Алгебра и анализ
Том 22 (2010), № 6

St. Petersburg Math. J.
Vol. 22 (2011), No. 6, Pages 1023–1049
S 1061-0022(2011)01182-3
Article electronically published on August 22, 2011

ON THE LINEAR PROBLEM ARISING IN THE STUDY OF A FREE BOUNDARY PROBLEM FOR THE NAVIER–STOKES EQUATIONS

V. A. SOLONNIKOV

Dedicated to Professor V. M. Babich on the occasion of his 80th birthday

Abstract. A problem under study arises as a result of linearization of a free boundary problem for Navier–Stokes equations governing the evolution of an isolated mass of a viscous incompressible capillary liquid.

§1. Introduction

The paper is devoted to the linear problem

$$\begin{cases} \boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\ \nabla\cdot\boldsymbol{v} = f(x,t) = \nabla\cdot\boldsymbol{F}(x,t), \quad x\in\mathcal{F},\ \ t>0, \\ T(\boldsymbol{v},p)\boldsymbol{N}(x) + \sigma\boldsymbol{N}(x)\mathfrak{L}\rho = \boldsymbol{d}(x,t), \\ \rho_t(x,t) + \boldsymbol{V}(x)\cdot\nabla_\tau\rho - \boldsymbol{v}(x,t)\cdot\boldsymbol{N}(x) = g(x,t), \quad x\in\mathcal{G}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x),\ \ x\in\mathcal{F}, \quad \rho(x,0) = \rho_0(x),\ \ x\in\mathcal{G}, \end{cases} \tag{1.1}$$

in a bounded domain $\mathcal{F}\subset\mathbb{R}^3$ with a smooth boundary $\mathcal{G}$. The unknowns are the vector field $\boldsymbol{v}(x,t) = (v_1,v_2,v_3)$ and the functions $p(x,t)$, $x\in\mathcal{F}$, and $\rho(x,t)$, $x\in\mathcal{G}$. By $T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$ we mean the stress tensor, $S(\boldsymbol{v}) = \big(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j}\big)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, $\boldsymbol{N}$ is the outward normal to $\mathcal{G}$, ν and σ are positive constants, and $\mathfrak{L}\rho = -\Delta_{\mathcal{G}}\rho + b(x)\rho$, where $\Delta_{\mathcal{G}}$ is the Laplace–Beltrami operator on $\mathcal{G}$ and $b(x)$ is a smooth function. Finally, $\boldsymbol{V}(x)$ is a vector field defined on $\mathcal{G}$ and ∇_τ is the tangential part of the gradient.

Problem (1.1) arises as a result of linearization of a free boundary problem for the Navier–Stokes equations governing the evolution of an isolated mass of a viscous incompressible capillary liquid. The latter was studied in the papers [1, 2] and others, where the method of the Lagrangian coordinates was used. This turned out to be especially fruitful in the case where the surface tension is not taken into account [3]. Problem (1.1) is obtained by applying the so-called Hanzawa coordinate transformation to the free boundary problem in order to write it in a fixed domain (see formula (5.2)). This transformation provides some technical advantages in the case of a capillary liquid with positive coefficient σ of the surface tension. We intend to apply the results of the present paper to the analysis of problems of magnetohydrodynamics.

In [4], problem (1.1) was studied in the Hölder spaces of functions.

2010 *Mathematics Subject Classification.* Primary 35Q35, 76D27.
Key words and phrases. Navier–Stokes equations, Laplace–Beltrami operator, linearization, free boundary problem.

The main result of the paper is a coercive estimate of the solution of problem (1.1) in anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}(Q_T)$ in a cylindrical domain $Q_T = \mathcal{F} \times (0,T)$. We recall the definition of these spaces. Let Ω be a domain in $\mathbb{R}^n$. The (isotropic) Sobolev space $W_2^l(\Omega)$ with $l > 0$ is the space of functions $u(x)$, $x \in \Omega$, with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{0\le|j|\le l} \|D^j u\|^2_{L_2(\Omega)} \equiv \sum_{0\le|j|\le l} \int_\Omega |D^j u(x)|^2\,dx$$

if $l = [l]$, i.e., l is an integer, and

$$\|u\|^2_{W_2^l(\Omega)} = \|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|j|=[l]} \int_\Omega\int_\Omega |D^j u(x) - D^j u(y)|^2 \frac{dx\,dy}{|x-y|^{n+2\lambda}}$$

if $l=[l]+\lambda$, $\lambda\in(0,1)$. As usual, $D^j u$ denotes a (generalized) partial derivative $\frac{\partial^{|j|}u}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}}$, where $j = (j_1, j_2, \dots, j_n)$ and $|j| = j_1 + \cdots + j_n$. The anisotropic space $W_2^{l,l/2}(Q_T)$, $Q_T = \Omega \times (0,T)$, can be defined as the space $L_2((0,T), W_2^l(\Omega)) \cap W_2^{l/2}((0,T), L_2(\Omega))$ supplied with the norm

$$\|u\|^2_{W_2^{l,l/2}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}\,dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{l/2}(0,T)}\,dx. \tag{1.2}$$

There exist many other equivalent norms in $W_2^{l,l/2}(Q_T)$; some of them will be used below. Sobolev spaces of functions given on smooth surfaces, in particular, on $\mathcal{G}$ and on $G_T = \mathcal{G}\times(0,T)$, are introduced in a standard way, with the help of local maps and partition of unity. We also find it convenient to introduce the spaces $W_2^{l,0}(Q_T) = L_2((0,T), W_2^l(\Omega))$ and $W_2^{0,l/2}(Q_T) = W_2^{l/2}((0,T), L_2(\Omega))$; the squares of norms in these spaces coincide, respectively, with the first and the second terms in (1.2). Finally, by $|u|_{l/2,r,Q_T}$ and $|u|_{l/2,r,G_T}$ we mean the norms of u in $W_2^{l/2}(0,T; W_2^r(\Omega))$ and $W_2^{l/2}(0,T; W_2^r(\mathcal{G}))$, respectively.

Theorem 1.1. *Assume that $l \in [0, 5/2)$, $l \ne 1/2, 1, 3/2$, and that the data of problem (1.1) possess the following regularity properties: $\boldsymbol{f} \in W_2^{l,l/2}(Q_T)$, $f \in W_2^{l+1,0}(Q_T)$, $f(x,t) = \nabla\cdot\boldsymbol{F}(x,t)$, $\boldsymbol{F} \in W_2^{0,1+l/2}(Q_T)$, $\boldsymbol{d}\cdot\boldsymbol{N} \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{1/2}(\mathcal{G}))$, $\boldsymbol{d} - \boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N}) \in W_2^{l+1/2,l/2+1/4}(G_T)$, $g \in W_2^{l+3/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{3/2}(\mathcal{G}))$, $\boldsymbol{v}_0 \in W_2^{l+1}(\mathcal{F}_1)$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$, where $T < \infty$, $Q_T = \mathcal{F}_1 \times (0,T)$, $G_T = \mathcal{G}\times(0,T)$. Assume also that $\boldsymbol{V} \in W_2^{l+3/2}(\mathcal{G})$. Finally, let the compatibility conditions*

$$\begin{aligned} &\nabla\cdot\boldsymbol{v}_0(x) = f(x,0), \quad x\in\mathcal{F}, \quad \text{if} \quad l<1/2,\\ &\nabla\cdot\boldsymbol{v}_0(x) = f(x,0), \quad x\in\mathcal{F}, \quad \nu\Pi_{\mathcal{G}}S(\boldsymbol{v}_0)\boldsymbol{N} = \Pi_{\mathcal{G}}\boldsymbol{d}(x,0), \quad x\in\mathcal{G}, \quad \text{if} \quad l>1/2 \end{aligned} \tag{1.3}$$

be satisfied, where $\Pi_{\mathcal{G}}\boldsymbol{d} = \boldsymbol{d} - \boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})$ is the projection of $\boldsymbol{d}$ to the tangent plane to $\mathcal{G}$. Then problem (1.1) has a unique solution $\boldsymbol{v}, p, \rho$ such that $\boldsymbol{v} \in W_2^{l+2,l/2+1}(Q_T)$, $\nabla p \in W_2^{l,l/2}(Q_T)$, $p \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{1/2}(\mathcal{G}))$, and the function ρ satisfies

$\rho \in W_2^{l+5/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))$, $\rho_t \in W_2^{l+3/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{3/2}(\mathcal{G})$, $\rho(\cdot,t) \in W_2^{l+2}(\mathcal{G})$ *for all* $t \in (0,T)$, *and this solution satisfies the inequality*

(1.4)
$$\begin{aligned} Y_T(\boldsymbol{v},p,\rho) \equiv{}& \|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} \\ &+ |p|_{l/2,1/2,G_T} + \|\rho\|_{W_2^{l+5/2,0}(G_T)} + |\rho|_{l/2,5/2,G_T} \\ &+ \|\rho_t\|_{W_2^{l+3/2,0}(G_T)} + |\rho_t|_{l/2,3/2,G_T} \\ \le{}& c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)} \\ &+ \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + |\boldsymbol{d}\cdot\boldsymbol{N}|_{l/2,1/2,G_T} \\ &+ \|g\|_{W_2^{l+3/2,0}(G_T)} + |g|_{l/2,3/2,G_T} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big) \\ \equiv{}& c(T)N_T. \end{aligned}$$

Moreover, if $g \in W_2^{l+3/2,l/2+3/4}(G_T)$, *then* $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$, *and*

(1.5)
$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + |p|_{l/2,1/2,G_T} \\ &\quad + \|\rho\|_{W_2^{l+5/2,0}(G_T)} + |\rho|_{l/2,5/2,G_T} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \\ &\le c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)} \\ &\quad + \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + |\boldsymbol{d}\cdot\boldsymbol{N}|_{l/2,1/2,G_T} \\ &\quad + \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big). \end{aligned}$$

The restriction $l < 5/2$ minimizes the order of compatibility of the initial and boundary data expressed by (1.3). The requirement $l \neq 1/2, 1, 3/2$ is technical; it is imposed to avoid the cases where the compatibility conditions (1.3) should be modified substantially (in this connection, see [5, 6]).

The imbedding theorems show that in the case where $f = 0$, $\boldsymbol{F} = 0$ the estimate (1.4) is coercive, i.e.,

$$N_T \le cY_T(\boldsymbol{v},p,\rho);$$

the same is true for (1.5). Hence, Theorem 1.1 guarantees the existence of a solution of problem (1.1) with maximal regularity properties.

By the trace theorem for the space $W_2^{l+2,l/2+1}(Q_T)$, we have $\boldsymbol{v}(\cdot,t) \in W_2^{l+1}(\mathcal{F})$, i.e., $\boldsymbol{v}$ is as smooth as $\boldsymbol{v}_0$. Proposition 4.1 implies that the same is true for ρ.

The proof of Theorem 1.1 is given in §§2–4. §5 contains a short discussion (without detailed proofs) of an application of Theorem 1.1 to the free boundary problem with initial domain of arbitrary shape and with an initial velocity vector field $\boldsymbol{v}_0(x)$ that need not be small.

§2. Parameter-dependent problem

As in [7], we consider the problem with a complex parameter s:

$$\begin{cases} s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x \in \mathcal{F}, \\ T(\boldsymbol{v},p)\boldsymbol{N} + \sigma\boldsymbol{N}\mathfrak{L}\rho = \boldsymbol{d}(x), \\ s\rho + \boldsymbol{V}(x)\cdot\nabla_\tau\rho - \boldsymbol{v}(x)\cdot\boldsymbol{N}(x) = g(x), \quad x \in \mathcal{G}. \end{cases} \tag{2.1}$$

The solution of (2.1) is also sought in the space of complex-valued functions.

Theorem 2.1. *Suppose* $\operatorname{Re} s \ge a \gg 1$, $\boldsymbol{f} \in W_2^l(\mathcal{F})$, $\boldsymbol{d} \in W_2^{l+1/2}(\mathcal{G})$, *and* $g \in W_2^{l+3/2}(\mathcal{G})$ *with* $l \in [0, 5/2)$. *Then problem* (2.1) *has a unique solution* $\boldsymbol{v} \in W_2^{2+l}(\mathcal{F})$, $p \in W_2^{1+l}(\mathcal{F})$, $\rho \in W_2^{l+5/2}(\mathcal{G})$, *and*

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{2+l}(\mathcal{F})} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathcal{F})} + \|p\|_{W_2^{l+1}(\mathcal{F})} + |s|^{l/2}\|p\|_{W_2^1(\mathcal{F})} \\ &\quad + |s|^{1+l/2}\|\rho\|_{W_2^{3/2}(\mathcal{G})} + |s|\,\|\rho\|_{W_2^{l+3/2}(\mathcal{G})} + |s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathcal{G})} + \|\rho\|_{W_2^{l+5/2}(\mathcal{G})} \\ &\quad \le c\big(\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathcal{F})} + |s|^{1/4+l/2}\|\boldsymbol{d} - \boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})\|_{L_2(\mathcal{G})} \\ &\qquad + \|\boldsymbol{d}\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathcal{G})} + |s|^{l/2}\|g\|_{W_2^{3/2}(\mathcal{G})} + \|g\|_{W_2^{l+3/2}(\mathcal{G})}\big) \end{aligned} \tag{2.2}$$

with constant independent of $|s|$ (*but, possibly, depending on* a).

Proof. We start with the proof of estimate (2.2). Without loss of generality, we may assume that $\boldsymbol{f}$ is divergence free, because any $\boldsymbol{f} \in L_2(\mathcal{F})$ can be decomposed into the orthogonal sum

$$\boldsymbol{f} = \boldsymbol{f}' + \nabla\varphi,$$

where $\boldsymbol{f}'$ is divergence free and φ is a solution of the Dirichlet problem

$$\nabla^2\varphi = \nabla\cdot\boldsymbol{f}, \quad x \in \mathcal{F}, \quad \varphi|_{\mathcal{G}} = 0.$$

Since

$$c_1\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} \le \|\nabla\varphi\|_{W_2^l(\mathcal{F})} + \|\boldsymbol{f}'\|_{W_2^l(\mathcal{F})} \le c_2\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})},$$

problem (2.1) is equivalent to a similar problem with $\boldsymbol{f}$ and p replaced by $\boldsymbol{f}'$ and $p' = p - \nabla\varphi$, respectively.

Step 1. We consider the following model problem in the half-space $\mathbb{R}^3_+ = \{x_3 > 0\}$:

$$\begin{cases} s\boldsymbol{v}(x) + (\boldsymbol{V}'\cdot\nabla')\boldsymbol{v}(x) - \nu\nabla^2\boldsymbol{v}(x) + \nabla p(x) = 0, \\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x_3 > 0, \\ \nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = b_j(x'), \quad j = 1,2, \\ -p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\Delta'\rho = b_3(x'), \\ s\rho + \boldsymbol{V}'\cdot\nabla'\rho + v_3(x) = g(x), \quad x_3 = 0, \end{cases} \tag{2.3}$$

where $\boldsymbol{V}'$ is a constant vector of the form $\boldsymbol{V}' = (V_1, V_2)$, $x' = (x_1, x_2)$, and $\nabla' = \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\big)$. Using the Fourier transformation in x_1, x_2, we reduce (2.3) to a boundary value problem on the half-axis $\mathbb{R}_+ = \{x_3 > 0\}$:

$$\begin{cases} \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + i\xi_j\widetilde{p} = 0, \quad j = 1,2, \\ \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_3 + \dfrac{d\widetilde{p}}{dx_3} = 0, \quad i\xi_1\widetilde{v}_1 + i\xi_2\widetilde{v}_2 + \dfrac{d\widetilde{v}_3}{dx_3} = 0, \quad x_3 > 0, \\ \nu\Big(\dfrac{d\widetilde{v}_j}{dx_3} + i\xi_j\widetilde{v}_3\Big) = \widetilde{b}_j, \quad j = 1,2, \\ -\widetilde{p} + 2\nu\dfrac{d\widetilde{v}_3}{dx_3} + \sigma|\xi|^2\widetilde{\rho} = \widetilde{b}_3, \\ s_1\widetilde{\rho} + \widetilde{v}_3 = \widetilde{g}, \quad x_3 = 0, \\ \widetilde{\boldsymbol{v}} \to 0, \quad \widetilde{p} \to 0 \quad (x_3 \to \infty), \end{cases} \tag{2.4}$$

where $\xi = (\xi_1, \xi_2)$, $r_1 = r_1(s,\xi) = \sqrt{s_1\nu^{-1} + |\xi|^2}$, $-\pi \le \arg\ r_1 < \pi$, and $s_1 = s + i\boldsymbol{V}'\cdot\xi$.

