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ASYMPTOTIC SOLUTIONS OF THE TWO-DIMENSIONAL MODEL WAVE EQUATION WITH DEGENERATING VELOCITY AND LOCALIZED INITIAL DATA

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Abstract. The Cauchy problem is considered for the two-dimensional wave equation with velocity $c = \sqrt{x_1}$ on the half-plane $\{x_1 \geq 0, x_2\}$, with initial data localized in a neighborhood of the point $(1, 0)$. This problem serves as a model problem in the theory of beach run-up of long small-amplitude surface waves excited by a spatially localized instantaneous source. The asymptotic expansion of the solution is constructed with respect to a small parameter equal to the ratio of the source linear size to the distance from the x_2-axis (the shoreline). The construction involves Maslov's canonical operator modified to cover the case of localized initial conditions. The relationship of the solution with the geometrical optics ray diagram corresponding to the problem is analyzed. The behavior of the solution near the x_2-axis is studied. Simple solution formulas are written out for special initial data.

Introduction

Consider the Cauchy problem for the degenerate wave equation

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(c^2(x) \operatorname{grad} u) = 0, \quad c = \sqrt{x_1}, \tag{0.1}$$

in the half-plane $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ with the initial conditions

$$u\big|_{t=0} = u_0(x), \quad u_t\big|_{t=0} = u_1(x) \tag{0.2}$$

of the form

$$u_0(x) = V\Big(\frac{x - a}{\mu}\Big), \quad u_1(x) = 0. \tag{0.3}$$

Here $V \in C_0^\infty(\mathbb{R}^2)$, μ is a small positive parameter, and $a \in \mathbb{R}^2_+$ is the point near which the initial perturbation is localized. In the following, we assume throughout that $a = (1, 0)$, which can always be achieved by a linear change of variables.

Problem (0.1)–(0.3) with a smooth nonvanishing velocity $c(x)$ arises in the theory of long water waves and describes the propagation of waves generated by a shock upheaval of a relatively small area of the sea bottom (the piston model of tsunami waves [1]–[3]). Then $u = u(x, t)$, $t \geq 0$, $x \in \mathbb{R}^2$, is the free surface elevation, $u_0(x)$ is the initial

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perturbation (corresponding to the bottom upheaval), and the velocity $c(x)$ depends on the slowly varying ocean depth $D(x)$ as $c(x) = \sqrt{gD(x)}$, where g is the acceleration due to gravity.

Efficient formulas for the asymptotic solution of this problem have been constructed quite recently in [4, 5] on the basis of Maslov's canonical operator [6] modified so as to employ a special Lagrangian manifold for an integral representation of the initial data [4, 7]. The solution structure is as to be expected: at the initial time, the solution is localized in a neighborhood of a point (see Figure 2), and then it becomes localized in a neighborhood of a time-dependent closed curve γ_t, referred to as the *solution front*. This front is defined as follows. Consider the trajectories of the two-degrees-of-freedom Hamiltonian system with Hamiltonian $H(p,x) = |p|c(x)$ in the phase space with coordinates $(p,x) = (p_1,p_2,x_1,x_2)$. The endpoints of the x-components of these trajectories form the front γ_t. At the beginning, γ_t is a smooth curve (almost a circle), and then it may exhibit turning points (focal points), self-intersection points, etc., because c is nonconstant. However, wave propagation can be described in this manner only in mid-ocean (far from the coast). (Note that our paper is purely mathematical; we use some terminology of fluid-wave theory solely for pictorial clarity.) When using a similar model for describing beach run-up (e.g., of tsunami waves) [3], the wave equation in (0.1) is only given in the domain bounded by the shoreline (rather than in the entire space $\mathbb{R}^2$), and the velocity vanishes on the boundary, so that the wave equation degenerates. Once the front reaches the shoreline, the asymptotic solutions constructed in [4, 5] make no sense, and one can ask how the solution behaves thereafter. Here, before trying to compute any asymptotic solution, one should refine the very statement of the problem. The question is, what (physically meaningful) conditions are needed on the boundary to make the problem well posed? Needless to say, the answer depends on how exactly the equation degenerates. In general, this question has been discussed by specialists in water wave theory, and it also proves quite reasonable in the one-dimensional case.

However valuable physical considerations may be (e.g., see [2]), we present a complete mathematical argument, which will come in handy when we subsequently define Maslov's canonical operator in our framework. What is important to us is an idea in the paper [8] (where further bibliography can be found), which deals with rapidly oscillating solutions in the one-dimensional case ($x \in [0,\infty)$) under the assumption that $c(x)/\sqrt{x}$ is a smooth positive function. That paper indicates the possibility of obtaining a well-posed problem by requiring the solution to have a finite energy integral. Note also that the corresponding one-dimensional beach run-up problem, even in the nonlinear setting, has been studied much better than the two-dimensional problem (see [3, 11, 14] and the bibliography therein).

In the present paper, we restrict ourselves to a two-dimensional problem in which the shoreline is straight and the depth is proportional to the distance from the shore, which results in (0.1)–(0.3). (It can always be ensured that $c(x) = \sqrt{x_1}$ by scaling the variables.) The results can be explained qualitatively with the use of Figures 1 and 2.

The solution localized for $t = 0$ in a neighborhood of the point $(1,0)$ (Figure 2, left) becomes, after some time, localized in a neighborhood of fronts varying in time in accordance with the Hamiltonian system with Hamiltonian $H(p,x) = |p|c(x)$ and given by formulas (2.3). For $t < 2$, the asymptotic expansion of the solution is given by the formulas in [4]. At time $t = 2$, the front touches the x_2-axis (the shoreline) and is reflected from it, as is the wave itself (Figure 1, middle). The wave amplitude in a neighborhood of the point of tangency proves to be bounded, but it is of larger order of magnitude as $\mu \to 0$ than the amplitude at the regular points of the front. Then the point of tangency splits into two, which move along the x_2-axis symmetrically, one up and one

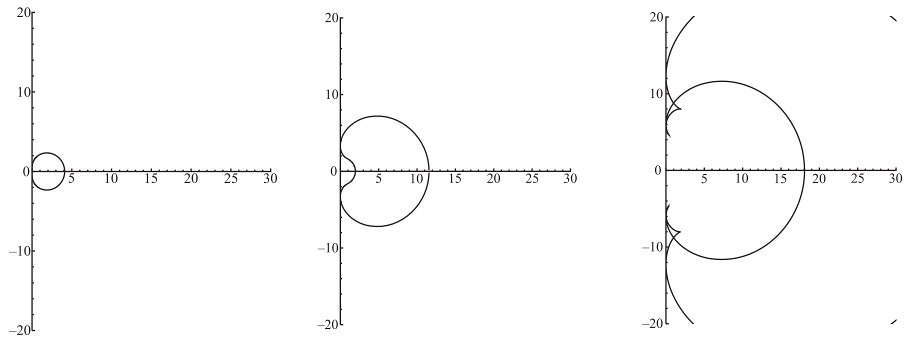

FIGURE 1. The fronts γ_t for various t. The formulas given in §4 correspond to the figure in the middle.

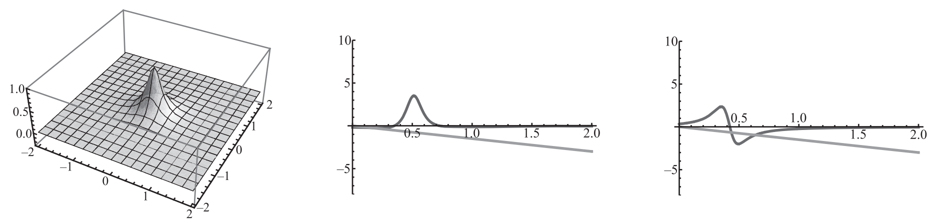

FIGURE 2. The solution at various t. The left figure corresponds to $t = 0$, while the middle and right figures describe the wave profiles before and after the reflection in the x_2-axis, respectively.

down (Figure 2, middle), and the corresponding components of the momentum at these points turn out to be unbounded. The front arc enclosed between these points consists of the points reflected by the x_2-axis, and the wave profile in a neighborhood of these points differs from that in a neighborhood of the points not yet reflected, because the reflection in the x_2-axis produces a jump in the Morse index. In this sense, the x_2-axis can be viewed as a space-time caustic [12], and the change in the form of the asymptotic behavior can be treated as an analog of the "metamorphosis of discontinuity" in linear hyperbolic systems [13]. The difference is that, in our case, the wave profile "remembers" the shape of the original source and also depends on the front points. If the incident wave profile has the form of a "cap", then the reflected wave has the shape of an "N"-wave (Figure 2, middle and right). (Here we use the terminology in [3, 11], where this process was studied in the one-dimensional case.) After that, the trajectories "carry the front points away into deep waters", where the reflection occurs again, this time resulting in "standard" space-time caustics. (The asymptotic behavior in a neighborhood of these caustics was described in [5]; see also [15].) Then the trajectories reach the x_2-axis, are again reflected (Figure 1, right), etc.

