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TRACE HARDY–SOBOLEV INEQUALITIES IN CONES

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Dedicated to V. M. Babich on the occasion of his 80th birthday

Abstract. Sharp constants are found for the trace Hardy–Sobolev inequalities in cones. The question as to whether these constants are attained is discussed.

§1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open cone; we assume that $G = \Omega \cap \mathbb{S}^{n-1}$ is a domain with strictly Lipschitz boundary on the unit sphere. Let $\dot{\mathcal{C}}^1(\Omega)$ denote the set of continuously differentiable functions whose support is bounded and separated away from the origin. For $1 \leq p \leq \infty$, denote by $\dot{W}_p^1(\Omega)$ the completion of $\dot{\mathcal{C}}^1(\Omega)$ with respect to the norm $\|\nabla v\|_{p,\Omega}$.

Consider a scale of weighted trace spaces $L_{q,\sigma}(\partial\Omega)$ with the norms

$$\|v\|_{q,\sigma,\partial\Omega} = \|r^{\sigma-1}v\|_{L_q(\partial\Omega)},$$

where $r = |x|$.

For $1 < p < \infty$ with $p \neq n$ and $\frac{1}{p} \leq \sigma \leq \min\{1, \frac{n}{p}\}$, we denote by $p_\sigma^{**} = \frac{(n-1)p}{n-\sigma p}$ the critical exponent for the trace embedding $\dot{W}_p^1(\Omega) \hookrightarrow L_{q,\sigma}(\partial\Omega)$:

$$(\mathrm{I}_\sigma) \qquad \lambda(p,\sigma,\Omega) = \inf_{v \in \dot{W}_p^1(\Omega)\setminus\{0\}} J_\sigma(v) \equiv \inf_{v \in \dot{W}_p^1(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p_\sigma^{**},\sigma,\partial\Omega}} > 0.$$

The case where $p < n$, $\sigma = 1$ leads to the trace Sobolev inequality; see, e.g., [M, 1.4.5]. In a natural way, the case of $\sigma = \frac{1}{p}$ will be called the trace Hardy inequality. The intermediate cases can be obtained from these cases by the Hölder inequality. For $p > n$ one should use the Morrey inequality, see [M, 1.4.5], instead of the trace Sobolev inequality.

We call (I_σ) *the trace Hardy–Sobolev inequality.*

Remark 1. For $p = n$ the infimum in (I_σ) equals zero. For $\sigma = 1$ this is a classical fact; for $\sigma < 1$ it suffices to consider the sequence

$$v_k(x) = \begin{cases} 0 & \text{if} \quad r \leq \frac{1}{k} \quad \text{or} \quad r \geq 2k, \\ kr - 1 & \text{if} \quad \frac{1}{k} < r < \frac{2}{k}, \\ 1 & \text{if} \quad \frac{2}{k} \leq r \leq k, \\ 2 - \frac{r}{k} & \text{if} \quad k < r < 2k. \end{cases}$$

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Since the embedding operator $\dot{W}_p^1(\Omega) \hookrightarrow L_{p^{**}_\sigma,\sigma}(\partial\Omega)$ is noncompact, the problem of attainability of the norm of this operator (i.e., the question concerning the existence of an extremal function in the embedding theorem) is nontrivial. The conventional Hardy–Sobolev inequality generated by the embedding $\dot{W}_p^1(\Omega) \hookrightarrow L_{p^*_\sigma,\sigma}(\Omega)$ (here $p^*_\sigma = \frac{np}{n-\sigma p}$, $0 \le \sigma \le \min\{1, \frac{n}{p}\}$) was treated in many papers; see, e.g., the recent survey [N] and further references therein. The problem for the trace embedding is not so well studied.

The structure of our paper is as follows. §§2 and 3 are devoted to the values of sharp constants, respectively, in the trace Hardy inequality and in the trace Sobolev inequality. In §4 we discuss the existence of a minimizer in the intermediate cases.

We recall some notation. For $p < n$ we put $p^* = p^*_1 = \frac{np}{n-p}$ (this is the usual critical Sobolev exponent) and $p^{**} = p^{**}_1 = \frac{(n-1)p}{n-p}$. Next, $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere in $\mathbb{R}^n$. Further, $\mathcal{B}$ is the Euler beta-function, while $\mathrm{P}^\mu_\nu(x)$ is the Legendre function (see, e.g., [GR, 8.7]).

§2. The "Hardy" case $\sigma = \frac{1}{p}$

Our first result provides the sharp constants for the trace Hardy inequality in cones.