It is convenient to exclude the function $\widetilde{\rho}$ from (2.4), writing this problem in the form

$$
\begin{cases}
\nu\Big(r_1^2-\dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j+i\xi_j\widetilde{p}=0, \quad j=1,2,\\
\nu\Big(r_1^2-\dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j+\dfrac{d\widetilde{p}}{dx_3}=0, \quad i\xi_1\widetilde{v}_1+i\xi_2\widetilde{v}_2+\dfrac{d\widetilde{v}_3}{dx_3}=0, \quad x_3>0,\\
\nu\Big(\dfrac{d\widetilde{v}_j}{dx_3}+i\xi_j\widetilde{v}_3\Big)=\widetilde{b}_j, \quad j=1,2,\\
-\widetilde{p}+2\nu\dfrac{d\widetilde{v}_3}{dx_3}-\dfrac{\sigma}{s_1}|\xi|^2\widetilde{v}_3=\widetilde{b}_3-\dfrac{\sigma}{s_1}|\xi|^2\widetilde{g}, \quad x_3=0,\\
\widetilde{\boldsymbol{v}}\to 0, \quad \widetilde{p}\to 0 \quad (x_3\to\infty).
\end{cases}
\tag{2.5}
$$

In the paper [2], an explicit formula for the solution of (2.5) was obtained; in particular, it was shown that, if $\operatorname{Re} s_1>0$, then

$$
\begin{aligned}
\widetilde{v}_i=&-\frac{1-\delta_{i3}}{\nu r_1}e_0(x_3)\widetilde{b}_i+\frac{e_0(x_3)}{\nu^2 r_1(r_1+|\xi|)P_1}\sum_{j=1}^3 U_{ij}\widetilde{b}_j+\frac{e_1(x_3)}{\nu^2(r_1+|\xi|)P_1}\sum_{j=1}^3 V_{ij}\widetilde{b}_j\\
&-\frac{\sigma|\xi|^2e_0(x_3)}{\nu^2 s_1 r_1(r_1+|\xi|)P_1}U_{i3}\widetilde{g}-\frac{\sigma|\xi|^2 e_1(x_3)}{\nu^2 s_1(r_1+|\xi|)P_1}V_{i3}\widetilde{g}, \quad i=1,2,3,
\end{aligned}
\tag{2.6}
$$

$$
\widetilde{p}=\frac{r_1 s_1}{\nu P_1}\Big[\Big(2\nu+\frac{\sigma\xi^2}{s_1 r_1}\Big)(i\xi_1\widetilde{b}_1+i\xi_2\widetilde{b}_2)-\nu\Big(r_1+\frac{\xi^2}{r_1}\Big)\Big(\widetilde{b}_3-\frac{\sigma}{s_1}|\xi|^2\widetilde{g}\Big)\Big]e^{-|\xi|x_3},
\tag{2.7}
$$

where

$$
e_0(x_3)=e^{-r_1x_3}, \quad e_1(x_3)=\frac{e^{-r_1x_3}-e^{-|\xi|x_3}}{r_1-|\xi|},
\tag{2.8}
$$

$$
P_1=(r_1^2+|\xi|^2)^2-4r_1|\xi|^2+\frac{\sigma}{\nu^2}|\xi|^3=\frac{s_1}{\nu}\Big(\frac{s_1}{\nu}+4|\xi|^2\Big(1-\frac{|\xi|}{r_1+|\xi|}\Big)+\frac{\sigma|\xi|^3}{\nu s_1}\Big),
\tag{2.9}
$$

and U_{ij}, V_{ij} are the entries of the matrices

$$
\mathcal{U}=\begin{pmatrix}
\xi_1^2((3r_1-|\xi|)s_1+\frac{\sigma}{\nu}|\xi|^2) & \xi_1\xi_2((3r_1-|\xi|)s_1+\frac{\sigma}{\nu}|\xi|^2) & i\xi_1 r_1 s_1(r_1-|\xi|)\\
\xi_1\xi_2((3r_1-|\xi|)s_1+\frac{\sigma}{\nu}|\xi|^2) & \xi_2^2((3r_1-|\xi|)s_1+\frac{\sigma}{\nu}|\xi|^2) & i\xi_1 r_1 s_1(r_1-|\xi|)\\
-i\xi_1 r_1 s_1(r_1-|\xi|) & -i\xi_2 r_1 s_1(r_1-|\xi|) & -|\xi| r_1 s_1(r_1+|\xi|)
\end{pmatrix},
$$

$$
\mathcal{V}=\begin{pmatrix}
-\xi_1^2(2r_1s_1+\frac{\sigma}{\nu}|\xi|^2) & -\xi_1\xi_2(2r_1s_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_1 s_1(r_1^2+|\xi|^2)\\
-\xi_1\xi_2(2r_1s_1+\frac{\sigma}{\nu}|\xi|^2) & -\xi_2^2(2r_1s_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_2 s_1(r_1^2+|\xi|^2)\\
-i\xi_1|\xi|(2r_1s_1+\frac{\sigma}{\nu}|\xi|^2) & -i\xi_2\xi|(2r_1s_1+\frac{\sigma}{\nu}|\xi|^2) & |\xi| s_1(r_1^2+|\xi|^2)
\end{pmatrix}.
$$

In [2] it was shown that for $\operatorname{Re} s_1\geq\gamma>0$ we have

$$
\begin{gathered}
\frac{\gamma^2}{\nu^2}+|s_1|\,|\xi|^2+|s_1|^2+\sigma|\xi|^3\leq c(\gamma)|P_1|,\\
\int_0^\infty\Big|\frac{d^j e_0(x_3)}{dx_3^j}\Big|^2\,dx_3\leq\frac{1}{\sqrt{2}}|r_1|^{2j-1},\\
\int_0^\infty\Big|\frac{d^j e_1(x_3)}{dx_3^j}\Big|^2\,dx_3\leq c\frac{|r_1|^{2j-1}+|\xi|^{2j-1}}{|r_1|^2},\\
\int_0^\infty\int_0^\infty\Big|\frac{d^j e_0(x_3+z)}{dx_3^j}-\frac{d^j e_0(x_3)}{dx_3^j}\Big|^2\frac{dx_3dz}{|z|^{1+2\kappa}}\leq c|r_1|^{2(j+\kappa)-1},\\
\int_0^\infty\int_0^\infty\Big|\frac{d^j e_1(x_3+z)}{dx_3^j}-\frac{d^j e_1(x_3)}{dx_3^j}\Big|^2\frac{dx_3dz}{|z|^{1+2\kappa}}\leq c\frac{|r_1|^{2(j+\kappa)j-1}+|\xi|^{2(j+\kappa)-1}}{|r_1|^2},
\end{gathered}
\tag{2.10}
$$

where $j \geq 0$, $\kappa \in (0,1)$. Moreover, if $\operatorname{Re} s \geq \gamma > 0$ and $\gamma > (2\nu)^{-1}|\boldsymbol{V}'|^2$, then

$$c|r_1(s_1,\xi)| \leq \sqrt{|s|+|\xi|^2} \leq c'|r_1(s_1,\xi)|.$$

Using the above inequalities and repeating the calculations in the proof of Theorem 3.1 in [2] (carried out in the case where $\boldsymbol{V}'=0$), we obtain

$$\begin{aligned}
&\|\widetilde{\boldsymbol{v}}\|^2_{\dot W_2^{l+2}(\mathbb{R}_+)} + |r(s,\xi)|^{2(l+2)}\|\widetilde{\boldsymbol{v}}\|^2_{L_2(\mathbb{R}_+)} + \|\widetilde{p}_{x_3}\|^2_{\dot W_2^{l}(\mathbb{R}_+)} + |r|^{2l}|\xi|^2\|\widetilde{p}\|^2_{L_2(\mathbb{R}_+)}\\
&\quad\leq c(\|\widetilde{\boldsymbol{v}}\|^2_{\dot W_2^{l+2}(\mathbb{R}_+)} + |r_1(s_1,\xi)|^{2(l+2)}\|\widetilde{\boldsymbol{v}}\|^2_{L_2(\mathbb{R}_+)} + \|\widetilde{p}_{x_3}\|_{L_2(\mathbb{R}_+)} + |r_1|^{2l}|\xi|^2\|\widetilde{p}\|^2_{L_2(\mathbb{R}_+)})\\
&\quad\leq c(|r_1|^{2l+1}(|\widetilde{b}_1|^2+|\widetilde{b}_2|^2) + |\xi|\,|r_1|^{2l}|\widetilde{b}_3|^2 + |r_1|^{2l}|\xi|^3|\widetilde{g}|^2)\\
&\quad\leq c(|r|^{2l+1}(|\widetilde{b}_1|^2+|\widetilde{b}_2|^2) + |\xi|\,|r|^{2l}|\widetilde{b}_3|^2 + |r|^{2l}|\xi|^3|\widetilde{g}|^2),
\end{aligned}$$

where $\|\cdot\|_{\dot W_2^l(\mathbb{R}^n)}$ is the principal part of the norm in $W_2^l(\mathbb{R}^n)$:

$$\|u\|^2_{\dot W_2^l(\mathbb{R}^n)} = \sum_{|j|=[l]}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|D^ju(x)-D^ju(y)|^2\frac{dx\,dy}{|x-y|^{n+2\lambda}}, \quad \lambda = l-[l]\in(0,1).$$

Now we integrate this inequality with respect to $\xi\in\mathbb{R}^2$ and use the Parceval formula. This leads to

(2.11)
$$\begin{aligned}
&\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{2+l}\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla p\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla p\|^2_{L_2(\mathbb{R}^3_+)}\\
&\quad\leq c\Big(\|\boldsymbol{b}\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l+1/2}\|\boldsymbol{b}'\|^2_{L_2(\mathbb{R}^2)} + |s|^l\|b_3\|^2_{W_2^{1/2}(\mathbb{R}^2)}\\
&\qquad\qquad + \|g\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^l\|g\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big).
\end{aligned}$$

We supplement (2.11) with estimates for $p|_{x_3=0}\equiv p(0)$ and ρ. By (2.7), we have

$$|\widetilde{p}(0)| \leq c(|\widetilde{\boldsymbol{b}}| + (1+|\xi|)|\widetilde{g}|), \tag{2.12}$$

which implies that

$$\begin{aligned}
\|p(0)\|_{W_2^{1/2}(\mathbb{R}^2)} &\leq c\Big(\|\boldsymbol{b}\|_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{3/2}(\mathbb{R}^2)}\Big),\\
\|p(0)\|_{W_2^{l+1/2}(\mathbb{R}^2)} &\leq c\Big(\|\boldsymbol{b}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}\Big).
\end{aligned} \tag{2.13}$$

To estimate the norms of ρ, we use the identities

$$\begin{aligned}
&s_1\widetilde{\rho} = \widetilde{g} - \widetilde{v}_3(0),\\
&\sigma|\xi|^2\widetilde{\rho} = \widetilde{b}_3 + \Big(\widetilde{p} - 2\nu\frac{d\widetilde{v}_3}{dx_3}\Big)\Big|_{x_3=0} = \widetilde{b}_3 + \Big(\widetilde{p}(0) + 2\nu\sum_{j=1}^2 i\xi_j\widetilde{v}_j(0)\Big).
\end{aligned} \tag{2.14}$$

Since

$$\widetilde{v}_3(0) = \frac{\sum_{j=1}^3 U_{3j}\widetilde{b}_j}{\nu^2 r_1(r_1+|\xi|)P_1} - \frac{\sigma\xi^2U_{33}\widetilde{g}}{\nu^2 s_1 r_1(r_1+|\xi|)P_1},$$

and, as a consequence,

$$|\widetilde{v}_3(0)| \leq c(|\xi|\,|r|^{-2}|\widetilde{\boldsymbol{b}}| + |\widetilde{g}|),$$

relations (2.14) imply that

$$\gamma|\widetilde{\rho}| \leq c(|\xi|\,|r|^{-2}|\widetilde{\boldsymbol{b}}| + |\widetilde{g}|), \quad \sigma|\xi|^2|\widetilde{\rho}| \leq c(|\widetilde{\boldsymbol{b}}| + (1+|\xi|)|\widetilde{g}|). \tag{2.15}$$

Hence,

$$\begin{aligned}
\|\rho\|_{W_2^{5/2}(\mathbb{R}^2)} &\leq c\Big(\|\boldsymbol{b}\|_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{3/2}(\mathbb{R}^2)}\Big),\\
\|\rho\|_{W_2^{l+5/2}(\mathbb{R}^2)} &\leq c\Big(\|\boldsymbol{b}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}\Big).
\end{aligned} \tag{2.16}$$

Now we pass to estimating $s\widetilde{\rho}$. By (2.14),

$$|s|\,|\widetilde{\rho}| \le |\boldsymbol{V}'|\,|\xi|\,|\widetilde{\rho}| + c(|\xi|\,|r|^{-2}|\widetilde{\boldsymbol{b}}| + |\widetilde{g}|),$$

which yields

$$\begin{aligned} |s|\,\|\rho\|_{W_2^{l+3/2}(\mathbb{R}^2)} &\le |\boldsymbol{V}'|\,\|\rho\|_{W_2^{l+5/2}(\mathbb{R}^2)} + c(\|\boldsymbol{b}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}),\\ |s|^{1+l/2}\|\rho\|_{W_2^{3/2}(\mathbb{R}^2)} &\le |\boldsymbol{V}'|\,|s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathbb{R}^2)} + c|s|^{l/2}(\|\boldsymbol{b}\|_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{3/2}(\mathbb{R}^2)}). \end{aligned} \tag{2.17}$$

Estimates (2.16), (2.17) show that

$$\begin{aligned} |s|^{l/2}\|p(0)\|_{W_2^{1/2}(\mathbb{R}^2)} &+ \|p(0)\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathbb{R}^2)} + \|\rho\|_{W_2^{l+5/2}(\mathbb{R}^2)}\\ &+ |s|\,\|\rho\|_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{1+l/2}\|\rho\|_{W_2^{3/2}(\mathbb{R}^2)}\\ \le c\Big(|s|^{l/2}\|\boldsymbol{b}\|_{W_2^{1/2}(\mathbb{R}^2)} &+ \|\boldsymbol{b}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|g\|_{W_2^{3/2}(\mathbb{R}^2)} + \|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}\Big). \end{aligned} \tag{2.18}$$

Step 2. Consider the problem

$$\begin{cases} s\boldsymbol{v} + (\boldsymbol{V}'\cdot\nabla')\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x),\\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x_3 > 0,\\ \nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = 0, \quad j = 1,2,\\ -p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\Delta'\rho = 0,\\ s\rho + \boldsymbol{V}'\cdot\nabla'\rho + v_3 = 0, \quad x_3 = 0. \end{cases} \tag{2.19}$$

Our goal is to construct the solution of (2.19) and to obtain an estimate similar to (2.11), (2.18). Without loss of generality, we may assume that $\nabla\cdot\boldsymbol{f} = 0$; otherwise we could decompose $\boldsymbol{f}$ in the sum of a divergence free and a potential vector field:

$$\boldsymbol{f} = \boldsymbol{f}' + \nabla\phi,$$

where ϕ is a solution of the Dirichlet problem

$$\nabla^2\phi(x) = \nabla\cdot\boldsymbol{f}(x), \quad x_3 > 0, \quad \phi|_{x_3=0} = 0.$$

Problem (2.19) is equivalent to a similar problem with $\boldsymbol{f}'$ instead of $\boldsymbol{f}$ and $p' = p - \phi$ instead of p.