We point out that this description of the front is related to the right choice of the boundary condition for equation (0.1) on the x_2-axis, which leads to a reasonable method for constructing the asymptotic expansion of the solution in a neighborhood of the x_2-axis on the basis of a modification of Maslov's canonical operator. (This is one of the main results of the paper.)

In §1, we describe the domain of the spatial part of our wave operator, prove that the Cauchy problem is well posed, and formulate the energy estimates. The case of localized initial data is considered in Subsection 1.3, where we outline the method of solution of this problem, which is implemented in Subsections 1.2 and 1.3. As in [4], our method involves Maslov's canonical operator. In particular, we represent the localized initial data in the form of an integral with respect to a parameter, by analogy with [4, 14]. Because of degeneration at the boundary, the trajectories of the Hamiltonian vector field go to infinity in the momentum direction in finite time. Hence, the formulas for the asymptotic solution inevitably involve the canonical operator applied to functions with noncompact support, which renders the standard definition of the canonical operator useless. A definition adequate to our aims is given in §2; geometrically, it is based on a partial compactification of the Lagrangian manifold by adding some points at infinity of the phase space. This results in a new type of charts on the Lagrangian manifold; the canonical operator in these charts is defined by Cauchy principal value integrals (cf. [9, 10]). The asymptotic expansion of the solution is constructed in §3, and §4 provides an example.

In the present paper, we only deal with the mathematical aspects of the problem and say nothing concerning the extent to which the model applies to the original physical problem. The case of more general shorelines and coordinate dependences of the depth requires additional studies.

§1. Cauchy problem for the wave equation with degenerating velocity

Equation (0.1) and the initial conditions (0.2) do not form a well-posed problem by themselves because of the boundary at $x_1 = 0$, where the equation degenerates. To obtain a well-posed problem, one should additionally specify the domain of the spatial part of the wave operator occurring in (0.1). This is done below. Note that this description can be interpreted as the finiteness of the energy integral [16] for the solutions in question and that problem (0.1)–(0.2) proves to be well posed in the corresponding energy spaces.

1.1. The operator L_0 and the selfadjoint extension L. Let $C_0^\infty(\bar{\mathbb{R}}^2_+)$ be the space of smooth functions compactly supported in the closed half-plane $\bar{\mathbb{R}}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$. (Unlike the elements of $C_0^\infty(\mathbb{R}^2_+)$, such functions are not necessarily zero in a neighborhood of the boundary of the half-plane.)

Proposition 1. *The operator L_0 defined in the Hilbert space $L_2(\mathbb{R}^2_+)$ by the differential expression $\ell = -\frac{\partial}{\partial x_1} x_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial^2}{\partial x_2^2}$ on the domain $D_0 = C_0^\infty(\bar{\mathbb{R}}^2_+)$ is positive and essentially selfadjoint.*

Proof. **1.** Let $u, v \in D_0$. Elementary integration by parts shows that (the integrated terms at $x_1 = 0$ are zero)

$$(u, \ell v) = (\ell u, v), \quad (u, \ell u) = \int_{\mathbb{R}^2_+} x_1 \Big(\Big| \frac{\partial u}{\partial x_1} \Big|^2 + \Big| \frac{\partial u}{\partial x_2} \Big|^2 \Big) \, dx_1 dx_2 \geq 0,$$

where $(\,\cdot\,,\,\cdot\,)$ is the inner product in $L_2(\mathbb{R}^2_+)$. Hence, L_0 is symmetric and positive.

2. We show that the deficiency indices $d_\pm(L_0)$ of L_0 are zero. Since L_0 is positive, it follows that $d_+(L_0) = d_-(L_0) = \dim\ker(L_0^* + 1)$, where L_0^* is the adjoint operator of L_0. Thus, it suffices to show that the equation $(L_0^*+1)v = 0$ does not have nontrivial solutions. Let $v = v(x_1, x_2)$ be a solution of this equation. Then, in particular, $(\ell + 1)v = 0$ in the sense of distributions in $\mathbb{R}^2_+$. It follows that the Fourier transform $w = w(x_1, p_2)$ of v with respect to x_2 satisfies the equation

$$x_1 \frac{d^2 w}{dx_1^2} + \frac{dw}{dx_1} - x_1 p_2^2 w - w = 0.$$

The general solution of this equation has the form (see [17])

$$w(x_1, p_2) = e^{-|p_2|x_1} U\Big(\frac{|p_2|+1}{2|p_2|}, 1, 2|p_2|x_1\Big),$$

where $U(\alpha, \beta, z)$ is an arbitrary solution of the confluent hypergeometric equation $zU''_{zz} + (\beta - z)U'_z - \alpha U = 0$. For $\beta = 1$, the latter has the fundamental system of solutions formed by the Kummer function $\Phi(\alpha, 1, z)$ and the Tricomi function $\Psi(\alpha, 1, z)$ (see [17, 6.1(1) and 6.5(5)]). Thus,

$$w(x_1, p_2) = e^{-|p_2|x_1} \Big[c_1(p_2)\Phi\Big(\frac{|p_2|+1}{2|p_2|}, 1, 2|p_2|x_1\Big) + c_2(p_2)\Psi\Big(\frac{|p_2|+1}{2|p_2|}, 1, 2|p_2|x_1\Big)\Big].$$

The asymptotic behavior as $z \to +\infty$ of the Kummer and Tricomi functions is as follows (see [17, 6.13(1) and 6.13(2)]): $\Phi(\alpha, 1, z) = \frac{1}{\Gamma(\alpha)} e^z z^\alpha (1 + O(z^{-1}))$; $\Psi(\alpha, 1, z) = z^{-\alpha}(1 + O(z^{-1}))$. Thus, $c_1(p_2) = 0$; otherwise, w (and hence v) would not lie in $L_2(\mathbb{R}^2_+)$. Next, from [17, 6.7(13)] it follows that the Tricomi function can be represented in the form $\Psi(\alpha, 1, z) = f_1(z)\ln z + f_2(z)$, where $f_1(z)$ and $f_2(z)$ are entire functions, $f_1(z) = \frac{\Phi(\alpha,1,z)}{\Gamma(\alpha)}$, and $f_1(0) = \frac{1}{\Gamma(\alpha)}$. Now an easy computation shows that if $u \in D_0$, then

$$(L_0 u, v) \equiv (\ell u, v) = \int_{-\infty}^{\infty} \frac{\overline{c_2(p_2)}\widetilde{u}(0, p_2)}{\Gamma\big(\frac{|p_2|+1}{2|p_2|}\big)}\, dp_2 + (u, \varphi),$$

where $\widetilde{u}(x_1, p_2)$ is the Fourier transform of u with respect to the second variable, the bar stands for complex conjugation, and φ is a continuous function. Since $v \in D(L_0^*)$, it follows that this expression is a continuous functional of u in the L^2-norm, which is possible only if $c_2(p_2) \equiv 0$. Thus, $v = 0$, and the proof of the proposition is complete. □

Let L be the closure of L_0, and let $D = D(L)$ be the domain of L. By Proposition 1, $L = L^* = L_0^*$. We want to give a convenient description of L and $D(L)$. Consider the differential expression

$$a\,:\,u \longmapsto -i\begin{pmatrix}\sqrt{x_1}\frac{\partial u}{\partial x_1}\\ \sqrt{x_1}\frac{\partial u}{\partial x_2}\end{pmatrix}$$

and the formally adjoint expression

$$a^*\,:\,\begin{pmatrix}u\\ v\end{pmatrix} \longmapsto -i\Big[\frac{\partial(\sqrt{x_1}u)}{\partial x_1} + \frac{\partial(\sqrt{x_1}v)}{\partial x_2}\Big].$$

Then $\ell = a^*a$, and integration by parts shows that

$$(u, \ell v) = (\ell u, v) = (au, av), \quad u, v \in D_0. \tag{1.1}$$

Let A_0 be the operator defined by a on D_0, and let A be the closure of A_0. (The closability of A_0 is obvious, because the domain of the adjoint operator A_0^* contains the set $C_0^\infty(\mathbb{R}^2_+) \oplus C_0^\infty(\mathbb{R}^2_+)$, which is dense in $L_2(\mathbb{R}^2_+) \oplus L_2(\mathbb{R}^2_+)$ and on which A_0^* is given by the differential expression a^*.)