Theorem 1. *Suppose $1 < p < \infty$, $p \ne n$, $\sigma = \frac{1}{p}$. Then the infimum in* (I_σ) *is not attained and equals* $(\Lambda^{(p)}(G))^{\frac{1}{p}}$, *where*

$$\Lambda^{(p)}(G) = \min_{v \in W_p^1(G)\setminus\{0\}} \frac{\displaystyle\int_G \Big(\Big(\tfrac{n-p}{p}\Big)^2 v^2 + |\nabla' v|^2\Big)^{\frac{p}{2}} d\Theta}{\displaystyle\int_{\partial G} |v|^p \, dS} \tag{1}$$

(∇' stands for the tangential gradient on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$).

Proof. First, the minimum in (1) is attained because the trace embedding $W_p^1(G) \hookrightarrow L_p(\partial G)$ is compact, and this minimum is positive because $p \ne n$. Let V denote the minimizer of (1) normalized in $L_p(\partial G)$. By a standard argument, V is positive in $\bar{G}$.

We define $U(r,\Theta) = r^{1-\frac{n}{p}} V(\Theta)$, where (r,Θ) stands for the spherical coordinates in Ω. Then for any $h \in \dot{\mathcal{C}}^1(\Omega)$ we have

$$\begin{aligned}
\int_\Omega |\nabla U|^{p-2} \nabla U \cdot \nabla h \, dx &= \int_\Omega |\nabla U|^{p-2} \Big(U_r h_r + \frac{1}{r^2} \nabla' U \cdot \nabla' h\Big) dx \\
&= \int_G \Big(\Big(\tfrac{n-p}{p}\Big)^2 V^2 + |\nabla' V|^2\Big)^{\frac{p-2}{2}} V \cdot \int_0^\infty \tfrac{p-n}{p} \, r^{\frac{n}{p}-1} h_r \, dr \, d\Theta \\
&\quad + \int_0^\infty \int_G \Big(\Big(\tfrac{n-p}{p}\Big)^2 V^2 + |\nabla' V|^2\Big)^{\frac{p-2}{2}} \nabla' V \cdot \nabla' h \; r^{\frac{n}{p}-2} \, d\Theta \, dr \\
&= \int_0^\infty \int_G \Big(\Big(\tfrac{n-p}{p}\Big)^2 V^2 + |\nabla' V|^2\Big)^{\frac{p-2}{2}} \Big(\Big(\tfrac{n-p}{p}\Big)^2 V h + \nabla' V \cdot \nabla' h\Big) r^{\frac{n}{p}-2} \, d\Theta \, dr \\
&\overset{(*)}{=} \Lambda^{(p)}(G) \int_0^\infty \int_{\partial G} V^{p-1} h \; r^{\frac{n}{p}-2} \, dS \, dr = \Lambda^{(p)}(G) \int_{\partial\Omega} \frac{U^{p-1}}{r^{p-1}} h \, d\Sigma
\end{aligned} \tag{2}$$

(identity $(*)$ follows from the weak Euler–Lagrange equation for V). Thus, U is a positive weak solution of the Steklov-type problem

$$-\Delta_p u \equiv -\mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in} \quad \Omega, \qquad |\nabla u|^{p-2} \frac{\partial u}{\partial \mathbf{n}} = \Lambda^{(p)}(G) \; \frac{u^{p-1}}{r^{p-1}} \quad \text{on} \quad \partial\Omega.$$

Now, the relation $\Lambda^{(p)}(G) \le \lambda^p(p, \frac{1}{p}, \Omega)$ follows from the generalized Picone identity (see [AH]). For any $u \in \dot{\mathcal{C}}^1(\Omega)$, we set $h = \frac{|u|^p}{U^{p-1}}$. Then (2) implies that

$$\begin{aligned}\Lambda^{(p)}(G)\int_{\partial\Omega}\frac{|u|^p}{r^{p-1}}\,d\Sigma &= \Lambda^{(p)}(G)\int_{\partial\Omega}\frac{U^{p-1}}{r^{p-1}}h\,d\Sigma = \int_\Omega |\nabla U|^{p-2}\nabla U\cdot\nabla h\,dx\\ &= \int_\Omega \Big(p|\nabla U|^{p-2}\nabla U\cdot\nabla u\frac{|u|^{p-2}u}{U^{p-1}} - (p-1)|\nabla U|^p\frac{|u|^p}{U^p}\Big)\,dx\\ &\overset{(**)}{\le} \int_\Omega \Big(p|\nabla u|\cdot|\nabla U|^{p-1}\frac{|u|^{p-1}}{U^{p-1}} - (p-1)|\nabla U|^p\frac{|u|^p}{U^p}\Big)\,dx\\ &\le \int_\Omega |\nabla u|^p\,dx.\end{aligned} \tag{3}$$

Here $(**)$ is the Cauchy inequality, and the last inequality follows from

$$x^p - pxy^{p-1} + (p-1)y^p \ge 0, \qquad x, y > 0. \tag{4}$$

By approximation, (3) is true for $u \in \dot{W}^1_p(\Omega)$.