Thus, we assume that $\boldsymbol{f}$ is divergence free and extend $\boldsymbol{f}$ to $\mathbb{R}^3$ with preservation of this property and of the regularity properties; namely, we require that $\nabla\cdot\boldsymbol{f}^* = 0$,

$$\|\boldsymbol{f}^*\|_{L_2(\mathbb{R}^3)} \le c\|\boldsymbol{f}\|_{L_2(\mathbb{R}^3_+)}, \quad \|\boldsymbol{f}^*\|_{W_2^l(\mathbb{R}^3)} \le c\|\boldsymbol{f}\|_{W_2^l(\mathbb{R}^3_+)},$$

where $\boldsymbol{f}^*$ is the extension of $\boldsymbol{f}$.

We define $\boldsymbol{u}$ as the solution of the system

$$s\boldsymbol{u} + (\boldsymbol{V}'\cdot\nabla)\boldsymbol{u} - \nu\nabla^2\boldsymbol{u} = \boldsymbol{f}^*(x), \quad x\in\mathbb{R}^3.$$

Taking the Fourier transform with respect to x_1, x_2, x_3, we obtain the solution in the form

$$\widetilde{\boldsymbol{u}}(\xi) = \frac{\widetilde{\boldsymbol{f}}^*}{s_1 + \nu\xi^2},$$

where $\xi = (\xi_1,\xi_2,\xi_3)$ is the dual variable and $s_1 = s + i\boldsymbol{V}'\cdot\xi$. It is clear that $\nabla\cdot\boldsymbol{u} = 0$. The corresponding pressure p vanishes. The vector field $\boldsymbol{u}$ satisfies the inequalities

$$|s|\,\|\boldsymbol{u}\|_{L_2(\mathbb{R}^3)} \le c\|\boldsymbol{f}^*\|_{L_2(\mathbb{R}^3)}, \quad \|\boldsymbol{u}\|_{W_2^{l+2}(\mathbb{R}^3)} \le c\|\boldsymbol{f}^*\|_{W_2^l(\mathbb{R}^3)},$$

and, as a consequence,

$$|s|^{1+l/2}\|\boldsymbol{u}\|_{L_2(\mathbb{R}^3)} + \|\boldsymbol{u}\|_{W_2^{l+2}(\mathbb{R}^3)} \le c(|s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathbb{R}^3)} + \|\boldsymbol{f}\|_{W_2^l(\mathbb{R}^3)}).$$

The difference $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ is a solution of (2.3) with

$$b_j(x) = -\nu\Big(\frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3}\Big), \quad j = 1,2,3, \quad g = -u_3.$$

Hence, $\boldsymbol{w}, p, \rho$ satisfy (2.11), (2.18). It follows that

$$\begin{aligned} &\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{l+2}\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla p\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla p\|^2_{L_2(\mathbb{R}^3_+)} \\ &\qquad + |s|^l\|p(0)\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|p(0)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|\rho\|^2_{W_2^{5/2}(\mathbb{R}^2)} \\ &\qquad + \|\rho\|^2_{W_2^{l+5/2}(\mathbb{R}^2)} + |s|^2\|\rho\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{2+l}\|\rho\|^2_{W_2^{3/2}(\mathbb{R}^2)} \\ &\le c\Big(\|\boldsymbol{f}\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\boldsymbol{f}\|^2_{L_2(\mathbb{R}^3_+)}\Big). \end{aligned} \tag{2.20}$$

Step 3. We consider the problem

$$\begin{cases} s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x_3 > 0, \\ \nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = b_j, \quad j = 1,2, \\ -p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\Delta'\rho = b_3(x'), \\ s\rho + \boldsymbol{V}'\cdot\nabla'\rho + v_3 = g(x'), \quad x_3 = 0. \end{cases} \tag{2.21}$$

The first equation can be written in the form

$$s\boldsymbol{v} + (\boldsymbol{V}'\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x) + (\boldsymbol{V}'\cdot\nabla)\boldsymbol{v},$$

and the term $(\boldsymbol{V}'\cdot\nabla)\boldsymbol{v}$ can be estimated by an interpolation inequality as follows:

$$\begin{aligned} \|(\boldsymbol{V}'\cdot\nabla)\boldsymbol{v}\|_{W_2^l(\mathbb{R}^3_+)} &\le c|s|^{-1/2}\Big(\|\boldsymbol{v}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)}\Big), \\ |s|^{l/2}\|(\boldsymbol{V}'\cdot\nabla)\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)} &\le c|s|^{-1/2}\Big(\|\boldsymbol{v}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)}\Big), \end{aligned}$$

whence

$$\|(\boldsymbol{V}'\cdot\nabla)\boldsymbol{v}\|_{W_2^l(\mathbb{R}^3_+)} + |s|^{l/2}\|(\boldsymbol{V}'\cdot\nabla)\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)} \le c|s|^{-1/2}\Big(\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)}\Big).$$

If $|s|$ is sufficiently large, then (2.11), (2.18), and (2.20) yield

$$\begin{aligned} &\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^l\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla p\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla p\|^2_{L_2(\mathbb{R}^3_+)} \\ &\quad + |s|^l\|p(0)\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|p(0)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|\rho\|^2_{W_2^{5/2}(\mathbb{R}^2)} + \|\rho\|^2_{W_2^{l+5/2}(\mathbb{R}^2)} \\ &\quad + |s|^2\|\rho\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{2+l}\|\rho\|^2_{W_2^{3/2}(\mathbb{R}^2)} \\ &\le c\Big(\|\boldsymbol{f}\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\boldsymbol{f}\|^2_{L_2(\mathbb{R}^3_+)} + \|\boldsymbol{b}\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l+1/2}\|\boldsymbol{b}'\|^2_{L_2(\mathbb{R}^2)} \\ &\qquad + |s|^l\|b_3\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^l\|g\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big). \end{aligned} \tag{2.22}$$

Step 4. We consider the problem

$$
\begin{cases} s\boldsymbol{v}(x) - \nu\nabla^2\boldsymbol{v}(x) + \nabla p(x) = 0, \\ \nabla\cdot\boldsymbol{v}(x) = h(x), \quad x\in\mathbb{R}^3_+, \\ \nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = 0, \quad j=1,2, \\ -p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\Delta'\rho = 0, \\ s\rho + (\boldsymbol{V}'\cdot\nabla)\rho + v_3 = 0, \quad x_3 = 0, \end{cases} \tag{2.23}
$$

under the assumption that h decays sufficiently rapidly at infinity, and

$$
h = \nabla\cdot\boldsymbol{H}(x) + h'(x) \tag{2.24}
$$

with compactly supported h'. We reduce (2.23) to (2.21). For this, we introduce $\boldsymbol{w} = \nabla\Phi(x)$, where Φ is a solution of the Dirichlet problem

$$
\nabla^2\Phi(x) = h(x), \quad x\in\mathbb{R}^3_+, \quad \Phi(x)|_{x_3=0} = 0. \tag{2.25}
$$

By the Green identity,

$$
\begin{aligned} \int_{\mathbb{R}^3_+} |\nabla\Phi(x)|^2\,dx &= -\int_{\mathbb{R}^3_+} \Phi(x)\nabla^2\Phi(x)\,dx = \int_{\mathbb{R}^3_+} (\nabla\Phi(x)\cdot\boldsymbol{H} - h'(x)\Phi(x))\,dx \\ &\le c\Big(\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)}\|\nabla\Phi\|_{L_2(\mathbb{R}^3_+)} + \|h'\|_{L_{6/5}(supp\ h')}\|\Phi\|_{L_6(\mathbb{R}^3_+)}\Big) \\ &\le c\|\nabla\Phi\|_{L_2(\mathbb{R}^3_+)}\Big(\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)} + \|h'\|_{L_2(\mathbb{R}^3_+)}\Big). \end{aligned} \tag{2.26}
$$

Moreover, the coercive estimate for problem (2.25) yields

$$
\|\nabla\Phi\|_{\dot W_2^{2+l}(\mathbb{R}^3_+)} \le c\|h\|_{\dot W_2^{1+l}(\mathbb{R}^3_+)},
$$

whence

$$
\begin{aligned} &\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)} \\ &\quad\le c|s|^{1+l/2}\Big(\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)} + \|h'\|_{L_2(\mathbb{R}^3_+)}\Big) + c\|h\|_{W_2^{1+l}(\mathbb{R}^3_+)}. \end{aligned} \tag{2.27}
$$

The functions $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{w}$, p, ρ represent the solution of problem (2.21) with the data

$$
\begin{aligned} &\boldsymbol{f} = -s\boldsymbol{w} + \nu\nabla^2\boldsymbol{w}, \\ &b_j = -\nu\Big(\frac{\partial w_j}{\partial x_3} + \frac{\partial w_3}{\partial x_j}\Big), \quad j=1,2, \quad b_3 = -2\nu\frac{\partial w_3}{\partial x_3}, \quad g = -w_3, \end{aligned}
$$

and they can be estimated by (2.22). Together with (2.27), this estimate yields

$$
\begin{aligned} &\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{2+l}\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla p\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla p\|^2_{L_2(\mathbb{R}^3_+)} \\ &\qquad + |s|^l\|p(0)\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|p(0)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|\rho\|^2_{W_2^{5/2}(\mathbb{R}^2)} \\ &\qquad + \|\rho\|^2_{W_2^{l+5/2}(\mathbb{R}^2)} + |s|^2\|\rho\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{2+l}\|\rho\|^2_{W_2^{3/2}(\mathbb{R}^2)} \\ &\quad\le c|s|^{2+l}\Big(\|\boldsymbol{H}\|^2_{L_2(\mathbb{R}^3_+)} + \|h'\|^2_{L_2(\mathbb{R}^3_+)}\Big) + c\|h\|^2_{W_2^{1+l}(\mathbb{R}^3_+)}. \end{aligned} \tag{2.28}
$$

Step 5. We estimate the solution of (2.1) in the vicinity of an arbitrary fixed point $x_0\in\mathcal{G}$ by Schauder's localization method. Without loss of generality, we may assume that $x_0 = 0$ and that the inward normal $-\boldsymbol{N}(0)$ is parallel to $\boldsymbol{e}_3$. Let $\zeta(x)$ be a smooth cutoff function equal to 1 for $|x| \le \delta/2$ and to zero in the domain $|x| \ge \delta$. The functions $\boldsymbol{w} = \zeta(x)\boldsymbol{v}(x)$, $\boldsymbol{q} = \zeta\boldsymbol{p}$, $r = \zeta\rho$ satisfy the equations

$$\begin{cases} s\boldsymbol{w} - \nu\nabla^2\boldsymbol{w} + \nabla q = \boldsymbol{f}(x)\zeta(x) + \boldsymbol{m}_1(\boldsymbol{v}, p), \\ \nabla\cdot\boldsymbol{w}(x) = \nabla\zeta\cdot\boldsymbol{v}(x), \quad x \in \mathcal{F}, \\ T(\boldsymbol{w}, q)\boldsymbol{N} - \sigma\boldsymbol{N}\Delta_{\mathcal{G}} r = \zeta(x)\boldsymbol{d}(x) + \boldsymbol{m}_2(\boldsymbol{v}, \rho), \\ sr(x) + \boldsymbol{V}'\cdot\nabla r - \boldsymbol{w}(x)\cdot\boldsymbol{N}(x) = \rho\boldsymbol{V}'\cdot\nabla\zeta + g(x)\zeta(x), \quad x \in \mathcal{G}, \end{cases} \tag{2.29}$$

where

$$\boldsymbol{m}_1(\boldsymbol{v}, p) = -2\nu\nabla\zeta(x)\cdot\nabla\boldsymbol{v} - \nu\boldsymbol{v}\nabla^2\zeta + p\nabla\zeta,$$
$$\boldsymbol{m}_2(\boldsymbol{v}, \rho) = \nu\Big(\boldsymbol{v}(x)\frac{\partial\zeta}{\partial N} + \nabla\zeta(x)(\boldsymbol{v}\cdot\boldsymbol{N})\Big) + \boldsymbol{N}\sigma\big(\zeta(x)\Delta_{\mathcal{G}}\rho - \Delta_{\mathcal{G}}(\zeta\rho) - b(x)\zeta(x)\rho(x)\big).$$

We assume that in the d-neighborhood of the origin ($d \ge 2\delta$) the surface $\mathcal{G}$ is given by the equation

$$x_3 = \phi(x'), \quad x' = (x_1, x_2).$$

The function ϕ is smooth and $\phi(0) = 0$, $\nabla\phi(0) = 0$, which implies that

$$|\nabla\phi(x')| \le c|x'|, \quad |\phi(x')| \le c|x'|^2 \tag{2.30}$$

for $|x'| \le d$. The components of $\boldsymbol{N}$ and the Laplace–Beltrami operator $\Delta_{\mathcal{G}}$ are expressed in terms of ϕ as follows:

$$N_\alpha = \frac{\phi_{y_\alpha}}{\sqrt{1+|\nabla\phi|^2}}, \quad \alpha = 1, 2, \quad N_3 = -\frac{1}{\sqrt{1+|\nabla\phi|^2}},$$
$$\Delta_{\mathcal{G}} = \frac{1}{\sqrt{1+|\nabla\phi|^2}}\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial y_\alpha}\Big(\delta_{\alpha\beta}\sqrt{1+|\nabla\phi|^2} - \frac{\phi_{y_\alpha}\phi_{y_\beta}}{\sqrt{1+|\nabla\phi|^2}}\Big)\frac{\partial}{\partial y_\beta}.$$

We make a change of variables in (2.29):

$$y = F(x) \,:\, y' = x', \quad y_3 = x_3 - \phi(x').$$

If d is sufficiently small, then the transformation F is invertible, establishing a one-to-one correspondence between the domain $K_d = \{|x| \le d,\ x \in \mathcal{F}\}$ and a certain subdomain D of $\mathbb{R}^3_+$. The operators ∇_x and $S(\boldsymbol{v})$ are transformed into $\widehat{\nabla} = \nabla_y - \frac{\partial}{\partial y_3}\nabla\phi(y')$ and $\widehat{S}(\boldsymbol{v}) = \widehat{\nabla}\boldsymbol{v} + (\widehat{\nabla}\boldsymbol{v})^T$, respectively, and we have

$$\nabla_x\cdot\boldsymbol{f}(x) = \widehat{\nabla}\cdot\boldsymbol{f}(x(y)) = \nabla_y\cdot\widehat{\boldsymbol{f}}(y),$$

where $\widehat{f}_i = f_i - \delta_{i3}\sum_{\alpha=1}^{2}\phi_{y_\alpha}f_\alpha$.

