu} (\varkappa_{\mathbf{V}, W} + \mu^{-1} h \mathcal{G}(k, h, \mu)).$$

Combining this with Proposition 1, we obtain Theorem 1.1.

§5. PROOF OF THEOREMS 1.2–1.4

First, we derive a general formula for the eigenfrequency in the critical case where

$$(5.1) \quad \varkappa_{\mathbf{V}, W} = \langle \mathcal{V}_h[1] \rangle = 0.$$

(By an eigenfrequency, we mean a solution of equation (2.6), in the notation (3.8).)

From (4.1), (5.1), (4.2), (3.8), (0.12), and the fact that N is arbitrary, it follows that

$$(5.2) \quad \varepsilon(\mu, h) \mathcal{F}_{\varepsilon(\mu, h)}(k) = -\mu^{-2} \langle (\mathcal{V}_h \tilde{A}(k)) (\mathcal{V}_h[1]) \rangle + \mu^{-3} h^3 \mathcal{G}(k, h, \mu),$$

where $\mathcal{G}(k, h, \mu)$ is a function holomorphic in k and uniformly bounded in μ and h .

First, we prove that

$$(5.3) \quad \langle (\mathcal{V}_h \tilde{A}(k)) (\mathcal{V}_h[1]) \rangle = -\frac{1}{2} \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^{\infty} |x - y| \mathcal{V}_h[1](y) dy \right) dx + h^2 k \mathcal{G}_1(k, h),$$

where $\mathcal{G}_1(k, h)$ is a function holomorphic in k and uniformly bounded in h . Using (4.3), we immediately obtain the equation

$$(5.4) \quad \langle (\mathcal{V}_h \tilde{A}(k)) (\mathcal{V}_h[1]) \rangle = -\frac{1}{2} \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^{\infty} |x - y| \mathcal{V}_h[1](y) dy \right) dx + k \tilde{\mathcal{G}}_1(k, h),$$

where $\tilde{\mathcal{G}}_1(k, h)$ is a function holomorphic in k and uniformly bounded in h that is defined by the equation

$$\tilde{\mathcal{G}}_1(k, h) = \frac{1}{2} \sum_{i=2}^{\infty} (-1)^i \frac{k^{i-2}}{i!} \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^{\infty} |x - y|^i \mathcal{V}_h[1](y) dy \right) dx.$$

Similarly to (4.8), it can easily be proved that

$$(5.5) \quad |\tilde{\mathcal{G}}_1(k, h)| \leq C_8 h^2.$$

Now, (5.4) and (5.5) lead to (5.3).

We prove that, in the critical case (5.1) in question, the following identity is valid:

$$(5.6) \quad \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^{\infty} |x-y| \mathcal{V}_h[1](y) dy \right) dx = -2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^x \mathcal{V}_h[1](t) dt \right)^2 dx.$$

Indeed,

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^{\infty} |x-y| \mathcal{V}_h[1](y) dy \right) dx \\ &= \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^x (x-y) \mathcal{V}_h[1](y) dy \right) dx + \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_x^{\infty} (y-x) \mathcal{V}_h[1](y) dy \right) dx. \end{aligned}$$

We change the order of integration in the second integral on the right:

$$\int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_x^{\infty} (y-x) \mathcal{V}_h[1](y) dy \right) dx = \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^y (y-x) \mathcal{V}_h[1](x) dx \right) dy.$$

The last two identities imply

$$\int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^{\infty} |x-y| \mathcal{V}_h[1](y) dy \right) dx = 2 \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^x (x-y) \mathcal{V}_h[1](y) dy \right) dx.$$

Furthermore, integration by parts yields

$$\int_{-\infty}^x (x-y) \mathcal{V}_h[1](y) dy = \int_{-\infty}^x \int_{-\infty}^y \mathcal{V}_h[1](t) dt dy.$$

From the last two equations, we obtain

$$(5.7) \quad \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^{\infty} |x-y| \mathcal{V}_h[1](y) dy \right) dx = 2 \int_{-\infty}^{\infty} U'(x) \int_{-\infty}^x U(y) dy dx,$$

where

$$U(z) = \int_{-\infty}^z \mathcal{V}_h[1](t) dt.$$

Integrating the right-hand side of identity (5.7) by parts and taking into account the fact that $\langle \mathcal{V}_h[1] \rangle = 0$, we obtain

$$(5.8) \quad \int_{-\infty}^{\infty} \mathcal{V}_h \left(\int_{-\infty}^x \int_{-\infty}^y \mathcal{V}_h[1](t) dt dy \right) dx = - \int_{-\infty}^{\infty} \left(\int_{-\infty}^x \mathcal{V}_h[1](t) dt \right)^2 dx.$$

The last two identities result in (5.6).