To prove that $\Lambda^{(p)}(G) = \lambda^p(p, \frac{1}{p}, \Omega)$, we consider the sequence

$$u_\delta(r,\Theta) = \begin{cases} r^{1-\frac{n}{p}+\delta}V(\Theta) & \text{if} \quad r \le 1,\\ r^{1-\frac{n}{p}-\delta}V(\Theta) & \text{if} \quad r \ge 1.\end{cases}$$

Clearly, $u_\delta \in \dot{W}^1_p(\Omega)$. A direct computation gives

$$\begin{aligned}\Lambda^{(p)}(G)\int_{\partial\Omega}\frac{|u_\delta|^p}{r^{p-1}}\,d\Sigma &= \frac{2}{p\delta}\cdot\Lambda^{(p)}(G)\int_{\partial G}V^p dS = \frac{2}{p\delta}\int_G\Big(\Big(\tfrac{n-p}{p}\Big)^2V^2 + |\nabla' V|^2\Big)^{\frac{p}{2}}dS\\ &= \frac{1}{p\delta}\int_G\Big[\Big(\Big(\tfrac{n-p}{p}+\delta\Big)^2V^2 + |\nabla' V|^2\Big)^{\frac{p}{2}} + \Big(\Big(\tfrac{n-p}{p}-\delta\Big)^2V^2 + |\nabla' V|^2\Big)^{\frac{p}{2}}\Big]\,dx + O(\delta)\\ &= \int_\Omega |\nabla u_\delta|^p\,dx + O(\delta),\end{aligned} \tag{5}$$

and the claim follows.

Finally, equality in $(**)$ means that $\nabla u \parallel \nabla U$, while equality in (4) means that $x = y$. These two facts show that

$$\frac{\nabla u}{u} = \frac{\nabla U}{U} \qquad \Longrightarrow \qquad u = cU$$

on the set $\{u \neq 0\}$ and, therefore, in the entire domain Ω. Since $U \notin \dot{W}^1_p(\Omega)$, equality in (3) is impossible. $\square$

Note that the minimizer of (1) is unique up to a multiplicative constant. There are many ways to prove this. The simplest one is to use convexity; see, e.g., [LL, Theorem 7.8] and [Kw, Proposition 4] for particular cases.

Lemma 1. *Let $F(t,x)$ be a function on $\mathbb{R}_+ \times \mathbb{R}^n$; assume that it is positive homogeneous of degree $p > 1$ and convex. If $1 \le q \le p$, then $F(v, \nabla v)$ is convex with respect to v^q on the set of positive functions.*

Proof. Set $f = v^q$; we need to prove that $\Phi(f) = F(f^{\frac{1}{q}}, \frac{1}{q}f^{\frac{1}{q}-1}\nabla f)$ is convex. Observe that the homogeneity of F implies the relations

$$\begin{aligned} D^2_{tt}F(s,y)\,s + \sum_k D^2_{tx_k}F(s,y)\,y_k &= (p-1)D_tF(s,y);\\ D^2_{tx_j}F(s,y)\,s + \sum_k D^2_{x_kx_j}F(s,y)\,y_k &= (p-1)D_{x_j}F(s,y), \quad j = 1,\dots,n.\end{aligned} \tag{6}$$

TABLE 1

n	3	4	5	6
$\lambda(2, \frac{1}{2}, \mathbb{R}^n_+)$	$\dfrac{2\pi}{\Gamma^2(\frac{1}{4})}$	$\sqrt{\dfrac{2}{\pi}}$	$\dfrac{\Gamma^2(\frac{1}{4})}{4\pi}$	$\sqrt{\dfrac{\pi}{2}}$

Using (6), by direct calculations we obtain

$$d^2\Phi(f;\ h) = \frac{f^{\frac{p}{q}-2}}{q^2} \cdot \left[\frac{p-q}{p-1} \cdot d^2F\Big(1, \tfrac{\nabla f}{qf};\ h, z^{(1)}\Big) + \frac{q-1}{p-1} \cdot d^2F\Big(1, \tfrac{\nabla f}{qf};\ 0, z^{(2)}\Big)\right], \tag{7}$$

where

$$z^{(1)} = \nabla h - \tfrac{(q-1)\nabla f}{qf} h; \qquad z^{(2)} = \nabla h - \tfrac{\nabla f}{f} h.$$

Since F is convex, the expression in (7) is nonnegative, and the claim follows. □

Now, uniqueness in (1) follows from the fact that the function $F(t,x) = (\alpha^2 t^2 + |x|^2)^{\frac{p}{2}}$ is strictly convex. Thus the vanishing of (7) implies that $\frac{\nabla h}{h} = \frac{\nabla f}{f}$, i.e., $h = cf$.

In particular, this means that if G is the spherical "hat"

$$G = \{(\theta, \phi_1, \dots, \phi_{n-2}) \in \mathbb{S}^{n-1} \ :\ 0 < \theta < \theta_*\}, \tag{8}$$

then the minimizer in (1) depends only on θ. For $p = 2$ this gives the opportunity to calculate $\Lambda^{(2)}(G)$ explicitly.