We write equations (2.29) in the variables $\{y\}$, keeping the old notation for all transformed functions. We have

$$\begin{cases} s\boldsymbol{w} - \nu\nabla^2\boldsymbol{w} + \nabla q = \boldsymbol{M}_1(\boldsymbol{w}, q) + \boldsymbol{m}_1(\boldsymbol{v}, p) + \zeta\boldsymbol{f}, \\ \nabla\cdot\boldsymbol{w} = (\nabla - \widehat{\nabla})\cdot\boldsymbol{w} + \widehat{\nabla}\zeta\cdot\boldsymbol{v}, \end{cases} \tag{2.31}$$

where $\nabla = \nabla_y$,

$$\boldsymbol{M}_1(\boldsymbol{w}, q) = \nu(\widehat{\nabla}^2 - \nabla^2)\boldsymbol{w} + (\nabla - \widehat{\nabla})q. \tag{2.32}$$

We note that the function $\nabla\zeta\cdot\boldsymbol{v}$ can be written in the form

$$\nabla\zeta\cdot\boldsymbol{v} = \frac{1}{s}\nabla\zeta\cdot(\nu\nabla^2\boldsymbol{v} - \nabla p + \boldsymbol{f}) = \nabla\cdot\boldsymbol{A}_s(\boldsymbol{v}, p) + a_s(\boldsymbol{v}, p) + \frac{1}{s}\nabla\zeta\cdot\boldsymbol{f}, \tag{2.33}$$

where

$$\begin{aligned} \boldsymbol{A}_s(\boldsymbol{v},p) &= \frac{1}{s}\big(\nu\nabla\boldsymbol{v}\nabla\zeta - p\nabla\zeta\big), \\ a_s(\boldsymbol{v},p) &= \frac{1}{s}\big(-\nu D^2\zeta : \nabla\boldsymbol{v} + p\nabla^2\zeta\big), \end{aligned} \tag{2.34}$$

$D^2\zeta = \big(\frac{\partial^2\zeta}{\partial x_i\partial x_j}\big)_{i,j=1,2,3}$, and $\nabla\boldsymbol{v} = \big(\frac{\partial v_i}{\partial x_j}\big)_{i,j=1,2,3}$. Consequently, $h \equiv (\nabla - \widehat{\nabla})\cdot\boldsymbol{w} + \widehat{\nabla}\zeta\cdot\boldsymbol{v}$ satisfies (2.24) with

$$\boldsymbol{H} = \boldsymbol{e}_3\sum_{\alpha=1}^{2}\phi_{y_\alpha}w_\alpha + \widehat{A}_s(\boldsymbol{v},p), \quad h' = a_s(\boldsymbol{v},p) + \frac{1}{s}\widehat{\nabla}\zeta\cdot\boldsymbol{f}. \tag{2.35}$$

We write the boundary condition $T\boldsymbol{N} - \sigma\boldsymbol{N}\Delta_{\mathcal{G}}r = \zeta\boldsymbol{d} + \boldsymbol{m}_2$ for the tangential and normal components separately; moreover, we can take only the first two components of the tangential part. This gives the following system of three equations:

$$\nu\Big(\sum_{i=1}^{3}\widehat{S}_{\alpha i}(\boldsymbol{w})N_i - N_\alpha(\boldsymbol{N}\cdot\widehat{S}(\boldsymbol{w})\boldsymbol{N})\Big) = \zeta(d_\alpha - N_\alpha(\boldsymbol{d}\cdot\boldsymbol{N})) + m_{2\alpha} - N_\alpha(\boldsymbol{N}\cdot\boldsymbol{m}_2), \quad \alpha = 1,2,$$

$$-q + \nu\boldsymbol{N}\cdot\widehat{S}(\boldsymbol{w})\boldsymbol{N} - \sigma\Delta_{\mathcal{G}}r = \zeta\boldsymbol{d}\cdot\boldsymbol{N} + \boldsymbol{m}_2\cdot\boldsymbol{N},$$

i.e.,

$$\begin{cases} \nu S_{\alpha 3}(\boldsymbol{w}) = L_\alpha(\boldsymbol{w}) + l_\alpha(\boldsymbol{v}) + \zeta d'_\alpha(y), \quad \alpha = 1,2, \\ -q + \nu S_{33}(\boldsymbol{w}) - \sigma\Delta' r = L_3(\boldsymbol{w}) + B'r + l_3(\boldsymbol{v}) + \zeta\boldsymbol{d}\cdot\boldsymbol{N}, \end{cases} \tag{2.36}$$

where $d'_\alpha = d_\alpha - N_\alpha(\boldsymbol{d}\cdot\boldsymbol{N})$,

$$\begin{aligned} L_\alpha(\boldsymbol{w}) &= \nu\Big(S_{\alpha 3} - \sum_{j=1}^{3}\widehat{S}_{\alpha j}N_j + N_\alpha(\boldsymbol{N}\cdot\widehat{S}(\boldsymbol{w})\boldsymbol{N})\Big), \\ L_3(\boldsymbol{w}) &= \nu\Big(S_{33}(\boldsymbol{w}) - \boldsymbol{N}\cdot\widehat{S}(\boldsymbol{w})\boldsymbol{N}\Big), \\ B'r &= -\sigma(\Delta' - \Delta_{\mathcal{G}})r, \\ l_\alpha(\boldsymbol{v}) &= m_{2\alpha}(\boldsymbol{v}) - N_\alpha(\boldsymbol{m}_2(\boldsymbol{v})\cdot\boldsymbol{N}), \\ l_3(\boldsymbol{v}) &= \boldsymbol{m}_2\cdot\boldsymbol{N}. \end{aligned}$$

Finally, we have

$$sr + \boldsymbol{V'}\cdot\nabla' r + w_3 = (w_3 + \boldsymbol{w}\cdot\boldsymbol{N}) + \rho\boldsymbol{V'}\cdot\nabla'\zeta + \zeta g. \tag{2.37}$$

Now, we extend $\boldsymbol{w}$, q, r by zero to $\mathbb{R}^3_+$ and $\mathbb{R}^2$ and regard (2.31), (2.36), (2.37) as a problem of the type (2.21) in the half-space. We estimate $\boldsymbol{w}, q, r$ with the help of (2.22), (2.28). Observe that, by (2.30), the leading coefficients of the operators $\boldsymbol{M}_1$, $\nabla - \widehat{\nabla}$, L_i, B' are small provided so is δ. By [2, Lemma 4.1],

$$\begin{aligned} \|\boldsymbol{M}_1\|_{W_2^l(\mathbb{R}^3_+)} &\le c\delta^\theta\Big(\|\boldsymbol{w}\|_{W_2^{2+l}(\mathbb{R}^3_+)} + \|\nabla q\|_{W_2^l(\mathbb{R}^3_+)}\Big) \\ &\quad + c(\theta)\Big(\|\boldsymbol{w}\|_{W_2^{l+1}(\mathbb{R}^3_+)} + \|\nabla q\|_{W_2^{l-1}(\mathbb{R}^3_+)}\Big), \\ |s|^{l/2}\|\boldsymbol{M}_1\|_{L_2(\mathbb{R}^3_+)} &\le c|s|^{l/2}\Big(\delta(\|\boldsymbol{w}\|_{W_2^l(\mathbb{R}^3_+)} + \|\nabla q\|_{L_2(\mathbb{R}^3_+)}) + \|\boldsymbol{w}\|_{W_2^1(\mathbb{R}^3_+)}\Big), \end{aligned}$$

where $\theta \in (0,1)$. By interpolation inequalities,

$$\|\boldsymbol{w}\|_{W_2^{l+1}(\mathbb{R}^3_+)} \le c\Big(|s|^{-1/2}\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1/2+l/2}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)}\Big),$$
$$\|\nabla q\|_{W_2^{l-1}(\mathbb{R}^3_+)} \le c\Big(|s|^{-1/2}\|\nabla q\|_{W_2^{l}(\mathbb{R}^3_+)} + |s|^{l/2-1/2}\|\nabla q\|_{L_2(\mathbb{R}^3_+)}\Big),$$
$$\|\boldsymbol{w}\|_{W_2^{1}(\mathbb{R}^3_+)} \le c\Big(|s|^{-l/2-1/2}\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1/2}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)}\Big),$$

whence

$$\|\boldsymbol{M}_1\|_{W_2^{l}(\mathbb{R}^3_+)} + |s|^{l/2}\|\boldsymbol{M}_1\|_{L_2(\mathbb{R}^3_+)} \le c(\delta^\theta + c(\delta)|s|^{-1/2}) \times\Big(\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + \|\nabla q\|_{W_2^{l}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)} + |s|^{l/2}\|\nabla q\|_{L_2(\mathbb{R}^3_+)}\Big).$$

In a similar way we obtain

$$\begin{aligned}
&\sum_{j=1}^{2}\Big(\|L_j(\boldsymbol{w})\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2+1/4}\|L_j(\boldsymbol{w})\|_{L_2(\mathbb{R}^2)}\Big)\\
&\qquad + \|L_3(\boldsymbol{w})\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|L_3(\boldsymbol{w})\|_{W_2^{1/2}(\mathbb{R}^2)}\\
&\qquad + \|w_3 + \boldsymbol{w}\cdot\boldsymbol{N}\|_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{l/2+3/4}\|w_3 + \boldsymbol{w}\cdot\boldsymbol{N}\|_{L_2(\mathbb{R}^2)}\\
&\qquad + \|(\nabla - \widehat{\nabla})\cdot\boldsymbol{w}\|_{W_2^{l+1}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{e}_3\sum_{\alpha=1}^{2}\phi_{y_\alpha}w_\alpha\|_{L_2(\mathbb{R}^3_+)}\\
&\quad \le c(\delta^\theta + c(\delta)|s|^{-1/2})\Big(\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{l/2+1}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)}\Big),\\
&\|B'r\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|B'r\|_{W_2^{1/2}(\mathbb{R}^2)}\\
&\quad \le c\delta^\theta\|r\|_{W_2^{l+5/2}(\mathbb{R}^2)} + c(\theta)\|r\|_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{l/2}\Big(c\delta^\theta\|r\|_{W_2^{5/2}(\mathbb{R}^2)} + c(\theta)\|r\|_{W_2^{3/2}(\mathbb{R}^2)}\Big)\\
&\quad \le c(\delta^\theta + c(\delta)|s|^{-1/2})\Big(\|r\|_{W_2^{l+5/2}(\mathbb{R}^2)} + |s|^{l/2}\|r\|_{W_2^{5/2}(\mathbb{R}^2)} + |s|^{1+l/2}\|r\|_{W_2^{3/2}(\mathbb{R}^2)}\Big).
\end{aligned}$$

Now we pass to estimating $\widehat{\nabla}\zeta\cdot\boldsymbol{v}$, $\boldsymbol{m}_1$, and $\boldsymbol{m}_2$. We have

$$\begin{aligned}
&\|\widehat{\nabla}\zeta\cdot\boldsymbol{v}\|_{W_2^{l+1}(\mathbb{R}^2)} + \|\boldsymbol{m}_1\|_{W_2^{l}(\mathbb{R}^3_+)} + |s|^{l/2}\|\boldsymbol{m}_1\|_{L_2(\mathbb{R}^3_+)}\\
&\qquad + \sum_{\alpha=1}^{2}\Big(\|m_{2\alpha} - N_\alpha(\boldsymbol{m}_2\cdot\boldsymbol{N})\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2+1/4}\|m_{2\alpha} - N_\alpha(\boldsymbol{m}_2\cdot\boldsymbol{N})\|_{L_2(\mathbb{R}^2)}\Big)\\
&\qquad + \|\boldsymbol{m}_2\cdot\boldsymbol{N}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|\boldsymbol{m}_2\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathbb{R}^2)}\\
&\quad \le c(\delta)\Big(\|\boldsymbol{v}\|_{W_2^{l+1}(K_{2\delta})} + |s|^{l/2}\|\boldsymbol{v}\|_{W_2^{1}(K_{2\delta})} + \|\boldsymbol{v}\|_{W_2^{l+1/2}(S_{2\delta})} + |s|^{l/2+1/4}\|\boldsymbol{v}\|_{L_2(S_{2\delta})}\\
&\qquad + |s|^{l/2}\|\boldsymbol{v}\|_{W_2^{1/2}(S_{2\delta})} + \|\rho\|_{W_2^{l+3/2}(S_{2\delta})} + |s|^{l/2}\|\rho\|_{W_2^{3/2}(S_{2\delta})}\\
&\qquad + \|p\|_{W_2^{l}(K_{2\delta})} + |s|^{l/2}\|p\|_{L_2(K_{2\delta})}\Big),
\end{aligned}$$

and moreover,

$$|s|^{1+l/2}\Big(\|\boldsymbol{A}_s\|_{L_2(\mathbb{R}^3_+)} + \|a_s\|_{L_2(\mathbb{R}^3_+)}\Big) \le c|s|^{l/2}\Big(\|\boldsymbol{v}\|_{W_2^{1}(K_{21\delta})} + \|p\|_{L_2(K_{2\delta})}\Big).$$

The coefficient $\delta^\theta + c(\delta)|s|^{-1/2}$ can be made arbitrarily small by the choice of a small δ and large $|s|$. In this case, it is not hard to verify that an application of (2.22), (2.28) to our problem (2.31), (2.36), (2.37) leads to the inequality

(2.38)
$$\begin{aligned}
&\|\boldsymbol{w}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{2+l}\|\boldsymbol{w}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla q\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla q\|^2_{L_2(\mathbb{R}^3_+)}\\
&\qquad + |s|^l\|q(0)\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|q(0)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|r\|^2_{W_2^{5/2}(\mathbb{R}^2)} + \|r\|^2_{W_2^{l+5/2}(\mathbb{R}^2)}\\
&\qquad + |s|^2\|r\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{2+l}\|r\|^2_{W_2^{3/2}(\mathbb{R}^2)}\\
&\le c\Big(\|\zeta\boldsymbol{f}\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\zeta\boldsymbol{f}\|^2_{L_2(\mathbb{R}^3_+)} + \|\zeta\boldsymbol{d}'\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l+1/2}\|\zeta\boldsymbol{d}'\|^2_{L_2(\mathbb{R}^2)}\\
&\qquad + \|\zeta\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|\zeta\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|\zeta g\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^l\|\zeta g\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big)\\
&\quad + c\Big(\|p\|^2_{W_2^l(K_{2\delta})} + |s|^l\|p\|^2_{L_2(K_{2\delta})} + \|\boldsymbol{v}\|^2_{W_2^{l+1}(K_{2\delta})} + |s|^l\|\boldsymbol{v}\|^2_{W_2^1(K_{2\delta})} + \|\boldsymbol{v}\|^2_{W_2^{l+1/2}(S_{2\delta})}\\
&\qquad + |s|^{l+1/2}\|\boldsymbol{v}\|^2_{L_2(S_{2\delta})} + |s|^l\|\boldsymbol{v}\|^2_{W_2^{1/2}(S_{2\delta})} + \|\rho\|^2_{W_2^{l+3/2}(S_{2\delta})} + |s|^l\|\rho\|^2_{W_2^{3/2}(S_{2\delta})}\Big).
\end{aligned}$$

Inequalities of this type can be obtained in a neighborhood of any point of $\mathcal{G}$ and also of any interior point of $\mathcal{F}$ if the distance of that point to $\mathcal{G}$ is larger that $\delta_1 > 0$ (in this case the norms of g and $\boldsymbol{d}$ do not occur in the estimate). If we cover $\mathcal{F}$ by a finite number of such neighborhoods and add estimates (2.38) together, we obtain

$$\begin{aligned}
&\|\boldsymbol{v}\|^2_{W_2^{2+l}(\mathcal{F})} + |s|^l\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})} + \|\nabla p\|^2_{W_2^l(\mathcal{F})} + |s|^l\|\nabla p\|^2_{L_2(\mathcal{F})}\\
&\qquad + \|p\|^2_{W_2^{l+1/2}(\mathcal{G})} + |s|^l\|p\|^2_{W_2^{1/2}(\mathcal{G})} + \|\rho\|^2_{W_2^{l+5/2}(\mathcal{G})} + |s|^l\|\rho\|^2_{W_2^{5/2}(\mathcal{G})}\\
&\qquad + |s|^2\|\rho\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^{2+l}\|\rho\|^2_{W_2^{3/2}(\mathcal{G})}\\
&\le c\Big(\|\boldsymbol{f}\|^2_{L_2(\mathcal{F})} + |s|^l\|\boldsymbol{f}\|^2_{L_2(\mathcal{F})} + \|\Pi_{\mathcal{G}}\boldsymbol{d}\|^2_{W_2^{l+1/2}\mathcal{G}} + |s|^{l+1/2}\|\Pi_{\mathcal{G}}\boldsymbol{d}\|^2_{L_2(\mathcal{G})}\\
&\qquad + \|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{l+1/2}(\mathcal{G})} + |s|^l\|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{1/2}(\mathcal{G})} + \|g\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^l\|g\|^2_{W_2^{3/2}(\mathcal{G})}\Big)\\
&\qquad + c\Big(\|p\|^2_{W_2^l(\mathcal{F})} + |s|^l\|p\|^2_{L_2(\mathcal{F})} + N_1^2(\boldsymbol{v}) + N_2^2(\rho)\Big),
\end{aligned} \tag{2.39}$$

where

$$N_1^2(\boldsymbol{v}) = \|\boldsymbol{v}\|^2_{W_2^{l+1}(\mathcal{F})} + |s|^l\|\boldsymbol{v}\|^2_{W_2^1(\mathcal{F})} + \|\boldsymbol{v}\|^2_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l+1/2}\|\boldsymbol{v}\|^2_{L_2(\mathcal{G})} + |s|^l\|\boldsymbol{v}\|^2_{W_2^{1/2}(\mathcal{G})},$$
$$N_2^2(\rho) = \|\rho\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^l\|\rho\|^2_{W_2^{3/2}(\mathcal{G})}.$$

At the next step we estimate p.