Proposition 2. *We have $L = A^*A$. The domain D of L coincides with the set of elements $u \in L_2(\mathbb{R}^2_+)$ such that*

$$\ell u \in L_2(\mathbb{R}^2_+), \quad au \in L_2(\mathbb{R}^2_+) \oplus L_2(\mathbb{R}^2_+), \tag{1.2}$$

where the derivatives are understood in the sense of distributions in $\mathbb{R}^2_+$.

Proof. **1.** We prove that $L = A^*A$. Let $u \in D$. By definition, there exists a sequence $u_n \in D_0$, $n = 0, 1, 2, \ldots$, such that $u_n \to u$ and $\ell u_n \to v = Lu$ in $L_2(\mathbb{R}^2_+)$ as $n \to \infty$. From (1.1) it follows that $\|au_n\|^2 = (u_n, \ell u_n) \to (u, Lu)$ as $n \to \infty$; in particular, the sequence au_n is bounded. Hence, it has a subsequence (which we again denote by au_n for brevity) weakly convergent to some element $w \in L_2(\mathbb{R}^2_+) \oplus L_2(\mathbb{R}^2_+)$, $au_n \xrightarrow{w} w$. Moreover, obviously we have

$$\|w\| \le \lim_{n\to\infty} \|au_n\| = \sqrt{(u, Lu)}. \tag{1.3}$$

Next, again by (1.1), $(au_n, au_k) = (u_n, \ell u_k) \to (u, Lu)$ as $n, k \to \infty$, so that for each $\varepsilon > 0$ there exists n_ε with

$$|(au_n, au_k)| \ge (u, Lu) - \varepsilon \quad \text{for } n, k > n_\varepsilon. \tag{1.4}$$

Take $k > n_\varepsilon$ and let $n \to \infty$ in (1.4). Since $au_n \xrightarrow{w} w$, we obtain $|(w, au_k)| \ge (u, Lu) - \varepsilon$ for $k > n_\varepsilon$. Next, let $k \to \infty$. We obtain $\|w\|^2 \ge (u, Lu) - \varepsilon$, whence $\|w\|^2 \ge (u, Lu)$, because ε is arbitrary. By combining this with inequality (1.3), we obtain $\|w\| = \sqrt{(u, Lu)} = \lim_{n\to\infty} \|au_n\|$, which means that the sequence au_n converges not only weakly but also strongly, $au_n \to w$.

Thus, we have shown that $u \in D(A)$ and $Au = w$. Now we prove that $w \in D(A^*)$ and $A^* w = Lu$. It suffices to show that $(w, a\xi) = (Lu, \xi)$, $\xi \in D_0$, because A is obtained from a by closure from D_0. But this is obvious: $(w, a\xi) = \lim_{n\to\infty}(au_n, a\xi) = \lim_{n\to\infty}(\ell u_n, \xi) = (Lu, \xi)$. Hence, $Lu = A^*Au$ for $u \in D$; i.e., $L \subset A^*A$. Since both operators are selfadjoint, we have $L = A^*A$, and the proof of the first claim is complete.

2. Now we prove the claim about the domain of L. First, let $u \in D$. Then, as has already been shown above, $u \in D(A)$ and relations (1.2) are valid. (This is a special case of the general fact that if P is a differential operator in L_2 and $Pu = v$ in L_2, then $Pu = v$ in the sense of distributions as well, because the set of smooth compactly supported functions, which is dense in L_2, is contained in the domain of the adjoint operator.)

Now let $u \in L_2(\mathbb{R}^2_+)$ satisfy (1.2). We claim that $u \in D$. Since $L = L^*$, it suffices to show that the functional $v \mapsto (u, \ell v)$, $v \in D_0$, is bounded. For this, we prove that conditions (1.2) imply the relation $(u, \ell v) = (\ell u, v)$, where ℓu is understood in the sense of distributions in $\mathbb{R}^2_+$. (Thus, in particular, $Lu = \ell u$.) First, we note that $(u, \ell v) = (u, a^*av)$. Next, let $\varphi(\tau)$ be a smooth function such that $\varphi(\tau) = 0$ for $\tau \le 1$ and $\varphi(\tau) = 1$ for $\tau \ge 2$. Then

$$\begin{aligned}
&\varphi(nx_1)av = \varphi(nx_1)\begin{pmatrix} \sqrt{x_1}\frac{\partial v}{\partial x_1} \\ \sqrt{x_1}\frac{\partial v}{\partial x_2} \end{pmatrix} \in C_0^\infty(\mathbb{R}^2_+) \oplus C_0^\infty(\mathbb{R}^2_+), \\
&\varphi(nx_1)av \to av \ \text{ in } \ L_2(\mathbb{R}^2_+) \oplus L_2(\mathbb{R}^2_+) \ \text{ as } \ n \to \infty, \\
a^*(\varphi(nx_1)av) &= \frac{\partial}{\partial x_1}\Big(x_1\varphi(nx_1)\frac{\partial v}{\partial x_1}\Big) + \frac{\partial}{\partial x_2}\Big(x_1\varphi(nx_1)\frac{\partial v}{\partial x_2}\Big) \\
&= \varphi(nx_1)\frac{\partial}{\partial x_1}\Big(x_1\frac{\partial v}{\partial x_1}\Big) + \varphi(nx_1)\frac{\partial}{\partial x_2}\Big(x_1\frac{\partial v}{\partial x_2}\Big) + nx_1\varphi'(nx_1)\frac{\partial v}{\partial x_1} \\
&= \varphi(nx_1)\Big[\frac{\partial}{\partial x_1}\Big(x_1\frac{\partial v}{\partial x_1}\Big) + x_1\frac{\partial^2 v}{\partial x_2^2}\Big] + \psi(nx_1)\frac{\partial v}{\partial x_1} \\
&\to a^*av \ \text{ in } \ L_2(\mathbb{R}^2_+) \ \text{ as } \ n \to \infty.
\end{aligned}$$

(Here $\psi(\tau) = \tau\varphi'(\tau)$.) Hence,

$$(u, \ell v) = (u, a^*av) = \lim_{n\to\infty}(u, a^*(\varphi(nx_1)av)) = \lim_{n\to\infty}(au, \varphi(nx_1)av) = (au, av),$$

where au is understood in the sense of distributions in $\mathbb{R}^2_+$ and, by the second and third conditions in (1.2), $au \in L_2(\mathbb{R}^2_+) \oplus L_2(\mathbb{R}^2_+)$.