Theorem 2. *Let G be the spherical "hat"* (8)*. Then*

$$\Lambda^{(2)}(G) = \frac{(n-2)^2}{4} \cdot \frac{P^{-\frac{n-1}{2}}_{-\frac{1}{2}}\big(\cos(\theta_*)\big)}{P^{-\frac{n-3}{2}}_{-\frac{1}{2}}\big(\cos(\theta_*)\big)}. \tag{9}$$

In particular, for $\theta_* = \frac{\pi}{2}$, which case corresponds to $\Omega = \mathbb{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$, we have

$$\Lambda^{(2)}(G) = \frac{2\Gamma^2\big(\frac{n}{4}\big)}{\Gamma^2\big(\frac{n-2}{4}\big)}. \tag{10}$$

Proof. By standard variational arguments, the minimizer in (1) for $p = 2$ is an eigenfunction of the Steklov problem (Δ' stands for the Beltrami operator on $\mathbb{S}^{n-1}$)

$$-\Delta' V + \tfrac{(n-2)^2}{4} V = 0 \ \text{ in } \ G, \qquad \frac{\partial V}{\partial \mathbf{n}} = \Lambda^{(2)}(G)\ V \ \text{ on } \ \partial G.$$

Since this minimizer is a "one-dimensional" function, this problem is reduced to

$$V'' + (n-2)\cot(\theta) V' - \tfrac{(n-2)^2}{4} V = 0, \quad |V(0)| < \infty; \quad V'(\theta_*) = \Lambda^{(2)}(G) V(\theta_*).$$

A solution bounded at the origin is proportional to $\sin^{-\frac{n-3}{2}}(\theta) \cdot P^{-\frac{n-3}{2}}_{-\frac{1}{2}}\big(\cos(\theta)\big)$ (see, e.g., [Km, 2.171]). Now, formula (9) follows from [GR, 8.733], and formula (10) from [GR, 8.756.1]. □

As an illustration, we list the sharp constants in the trace Hardy inequality for the half-spaces of small dimensions. It is easily seen that $\lambda(2, \frac{1}{2}, \mathbb{R}^n_+) \cdot \lambda(2, \frac{1}{2}, \mathbb{R}^{n+2}_+) = \frac{n-2}{2}$.

§3. The "Sobolev" case $\sigma = 1$

The next theorem deals with the trace Sobolev inequality in circular cones.

Theorem 3. *Let Ω be a convex circular cone, i.e., let G be a spherical "hat" (8), $\theta_* \le \frac{\pi}{2}$. Suppose $1 < p < n$ and $\sigma = 1$. Then the function*

$$w(x) = |x - \mathbf{e}|^{-\frac{n-p}{p-1}}, \qquad \mathbf{e} = (0, \dots, 0, -1), \tag{11}$$

provides the minimum of J_σ.

Remark 2. For the case of the half-space ($\theta_* = \frac{\pi}{2}$), this statement was proved in the remarkable paper [Nt]; see also [E] for $p = 2$.

Proof. We repeat the proof of [Nt, Theorem 1] almost literally. The method is based on the mass transportation approach (generalized Monge–Kantorovich problem). Consider two probability measures on Ω with smooth densities F and G whose supports are bounded and separated away from the origin. Then (see [Br] and [MC]) there exists the so-called *Brenier map* $T = \nabla\varphi$ such that for all measurable functions ψ we have

$$\int_\Omega \psi(x) G(x)\, dx = \int_\Omega \psi(T(x)) F(x)\, dx. \tag{12}$$

Moreover, the function φ is convex and satisfies the Monge–Ampère equation

$$F(x) = G(\nabla\varphi(x)) \cdot \det(D^2\varphi(x)) \tag{13}$$

almost everywhere with respect to the measure $F dx$. Here $D^2\varphi$ is the Hessian matrix of φ, which exists a.e. by A. D. Aleksandrov's theorem.

By (12),

$$\int_\Omega G^{1-\frac{1}{n}}(x)\, dx = \int_\Omega G^{-\frac{1}{n}}(\nabla\varphi(x)) F(x)\, dx.$$

Using (13) and the Hadamard inequality, we obtain

$$\int_\Omega G^{1-\frac{1}{n}}(x)\, dx = \int_\Omega \det^{\frac{1}{n}}(D^2\varphi(x)) F^{1-\frac{1}{n}}(x)\, dx \le \frac{1}{n} \int_\Omega \Delta\varphi(x) F^{1-\frac{1}{n}}(x)\, dx. \tag{14}$$

Since φ is convex, on the right-hand side of (14) we can replace $\Delta\varphi$, understood in (14) as calculated a.e., by the full distributional Laplacian.