Step 6. We have assumed that $\boldsymbol{f}$ is divergence free. Hence, p can be regarded as a solution of the problem

$$\nabla^2 p = 0, \quad x\in\mathcal{F}, \quad p = \nu\boldsymbol{N}\cdot\boldsymbol{S}(\boldsymbol{v})\boldsymbol{N} + \sigma\mathfrak{L}\rho - \boldsymbol{d}\cdot\boldsymbol{N}, \quad x\in\mathcal{G}.$$

It is well known that

$$\begin{aligned}
\|p\|_{L_2(\mathcal{F})} &\le c\|\nu\boldsymbol{N}\cdot\boldsymbol{S}(\boldsymbol{v})\boldsymbol{N} + \sigma\mathfrak{L}\rho - \boldsymbol{d}\cdot\boldsymbol{N}\|_{L_2(\mathcal{G})}\\
&\le c\Big(\|\nabla\boldsymbol{v}\|_{L_2(\mathcal{G})} + \|\rho\|_{W_2^2(\mathcal{G})} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{L_2(\mathcal{G})}\Big).
\end{aligned}$$

By the interpolation inequality

$$\|p\|^2_{W_2^l(\mathcal{F})} \le \epsilon\|\nabla p\|^2_{W_2^l(\mathcal{F})} + c(\epsilon)\|p\|^2_{L_2(\mathcal{F})},$$

we have

$$\begin{aligned}
&\|p\|^2_{W_2^l(\mathcal{F})} + |s|^l\|p\|^2_{L_2(\mathcal{F})}\\
&\qquad \le \epsilon\|\nabla p\|^2_{W_2^l(\mathcal{F})} + c(\epsilon)|s|^l\Big(\|\nabla\boldsymbol{v}\|^2_{L_2(\mathcal{G})} + \|\rho\|^2_{W_2^2(\mathcal{G})} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{L_2(\mathcal{G})}\Big)
\end{aligned} \tag{2.40}$$

and

$$|s|^l\|\rho\|^2_{W_2^2(\mathcal{G})} \le c|s|^l\Big(|s|^{-1/2}\|\rho\|^2_{W_2^{5/2}(\mathcal{G})} + |s|^{1/2}\|\rho\|^2_{W_2^{3/2}(\mathcal{G})}\Big),$$
$$|s|^l\|\nabla \boldsymbol{v}\|^2_{L_2(\mathcal{G})} \le c|s|^l\Big(|s|^{-l-1/2}\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathcal{F})} + |s|^{3/2}\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})}\Big).$$

Estimating the expressions $N_1^2(\boldsymbol{v})$ and $N_2^2(\rho)$ in a similar way, we show that (2.39) and (2.40) imply (2.2) in the case of large $|s|$.

The solvability of problem (2.1) is established in §3. □

§3. End of the proof of Theorem 2.1

Continuing the proof of Theorem 2.1, we establish the solvability of the problem (2.1). We use the method applied in [7] to the analysis of parabolic initial-boundary value problems and in [2] to the evolution Stokes problem similar to (1.1).

We need the following auxiliary proposition.

Proposition 3.1. *For arbitrary* $\boldsymbol{f} \in W_2^l(\mathcal{F})$ *and* $\boldsymbol{d} \in W_2^{l+1/2}(\mathcal{G})$, *the problem*

$$\begin{cases} s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x\in\mathcal{F}, \\ T(\boldsymbol{v},p)\boldsymbol{N} = \boldsymbol{d}(x), \quad x \in \mathcal{G}, \end{cases} \tag{3.1}$$

with $\operatorname{Re} s \gg 1$ *has a unique solution* $\boldsymbol{v} \in W_2^{2+l}(\mathcal{F})$, $p \in W_2^{l+1}(\mathcal{F})$, *and this solution satisfies the inequality*

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{l+2}(\mathcal{F})} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathcal{F})} + \|\nabla p\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\nabla p\|_{L_2(\mathcal{F})} \\ &\qquad + \|p\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|p\|_{W_2^{l/2}(\mathcal{G})} \\ &\le c\Big(\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathcal{F})} + \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{W_2^{l+1/2}(\mathcal{G})} \\ &\qquad + |s|^{l/2+1/4}\|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{L_2(\mathcal{G})} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathcal{G})}\Big). \end{aligned} \tag{3.2}$$

This theorem was proved in [8]; see also [9].

We consider problem (2.1) with $\boldsymbol{f} = 0$ and $\boldsymbol{d} = 0$. Let $\{\varphi_k\}$, $k = 1,2,\dots,$ be a sufficiently "fine" smooth partition of unity, $\sum_k \varphi_k(x) = 1$, defined on $\mathcal{G}$ and in a certain neighborhood of $\mathcal{G}$. We may assume that $\operatorname{supp}\varphi_i \subset K_\delta^{(i)}$, where $K_\delta^{(i)}$ is a ball $|x-x_i| \le \delta$, $x_i \in \mathcal{G}$. We also assume that there exist smooth functions $\psi_i(x)$ with $\operatorname{supp}\psi_i \subset K_\delta^i$ such that $\psi_i(x)\varphi_i(x) = \varphi_i(x)$. We suppose that

$$|D^j\varphi_i(x)| + |D^j\psi_i(x)| \le c\delta^{-|j|}$$

and that each point x can belong to at most M_0 balls $K_\delta^{(i)}$ with M_0 independent of δ.

Let $y_3 = \phi_i(y')$, $y' = (y_1, y_2)$, be the equation of $\mathcal{G}$ in a neighborhood of the point x_i in a local Cartesian coordinate system $y = (y_1, y_2, y_3)$ with center at x_i and with the y_3-axis directed along the vector $-\boldsymbol{N}(x_i)$. It is clear that $y = C_i(x - x_i)$, where C_i is an orthogonal matrix. Without loss of generality it may be assumed that ϕ_i is defined on the entire plane $y_3 = 0$ (i.e., on the tangent plane to $\mathcal{G}$ at the point x_i) and satisfies (2.30) near the origin. The transformation $z_1 = y_1, z_2 = y_2, z_3 = y_3 - \phi_i(y')$ "rectifies" $\mathcal{G}$ near x_i. We denote by $Z_j(x)$ the composition of this transformation with $y = C_i(x-x_i)$.

Now we describe briefly the method to be used to prove the solvability of the problem

$$
\begin{cases}
s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\
\nabla\cdot\boldsymbol{v}(x) = 0, \quad x\in\mathcal{F}, \\
T(\boldsymbol{v},p)\boldsymbol{N} + \sigma\boldsymbol{N}\mathfrak{L}\rho = 0, \\
s\rho + \boldsymbol{V}\cdot\nabla_\tau\rho - \boldsymbol{v}(x)\cdot\boldsymbol{N}(x) = g(x), \quad x\in\mathcal{G}.
\end{cases} \tag{3.3}
$$

We construct a linear operator R that takes every function $g\in W_2^{l+3/2}(\mathcal{G})$ to an element $U=(\boldsymbol{v},p,\rho)$, where $\boldsymbol{v}$ is a divergence free vector field belonging to $W_2^{l+2}(\mathcal{F})$, $p\in W_2^{l+1}(\mathcal{F})$ and $\rho\in W_2^{l+5/2}(\mathcal{G})$, such that

$$
\begin{cases}
s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\
\nabla\cdot\boldsymbol{v}(x) = 0, \quad x\in\mathcal{F}, \\
T(\boldsymbol{v},p)\boldsymbol{N} + \sigma\boldsymbol{N}\mathfrak{L}\rho = 0, \\
s\rho + \boldsymbol{V}\cdot\nabla_\tau\rho - \boldsymbol{v}(x)\cdot\boldsymbol{N}(x) = g(x) + Ag(x), \quad x\in\mathcal{G},
\end{cases} \tag{3.4}
$$

where A is a continuous linear operator in $W_2^{l+3/2}(\mathcal{G})$, and the operator $I+A$ is invertible. Then $U=R(I+A)^{-1}g$ is a solution of (3.3), as required.

We define Rg as the sum of three terms:

$$
Rg = R_1g + R_2g + R_3g = U_1 + U_2 + U_3, \quad U_j = (\boldsymbol{v}^{(j)}, p^{(j)}, \rho^{(j)}).
$$

We set

$$
R_1g = \sum_k \psi_k(x)(\boldsymbol{v}_k(x), p_k(x), \rho_k(x)),
$$

where

$$
\boldsymbol{v}_k(x) = C_k^{-1}\boldsymbol{u}_k(Z_kx), \quad p(x) = q_k(Z_kx), \quad \rho_k(x) = r_k(Z_kx),
$$

and $(\boldsymbol{u}_k, q_k, r_k)$ is a solution of the half-space problem

$$
\begin{cases}
s\boldsymbol{u}_k(z) - \nu\nabla_z^2\boldsymbol{u}_k(z) + \nabla_z q_k(z) = 0, \\
\nabla\cdot\boldsymbol{u}_k(z) = 0, \quad z\in\mathbb{R}^3_+ = \{z_3>0\}, \\
\dfrac{\partial u_{k3}}{\partial z_j} + \dfrac{\partial u_{kj}}{\partial z_3} = 0, \quad j=1,2, \\
-q_k + 2\nu\dfrac{\partial u_{k3}}{\partial z_3} - \sigma\Delta'_z r_k = 0, \\
sr_k(z) + \boldsymbol{V}_k\cdot\nabla'_z r_k(z) + \boldsymbol{u}_{k3} = g(z)\varphi_k(z), \quad z_3 = 0,
\end{cases} \tag{3.5}
$$

with $\boldsymbol{V}_k = C_k\boldsymbol{V}(x_k)$, $z=Z_j(x)$.

It is clear that $\boldsymbol{v}^{(1)}\in W_2^{l+2}(\mathcal{F})$, $p^{(1)}\in W_2^{l+1}(\mathcal{F})$, $\rho^{(1)}\in W_2^{l+5/2}(\mathcal{G})$. We set

$$
\begin{cases}
s\boldsymbol{v}^{(1)} - \nu\nabla^2\boldsymbol{v}^{(1)} + \nabla p^{(1)} \equiv \boldsymbol{f}_1(x), \\
\nabla\cdot\boldsymbol{v}^{(1)}(x) \equiv f_1(x), \quad x\in\mathcal{F}, \\
T(\boldsymbol{v}^{(1)},p^{(1)})\boldsymbol{N} + \sigma\boldsymbol{N}\mathfrak{L}\rho^{(1)}(x) \equiv \boldsymbol{d}_1(x), \\
s\rho^{(1)} + \boldsymbol{V}(x)\cdot\nabla_\tau\rho^{(1)} - \boldsymbol{v}^{(1)}(x)\cdot\boldsymbol{N}(x) \equiv g(x) + g_1(x), \quad x\in\mathcal{G},
\end{cases} \tag{3.6}
$$

and we define $R_2g = (\boldsymbol{v}^{(2)}, 0, 0)$, $\boldsymbol{v}^{(2)} = \nabla\Phi(x)$, where Φ is a solution of the Dirichlet problem

$$
\nabla^2\Phi(x) = -f_1(x), \quad x\in\mathcal{F}, \quad \Phi|_{\mathcal{G}} = 0.
$$

Finally, $R_3 g = (\boldsymbol{v}^{(3)}, p^{(3)}, 0)$ is a solution of

$$\begin{cases} s\boldsymbol{v}^{(3)} - \nu\nabla^2\boldsymbol{v}^{(3)} + \nabla p^{(3)} = \boldsymbol{f}_2(x), \\ \nabla\cdot\boldsymbol{v}^{(3)}(x) = 0, \quad x\in\mathcal{F}, \\ T(\boldsymbol{v}^{(3)}, p^{(3)})\boldsymbol{N} = \boldsymbol{d}_2(x), \end{cases}$$

where

$$\boldsymbol{f}_2 = -\boldsymbol{f}_1 - (s\boldsymbol{v}^{(2)} - \nu\nabla^2\boldsymbol{v}^{(2)}),$$
$$\boldsymbol{d}_2 = -\boldsymbol{d}_1 - S(\boldsymbol{v}^{(2)})\boldsymbol{N}.$$

Then the element $Rg = (\boldsymbol{v}, p, \rho)$ is a solution of (3.4) with $Ag = g_1 + \boldsymbol{v}^{(2)}\cdot\boldsymbol{N} + \boldsymbol{v}^{(3)}\cdot\boldsymbol{N}$. We have $\boldsymbol{v}^{(2)} \in W_2^{2+l}(\mathcal{F})$, $\boldsymbol{f}_2 \in W_2^l(\mathcal{F})$, $\boldsymbol{d}_2 \in W_2^{l+1/2}(\mathcal{G})$, $\boldsymbol{v}^{(3)} \in W_2^{l+2}(\mathcal{F})$, $p^{(3)} \in W_2^{l+1}(\mathcal{F})$, $Ag = g_1 - \boldsymbol{v}^{(2)}\cdot\boldsymbol{N} - \boldsymbol{v}^{(3)}\cdot\boldsymbol{N} \in W_2^{l+3/2}(\mathcal{G})$.

It remains to prove that $I + A$ is invertible.