Next, let $\varepsilon > 0$ be sufficiently small, and let $\chi_\varepsilon(\tau) \in C_0^\infty(\mathbb{R})$ be a function such that $0 \le \chi_\varepsilon(\tau) \le 1$, $\chi_\varepsilon(\tau) = 0$ for $x \notin (\varepsilon, \sqrt{\varepsilon})$, and $\chi_\varepsilon(\tau) = 1$ for $x \in (2\varepsilon, \sqrt{\varepsilon}/2)$. Let

$C_\varepsilon = \int_0^\infty \frac{\chi_\varepsilon(\tau)\,d\tau}{\tau|\ln\tau|}$. We have

$$\ln 2 - \ln\Big(\frac{|\ln\varepsilon| + \ln 4}{|\ln\varepsilon| - \ln 2}\Big) = \int_{2\varepsilon}^{\sqrt{\varepsilon}/2} \frac{d\tau}{\tau|\ln\tau|} \le C_\varepsilon \le \int_\varepsilon^{\sqrt{\varepsilon}} \frac{d\tau}{\tau|\ln\tau|} = \ln 2,$$

whence $\lim_{\varepsilon\to 0} C_\varepsilon = \ln 2$. Finally, set $\varphi_\varepsilon(\tau) = C_\varepsilon^{-1}\int_0^\tau \frac{\chi_\varepsilon(\theta)\,d\theta}{\theta|\ln\theta|}$, so that $0 \le \varphi_\varepsilon(\tau) \le 1$, $\varphi_\varepsilon(\tau) = 0$ for $x \le \varepsilon$, and $\varphi_\varepsilon(\tau) = 1$ for $x > \sqrt{\varepsilon}$. Obviously, $\varphi_\varepsilon(x_1)v \to v$ in $L_2(\mathbb{R}^2_+)$ as $\varepsilon \to 0$. Next,

$$a\varphi_\varepsilon(x_1)v = \varphi_\varepsilon(x_1)av + \begin{pmatrix} \sqrt{x_1}\varphi'_\varepsilon(x_1)v \\ 0 \end{pmatrix}.$$

On the other hand,

$$\|\sqrt{x_1}\varphi'_\varepsilon(x_1)v\|^2 = \int_{\mathbb{R}^2_+} x_1(\varphi'_\varepsilon(x_1))^2|v(x_1,x_2)|^2\,dx_1dx_2 \le CC_\varepsilon^{-2}\int_\varepsilon^{\sqrt{\varepsilon}} \frac{d\tau}{\tau(\ln\tau)^2} = \frac{CC_\varepsilon^{-2}}{2|\ln\varepsilon|},$$

where the constant C depends on the maximum of the modulus of v and the diameter of the support of v. Since $\lim_{\varepsilon\to 0} C_\varepsilon = \ln 2$, it follows that the last expression tends to zero, and we see that $a\varphi_\varepsilon(x_1)v \to av$ in $L_2(\mathbb{R}^2_+) \oplus L_2(\mathbb{R}^2_+)$ as $\varepsilon \to 0$. Now we can write $(au, av) = \lim_{\varepsilon\to 0}(au, a\varphi_\varepsilon(x_1)v) = \lim_{\varepsilon\to 0}(a^*au, \varphi_\varepsilon(x_1)v) = (a^*au, v) = (\ell u, v)$, as desired. The proof of the proposition is complete. □

Remark 1. The operator $L = A^*A$ is strictly positive, i.e., $L \ge 0$ and $\ker L = \ker A = \{0\}$. (The latter is obvious; indeed, the equation $Au = 0$ implies that $u_{x_1} = u_{x_2} = 0$, so that u is a constant, which is necessarily zero because $u \in L_2(\mathbb{R}^2_+)$.)

1.2. Statement of the Cauchy problem and its well-posedness. Now the statement of a well-posed problem for the degenerate wave equation (0.1) is obvious. Namely, consider the abstract hyperbolic problem

$$\frac{d^2u(t)}{dt^2} + Lu(t) = 0, \quad t \in [0,T], \tag{1.5}$$

$$u(t)\big|_{t=0} = u_0, \quad \frac{du(t)}{dt}\Big|_{t=0} = u_1 \tag{1.6}$$

in $L_2(\mathbb{R}^2_+)$, where L is the selfadjoint operator constructed in the preceding subsection. A *strong solution* of problem (1.5), (1.6) on the interval $[0,T]$ is a function $u \in C^2([0,T], L_2(\mathbb{R}^2_+))$ such that the initial conditions (1.6) are satisfied, $u(t) \in D(L)$ for $t \in [0,T]$, and equation (1.5) is satisfied for all $t \in [0,T]$.

Theorem 1. *For any $u_0 \in D(L)$ and $u_1 \in D(A)$, the Cauchy problem* (1.5), (1.6) *has a unique strong solution, and the energy conservation law*

$$J^2(t) = J^2(0), \quad t \in [0,T], \tag{1.7}$$

holds, where

$$J^2(t) = \frac{1}{2}\bigg(\bigg\|\frac{du(t)}{dt}\bigg\|^2 + \|Au(t)\|^2\bigg) \equiv \frac{1}{2}\int_{\mathbb{R}^2_+}\bigg[\bigg|\frac{\partial u(x,t)}{\partial t}\bigg|^2 + x_1\bigg|\frac{\partial u(x,t)}{\partial x}\bigg|^2\bigg]dx_1dx_2 \tag{1.8}$$

is the energy integral.

Proof. By [18, p. 78 of the Russian edition, Lemma 1], we have $D(L^{1/2}) = D(A)$. It remains to apply [18, p. 184 of the Russian edition, Theorems 5 and 6]. □

We also write down the energy relation for the case where the right-hand side of the equation is nonzero. (We shall need this in what follows when constructing asymptotic expansions.)

Proposition 3 (cf. [16, p. 480 of the Russian edition, equation (4)])**.** *Let $u(t)$ be a strong solution of the equation $\frac{d^2u(t)}{dt^2}+Lu(t)=F(t)$, $t\in[0,T]$, where $F(t)\in C([0,T],L_2(\mathbb{R}^2_+))$. Then*

$$J^2(t)=J^2(0)+\int_0^t\left(F(t),\frac{du(t)}{dt}\right)dt,\quad t\in[0,T], \tag{1.9}$$

where $(\,\cdot\,,\,\cdot\,)$ is the inner product in $L_2(\mathbb{R}^2_+)$.

We omit the trivial proof of this proposition.

Corollary 1 (Energy estimates)**.** *If the assumptions of Proposition 3 are satisfied, then*

$$\begin{aligned}\|u(t)\|&\le\|u(0)\|+C\big(J(0)+\sup_{\tau\in[0,T]}\|F(\tau)\|\big),\\ J(t)&\le C\big(J(0)+\sup_{\tau\in[0,T]}\|F(\tau)\|\big),\end{aligned}\qquad t\in[0,T], \tag{1.10}$$

with some constant $C=C(T)$.

Proof. Relation (1.9) and definition (1.8) of the energy integral imply that

$$J^2(t)\le J^2(0)+\int_0^t\|F(\tau)\|\left\|\frac{du}{dt}(\tau)\right\|d\tau\le J^2(0)+\sqrt{2}\int_0^t\|F(\tau)\|\,J(\tau)\,d\tau,$$

and consequently,

$$J_*^2\le J^2(0)+\sqrt{2}TF_*J_*,\quad\text{where }J_*=\sup_{\tau\in[0,T]}J(t),\quad F_*=\sup_{\tau\in[0,T]}\|F(t)\|\,.$$

Hence,

$$\begin{aligned}J_*&\le\frac{\sqrt{2}TF_*+\sqrt{2T^2F_*^2+4J^2(0)}}{2}\le C_1(F_*+J(0)),\\ \|u(t)-u(0)\|&\le\int_0^t\left\|\frac{du}{dt}(\tau)\right\|d\tau\le TJ_*\le C_1T(F_*+J(0)),\end{aligned}$$

and it remains to set $C=C_1\max\{1,T\}$. ☐

1.3. Problem with localized initial data: outline of the asymptotic analysis. Consider the Cauchy problem for equation (1.5) with the special initial data (0.2)–(0.3):

$$\frac{d^2u(t)}{dt^2}+Lu(t)=0,\quad u\big|_{t=0}=V\left(\frac{x-a}{\mu}\right),\quad u_t\big|_{t=0}=0. \tag{1.11}$$

Our aim is to construct the asymptotic expansion of the solution of problem (1.11) as $\mu\to0$. The general outline of our construction is as follows. First, we express the initial condition in the form of an integral of a rapidly oscillating function (given by Maslov's canonical operator on a special Lagrangian manifold) with respect to a parameter. Then, for each parameter value, we obtain an asymptotic solution of the corresponding Cauchy problem by the canonical operator method, so that the asymptotic solution of problem (1.11) will be expressed as the integral of these solutions with respect to the parameter. Finally, we use the energy estimates to prove that the resulting asymptotic solution gives indeed the asymptotic expansion of the solution of problem (1.11). We present a more detailed outline of each of these stages.