Integrating by parts, we get

$$n \int_\Omega G^{1-\frac{1}{n}}(x)\, dx \le \int_{\partial\Omega} F^{1-\frac{1}{n}}(x) \langle \nabla\varphi(x), \mathbf{n}\rangle\, d\Sigma - \int_\Omega \langle \nabla\varphi(x), \nabla(F^{1-\frac{1}{n}})(x)\rangle\, dx.$$

By the definition of the Brenier map, $\nabla\varphi(x) \in \Omega$ for all $x \in \Omega$. Therefore, $\langle \nabla\varphi(x), \mathbf{n}\rangle \le 0$ on $\partial\Omega$, and

$$n \int_\Omega G^{1-\frac{1}{n}}(x)\, dx \le - \int_\Omega \langle \nabla\varphi(x), \nabla(F^{1-\frac{1}{n}})(x)\rangle\, dx. \tag{15}$$

Adding the integral

$$\int_\Omega \langle \mathbf{e}, \nabla(F^{1-\frac{1}{n}})(x)\rangle\, dx = \int_{\partial\Omega} F^{1-\frac{1}{n}}(x) \langle \mathbf{e}, \mathbf{n}\rangle\, d\Sigma = \sin(\theta_*) \int_{\partial\Omega} F^{1-\frac{1}{n}}(x)\, d\Sigma$$

to both parts of (15), we arrive at the inequality

$$\sin(\theta_*) \int_{\partial\Omega} F^{1-\frac{1}{n}}(x)\, d\Sigma + n \int_\Omega G^{1-\frac{1}{n}}(x)\, dx \le \int_\Omega \langle \mathbf{e} - \nabla\varphi(x), \nabla(F^{1-\frac{1}{n}})(x)\rangle\, dx. \tag{16}$$

Put $F = v^{p^*}$, $G = u^{p^*}$. Then $\|v\|_{p^*,\Omega} = \|u\|_{p^*,\Omega} = 1$, and (16) becomes

$$\sin(\theta_*) \|v\|^{p^{**}}_{p^{**},\partial\Omega} \le \frac{(n-1)p}{n-p} \int_\Omega v^{\frac{n(p-1)}{n-p}}(x) \langle \mathbf{e} - \nabla\varphi(x), \nabla v(x)\rangle\, dx - n\|u\|^{p^{**}}_{p^{**},\Omega}.$$

By the Hölder inequality and (12),

$$\begin{aligned}\sin(\theta_*)\|v\|^{p^{**}}_{p^{**},\partial\Omega} &\le \frac{(n-1)p}{n-p}\,\|\nabla v\|_{p,\Omega}\bigg(\int_\Omega v^{p^*}(x)|\mathbf{e}-\nabla\varphi(x)|^{p'}\,dx\bigg)^{\frac{1}{p'}} - n\|u\|^{p^{**}}_{p^{**},\Omega}\\ &= \frac{(n-1)p}{n-p}\,\|\nabla v\|_{p,\Omega}\bigg(\int_\Omega u^{p^*}(x)|\mathbf{e}-x|^{p'}\,dx\bigg)^{\frac{1}{p'}} - n\|u\|^{p^{**}}_{p^{**},\Omega}.\end{aligned} \tag{17}$$

Note that the two sides of (17) do not involve the Brenier map. Hence, by approximation, this inequality remains valid for all $u, v \in \dot W^1_p(\Omega)$ normalized in $L_{p^*}(\Omega)$.

Now we specify (17) by setting $u = Cw$, where w is defined in (11) and $C = \|w\|^{-1}_{p^*,\Omega}$ is the normalization constant. Then for any $v \in \dot W^1_p(\Omega)$ such that $\|v\|_{p^*,\Omega} = 1$, we have

$$\|v\|^{p^{**}}_{p^{**},\partial\Omega} \le A\|\nabla v\|_{p,\Omega} - B, \tag{18}$$

where

$$A = \frac{(n-1)p\,C^{\frac{n(p-1)}{n-p}}}{(n-p)\sin(\theta_*)}\cdot \mathcal{I}^{\frac{1}{p'}}, \quad B = \frac{n\,C^{p^{**}}}{\sin(\theta_*)}\cdot\mathcal{I}, \quad \mathcal{I} = \|w\|^{p^{**}}_{p^{**},\Omega} = \int_\Omega \frac{dx}{|x-\mathbf{e}|^{(n-1)p'}}.$$

For arbitrary $v \in \dot W^1_p(\Omega)$ (without normalization), (18) can be rewritten as follows:

$$\bigg(\frac{K(v)}{J_1(v)}\bigg)^{p^{**}} \le AK(v) - B \quad\Longleftrightarrow\quad J_1^{p^{**}}(v) \ge \mathcal{F}(K(v)) \equiv \frac{K^{p^{**}}(v)}{AK(v)-B},$$

where

$$J_1(v) = \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^{**},\partial\Omega}}, \qquad K(v) = \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^*,\Omega}}.$$