By (5.2), (5.3), and (5.6), we have

$$(5.9) \quad \begin{aligned} \varepsilon(\mu, h) \mathcal{F}_{\varepsilon(\mu, h)}(k) &= -\mu^{-2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^x \mathcal{V}_h[1](t) dt \right)^2 dx \\ &\quad - \mu^{-2} h^2 k \mathcal{G}_1(k, h) + \mu^{-3} h^3 \mathcal{G}(k, h, \mu). \end{aligned}$$

Using the definition of \mathcal{V}_h and the fact that

$$\int_{-\infty}^x V_j \left(\frac{t-x_j}{h} \right) dt = h \int_{-\infty}^{(x-x_j)h^{-1}} V_j(t) dt,$$

we represent (5.9) in the form

$$(5.10) \quad \varepsilon(\mu, h) \mathcal{F}_{\varepsilon(\mu, h)}(k) = -\mu^{-2} h^2 (\tilde{\mathcal{Z}}_{\mathbf{V}, W} + k \mathcal{G}_1(k, h) - \mu^{-1} h \mathcal{G}(k, h, \mu)),$$

where

$$(5.11) \quad \tilde{\mathcal{Z}}_{\mathbf{V},W} := \int_{-\infty}^{\infty} \left(\sum_{j=1}^n \int_{-\infty}^{(x-x_j)h^{-1}} V_j(t) dt + \int_{-\infty}^x W(t) dt \right)^2 dx.$$

From (5.10), it follows that (2.6) takes the form

$$k = \frac{h^2}{2\mu^2} (\tilde{\mathcal{Z}}_{\mathbf{V},W} + k\mathcal{G}_1(k, h) - \mu^{-1}h\mathcal{G}(k, h, \mu)).$$

Hence,

$$(5.12) \quad k_{\varepsilon(\mu,h)} = \frac{h^2}{2\mu^2} \tilde{\mathcal{Z}}_{\mathbf{V},W} + \mathcal{O}(\mu^{-3}h^3).$$

Proof of Theorem 1.2. In the case in question, we have

$$\langle W \rangle = 0, \quad \langle V_j \rangle = 0, \quad j = 1, \dots, n.$$

Obviously, that there exist positive numbers c_j such that $\text{supp } V_j(x) \subset (-c_j, c_j)$. Therefore, for small h we have

$$(5.13) \quad \begin{aligned} & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{(x-x_j)h^{-1}} V_j(t) dt \right)^2 dx = \int_{x_j-hc_j}^{x_j+hc_j} \left(\int_{-\infty}^{(x-x_j)h^{-1}} V_j(t) dt \right)^2 dx = \mathcal{O}(h), \\ & \int_{-\infty}^{\infty} \int_{-\infty}^{(x-x_j)h^{-1}} V_j(t) dt \int_p^x W(t) dt dx = \mathcal{O}(h), \\ & \int_{-\infty}^{\infty} \int_{-\infty}^{(x-x_i)h^{-1}} V_i(t) dt \int_{-\infty}^{(x-x_j)h^{-1}} V_j(t) dt dx = 0, \quad i \neq j. \end{aligned}$$

By (5.11)–(5.13), it follows that

$$k_{\varepsilon} = \frac{1}{2}\mu^{-2}h^2 \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^x W(t) dt \right)^2 dx \right) + \mathcal{O}(\mu^{-2}h^3 + \mu^{-3}h^3),$$

and it remains to refer to Proposition 1. Theorem 1.2 is proved. □

Proof of Theorem 1.3. In the case in question, we have

$$\mathcal{V}_h[1] = V\left(\frac{x}{h}\right) + hW(x), \quad \text{supp } V \subset (-c_0, c_0), \quad \text{supp } W = [p_1, p_2],$$

and formula (5.11) takes the form

$$\tilde{\mathcal{Z}}_{\mathbf{V},W} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{xh^{-1}} V(t) dt + \int_{-\infty}^x W(t) dt \right)^2 dx.$$