1.3.1. Representation of the initial data. In [7, 4] it was shown that a rapidly decaying function of the form $V((x-a)/\mu)$ admits the integral representation

$$V\left(\frac{x-a}{\mu}\right)=\sqrt{\frac{\mu}{2\pi i}}\int_0^\infty K^{\mu/\rho}_{\Lambda_0^2}\big[\sqrt{\rho}\,\widetilde{V}(\rho\mathbf{n}(\psi))e(\alpha)\big]\,d\rho+O(\mu), \tag{1.12}$$

where $\mathbf{n}(\psi)=(\cos\psi,\sin\psi)$, $K^h_{\Lambda_0^2}$ is Maslov's canonical operator on the two-dimensional Lagrangian manifold $\Lambda_0^2=\{p=\mathbf{n}(\psi),\ x=a+\alpha\mathbf{n}(\psi)\}$, $\widetilde{V}(p)$ is the Fourier transform of the function $V(y)$, and $e(\alpha)$ is a cutoff function supported on a neighborhood of zero. We also use a similar representation in which Λ_0^2 is replaced by the Lagrangian manifold Λ that passes through the circle $\{p=\mathbf{n}(\psi),x=a\}$ and, in addition, is invariant with respect to the Hamiltonian vector field corresponding to the Hamiltonian $H(x,p)=\sqrt{x_1}|p|$ of our problem. This new Lagrangian manifold has the advantage that the solution is represented by the canonical operator on the same manifold for each t, which simplifies the study dramatically. This representation of the initial condition will be derived in Subsection 3.1.

1.3.2. Construction of the asymptotic solution. The main difficulty in the construction of the asymptotic solution by the canonical operator method is that, owing to the degeneracy of the problem, the trajectories of the Hamiltonian vector field go to infinity (with respect to the variable p_1) in finite time. Accordingly, the support of the amplitude (i.e., the function to which the canonical operator should be applied) becomes noncompact, and the traditional definition of the canonical operator (see, e.g., [6]) does not work in this situation. If, nevertheless, we formally write out this definition, then the integrals with respect to p_1 occurring in this definition become improper, and the standard regularization of these integrals as oscillatory integrals gives a function with a logarithmic singularity in the variable x_1. This function does not lie in the domain of L. A similar difficulty was encountered in the one-dimensional case (see [8, 14]): it was observed that, to regularize the integral properly, say, as $p\to\infty$, one should use the second branch of the Lagrangian manifold, on which $p\to-\infty$. (This results in an integral in the sense of the Cauchy principal value.)

The computations in the one-dimensional case are rather simple, and they were readily carried out in the cited papers without explicitly indicating their geometrical meaning. In the present paper, we point out the following simple and important geometric fact: the invariant Lagrangian manifold Λ can (and should!) be treated as a smooth submanifold of the "extended" phase space obtained by the replacement of the line $\mathbb{R}_{p_1}$ of the momentum variable p_1 by the circle $\bar{\mathbb{R}}_{p_1}=\mathbb{R}_{p_1}\cup\{\infty\}$. (With this interpretation, the support of the amplitude is compact for all t.) Then, in addition to canonical charts used in the standard construction of the canonical operator, a new type of canonical charts arises (namely, the charts containing points at infinity), and the canonical operator should be defined in these charts in such a way that, being applied to the amplitude, it gives a function lying in the domain of L. This is achieved with the use of Cauchy type integrals, and the condition that the resulting function should belong to $D(L)$ leads to the appearance of a phase factor in the passage through the point at infinity. This phase factor is similar to Maslov's index, which arises in the passage across the cycle of singularities.

The geometry of the Lagrangian manifold and the construction of the canonical operator are described in detail in §2. Note that only the symplectic form and the Jacobians have singularities at the points at infinity in this construction, while the action, the amplitudes, and the coefficients in the transport equations are smooth. Apparently, the situation is the same in the more general case where the velocity in the wave equation is proportional to the square root of the distance from the shore with a smooth nonvanishing coefficient of proportionality.

1.3.3. Asymptotic solution and the asymptotic expansion of the solution. The asymptotic solution is constructed in Subsection 3.2 by the above method and is a linear combination of functions of the form $u(t,\mu) = \sqrt{\mu}\int_\mu^\infty K_\Lambda^{\mu/\rho}[\varphi_\rho(t,\mu/\rho)]\,d\rho$ (we omit the argument x of u), where $\varphi_\rho(t,h)$ is a compactly supported function on Λ such that it smoothly depends on the parameters ρ and t, is a polynomial of degree N in h, and, together with all derivatives, rapidly decays as $\rho \to \infty$. Let us substitute this solution into the equation in problem (1.11). How small can we make the discrepancy between the left- and right-hand sides? If the discrepancy resulting from the substitution of the function $K_\Lambda^h[\varphi_\rho(t,h)]$ into the equation is $O(h^N)$, $N > 1$, uniformly with respect to ρ, then the discrepancy $r(t,\mu)$ given by the function $u(t,\mu)$ itself satisfies

$$\|r(t,\mu)\|_{L_2(\mathbb{R}^2_+)} \le C\sqrt{\mu}\int_\mu^\infty \left(\frac{\mu}{\rho}\right)^N d\rho = \frac{C\mu^{3/2}}{N-1}.$$

In other words, we cannot achieve an estimate better than $O(\mu^{3/2})$ regardless of how many terms of the asymptotic expansion we take. This is a well-known phenomenon related to the fact that the asymptotic solution does not take the contribution of long-wave harmonics into account. Nevertheless, long expansions permit improving estimates for the derivatives. Indeed, the standard estimates of derivatives of rapidly oscillating functions have the form

$$f(x,h) = O(h^N) \overset{\text{def}}{\Longleftrightarrow} \left\|\frac{\partial^\beta f}{\partial x^\beta}\right\|_{L_2(\mathbb{R}^2_+)} = O(h^{N-|\beta|}),$$

whence

$$\left\|\frac{\partial^\beta r(t,\mu)}{\partial x^\beta}\right\|_{L_2(\mathbb{R}^2_+)} \le C\sqrt{\mu}\int_\mu^\infty \left(\frac{\mu}{\rho}\right)^{N-\beta} d\rho = \frac{C\mu^{3/2}}{N-\beta-1}$$

provided that $N > \beta + 1$. Thus, if we take N terms of the expansion, then the discrepancy, together with the derivatives of order not exceeding $N-2$, is $O(\mu^{3/2})$. Next, in Subsection 3.1 it is shown that the discrepancy in the initial conditions can be forced to satisfy the same estimates provided that sufficiently many terms of the expansion are taken. This permits one to estimate the energy norm of the difference between the asymptotic solution and the exact solution by using Corollary 1. (The additional factor x_1 in the integrals defining the energy norm does not affect the estimate, because the functions given by the canonical operator decay rapidly as $x \to \infty$ and are $O(h^\infty)$ outside the projections of the supports of the amplitudes onto the x-plane.) Moreover, for $N > 4$ we see that the same estimate ($O(\mu^{3/2})$) remains valid if we apply the operator L to the difference between the asymptotic and the exact solution.

Now embedding theorems permit us to conclude that, for arbitrary fixed $\varepsilon > 0$, the difference between the asymptotic and the exact solution is small ($O(\mu^{3/2})$) in the sup-norm in the domain $x_1 \ge \varepsilon$, where the energy norm is equivalent to the conventional Sobolev norm. In the nonsingular chart, this conclusion remains valid (with $O(\mu^{3/2})$ replaced by $O(\mu)$) even if we take only the leading term of the asymptotic expansion in the canonical operator, because the contribution of all other terms is $O(\mu)$.

We proceed to a more detailed exposition of some parts of our construction.

§2. Invariant Lagrangian manifold and the canonical operator

2.1. Phase space and the Lagrangian manifold Λ. We define the phase space Φ of the problem in question as follows:

$$\Phi = \bar{\mathbb{R}}_{p_1} \times \mathbb{R}^3_{p_2,x_1,x_2} \equiv \mathbb{S}^1 \times \mathbb{R}^3, \tag{2.1}$$

where $\bar{\mathbb{R}}_{p_1} = \mathbb{R} \cap \{\infty\} \simeq \mathbb{S}^1$ is the extended real line of the variable p_1, diffeomorphic to a circle. (We use $q = 1/p_1$ as a coordinate on this line in a neighborhood of the point at infinity.) The phase space Φ is equipped with the symplectic form $\omega^2 = dp_1 \wedge dx_1 + dp_2 \wedge dx_2$, which has a singularity on the set $\Gamma = \{\infty\} \times \mathbb{R}^3$ of points at infinity; in a neighborhood of this set, the form can be written as $\omega_2 = \frac{dx_1 \wedge dq}{q^2} + dp_2 \wedge dx_2$.