By elementary calculus, the function $\mathcal{F}$ achieves its minimum at the point

$$\frac{p(n-1)B}{n(p-1)A} = \frac{n-p}{p-1}\,C\mathcal{I}^{\frac{1}{p}} = K(w),$$

and therefore,

$$J_1^{p^{**}}(v) \ge \frac{K^{p^{**}}(w)}{AK(w)-B} = \Big(\frac{n-p}{p-1}\Big)^{\frac{n(p-1)}{n-p}}\mathcal{I}^{\frac{p-1}{n-p}}\sin(\theta_*). \tag{19}$$

If $v = u = Cw$, then the Brenier map is the identity. Direct calculations show that all the inequalities become equalities, and the statement follows. □

Thus, the right-hand side of (19) equals $\lambda^{p^{**}}(p,1,\Omega)$. By [GR, 3.252.10], [GR, 8.733.1], and [GR, 8.335.1], we have

$$\begin{aligned}\mathcal{I} &= \omega_{n-2}\int_0^{\theta_*}\int_0^\infty \frac{r^{n-1}\sin^{n-2}(\theta)\,dr\,d\theta}{(r^2+2r\cos(\theta)+1)^a}\\ &= \omega_{n-2}2^{a-\frac12}\Gamma\Big(a+\tfrac12\Big)\mathcal{B}(n,2a-n)\int_0^{\theta_*}\sin^{n-a-\frac32}(\theta)\,P^{\frac12-a}_{n-a-\frac12}\big(\cos(\theta)\big)\,d\theta\\ &= \pi^{\frac n2-1}2^{a-\frac32}\Gamma\Big(a-\tfrac{n-1}{2}\Big)\mathcal{B}\Big(\tfrac n2, a-\tfrac n2\Big)\sin^{n-a-\frac12}(\theta_*)P^{\frac12-a}_{n-a-\frac32}\big(\cos(\theta_*)\big),\end{aligned}$$

where $a = \frac{(n-1)p}{2(p-1)}$.

In particular, for $\theta_* = \frac{\pi}{2}$ we obtain

$$\lambda(p,1,\mathbb{R}^n_+) = \Big(\tfrac{n-p}{p-1}\Big)^{\frac{1}{p'}}\Big(\frac{\omega_{n-2}}{2}\cdot\mathcal{B}\Big(\tfrac{n-1}{2},\tfrac{n-1}{2(p-1)}\Big)\Big)^{\frac{1}{(n-1)p'}}.$$

Now we show that for nonconvex cones, Theorem 3 fails.

Theorem 4. *Let Ω be a nonconvex circular cone, i.e., let G be a spherical "hat"* (8), $\theta_* > \frac{\pi}{2}$. *Suppose* $1 < p < n$ *and* $\sigma = 1$. *Then the function* (11) *does not provide the minimum to the functional J_σ, though it is a stationary point.*

Proof. Direct calculations show that w is a positive solution of the Neumann problem

$$-\Delta_p u = 0 \quad \text{in} \quad \Omega, \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \mathbf{n}} = \gamma u^{p^{**}-1} \quad \text{on} \quad \partial\Omega, \tag{20}$$

where $\gamma = J_1^{p^{**}}(w)/\|\nabla w\|_{p,\Omega}^{p^{**}-p}$.

Let $\Omega_1 \subset \Omega$ be a half-space such that $\partial\Omega_1$ is a hyperplane tangent to $\partial\Omega$. Then w solves the same problem (20) in Ω_1.

By the dilation invariance of the quotient J_1, the function w provides the minimum to J_1 in Ω_1. Thus, $\gamma = \lambda^{p^{**}}(p,1,\mathbb{R}^n_+)/\|\nabla w\|_{p,\Omega_1}^{p^{**}-p}$, and $\Omega_1 \subset \Omega$ implies that $J_1(w) > \lambda(p,1,\mathbb{R}^n_+)$.

Now we claim that $\lambda(p,1,\Omega) \le \lambda(p,1,\mathbb{R}^n_+)$. Indeed, consider $w_k(x) = |x - x^k|^{-\frac{n-p}{p-1}}$, where $x^k \notin \bar{\Omega}$, $x^k \to x^0 \neq 0$, and $x^0 \in \partial\Omega$. Since the neighborhood of x^0 in the large scale looks like a half-space, the dilation invariance of the quotient J_σ yields $\lim_k J_1(w_k) = \lambda(p,1,\mathbb{R}^n_+)$, and the statement follows. □

Remark 3. One can see from the proof that Theorem 3 remains valid for any convex cone Ω if its support hyperplanes at almost every point have a constant angle with the axis x_n. The simplest example of such a cone is a dihedral angle less than a half-space. Another interesting example is a cone such that G is an *arbitrary* simplex in $\mathbb{S}^{n-1}$.

Similarly, Theorem 4 remains valid if Ω is a complement of such a convex cone.