First, we consider the case where $p_2 < 0$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{xh^{-1}} V(t) dt + \int_{-\infty}^x W(t) dt \right)^2 dx \\ &= \int_{p_1}^{p_2} \left(\int_{p_1}^x W(t) dt \right)^2 dx - (p_2 + hc_0)\langle W \rangle^2 + \int_{-c_0h}^{c_0h} \left(\int_{-\infty}^{xh^{-1}} V(t) dt + \langle W \rangle \right)^2 dx \\ &= \int_{\text{supp } W} \left(\int_p^x W(t) dt \right)^2 dx + q\langle W \rangle^2 + \mathcal{O}(h). \end{aligned}$$

Using (5.11), (5.12), and the above identity, we get

$$k_{\varepsilon} = \frac{1}{2}\mu^{-2}h^2 \left(\int_{\text{supp } W} \left(\int_p^x W(t) dt \right)^2 dx + q\langle W \rangle^2 \right) + \mathcal{O}(\mu^{-2}h^3 + \mu^{-3}h^3).$$

In turn, the latter identity and Proposition 1 imply the claim of the theorem in the case where $p_2 < 0$.

The case of $p_1 > 0$ reduces to the above by the change of x by $-x$.

Now, we pass to the case where $0 \in \text{supp } W$. Since $\langle V \rangle = -\langle W \rangle$, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\frac{x}{h}} V(t) dt + \int_{-\infty}^x W(t) dt \right)^2 dx \\ &= \int_{-\infty}^{-c_0 h} \left(\int_{-\infty}^x W(t) dt \right)^2 dx + \int_{-c_0 h}^{c_0 h} \left(\int_{-\infty}^{x h^{-1}} V(t) dt + \int_{-\infty}^x W(t) dt \right)^2 dx \\ &+ \int_{c_0 h}^{\infty} \left(\langle V \rangle + \int_{-\infty}^x W(t) dt \right)^2 dx \\ &= \int_{-\infty}^0 \left(\int_{-\infty}^x W(t) dt \right)^2 dx + \int_0^{\infty} \left(\int_x^{\infty} W(t) dt \right)^2 dx + \mathcal{O}(h). \end{aligned}$$

Now, (5.11) and (5.12) show that

$$\begin{aligned} k_\varepsilon &= \frac{1}{2} \mu^{-2} h^2 \left(\int_{-\infty}^0 \left(\int_{-\infty}^x W(t) dt \right)^2 dx + \int_0^{\infty} \left(\int_x^{\infty} W(t) dt \right)^2 dx \right) \\ &+ \mathcal{O}(\mu^{-2} h^3 + \mu^{-3} h^3). \end{aligned}$$

Combining this with Proposition 1, we see that the theorem is valid in the case where $0 \in \text{supp } W$. Theorem 1.3 is proved completely. \square

Proof of Theorem 1.4. In the case under consideration, formula (5.11) takes the form

$$(5.14) \quad \tilde{\chi}_{\mathbf{V}, W} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{(x-x_1)h^{-1}} V(t) dt - \int_{-\infty}^{(x-x_2)h^{-1}} V(t) dt \right)^2 dx.$$

Since

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{(x-x_1)h^{-1}} V(t) dt - \int_{-\infty}^{(x-x_2)h^{-1}} V(t) dt \right)^2 dx = \int_{x_1}^{x_2} \left(\int_{-\infty}^x V(t) dt \right)^2 dx + \mathcal{O}(h),$$

we have

$$k_\varepsilon = \frac{1}{2} \mu^{-2} h^2 \left(\int_{x_1}^{x_2} \left(\int_{-\infty}^x V(t) dt \right)^2 dx \right) + \mathcal{O}(\mu^{-2} h^3 + \mu^{-3} h^3)$$

by (5.14) and (5.12). Applying Proposition 1, we complete the proof. \square

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Received 11/JUL/2010

Translated by B. M. BEKKER