We construct a Lagrangian manifold Λ in Φ passing through the circle $\Lambda_0 = \{(x,p) \in \mathbb{R}^4 : x = a, |p| = 1\}$, $a = (1,0)$, and invariant with respect to the Hamiltonian vector field with Hamiltonian $H(x,p) = \sqrt{x_1}|p|$. We parametrize the manifold Λ_0 by the variable $\psi \in \mathbb{S}^1 = \mathbb{R} \bmod 2\pi$ by setting $p = \mathbf{n}(\psi) \equiv (\cos\psi, \sin\psi)$. The Hamiltonian system has the form

$$\dot{x}_j = \frac{p_j}{|p|}\sqrt{x_1}, \quad j = 1,2, \quad \dot{p}_1 = -\frac{|p|}{2\sqrt{x_1}}, \quad \dot{p}_2 = 0. \tag{2.2}$$

Its solution with initial conditions on Λ_0 is described by the formulas

$$\begin{aligned} &X_1(\psi,\tau) = \frac{\sin^2\phi}{\sin^2\psi}, \qquad X_2(\psi,\tau) = \frac{\tau}{2\sin\psi} + \frac{\sin 2\psi - \sin 2\phi}{2\sin^2\psi}, \\ &P_1(\psi,\tau) = \sin\psi\cot\phi, \quad P_2(\psi,\tau) = \sin\psi \end{aligned} \tag{2.3}$$

for $\psi \neq 0, \pi$, where τ is the time along the trajectories of the Hamiltonian system (proper time) and

$$\phi = \psi + \frac{\tau\sin\psi}{2}, \tag{2.4}$$

and by the formulas

$$X_1(\psi,\tau) = \left(1 \pm \frac{\tau}{2}\right)^2, \; P_1(\psi,\tau) = \left(1 \pm \frac{\tau}{2}\right)^{-1}, \; X_2(\psi,\tau) = P_2(\psi,\tau) = 0 \tag{2.5}$$

for $\psi = 0, \pi$. (The upper sign corresponds to $\psi = 0$, and the lower sign corresponds to $\psi = \pi$.) The functions (2.3)–(2.5) are everywhere smooth except for the function $P_1(\psi,\tau)$, which is infinite on the curves

$$\tau_k = \tau_k(\psi)\frac{2(\pi k - \psi)}{\sin\psi}, \quad k = 0, \pm 1, \pm 2, \dots . \tag{2.6}$$

However, the following proposition shows that these functions specify an everywhere smooth mapping into the phase space Φ. Note also that, for each fixed τ, these functions determine the solution fronts in the phase space, and their x-components determine the solution fronts in the half-plane $x_1 \geq 0$.

Proposition 4. *The functions* (2.3)–(2.5) *specify a smooth immersion of the infinite cylinder* $\mathbb{S}^1 \times \mathbb{R}$ *in* Φ.

Proof. First, we prove that the mapping in question is smooth. Near the curves (2.6), we can use the variable q instead of p_1 and verify that the resulting function $q = q(\psi,\tau)$ is smooth; this can be done by a straightforward computation. However, we give an alternative, more geometric argument, which can readily be generalized to the case where the velocity is proportional to $\sqrt{x_1}$ with a smooth nonconstant coefficient. We make the change of variables $p_1 = 1/q$ in the Hamiltonian system (2.2) and take into account the fact that the Hamiltonian is constant on the trajectories of the Hamiltonian system (and is equal to unity, precisely as on the original manifold Λ_0). Then the system acquires the form

$$\dot{x}_1 = \frac{q}{1 + q^2p_2^2}, \quad \dot{x}_2 = p_2x_1, \quad \dot{q} = \frac{1}{2}(1 + q^2p_2^2), \quad \dot{p}_2 = 0. \tag{2.7}$$

This system has smooth right-hand sides, so that the smooth dependence of the solutions on the parameters (ψ, τ) readily follows from the main theorems of the theory of ordinary differential equations solved for the derivative. The Hamiltonian vector field vanishes nowhere on these trajectories and is not tangent to the initial manifold, whence it follows that the mapping in question has full rank. $\square$

2.2. Canonical charts, action, and Jacobians on Λ. Here we introduce some objects needed in the definition of the canonical operator. We define the (nonsingular) action function $S(\psi, \tau)$ on the Lagrangian manifold Λ by the conditions

$$S\big|_{\Lambda_0} = 0, \quad dS = p\,dx \equiv p_1\,dx_1 + p_2\,dx_2. \tag{2.8}$$

One can compute S by integrating the form $p\,dx$ along trajectories of the Hamiltonian vector field. Since the Hamiltonian $H(x, p)$ is homogeneous of degree 1, the Euler identity shows that $p\dot{x} = H(x, p) = 1$ on these trajectories, and we finally obtain $S(\psi, \tau) = \tau$. Now we cover Λ by canonical charts and introduce phase functions and Jacobians in these charts. By the lemma on local coordinates (see, e.g., [19]), the set $\Lambda \setminus \Gamma$ can be covered by canonical charts of various types with coordinates (x_1, x_2), (p_1, x_2), (x_1, p_2), or (p_1, p_2). The phase functions and Jacobians in these charts are defined in the standard way. It remains to define canonical charts in neighborhoods of points of the set Γ.

Lemma 1. *An arbitrary point in Γ has a neighborhood where either (q, x_2) or (q, p_2) can be taken for coordinates on Λ.*

Proof. This can be proved by a straightforward computation on the basis of formulas (2.3) and (2.5). We prefer to give a more geometric proof similar to that of the lemma on local coordinates in [19]. In a neighborhood of Γ, the condition that Λ is Lagrangian can be written in the form $dq \wedge dx_1 - q^2\,dp_2 \wedge dx_2 = 0$. Note that $\dot{q} = \frac{1}{2}$ on Γ, so that $dq \neq 0$.

On the other hand, the Lagrangian property degenerates on Γ into $dq \wedge dx_1 = 0$, so that dq and dx_1 are linearly dependent there. Since Λ is immersed in Φ, it follows that at least one of the differentials dp_2 and dx_2 is linearly independent of dq, and the proof of the lemma is complete. $\square$

Thus, the set $\Gamma \in \Lambda$ can be covered by charts with coordinates (q, x_2) or (q, p_2). We consider both cases.

2.2.1. Charts with coordinates (q, x_2). We define the phase function in these charts by the formula $S(q, x_2) = \tau - p_1 x_1$. Since the Hamiltonian is equal to unity on the trajectories in question, we have $S(q, x_2) = \tau - \frac{p_1}{p^2} = \tau - \frac{1}{1+qp_2^2}$. Thus, the phase function is smooth.

We introduce the Jacobian $J(q, x_2) = \det \frac{\partial(p_1, x_2)}{\partial(\psi, \tau)}$. We have

$$J(q, x_2) = \frac{\partial p_1}{\partial q} \det \frac{\partial(q, x_2)}{\partial(\psi, \tau)} = -\frac{1}{q^2} \det \frac{\partial(q, x_2)}{\partial(\psi, \tau)}.$$

Since the second factor is smooth and does not vanish on the chart, we see that $J(q, x_2)$ has a singularity of type $1/q^2$ on Γ.

2.2.2. Charts with coordinates (q, p_2). We define the phase function in these charts by the formula $S(q, p_2) = \tau - p_1 x_1 - p_2 x_2$. Then, by analogy with the preceding argument, $S(q, p_2) = \tau - \frac{p_1}{p^2} - p_2 x_2 = \tau - \frac{1}{1+qp_2^2} - p_2 x_2$. Thus, the phase function is also smooth in this case. Next, for the Jacobian $J(q, p_2) = \det \frac{\partial(p_1, x_2)}{\partial(\psi, p)}$ we have $J(q, p_2) = \frac{\partial p_1}{\partial q} \det \frac{\partial(q, p_2)}{\partial(\psi, \tau)} = -\frac{1}{q^2} \det \frac{\partial(q, p_2)}{\partial(\psi, \tau)}$. Thus, $J(q, x_2)$ has a singularity of type $1/q^2$ on Γ in this case as well.