Remark 4. The claims of Theorems 3 and 4 hold true also for a more general definition of the norm in the numerator of J_1. Namely, consider an arbitrary norm $|||x|||$ in $\mathbb{R}^n$ and replace $\|\nabla v\|_{p,\Omega}$ by

$$|||\nabla v|||_{p,\Omega} = \left(\int_\Omega |||\nabla v|||^p \, dx\right)^{\frac{1}{p}}.$$

Then Theorems 3 and 4 remain valid if we replace the Euclidean norm in (11) by the dual norm

$$|||x|||_o = \sup_{|||\xi|||\le 1} \langle x, \xi\rangle = \sup_{\xi \neq 0} \frac{\langle x, \xi\rangle}{|||\xi|||}.$$

The proofs run without changes.

§4. Intermediate case

Now we consider the case where $\frac{1}{p} < \sigma < \min\{1, \frac{n}{p}\}$. For the conventional Hardy–Sobolev inequality, the corresponding sharp constants in cones are attained (see [N], [N1]). This is the case also for the trace embedding.

Theorem 5. *Let* $1 < p < \infty$, $p \neq n$, $\frac{1}{p} < \sigma < \min\{1, \frac{n}{p}\}$. *Then the infimum in* (I_σ) *is attained.*

Remark 5. Standard arguments show that under suitable normalization the minimizer of J_σ is a positive solution of the Neumann problem

$$-\Delta_p u = 0 \quad \text{in} \quad \Omega, \qquad |\nabla u|^{p-2}\frac{\partial u}{\partial \mathbf{n}} = \frac{u^{p_\sigma^{**}-1}}{r^{(1-\sigma)p_\sigma^{**}}} \quad \text{on} \quad \partial\Omega. \tag{21}$$

Proof. Consider a minimizing sequence $\{v_k\}$ for the quotient J_σ. Since this quotient is homogeneous of degree zero and invariant with respect to dilations, we can ensure the relations

$$\int_\Omega |\nabla v_k|^p\, dx = 1, \qquad \int_{B_1\cap\Omega} |\nabla v_k|^p\, dx = \frac{1}{2}. \tag{22}$$

There is no loss of generality in assuming that $v_k \rightharpoonup v$ in $\dot W_p^1(\Omega)$.

By the P.-L. Lions concentration-compactness principle (see [Ls, Part 1] and also [N1, Lemma 3.1]), we have

$$\begin{aligned} &|r^{\sigma-1}v_k|^{p_\sigma^{**}} \rightharpoonup |r^{\sigma-1}v|^{p_\sigma^{**}} + \alpha_0\delta_0(x) + \alpha_\infty\delta_\infty(x),\\ &|\nabla v_k|^p \rightharpoonup \mu \geq |\nabla v|^p + \beta_0\delta_0(x) + \beta_\infty\delta_\infty(x), \end{aligned} \tag{23}$$

where

$$\alpha_0, \alpha_\infty \geq 0, \quad \beta_0 = \lambda^p(p,\sigma,\Omega)\alpha_0^{p/p^{**}}, \quad \beta_\infty = \lambda^p(p,\sigma,\Omega)\alpha_\infty^{p/p^{**}}.$$

The second convergence in (23) is in the sense of measures on the compact set $\mathcal{K} = \bar\Omega \cup \{\infty\}$, and the first is in the sense of measures on $\partial\mathcal{K}$.

Since $\{v_k\}$ is a minimizing sequence, we can repeat the arguments in [N1, Theorem 3.1] to obtain the following alternative: either v is a minimizer of $J_\sigma(v)$ and $\alpha_0 = \alpha_\infty = 0$, or $v = 0$; in the latter case, $\beta_0 = 1$ or $\beta_\infty = 1$. However, the second relation in (22) allows only the first possibility, and the statement follows. □

Let $\Omega = \mathbb{R}^n_+$. Then we could expect that the least energy solution of (21) is symmetric. For $p < n$ this is the case, the minimizer of $J_\sigma(u)$ is radially symmetric in x', i.e., $u = u(|x'|, x_n)$. This follows from the properties of the Schwarz symmetrization with respect to the x'-variables (or from the properties of the Steiner symmetrization with respect to x_1 for $n = 2$). Indeed, this transformation reduces the numerator in J_σ (see, e.g., [PS, Chapter 7]) and enlarges the denominator (see, e.g., [LL, Chapter 3]). Moreover, u is symmetrically decreasing in x' for every $x_n > 0$ and decreasing in x_n for any $x' \in \mathbb{R}^{n-1}$ (the latter claim can be proved by monotone rearrangement with respect to x_n).

For $p > n$ the symmetrization arguments do not work because of the condition $u(0) = 0$. It turns out that the symmetry of the minimizer really breaks in this case. For the conventional Hardy–Sobolev inequality in $\mathbb{R}^n \setminus \{0\}$, this effect was established in [N1, Theorem 4.1].