We compute the functions $\boldsymbol{f}_1$, f_1, $\boldsymbol{d}_1$, g_1 occurring in (3.6). Since

$$\boldsymbol{v}^{(1)}(x) = \sum_k \psi_k(x)\boldsymbol{v}_k(x), \quad \boldsymbol{p}^{(1)}(x) = \sum_k \psi_k(x)p_k(x), \quad x\in\mathcal{F},$$
$$\rho^{(1)}(x) = \sum_k \psi_k(x)\rho_k(x), \quad x\in\mathcal{G},$$

we have

$$\begin{aligned} \boldsymbol{f}_1(x) &= \sum_k \psi_k(x)(s\boldsymbol{v}_k - \nu\nabla^2\boldsymbol{v}_k + \nabla p_k) - \nu\sum_k(\nabla^2(\psi_k\boldsymbol{v}_k) - \psi_k\nabla^2\boldsymbol{v}_k) \\ &\qquad + \sum_k(\nabla(\psi_k p_k) - \psi_k\nabla p_k), \\ f_1(x) &= \sum_k(\nabla\psi_k\cdot\boldsymbol{v}_k + \psi_k\nabla\cdot\boldsymbol{v}_k(x)), \\ \Pi_{\mathcal{G}}\boldsymbol{d}_1 &= \sum_k \psi_k\Pi_{\mathcal{G}}S(\boldsymbol{v}_k)\boldsymbol{N} + \sum_k \Pi_{\mathcal{G}}(S(\psi_k\boldsymbol{v}_k) - \psi_k S(\boldsymbol{v}_k))\boldsymbol{N}, \\ \boldsymbol{d}_1\cdot\boldsymbol{N} &= \sum_k \psi_k(-p_k + \nu\boldsymbol{N}\cdot S(\boldsymbol{v}_k)\boldsymbol{N} + \sigma\mathfrak{L}\rho_k) \\ &\qquad + \nu\sum_k \boldsymbol{N}\cdot(S(\psi_k\boldsymbol{v}_k) - \psi_k S(\boldsymbol{v}_k))\boldsymbol{N} + \sigma\sum_k(\mathfrak{L}(\psi_k\rho_k) - \psi_k\mathfrak{L}\rho_k). \end{aligned}$$

Consider the leading terms in the above formulas. By (3.5),

$$\begin{gathered} s\boldsymbol{v}_k - \nu\nabla_x^2\boldsymbol{v}_k + \nabla_x p_k = C_k^{-1}(\nu(\nabla_y^2 - \nabla_z^2)\boldsymbol{u}_k + (\nabla_y - \nabla_z)q_k), \\ \nabla\cdot\boldsymbol{v}_k = (\nabla_y - \nabla_z)\cdot\boldsymbol{u}_k. \end{gathered} \tag{3.7}$$

Moreover, $\Pi_{\mathcal{G}}S_x(\boldsymbol{v}_k)\boldsymbol{N} = C_k^{-1}\Pi_k S_y(\boldsymbol{u}_k)\boldsymbol{N}_k$, where $\Pi_k\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{N}_k(\boldsymbol{N}_k\cdot\boldsymbol{f})$ and $\boldsymbol{N}_k = C_k\boldsymbol{N}$ is the vector whose components are given by

$$N_{kj}(y) = \frac{\phi_{ky_j}}{\sqrt{1+|\nabla\phi_k|^2}}, \quad j=1,2, \quad N_{k3} = -\frac{1}{\sqrt{1+|\nabla\phi_k|^2}}$$

in a neighborhood of the origin. Hence,

$$\begin{gathered} \Pi_{\mathcal{G}}S_x(\boldsymbol{v}_k)\boldsymbol{N} = \Pi_{\mathcal{G}}\Big(\Pi_{\mathcal{G}}C_k^{-1}(S_y(\boldsymbol{u}_k)\boldsymbol{N}_k - \sum_{j=1}^2 \boldsymbol{e}_j S_{zj3}(\boldsymbol{u}_k))\Big), \\ -p_k + \nu\boldsymbol{N}\cdot S_x(\boldsymbol{v}_k)\boldsymbol{N} + \sigma\mathfrak{L}\rho_k = \nu\Big(\boldsymbol{N}_k\cdot S_y(\boldsymbol{u}_k)\boldsymbol{N}_k - 2\frac{\partial u_{k3}}{\partial z_3}\Big) + \sigma(\mathfrak{L} + \Delta_z')r_k, \end{gathered} \tag{3.8}$$

where $S_y(\boldsymbol{u}) = (\nabla_y \boldsymbol{u}) + (\nabla_y \boldsymbol{u})^T$. Finally, the identity

$$s\rho^{(1)} + \boldsymbol{V}(x)\cdot\nabla_\tau\rho^{(1)} - \boldsymbol{v}^{(1)}\cdot\boldsymbol{N} \\ = \sum_k \psi_k(s\rho_k + \boldsymbol{V}(x)\cdot\nabla_\tau\rho_k - \boldsymbol{v}_k\cdot\boldsymbol{N}) + \sum_k (\boldsymbol{V}(x)\cdot\nabla_\tau\psi_k)\rho_k$$

shows that

$$g_1 = \sum_k \psi_k((\boldsymbol{V}_k(y)\cdot\nabla_{y\tau} - \boldsymbol{V}_k(0)\cdot\nabla'_z)r_k - (\boldsymbol{u}_k\cdot\boldsymbol{N} + \boldsymbol{u}_{k3})) + \sum_k (\boldsymbol{V}(x)\cdot\nabla_\tau\psi_k)\rho_k \tag{3.9}$$

with $\boldsymbol{V}_k = C_k\boldsymbol{V}$.

A simple calculation yields

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_3}\phi_{k,j}(y'), \quad \phi_{k,j} = (1-\delta_{3j})\frac{\partial \phi_k}{\partial y_j},$$

so that

$$\nabla_y\cdot\boldsymbol{u}_k - \nabla_z\cdot\boldsymbol{u}_k = \sum_{j=1}^{2}\phi_{k,j}\frac{\partial u_{kj}}{\partial y_3}.$$

Hence,

$$\begin{aligned} f_1(x) &= \sum_k (\nabla\psi_k\cdot\boldsymbol{v}_k + \psi_k(\nabla_y - \nabla_z)\cdot\boldsymbol{u}_k) \\ &= \sum_k \nabla\psi_k\cdot\boldsymbol{v}_k + \sum_k\sum_{m,j=1}^{3}\psi_k C_{3m}\frac{\partial}{\partial x_m}\phi_{k,j}u_{kj} \\ &= \sum_{m,j=1}^{3}\frac{\partial}{\partial x_m}\sum_k \psi_k C_{3m}\phi_{k,j}u_{kj} + \sum_k \boldsymbol{\chi}_k\cdot\boldsymbol{u}_k, \end{aligned}$$

where $\boldsymbol{\chi}_k$ is the vector field with the components

$$\chi_{kj} = \sum_{m=1}^{3} C_{jm}\frac{\partial\psi_k}{\partial x_m} - \sum_{m=1}^{3}\frac{\partial}{\partial x_m}(\psi_k C_{3m}\phi_{k,j}).$$

Since

$$\boldsymbol{u}_k(z,t) = \frac{1}{s}(\nu\nabla_z^2\boldsymbol{u}_k - \nabla_z q_k) = \frac{1}{s}\sum_{j,m=1}^{3}\frac{\partial}{\partial x_m}(C_{jm} - C_{3m}\phi_{k,j})\Big(\nu\frac{\partial\boldsymbol{u}_k}{\partial z_j} - \boldsymbol{e}_j q_k\Big),$$

we have

$$f_1 = \nabla\cdot\boldsymbol{F} + F'$$

with

$$\begin{aligned} F_m(x) &= \sum_k \boldsymbol{\chi}_k\cdot\sum_{j=1}^{3}\frac{1}{s}(C_{jm} - C_{3m}\phi_{k,j})\Big(\nu\frac{\partial\boldsymbol{u}_k}{\partial z_j} - \boldsymbol{e}_j q_k\Big) \\ &\quad + \sum_k \psi_k(x)C_{3m}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\phi_{k,\alpha}u_{k\alpha}, \\ F'(x) &= -\sum_k\sum_{m=1}^{3}\frac{\partial\boldsymbol{\chi}_k(x)}{\partial x_m}\cdot\sum_{j=1}^{3}\frac{1}{s}(C_{jm} - C_{3m}\phi_{k,j})\Big(\nu\frac{\partial\boldsymbol{u}_k}{\partial z_j} - \boldsymbol{e}_j q_k\Big). \end{aligned} \tag{3.10}$$

Now we pass to estimates. Since every point of $\mathcal{F}\cap\mathcal{G}$ belongs to at most M_0 domains $K_\delta^{(i)}$, the functions of the form $f(x)=\sum_j f_j(x)$, $\operatorname{supp} f_j\subset K_\delta^{(j)}$, satisfy the inequality

$$\|f\|^2_{W_2^r(\mathcal{F})}\le c_0\sum_j\|f_j\|^2_{W_2^r(\mathcal{F})}$$

with c_0 independent of δ. By (2.16) and (2.28),

$$\begin{aligned}\|\boldsymbol{u}_k\|^2_{W_2^{2+l}(\mathbb{R}^3_+)}&+|s|^{2+l}\|\boldsymbol{u}_k\|^2_{L_2(\mathbb{R}^3_+)}+\|\nabla q_k\|^2_{W_2^l(\mathbb{R}^3_+)}+|s|^l\|\nabla q_k\|^2_{L_2(\mathbb{R}^3_+)}\\&+\|q_k\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}+|s|^l\|q_k\|^2_{W_2^{1/2}(\mathbb{R}^2)}+\|r_k\|^2_{W_2^{l+5/2}(\mathbb{R}^2)}+|s|^l\|r_k\|^2_{W_2^{5/2}(\mathbb{R}^2)}\\&+|s|^2\|r_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^{2+l}\|r_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\\\le c\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}&+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}.\end{aligned}$$

Observe that in (3.7)–(3.10) we have linear differential expressions with respect to $\boldsymbol{u}_k$, q_k, r_k, whose leading coefficients are small in $K_\delta^{(k)}$. Hence, we can use Lemma 4.1 in [2] to obtain the inequality

$$\begin{aligned}\|\boldsymbol{f}_1\|^2_{W_2^l(\mathcal{F})}&+|s|^l\|\boldsymbol{f}_1\|^2_{L_2(\mathcal{F})}+\|f_1\|^2_{W_2^{l+1}(\mathcal{F})}\\&+|s|^{2+l}(\|\boldsymbol{F}\|^2_{L_2(\mathcal{F})}+\|F'\|^2_{L_2(\mathcal{F})})+\|\Pi_{\mathcal{G}}\boldsymbol{d}_1\|^2_{W_2^{l+1/2}(\mathcal{G})}\\&+|s|^{l+1/2}\|\Pi_{\mathcal{G}}\boldsymbol{d}_1\|^2_{L_2(\mathcal{G})}+\|\boldsymbol{d}_1\cdot\boldsymbol{N}\|^2_{W_2^{l+1/2}(\mathcal{G})}+|s|^l\|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{1/2}(\mathcal{G})}\\&+\|g_1\|^2_{W_2^{l+3/2}(\mathcal{G})}+|s|^{l+3/2}\|g_1\|^2_{L_2(\mathcal{G})}\\\le c(\delta^\theta&+c(\delta)|s|^{-1/2})^2\sum_k\Big(\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\|\boldsymbol{v}^{(2)}\|^2_{W_2^{2+l}(\mathcal{F})}&+|s|^{2+l}\|\boldsymbol{v}^{(2)}\|^2_{L_2(\mathcal{F})}\\&\le c\Big(\|f_1\|^2_{W_2^{1+l}(\mathcal{F})}+|s|^{2+l}(\|\boldsymbol{F}\|^2_{L_2(\mathcal{F})}+\|F'\|^2_{L_2(\mathcal{F})})\Big)\\&\le c(\delta^\theta+c(\delta)|s|^{-1/2})^2\sum_k\Big(\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big)\end{aligned}$$

and, by Proposition 3.1,

$$\begin{aligned}\|\boldsymbol{v}^{(3)}\|^2_{W_2^{2+l}(\mathcal{F})}&+|s|^{2+l}\|\boldsymbol{v}^{(3)}\|^2_{L_2(\mathcal{F})}+\|\nabla p^{(3)}\|^2_{W_2^l(\mathcal{F})}\\&+|s|^l\|\nabla p^{(3)}\|^2_{L_2(\mathcal{F})}+\|p^{(3)}\|^2_{W_2^{l+1/2}(\mathcal{G})}+|s|^l\|p^{(3)}\|^2_{W_2^{1/2}(\mathcal{G})}\\\le c\Big(&\|\boldsymbol{f}_2\|^2_{W_2^l(\mathcal{F})}+|s|^l\|\boldsymbol{f}_2\|^2_{L_2(\mathcal{F})}+\|\Pi_{\mathcal{G}}\boldsymbol{d}_2\|^2_{W_2^{l+1/2}(\mathcal{G})}\\&+|s|^{l+1/2}\|\Pi_{\mathcal{G}}\boldsymbol{d}_1\|^2_{L_2(\mathcal{G})}+\|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{l+1/2}(\mathcal{G})}+|s|^l\|\boldsymbol{d}_2\cdot\boldsymbol{N}\|^2_{W_2^{1/2}(\mathcal{G})}\Big)\\\le c(\delta^\theta&+c(\delta)|s|^{-1/2})^2\sum_k\Big(\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}\|Ag\|^2_{W_2^{l+3/2}(\mathcal{G})}&+|s|^l\|Ag\|^2_{W_2^{3/2}(\mathcal{G})}\\&\le c\Big(\|g_1+\boldsymbol{v}^{(2)}\cdot\boldsymbol{N}+\boldsymbol{v}^{(3)}\cdot\boldsymbol{N}\|^2_{W_2^{l+3/2}(\mathcal{G})}+|s|^l\|g_1+\boldsymbol{v}^{(2)}\cdot\boldsymbol{N}+\boldsymbol{v}^{(3)}\cdot\boldsymbol{N}\|^2_{W_2^{3/2}(\mathcal{G})}\Big)\\&\le c(\delta^\theta+c(\delta)|s|^{-1/2})^2\sum_k\Big(\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big).\end{aligned}$$

It can be verified that the expression on the right does not exceed

$$c_1(\delta^\theta + c_1(\delta)|s|^{-1/2})^2(\|g\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^l\|g\|^2_{W_2^{3/2}(\mathcal{G})}),$$

which shows that A is a contraction operator in the case of small δ and large $|s|$. This completes the proof of the solvability of problem (3.3).

The solution of (2.1) can be constructed as the sum

$$\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2, \quad p = \pi_1 + \pi_2,$$

where $(\boldsymbol{w}_1, \pi_1)$ is a solution of (3.1) and $(\boldsymbol{w}_2, \pi_2, \rho)$ is a solution of (3.3) with g replaced by $g + \boldsymbol{w}_1 \cdot \boldsymbol{N}$. Theorem 2.1 is proved.

Remark 1. We have assumed that $\operatorname{Re} s$ is a sufficiently large positive number. In fact, the claim of Theorem 2.1 is true for $\operatorname{Re} s > a$, where a is determined by the spectrum of problem (2.1). It is well known (see [10, 4]) that if $\boldsymbol{d} = 0$, then this problem can be written in the form

$$(sI - \mathcal{A})U = G, \tag{3.11}$$

where $U = (\boldsymbol{v}, \rho)^T$, $G = (\boldsymbol{f}, g)^T$, I is the (2×2)-unit matrix and $\mathcal{A}$ is the (2×2)-matrix operator

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with the entries defined by

$$A_{11}\boldsymbol{v} = \nu\nabla^2\boldsymbol{v} - p_1(\boldsymbol{v}), \qquad A_{12}\rho = -p_2(\rho),$$
$$A_{21}\boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{N}|_{\mathcal{G}}, \qquad A_{22}\rho = -\boldsymbol{V} \cdot \nabla_\tau \rho|_{\mathcal{G}}.$$

By $p_1(\boldsymbol{v})$ and $p_2(\rho)$ we mean harmonic functions in $\mathcal{F}$ satisfying the boundary conditions

$$p_1(\boldsymbol{v}) = \nu\boldsymbol{N} \cdot S(\boldsymbol{v})\boldsymbol{N}, \quad p_2(\rho) = -\sigma\mathfrak{L}\rho$$

on $\mathcal{G}$ (hence, the pressure as an independent function is excluded). The domain of $\mathcal{A}$ is the subspace of $W_2^2(\mathcal{F}) \times W_2^{5/2}(\mathcal{G})$ defined by the conditions $\nabla \cdot \boldsymbol{v} = 0$, $\Pi_{\mathcal{G}} S(\boldsymbol{v})\boldsymbol{N}|_{\mathcal{G}} = 0$. By Theorem 2.2, for large positive $\operatorname{Re} s$, equation (3.11) is solvable, and inequality (3.7) with $l = 0$ yields

$$\|(sI - \mathcal{A})^{-1}G\|_D \le c\|G\|_X,$$

where $D = W_2^2(\mathcal{F}) \times W_2^{5/2}(\mathcal{G})$ and $X = L_2(\mathcal{F}) \times W_2^{3/2}(\mathcal{G})$. Therefore, $(sI - \mathcal{A})^{-1}$ is compact and the spectrum of $\mathcal{A}$ consists of eigenvalues with the only accumulation point at infinity (in the left complex half-plane). There may exist at most finitely many points of the spectrum in the right half-plane. Let a_0 be the upper bound of the real parts of these points. Proposition 3.1 holds true for $\operatorname{Re} s > 0$; hence, in Theorem 2.1, we can require that $\operatorname{Re} s > \max(0, a_0) = a$.