2.3. Definition of the canonical operator on Λ. Now we are in a position to define the canonical operator on Λ. Consider a locally finite cover of Λ by canonical charts U_j. The canonical operator K_Λ^h acts on a compactly supported function φ on Λ by the formula $K_\Lambda^h\varphi = \sum_j K_j^h\varphi$, where K_j^h is the local canonical operator in the chart U_j. The definition of local canonical operators in the charts that do not meet Γ is standard (see, e.g., [6]). Therefore, we only dwell on the definitions for the charts of the two above-mentioned types in neighborhoods of points of Γ.

2.3.1. Charts with coordinates (q, x_2). We define the local canonical operator in such a chart by the formula

$$[K_j^h\varphi](x,h) = \sqrt{\frac{i}{2\pi h}}\int e^{\frac{i}{h}\left(\frac{x_1}{q}+S(q,x_2)\right)}\frac{\varphi(q,x_2)}{\sqrt{J(q,x_2)}}\frac{dq}{q^2}, \tag{2.9}$$

where $\varphi = \varphi(q, x_2)$ is a smooth function on Λ compactly supported in the chart and expressed via the local coordinates of the chart; we assume that $\arg i = \frac{\pi}{2}$, and, when computing the square root, the argument of the Jacobian is chosen as follows. Note that the Jacobian does not change its sign in the passage through the point $q = 0$ (at the point itself, the Jacobian is infinite), because it is proportional to $1/q^2$ with a nonzero coefficient. We fix the argument of the Jacobian somehow for $q < 0$ and define it for $q > 0$ by the rule

$$\arg J(q,x_2)|_{q>0} = \arg J(q,x_2)|_{q<0} + 2\pi. \tag{2.10}$$

Then it turns out that the integrand has a singularity of type $1/q$ with a smooth coefficient at the point $q = 0$. A straightforward computation proves the following lemma.

Lemma 2. *The integral* (2.9) *treated as a singular integral in the sense of the Cauchy principal value determines a function $[K_j^h\varphi](x,h)$ smooth up to the boundary $x_1 = 0$ in the half-plane $x_1 \geq 0$. In particular, this function lies in the domain of the operator L.*

The increment (2.10) of the argument gives rise to a contribution to the Maslov index on Λ caused by the "reflection" of solutions in the boundary $x_1 = 0$.

2.3.2. Charts with coordinates (q, p_2). In this case, the situation is completely similar. The local canonical operator has the form

$$[K_j^h\varphi](x,h) = \frac{i}{2\pi h}\iint e^{\frac{i}{h}\left(\frac{x_1}{q}+x_2p_2+S(q,p_2)\right)}\frac{\varphi(q,p_2)}{\sqrt{J(q,p_2)}}\frac{dq\,dp_2}{q^2}; \tag{2.11}$$

i.e., it differs from (2.9) in that there is an additional integration variable p_2. Lemma 2 remains valid in this case.

§3. Construction of the asymptotic expansion of the solution

3.1. Representation of the initial data via the canonical operator. Let K_Λ^h be the above-constructed Maslov canonical operator [6] with small parameter h on the Lagrangian manifold Λ.

Theorem 2. *There exists a smooth compactly supported function $\phi_{\mu\rho} = \phi_{\mu\rho}(\tau,\psi)$ on Λ smoothly depending on the parameters $\rho \in [0,\infty)$ and $\mu \in [0,1)$ and possessing the following properties*:

(1) *together with all its derivatives, $\phi_{\mu\rho}$ decays rapidly as $\rho \to \infty$;*
(2) *$\phi_{0\rho}(\alpha,\psi) = \rho\widetilde{V}(\rho\mathbf{n}(\psi))\chi(\tau)$, where $\widetilde{V}$ is the Fourier transform of the function $V(y)$ and $\chi(\tau)$ is a smooth compactly supported cutoff function equal to unity in a neighborhood of zero*;

(3) *we have the representation*

$$V\Big(\frac{x}{\mu}\Big) = \Big(\frac{-i}{2\pi}\Big)^{1/2} \int_{\mu}^{\infty} \sqrt{\frac{\mu}{\rho}} K_{\Lambda}^{\mu/\rho}(\phi_{\mu\rho})\, d\rho + R_{\mu}(x_1, x_2), \tag{3.1}$$

where the remainder $R_\mu(x_1,x_2)$ satisfies the estimates

$$\|R_\mu(x_1,x_2)\|_{H^s(\mathbb{R}^2)} \le C_s \mu^{3/2}, \quad s = 0,1,2,\dots.$$

Proof. The proof relies on Taylor series expansions and is rather cumbersome, and we omit it for lack of space. □

3.2. Solution of problem (1.11)**.** Now problem (1.11) can be solved as follows. We represent the initial condition in the form of an integral with respect to the parameter ρ by Theorem 2. For each value of ρ, we construct the solution of the corresponding problem with rapidly oscillating initial data by using the canonical operator defined in §2 on the manifold $\Lambda_t = \Lambda$ with coordinates $\tau_t = \tau + t, \psi$. Finally, we integrate with respect to ρ. The argument in Subsection 1.3.3, combined with the constructions in [4], implies the following theorem.

Theorem 3. *The construction described above provides the solution of the Cauchy problem* (1.11) *in the form* $\big(\frac{-i}{2\pi}\big)^{1/2} \int_\mu^\infty \sqrt{\frac{\mu}{\rho}} K_{\Lambda_t}^{\mu/\rho}(\widetilde{V}(\rho n(\psi)))\, d\rho$ *modulo* $O(\mu^{3/2})$ *both in the scale of energy spaces and* (*in any half-plane of the form* $x_1 > \varepsilon > 0$) *in the uniform norm.*

§4. Example

We consider an example of an initial perturbation V (see [20, 21, 22, 5]) such that the asymptotic expansion of the solution in a neighborhood of the x_2-axis and the metamorphosis of the wave profile can be described by elementary algebraic functions. The argument in this section is only physically rigorous. We have already mentioned that the asymptotic expansion constructed here for the solution is localized in a neighborhood of the solution front, which is determined by formulas (2.3) (see Figure 1). We restrict ourselves to the case shown in Figure 1 in the middle and present the asymptotic expansion of the solution in a neighborhood of the point $x^+ = (x_1 = 0, x_2 = x_2^+(\tau) = X_2(\tau, \psi^+(\tau)))$, where the front is tangent to the half-line $\{x_2 > 0\}$. Take the chart with coordinates (x_1, p_2). Then the general form of the solution of our Cauchy problem, together with formula (2.9) rewritten in these coordinates, results in the formula

$$u \approx -\operatorname{Re}\bigg(\frac{i}{\pi}\int_0^\infty d\rho\bigg(\int_{-\infty}^{\infty} \frac{\rho\widetilde{V}(\rho\mathbf{n}(\psi))\exp\frac{i\rho(\tau+p_1(x_1-X_1(\tau+t)))}{\mu}}{\sqrt{J(\tau+t,\psi)}}\mathbf{e}_t(\tau,\psi)\,dp_1\bigg)\bigg). \tag{4.1}$$