Theorem 6. *Let G be a spherical "hat" (8). Then for any $p > n$ there exists $\widehat\sigma(n,p,\theta_*) < \frac{n}{p}$ such that for $\sigma > \widehat\sigma$ no symmetric function $u(|x'|, x_n)$ gives the minimum to the functional J_σ.*

Proof. Let $u(|x'|, x_n)$ be a function providing the minimum to the functional J_σ over the set of symmetric functions in $\dot W_p^1(\Omega)$. Without loss of generality, we assume that $\|u\|_{p_\sigma^{**},\sigma,\partial\Omega} = 1$. By the principle of symmetric criticality, see [P], $dJ_\sigma(u;h) = 0$ for any variation $h \in \dot W_p^1(\Omega)$.

As in [N2, Theorem 1.3], the second differential of J_σ at the point u can be written as follows:

(24)

$$\begin{aligned} J_\sigma^{p-1}(u)\cdot d^2J_\sigma(u;\ h) &= \int_\Omega |\nabla u|^{p-4}\big((p-2)\langle\nabla u, \nabla h\rangle^2 + |\nabla u|^2|\nabla h|^2\big)\, dx\\ &- J_\sigma^p(u)\cdot\Big[(p-p_\sigma^{**})\cdot\Big(\int_{\partial\Omega}\frac{|u|^{p_\sigma^{**}-2}uh}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma\Big)^2 + (p_\sigma^{**}-1)\cdot\int_{\partial\Omega}\frac{|u|^{p_\sigma^{**}-2}h^2}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma\Big]. \end{aligned}$$

Now we set $h(x) = u(|x'|, x_n)f(x)$, where $f(x) = \frac{x_1}{r}$. By the symmetry of u, we have $\int_{\partial\Omega} \frac{|u|^{p_\sigma^{**}-2}uh}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma = 0$. Next,

$$\langle \nabla u, \nabla h\rangle^2 = |\nabla u|^2 \langle \nabla u, \nabla(uf^2)\rangle + u^2\langle \nabla u, \nabla f\rangle^2, \qquad |\nabla h|^2 = \langle \nabla u, \nabla(uf^2)\rangle + u^2|\nabla f|^2.$$

Substituting these formulas into (24) and using the relation $dJ_\sigma(u; uf^2) = 0$, we obtain

$$J_\sigma^{p-1}(u)\cdot d^2J_\sigma(u;\ h) = \int_\Omega |\nabla u|^{p-4}u^2\left((p-2)\langle \nabla u, \nabla f\rangle^2 + |\nabla u|^2|\nabla f|^2\right)\, dx$$
$$- J_\sigma^p(u)\cdot (p_\sigma^{**} - p)\cdot \int_{\partial\Omega} \frac{|u|^{p_\sigma^{**}}f^2}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma.$$

Note that

$$\int_{\partial\Omega} \frac{|u|^{p_\sigma^{**}}}{r^{(1-\sigma)p_\sigma^{**}}}\ \frac{x_1^2}{r^2}\, d\Sigma = \frac{1}{n-1}\int_{\partial\Omega} \frac{|u|^{p_\sigma^{**}}}{r^{(1-\sigma)p_\sigma^{**}}}\ \frac{|x'|^2}{r^2}\, d\Sigma$$
$$= \frac{\sin^2(\theta_*)}{n-1}\int_{\partial\Omega} \frac{|u|^{p_\sigma^{**}}}{r^{(1-\sigma)p_\sigma^{**}}}\, d\Sigma = \frac{\sin^2(\theta_*)}{n-1}$$

(the last identity is the normalization condition for u). Since $|\nabla f| \le \frac{1}{r}$, we get

$$J_\sigma^{p-1}(u)\cdot d^2J_\sigma(u;\ h) \le (p-1)\cdot \int_\Omega |\nabla u|^{p-2}\frac{u^2}{r^2}\, dx - J_\sigma^p(u)\cdot (p_\sigma^{**} - p)\cdot \frac{\sin^2(\theta_*)}{n-1}. \tag{25}$$

Finally, we estimate the integral in (25) by the Hölder and Hardy inequalities, arriving at

$$d^2J_\sigma(u;\ h) \le J_\sigma(u)\cdot \left[(p-1)\left(\tfrac{p}{p-n}\right)^2 - (p_\sigma^{**} - p)\cdot \frac{\sin^2(\theta_*)}{n-1}\right].$$

The quantity in the square brackets is negative for σ close to $\frac{n}{p}$, and the statement follows. □

Corollary. *For $p > n$ and $\widehat{\sigma} < \sigma < \frac{n}{p}$, problem* (21) *in a circular cone has at least two nonequivalent positive solutions.*

Proof. The first solution is a global minimizer of J_σ (under suitable normalization), and the second is a minimizer over the set of symmetric functions. □

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