Remark 2. The interpolation inequality

$$|s|^{l/2+3/4}\|\nabla_\tau\rho\|_{L_2(\mathcal{G})} \le \delta\|\rho\|_{W_2^{l+5/2}(\mathcal{G})} + c(\delta)|s|^{1/2}\|\rho\|_{L_2(\mathcal{G})}, \quad \delta \ll 1,$$

and the equation $s\rho + \boldsymbol{V} \cdot \nabla_\tau\rho - \boldsymbol{v} \cdot \boldsymbol{N} = g$ imply that, along with (2.2), the solution of problem (2.1) satisfies the inequality

(3.12)
$$\begin{aligned}
&\|\boldsymbol{v}\|_{W_2^{2+l}(\mathcal{F})} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathcal{F})} + \|p\|_{W_2^{l+1}(\mathcal{F})} + |s|^{l/2}\|p\|_{W_2^1(\mathcal{F})} \\
&\quad + |s|^{1+l/2+3/4}\|\rho\|_{L_2(\mathcal{G})} + |s|\,\|\rho\|_{W_2^{l+3/2}(\mathcal{G})} + |s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathcal{G})} + \|\rho\|_{W_2^{l+5/2}(\mathcal{G})} \\
&\le c\Big(\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathcal{F})} + |s|^{1/4+l/2}\|\boldsymbol{d} - \boldsymbol{N}(\boldsymbol{d} \cdot \boldsymbol{N})\|_{L_2(\mathcal{G})} \\
&\qquad + \|\boldsymbol{d}\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|\boldsymbol{d} \cdot \boldsymbol{N}\|_{W_2^{1/2}(\mathcal{G})} + |s|^{l/2+3/4}\|g\|_{L_2(\mathcal{G})} + \|g\|_{W_2^{l+3/2}(\mathcal{G})}\Big).
\end{aligned}$$

§4. Proof of Theorem 1.1

We start with the following auxiliary proposition.

Proposition 4.1. *Let l be as in Theorem* 1.1. *If $\rho \in W_2^{l+5/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))$, and $\rho_t \in W_2^{l+3/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{3/2}(\mathcal{G}))$, then*

$$\|\rho(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})} \le c\big(\|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_t\|_{W_2^{l+3/2,0}(G_T)}\big), \tag{4.1}$$

and for $l>1$ we have

$$\|\rho_t(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})} \le c\big(\|\rho_t\|_{W_2^{l+3/2,0}(G_T)} + |\rho_t|_{l/2,3/2,G_T}\big). \tag{4.2}$$

If $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$, $l>1/2$, then

$$\|\rho_t(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})} \le c\|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)}. \tag{4.3}$$

For arbitrary functions $\rho_0 \in W_2^{l+2}(\mathcal{G})$ and $\rho_1 \in W_2^{l+1/2}(\mathcal{G})$, there exists a function

$$\rho \in W_2^{l+5/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))$$

with $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$ such that

$$\rho(x,0)=\rho_0(x), \quad \rho_t(x,0)=\rho_1(x)$$

and

$$\begin{aligned}\|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + |\rho|_{l/2,5/2,G_T} \\ \le c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_1\|_{W_2^{l+2}(\mathcal{G})}\Big).\end{aligned} \tag{4.4}$$

Proof. Inequalities (4.1)–(4.3) are consequences of the trace theorems for isotropic and anisotropic Sobolev–Slobodetskiĭ spaces. We turn to the second statement of the proposition. By the Slobodetskiĭ inverse trace theorem [11], we can construct $r_1 \in W_2^{l+5/2}(G_T)$ such that $r_1(x,0)=\rho_0(x)$, $r_{1t}(x,0)=0$, and

$$\|r_1\|_{W_2^{l+5/2}(G_T)} \le c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}.$$

By a similar theorem in the anisotropic case, there exists $r_2 \in W_2^{7/2+l,7/4+l/2}(G_T)$ such that $r_2(x,0)=0$, $r_{2t}(x,0)=\rho_1(x)$, and

$$\|r_2\|_{W_2^{l+7/2,l/2+7/4}(G_T)} \le c\|\rho_1\|_{W_2^{l+1/2}(\mathcal{G})}.$$

It is easily verified that $\rho=r_1+r_2$ possesses all the necessary properties. The proposition is proved. □

Proof of Theorem 1. We reduce (2.1) to a similar problem with zero divergence by constructing an auxiliary vector field $\boldsymbol{u}_1(x,t)=\nabla\Phi(x,t)$, where Φ is a solution of the Dirichlet problem

$$\nabla^2\Phi(x,t)=f(x,t), \quad x\in\mathcal{F}, \quad \Phi(x,t)|_{x\in\mathcal{G}}=0.$$

This function satisfies the inequality

$$\|\Phi\|_{W_2^{l+3,0}(Q_T)} \le c\|f\|_{W_2^{l+1,0}(Q_T)}; \tag{4.5}$$

moreover, since

$$\nabla^2\Phi_t(x,t)=f_t(x,t)=\nabla\cdot\boldsymbol{F}_t(x,t), \quad x\in\mathcal{F}, \quad \Phi_t(x,t)=0, \quad x\in\mathcal{G},$$

we have

$$\|\nabla\Phi_t\|_{W_2^{0,l/2}(Q_T)} \le c\|\boldsymbol{F}_t\|_{W_2^{0,l/2}(Q_T)}, \tag{4.6}$$

whence

$$\|\boldsymbol{u}_1\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\Big(\|f\|_{W_2^{l+1}(Q_T)} + \|\boldsymbol{F}_t\|_{W_2^{0,l/2}(Q_T)}\Big). \tag{4.7}$$

For $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{u}_1$, p, ρ we obtain

$$\begin{cases} \boldsymbol{v}_{1t} - \nu\nabla^2\boldsymbol{v}_1 + \nabla p = \boldsymbol{f}_1(x,t), \\ \nabla\cdot\boldsymbol{v}_1(x,t) = 0, \quad x\in\mathcal{F}, \quad t>0, \\ T(\boldsymbol{v}_1,p)\boldsymbol{N} + \sigma\boldsymbol{N}\mathfrak{L}\rho = \boldsymbol{d}_1(x,t), \\ \rho_t + \boldsymbol{V}\cdot\nabla_\tau\rho - \boldsymbol{v}_1(x,t)\cdot\boldsymbol{N}(x) = g_1(x,t), \quad x\in\mathcal{G}, \\ \boldsymbol{v}_1(x,0) = \boldsymbol{v}_0 - \boldsymbol{u}_1(x,0) \equiv \boldsymbol{w}_0(x), \quad x\in\mathcal{F}, \ \ \rho(x,0) = \rho_0(x), \ \ x\in\mathcal{G}, \end{cases} \tag{4.8}$$

where

$$\begin{cases} \boldsymbol{f}_1 = \boldsymbol{f} - \boldsymbol{u}_{1t} + \nu\nabla^2\boldsymbol{u}_1, \\ \boldsymbol{d}_1 = \boldsymbol{d} - \nu S(\boldsymbol{u}_1)\boldsymbol{N}, \quad g_1 = g + \boldsymbol{u}_1\cdot\boldsymbol{N}. \end{cases} \tag{4.9}$$

In particular,

$$\boldsymbol{d}_1\cdot\boldsymbol{N} = \boldsymbol{d}\cdot\boldsymbol{N} - \nu\boldsymbol{N}\cdot S(\boldsymbol{u}_1)\boldsymbol{N}|_{x\in\mathcal{G}}.$$

Now we reduce (4.8) to a similar problem with zero initial data. If $l<1$, we introduce a solenoidal vector field $\boldsymbol{u}_2(x,t)$ such that $\boldsymbol{u}_2(x,0) = \boldsymbol{w}_0(x)$ and

$$\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})}.$$

In the case where $l>1$, we also compute

$$\boldsymbol{v}_{1t}(x,0)|_{t=0} = \nu\nabla^2\boldsymbol{w}_0 - \nabla p_0(x) + \boldsymbol{f}_1(x,0) \equiv \boldsymbol{w}_1(x),$$

where p_0 is a solution of the problem

$$\begin{cases} \nabla^2 p_0(x) = \nabla\cdot\boldsymbol{f}_1(x,0), \quad x\in\mathcal{F}, \\ p_0(x) = \nu\boldsymbol{N}\cdot S(\boldsymbol{w}_0)\boldsymbol{N} + \sigma\mathfrak{L}\rho_0 - \boldsymbol{d}_1(x,0)\cdot\boldsymbol{N}, \quad x\in\mathcal{G}. \end{cases}$$

This solution satisfies the inequality

$$\begin{aligned} \|p_0\|_{W_2^l(\mathcal{F})} \le c\Big(&\|\boldsymbol{f}_1(\cdot,0)\|_{W_2^{l-1}(\mathcal{F})} + \|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} \\ &+ \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + \|\boldsymbol{d}_1(\cdot,0)\cdot\boldsymbol{N}\|_{W_2^{l-1/2}(\mathcal{G})}\Big), \end{aligned}$$

whence

$$\begin{aligned} \|\boldsymbol{w}_1\|_{W_2^{l-1}(\mathcal{F})} \le c\Big(&\|\boldsymbol{f}_1\|_{W_2^{l-1}(\mathcal{F})} + \|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} \\ &+ \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + \|\boldsymbol{d}_1\cdot\boldsymbol{N}\|_{W_2^{l-1/2}(\mathcal{G})}\Big). \end{aligned} \tag{4.10}$$

We find a solenoidal vector field $\boldsymbol{u}_2(x,t)$ such that

$$\boldsymbol{u}_2(x,0) = \boldsymbol{w}_0(x), \quad \boldsymbol{u}_{2t}(x,0) = \boldsymbol{w}_1(x)$$

and

$$\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{w}_1\|_{W_2^{l-1}(\mathcal{F})}\Big). \tag{4.11}$$

Moreover, we construct $p_1(x,t)$ and $\rho_1(x,t)$ such that $p_1(x,0) = p_0(x)$,

$$\rho_1(x,0) = \rho_0(x), \quad \rho_{1t}(x,0) = -\boldsymbol{V}(x)\cdot\nabla_\tau\rho_0 - \boldsymbol{w}_0(x)\cdot\boldsymbol{N} + g_1(x,0) \equiv \rho_1'(x) \tag{4.12}$$

and

$$
\|p_1\|_{W_2^{l+1,l/2+1/2}(Q_T)} \le c\|p_0\|_{W_2^l(\mathcal{F})},
$$

$$
(4.13) \qquad \|\rho_1\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_{1t}\|_{W_2^{l+3/2,l/2+1/4}(G_T)} + |\rho|_{l/2,5/2,G_T} \le c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_1'\|_{W_2^{l+1/2}(\mathcal{G})}\Big).
$$

The construction of ρ_1 is described in Proposition 4.1. The construction of $\boldsymbol{u}_2$ is carried out in the following way. We find $\boldsymbol{w}_0(x,0)$ and $\boldsymbol{w}_1(x,0)$, $x \in \mathbb{R}^3$, in the form $\boldsymbol{w}_i(x,0) = \boldsymbol{\xi}_i + \boldsymbol{\eta}_i$, where $\boldsymbol{\xi}_i$ is an extension of $\boldsymbol{w}_i(x)$ to $\mathbb{R}^3$ with preservation of the class; we assume that $\boldsymbol{\xi}_i$ has compact support. Then, using the result of Bogovskiĭ [12], we can find $\boldsymbol{\eta}_i$, also with compact support, satisfying the equation $\nabla \cdot \boldsymbol{\eta}_i = -\nabla \cdot \boldsymbol{\xi}_i$ and the inequalities

$$
\|\boldsymbol{\eta}_1\|_{W_2^{l+1}(\mathbb{R}^3)} \le c\|\boldsymbol{\xi}_1\|_{W_2^{l+1}(\mathbb{R}^3)} \le c\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega)},
$$
$$
\|\boldsymbol{\eta}_2\|_{W_2^{l-1}(\mathbb{R}^3)} \le c\|\boldsymbol{\xi}_2\|_{W_2^{l-1}(\mathbb{R}^3)} \le c\|\boldsymbol{w}_1\|_{W_2^{l-1}(\Omega)}.
$$

Finally, we introduce the vector field $\boldsymbol{u}_2(x,t)$ satisfying

$$
\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(\mathbb{R}^3_+\times\mathbb{R}_+)} \le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathbb{R}^3)} + \|\boldsymbol{w}_1\|_{W_2^{l-1}(\mathbb{R}^3)}\Big) \le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{w}_1\|_{W_2^{l-1}(\mathcal{F})}\Big).
$$

Usually, $\boldsymbol{u}_2$ is expressed in terms of $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ as a sum of convolution integrals (with respect to x_i); then it is divergence free.

For $\boldsymbol{v}_2 = \boldsymbol{v}_1 - \boldsymbol{u}_2$, $p_2 = p - p_1$, $\rho_2 = \rho - \rho_1$ we have

$$
(4.14) \qquad \begin{cases} \boldsymbol{v}_{2t} - \nu\nabla^2\boldsymbol{v}_2 + \nabla p_2 = \boldsymbol{f}_2(x,t), \\ \nabla\cdot\boldsymbol{v}_2(x,t) = 0, \qquad x\in\mathcal{F}, \quad t>0, \\ T(\boldsymbol{v}_2,p_2)\boldsymbol{N} + \sigma\boldsymbol{N}\mathfrak{L}\rho_2 = \boldsymbol{d}_2(x,t), \\ \rho_{2t} + \boldsymbol{V}\cdot\nabla_\tau\rho_2 = \boldsymbol{v}_2(x,t)\cdot\boldsymbol{N}(x) + g_2(x,t), \quad x\in\mathcal{G}, \\ \boldsymbol{v}_2(x,0) = 0, \quad x\in\mathcal{F}, \quad r_2(x,0) = 0, \quad x\in\mathcal{G}, \end{cases}
$$

where

$$
\boldsymbol{f}_2 = \boldsymbol{f}_1 - \Big(\boldsymbol{u}_{2t} - \nu\nabla^2\boldsymbol{u}_2 + \nabla p_1\Big),
$$
$$
\boldsymbol{d}_2 = \boldsymbol{d}_1 - (T(\boldsymbol{u}_2,p_1)\boldsymbol{N} + \sigma\mathfrak{L}\rho_1\boldsymbol{N}),
$$
$$
g_2 = g_1 - \boldsymbol{V}\cdot\nabla\rho_1 + \boldsymbol{w}_1\cdot\boldsymbol{N} - \rho_{1t}.
$$

Since $\boldsymbol{d}_2$, g_2 (and also $\boldsymbol{f}_2$ for $l>1$) vanish for $t=0$ and $l<5/2$, we can extend these functions by zero to the domain $t<0$ with preservation of the class, after which we extend them, also with preservation of the class, to the domain $t>T$. Then we apply the Laplace transformation, as in [7], assuming that $\operatorname{Re} s \equiv a$ is sufficiently large (s is the dual variable). Problem (4.14) reshapes to (3.1), whose solvability was proved in Theorem 2.1. The inverse Laplace transform yields the solution of (4.14) defined in an infinite time interval $(-\infty,+\infty)$. Using estimate (2.4) and the Parceval identity, we obtain an estimate of this solution in weighted Sobolev spaces with the weight e^{-at}. It follows that