Here $\mathbf{e}_t$ is a cutoff function, $(\tau(t,p_1,x_2), \psi(t,p_1,x_2))$ is a solution of the system of equations $P_1(\tau+t,\psi) = p_1$, $X_1(\tau+t,\psi) = x_1$, the Jacobian is given by the formula

$$J(\tau,\psi) = \det\frac{\partial(\mathrm{P}_1, \mathrm{X}_2)}{\partial(\tau,\psi)} = \frac{4(2+\tau\cos\psi)\sin\psi + \cos\psi(-6\sin 2\Phi + \sin 4\Phi)}{8\sin\psi\sin^2\Phi}, \tag{4.2}$$

and the square root of the Jacobian is taken as indicated in §2. Clearly, by setting $\tau' = \tau + t$ we can pass from the variable τ to τ' and, omitting the primes, rewrite formula (4.1) as

$$u \approx -\operatorname{Re}\bigg(\frac{i}{\pi}\int_0^\infty d\rho\bigg(\int_{-\infty}^{\infty} \frac{\rho\widetilde{V}(\rho\mathbf{n}(\psi))\exp\frac{i\rho(\tau-t+p_1(x_1-X_1(\tau)))}{\mu}\mathbf{e}_t(\tau,\psi)}{\sqrt{J(\tau,\psi)}}\,dp_1\bigg)\bigg), \tag{4.3}$$

where $\mathbf{e}_t(\tau,\psi)$ is a cutoff function equal to 1 in some neighborhood of the point $(\tau,\psi) = (t,\psi^+)$ and zero outside a larger neighborhood and $(\tau(p_1,x_2),\psi(p_1,x_2))$ is the solution of the system of equations

$$P_1(\tau,\psi) = p_1, \quad X_2(\tau,\psi) = x_2. \tag{4.4}$$

Let V be a source with elliptic cross section rotated by an angle θ:

$$V(y) = V^0(T(\theta)y), \quad T(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad V^0(y) = \frac{1}{(1+(\frac{y_1}{b_1})^2+(\frac{y_2}{b_2})^2)^{3/2}};$$

then $\widetilde{V} = b_1 b_2 \exp(-\rho\beta(\psi-\theta))$ and $\beta(\phi) = \sqrt{b_1^2\cos^2\phi + b_2^2\sin^2\phi}$. The function $V(y)$ is not compactly supported, and to make the argument rigorous one should multiply $V(\frac{x-a}{\mu})$ by a cutoff function $e(x-a)$. However, $V(\frac{x-a}{\mu})$ decays rapidly, and omitting the corresponding "tail" when computing the Fourier transform gives only a small correction to the asymptotic solution. Having this in mind, changing the order of integration in (4.3), and integrating with respect to ρ, we obtain

$$u \approx \frac{\mu b_1 b_2}{\pi}\operatorname{Re}\left(i\int \frac{\mathbf{e}_t(\tau-t,\psi)}{(\tau-t+p_1(x_1-X_1(\tau,\psi))+i\mu\beta)^2\sqrt{J(\tau,\psi)}}\,dp_1\right). \tag{4.5}$$

From the last integral it is seen that the asymptotic behavior of u as $\mu\to+0$ is determined by a neighborhood of the points p_1 corresponding to the zeros of the function $\tau(p_1,x_2)-t+p_1(x_1-X_1(\tau(p_1,x_2),\psi(p_1,x_2)))$. Since we are interested in points (x_1,x_2) lying near the front points close to the x_2-axis, which means that x_1 and $X_1(\tau(p_1,x_2))$ are small and $\tau(p_1,x_2)-t$ is bounded, it follows that the corresponding values of p_1 are large. Let us study the behavior of solutions of system (4.4) for large p_1. We rewrite (4.4) and the formula determining Φ (see (2.3)) in the form

$$\tan\phi = \frac{\sin\psi}{p_1}, \quad \tau = 2x_2\sin\psi - \frac{2}{\sin\psi}\left(\sin\psi\cos\psi - \frac{\tan\phi}{1+\tan^2\phi}\right), \quad \tau = \frac{2(\phi-\psi)}{\sin\psi}. \tag{4.6}$$

From this, we obtain one equation for the function $\psi(p_1,x_2)$ and a formula for τ:

$$\psi = \pi - x_2\sin^2\psi + \cos\psi\sin\psi + \arctan(\sin\psi/p_1) - \frac{p_1\sin\psi}{p_1^2+\sin^2\psi},$$
$$\tau = \frac{2\pi + 2\arctan(\sin\psi/p_1) - 2\psi}{\sin\psi},$$

where the principal value of $\arctan z$ is used, which corresponds to the neighborhood in question of singular points. Perturbation theory gives

$$\psi = \psi^0(x_2) + O\left(\frac{1}{p_1^2}\right), \quad \tau = \tau^0(x_2) + \frac{2}{p_1} + O\left(\frac{1}{p_1^2}\right), \tag{4.7}$$

where $\psi^0(x_2)\in(\pi,2\pi)$ is the solution of the equation $\psi = \pi - x_2\sin^2\psi + \cos\psi\sin\psi$ and $\tau^0(x_2) = \frac{2\pi-2\psi^0}{\sin\psi^0}$. Next,

$$(\tau - t + p_1(x_1 - X_1(\tau,\psi))) + i\mu\beta = p_1x_1 + \tau - t - \frac{p_1}{p_1^2+\sin^2\psi} + i\mu\beta(\psi),$$
$$J = \frac{p_1^2}{2\sin^2\psi}(1+\tau\cos\psi-\cot\psi) + 1 + \frac{\tau\cos\psi}{2} - \cot\psi.$$

Thus, the integral (4.5) acquires the form

$$\frac{\mu b_1 b_2}{\pi}\operatorname{Re}\left(i\int_{-\infty}^{\infty}\frac{\mathbf{e}_t(\tau-t,\psi)}{(p_1x_1+\tau-t-\frac{p_1}{p_1^2+\sin^2\psi}+i\mu\beta(\psi-\theta))^2\sqrt{J}}\,dp_1\right). \tag{4.8}$$

Note that the value $p_1 = 0$ lies outside the support of the function $\mathbf{e}_t(\tau, \psi)$. Now we take into account the fact that μ is a small parameter and make the change of variables $z = x_1 p_1$ in the integral (4.8) and the expansions (4.7). Then, clearly, the main contribution to the integral is from those p_1 for which the expression $p_1 x_1 + \tau - t$ is small. Since x_1 is small, we see that p_1 is sufficiently large. Note that if we make the change of variables $z = p_1 x_1/\mu$ and use the expansions (4.7), then the integrands can be expanded in powers of the variable $x_1/(z\mu)$. Hence, we can use the expansions (4.7) in the integrand, which essentially means that we deal with an asymptotic expansion in powers of x/μ. We can also omit the cutoff function and understand the resulting integrals as integrals in the sense of the Cauchy principal value. By retaining quite a few summands, we enlarge the domain of the variables (x_1, x_2) where the asymptotic expansion of the integral to be studied can be used. It should also be kept in mind that an extra term should be retained in the denominator of the integrand, because this denominator involves the important small term $i\mu\beta$.

A more detailed analysis of this argument, close to the ideas contained in the monographs [9, 10], shows that, in order to find the leading term of the asymptotic expansion in the domain $x_1 = O(\sqrt{\mu})$, one should retain the $O(1/p_1)$ terms in the first factor in the denominator of the integrand and the $O(1)$ terms in the integrand. Then, up to infinitesimals of higher order with respect to the parameter μ, we can rewrite the integral in the form

$$\frac{\sqrt{2}\sin\psi^0}{1+\tau^0\cos\psi^0-\cot\psi^0}\ \text{p.v.}\int_{-\infty}^{\infty}\frac{1}{p_1(p_1x_1+\tau^0-t+i\mu\beta^0+1/p_1)^2}\,dp_1, \tag{4.9}$$

where $\beta^0 = \beta(\psi^0 - \theta)$. We also assume that t ranges in a neighborhood of $\tau^0(x_2)$. Apart from the pole $p_1^0 = \frac{1}{2x_1}(t - \tau^0 - i\beta^0\mu - \sqrt{(t-\tau^0-i\beta^0\mu)^2 - 4x_1})$, all other poles of the integrand give an $O(1)$ contribution to the integral, and we do not take them into account (they can arise artificially in the passage from (4.8) to (4.9)), and the contribution of the pole p_1^0 is $\frac{(\tau^0-t+i\beta^0\mu)}{(-4x_1+(\tau^0-t+i\beta^0\mu)^2)^{3/2}}$. Finally, we have the following asymptotic representation of the solution of the original problem in a neighborhood of the point $(x_1, x_2) = (0, X_2(t))$ of tangency of the front:

$$u \approx \frac{2\mu\sqrt{2b_1b_2}\sin\psi^0}{1+\tau^0\cos\psi^0-\cot\psi^0}\ \operatorname{Re}\frac{e^{3i\pi/4}\big(\tau^0(x_2)-t+i\mu\beta(\psi^0(x_2)-\theta)\big)}{(-4x_1+(\tau^0(x_2)-t+i\mu\beta(\psi^0(x_2)-\theta))^2)^{3/2}},$$

where the argument of the continuous function in the denominator lies on the interval $(-\pi/2, \pi/2)$. For each x_2, this function coincides (up to a factor) with the (exact) solution obtained in [14] for the one-dimensional wave equation with velocity $c = \sqrt{x}$. The analysis of this function results in the pictures shown in Figure 2.

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