(4.15)
$$\begin{aligned}
&\|\boldsymbol{v}_2\|_{W_2^{l+2,l/2+1}(Q_{-\infty,T})} + \|\nabla p_2\|_{W_2^{l,l/2}(Q_{-\infty,T})} + \|p_2\|_{W_2^{l+1/2,0}(G_{-\infty,T})}\\
&\qquad + |p_2|_{l/2,1/2,G_{-\infty,T}} + \|\rho_2\|_{W_2^{l+5/2,0}(G_{-\infty,T})} + |\rho|_{l/2,5/2,G_{-\infty,T}}\\
&\qquad + \|\rho_{2t}\|_{W_2^{l+3/2,0}(G_{-\infty,T})} + |\rho_{2t}|_{l/2,3/2,G_{-\infty,T}}\\
&\le c\Big(\|\boldsymbol{f}_2\|_{W_2^{l,l/2}(Q_{-\infty,T})} + \|\boldsymbol{d}_2 - \boldsymbol{N}(\boldsymbol{d}_2\cdot\boldsymbol{N})\|_{W_2^{l+1/2,l/2+1/4}(G_{-\infty,T})}\\
&\qquad + \|\boldsymbol{d}_2\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_{-\infty,T})} + |\boldsymbol{d}_2\cdot\boldsymbol{N}|_{l/2,1/2,G_{-\infty,T}}\\
&\qquad 22 + \|g_2\|_{W_2^{l+3/2,0}(G_{-\infty,T})} + |g_2|_{l/2,3/2,G_{-\infty,T}}\Big),
\end{aligned}$$

where $Q_{-\infty,T} = \mathcal{F}\times(-\infty,T)$, $G_{-\infty,T} = \mathcal{G}\times(-\infty,T)$. All functions in (4.15) vanish for $t<0$, and the constant c is bounded for finite T. Using (4.15), it is easy to deduce the estimate

(4.16)
$$\begin{aligned}
&Y_T(\boldsymbol{v}_2,q_2,\rho_2) \le c(T)\Big(\|\boldsymbol{f}_2\|_{W_2^{l,l/2}(Q_T)} + \|\boldsymbol{d}_2 - \boldsymbol{N}(\boldsymbol{d}_2\cdot\boldsymbol{N})\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
&\quad + \|\boldsymbol{d}_2\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + |\boldsymbol{d}_2\cdot\boldsymbol{N}|_{l/2,1/2,G_T} + \|g_2\|_{W_2^{l+3/2,0}(G_T)} + |g_2|_{l/2,3/2,G_T}\Big),
\end{aligned}$$

and inequality (1.4) follows from (4.16), (4.11), and (4.7). This completes the proof of Theorem 1.1. □

For an application of this theorem to the proof of the local solvability of a nonlinear problem (see §5), it is important to be sure that the constant in the basic inequality (1.4) remains bounded for small T. In fact, this is not always the case, because the norm $\|u\|_{W_2^l(-\infty,T)}$, $l=[l]+\lambda$, $0<\lambda<1$, of the function $u(t)$ vanishing for $t<0$ is equivalent to

$$\Big(\|u\|^2_{W_2^l(0,T)} + \int_0^T \frac{|D_t^{[l]}u(t)|^2}{t^{2\lambda}}\,dt\Big)^{1/2}$$

(in this connection see [13] and [6, Chapter 4]). If $\lambda>1/2$ and $D_t^{[l]}u|_{t=0}=0$, then

$$\int_0^T \frac{|D_t^{[l]}u(t)|^2}{t^{2\lambda}}\,dt \le c\int_0^T \frac{dh}{h^{1+2\lambda}}\int_h^T |D_t^{[l]}u(t-h) - D_t^{[l]}u(t)|^2\,dt$$

with constant independent of T. If $\lambda<1/2$, then we have

$$\begin{aligned}
c_1\Big(\|u\|^2_{W_2^l(0,T)} + \int_0^T \frac{|D_t^{[l]}u(t)|^2}{t^{2\lambda}}\,dt\Big) &\le \|u\|^2_{W_2^l(0,T)} + \frac{1}{T^{2\lambda}}\int_0^T |D_t^{[l]}u(t)|^2\,dt\\
&\le c_2\Big(\|u\|^2_{W_2^l(0,T)} + \int_0^T \frac{|D_t^{[l]}u(t)|^2}{t^{2\lambda}}\,dt\Big),
\end{aligned}$$

where the constants are also independent of T. Hence, the constant $c(T)$ in (4.16) becomes uniformly bounded for finite T if all the $W_2^{l/2}(0,T)$-norms in this inequality are replaced with the $\widehat{W}_2^{l/2}(0,T)$-norms defined by

(4.17)
$$\begin{aligned}
&\|u\|_{\widehat{W}_2^{l/2}(0,T)} = \|u\|_{W_2^{l/2}(0,T)} \quad \text{if} \quad \lambda>1/2,\\
&\|u\|_{\widehat{W}_2^{l/2}(0,T)} = \Big(\|u\|^2_{W_2^{l/2}(0,T)} + \frac{1}{T^{2\lambda}}\int_0^T |D_t^{[l/2]}u(t)|^2\,dt\Big)^{1/2} \quad \text{if} \quad \lambda<1/2
\end{aligned}$$

(here λ is the fractional part of $l/2$). As a consequence of (4.15), we have

$$
\begin{aligned}
&\|\boldsymbol{v}_2\|_{\widehat{W}_2^{l+2,l/2+1}(Q_T)} + \|\nabla p_2\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|p_2\|_{W_2^{l+1/2,0}(G_T)} \\
&\qquad + \langle p_2\rangle_{l/2,1/2,G_T} + \|\rho_2\|_{W_2^{l+5/2,0}(G_T)} + \langle \rho_2\rangle_{l/2,5/2,G_T} \\
&\qquad + \|\rho_{2t}\|_{W_2^{l+3/2,0}(G_T)} + \langle \rho_{2t}\rangle_{l/2,3/2,G_T} + \langle \rho\rangle_{l/2,5/2,G_T} \\
&\le c\Big(\|\boldsymbol{f}_2\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|\boldsymbol{d}_2 - \boldsymbol{N}(\boldsymbol{d}_2\cdot\boldsymbol{N})\|_{\widehat{W}_2^{l+1/2,l/2+1/4}(G_T)} \\
&\qquad + \|\boldsymbol{d}_2\cdot\boldsymbol{N}\|_{\widehat{W}_2^{l+1/2,0}(G_T)} + \langle \boldsymbol{d}_2\cdot\boldsymbol{N}\rangle_{l/2,1/2,G_T} \\
&\qquad + \|g_2\|_{\widehat{W}_2^{l+3/2,0}(G_T)} + \langle g_2\rangle_{l/2,3/2,G_T}\Big),
\end{aligned} \tag{4.18}
$$

where $\langle\,\cdot\,\rangle_{l/2,r,G_T}$ is the norm in $\widehat{W}^{l/2}(0,T;W_2^r(\mathcal{G}))$. By $\widehat{W}_2^{l,l/2}(Q_T)$ we mean the space with the modified norm (1.2):

$$
\|u\|^2_{\widehat{W}_2^{l,l/2}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}\,dt + \int_\Omega \|u(x,\cdot)\|^2_{\widehat{W}_2^{l/2}(0,T)}\,dx. \tag{4.19}
$$

Clearly, the norms (1.2) and (4.19) are equivalent.

Now we turn to inequality (4.11). We can set $T=\infty$ in (4.11); it is also possible to assume that $\boldsymbol{u}_2$ vanishes for $t>t_0$. We use the following inequality (see [13, Lemma 2]):

$$
\int_0^\infty \frac{|v(t)|^2}{t^{2\lambda}}\,dt \le c\int_0^\infty \frac{dh}{h^{1+2\lambda}}\int_h^\infty |v(t-h)-v(y)|^2\,dt; \tag{4.20}
$$

which is valid for $\lambda\in(0,1/2)$ and for $\lambda\in(1/2,1)$, $v(0)=0$. It follows that the norm $\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(Q_T)}$ in (4.11) can be replaced with $\|\boldsymbol{u}_2\|_{\widehat{W}_2^{l+2,l/2+1}(Q_T)}$. The same is true for inequalities (4.6), (4.7). Consequently, along with (1.4), (1.5) we have

$$
\begin{aligned}
\widehat{Y}_T(\boldsymbol{v},p,\rho) &\equiv \|\boldsymbol{v}\|_{\widehat{W}_2^{l+2,l/2+1}(Q_T)} + \|\nabla p\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} \\
&\qquad + \langle p\rangle_{l/2,1/2,G_T} + \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \langle\rho\rangle_{l/2,5/2,G_T} \\
&\qquad + \|\rho_t\|_{W_2^{l+3/2,0}(G_T)} + \langle\rho_t\rangle_{l/2,3/2,G_T} + \langle\rho\rangle_{l/2,5/2,G_T} \\
&\le c\Big(\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widehat{W}_2^{0,1+l/2}(Q_T)} \\
&\qquad + \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{\widehat{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + \langle\boldsymbol{d}\cdot\boldsymbol{N}\rangle_{l/2,1/2,G_T} \\
&\qquad + \|g\|_{W_2^{l+3/2,0}(G_T)} + \langle g\rangle_{l/2,3/2,G_T} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big),
\end{aligned} \tag{4.21}
$$

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{\widehat{W}_2^{l+2,l/2+1}(Q_T)} + \|\nabla p\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + \langle p\rangle_{l/2,1/2,G_T} \\
&\qquad + \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \langle\rho\rangle_{l/2,5/2,G_T} + \|\rho_t\|_{\widehat{W}_2^{l+3/2,l/2+3/4}(G_T)} + \langle\rho\rangle_{l/2,5/2,G_T} \\
&\le c\Big(\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widehat{W}_2^{0,1+l/2}(Q_T)} \\
&\qquad + \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{\widehat{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + \langle\boldsymbol{d}\cdot\boldsymbol{N}\rangle_{l/2,1/2,G_T} \\
&\qquad + \|g\|_{\widehat{W}_2^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big),
\end{aligned} \tag{4.22}
$$

where the constants are independent of T.

§5. On the free boundary problem

Theorem 1.1 provides an analytical basis for the proof of the solvability of the free boundary problem governing the motion of an isolated liquid mass:

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{v} = 0, \quad x\in\Omega_t, \ t>0, \\ T(\boldsymbol{v},p)\boldsymbol{n}(x) = \sigma H\boldsymbol{n}(x), \\ V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x\in\Gamma_t, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x\in\Omega_0. \end{cases} \tag{5.1}$$

Unknown are the domain Ω_t with the boundary Γ_t for $t>0$, $\boldsymbol{v}(x,t)$, and $p(x,t)$, $x\in\Omega_t$. The domain Ω_0 is given. By $\boldsymbol{n}$ we mean the outward normal to Γ_t, V_n is the velocity of the evolution of Γ_t in the normal direction and H is the doubled mean curvature of Γ_t.

We assume that Γ_0 is close to a smooth closed surface $\mathcal{G}$ of arbitrary shape, so that Γ_0 can be regarded as a normal perturbation of $\mathcal{G}$:

$$\Gamma_0 = \{x = y + \boldsymbol{N}(y)\rho_0(y),\ y\in\mathcal{G}\},$$

where $\boldsymbol{N}(y)$ is the outward unit normal to $\mathcal{G}$ and ρ_0 is a given small function. We denote by $\mathcal{F}$ the domain bounded by $\mathcal{G}$. Also, we assume that, at least for small t, Γ_t is close to $\mathcal{G}$, too, and can be given by the equation $x = y + \boldsymbol{N}(y)\rho(y,t)$, $y\in\mathcal{G}$, with an unknown function $\rho(y,t)$.

As usual, the free boundary problem (5.1) is written as a nonlinear problem in a given domain, which is achieved by mapping Ω_t onto this domain. We use the transformation

$$x = y + \boldsymbol{N}^*(y)\rho^*(y) \equiv e_\rho(y)\,:\,\mathcal{F}\to\Omega_t, \tag{5.2}$$

where ρ^* and $\boldsymbol{N}^*$ are extensions of ρ and $\boldsymbol{N}$ from $\mathcal{G}$ to $\mathcal{F}$ such that $\boldsymbol{N}^*$ is a sufficiently regular vector field and ρ^* is a function with a small C^1-norm. This guarantees the invertibility of e_ρ.

Denoting by $\mathcal{L} = \mathcal{L}(y,\rho^*)$ the Jacobi matrix of the transformation $x = e_\rho(y)$, we set $L = \det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$. By $l_{ij}(y,\rho^*)$, $l^{ij}(y,\rho^*)$, $\widehat{L}_{ij}(y,\rho^*)$ we denote the entries of $\mathcal{L}$, $\mathcal{L}^{-1}$, $\widehat{\mathcal{L}}$. The transformation (5.2) converts the operator ∇_x of the gradient with respect to x to $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla$, $\nabla = \nabla_y$. Equations (5.1) take the form

$$\begin{cases} \boldsymbol{u}_t(y,t) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u},q,\rho), \\ \nabla\cdot\boldsymbol{u} = l_2(\boldsymbol{u},\rho), \quad y\in\mathcal{F}, t>0, \\ \Pi_{\mathcal{G}} S(\boldsymbol{u})\boldsymbol{N}(y) = \boldsymbol{l}_3(\boldsymbol{u},\rho), \\ -q + \nu\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}(y) + \sigma\mathfrak{L}\rho = l_4(\boldsymbol{u},\rho) + l_5(\rho) + \sigma\mathcal{H}(y), \\ \rho_t + \boldsymbol{V}(y)\cdot\nabla_\tau\rho - \boldsymbol{u}\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u},\rho), \quad y\in\mathcal{G}, \\ \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y\in\mathcal{F}, \quad \rho(y,0) = \rho_0(y), \quad y\in\mathcal{G}, \end{cases} \tag{5.3}$$

where $\mathcal{H}$ is the doubled mean curvature of $\mathcal{G}$, and

$$
(5.4)\quad \begin{cases}
\boldsymbol{l}_1(\boldsymbol{u},q,\rho) = \nu(\widetilde{\nabla}^2-\nabla^2)\boldsymbol{u} + (\nabla-\widetilde{\nabla})q + \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u} - (\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\\
l_2(\boldsymbol{u},\rho) = (I-\widehat{\mathcal{L}}^T)\nabla\cdot\boldsymbol{u} = \nabla\cdot(I-\widehat{\mathcal{L}})\boldsymbol{u},\\
\boldsymbol{l}_3(\boldsymbol{u},\rho) = \Pi_{\mathcal{G}}(\Pi_{\mathcal{G}}S(\boldsymbol{u})\boldsymbol{N}(y) - \Pi\widetilde{S}(\boldsymbol{u})\boldsymbol{n}(e_\rho(y))),\\
l_4(\boldsymbol{u},\rho) = \nu(\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{n}\cdot\widetilde{S}(\boldsymbol{u})\boldsymbol{n}),\\
l_5(\rho) = \sigma\displaystyle\int_0^1(1-s)\frac{d}{ds^2}\frac{\mathcal{L}^T(y,s\rho)\boldsymbol{N}}{|\mathcal{L}^T(y,s\rho)\boldsymbol{N}|}\,ds,\\
l_6(\boldsymbol{u},\rho) = \Big(\dfrac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{\Lambda(y,\rho)} - \boldsymbol{N} + \nabla_\tau\rho\Big)\cdot\boldsymbol{u} + (\boldsymbol{V}(y)-\boldsymbol{u}(u,t))\cdot\nabla_\tau\rho, \quad y\in\mathcal{G}.
\end{cases}
$$

By $\widetilde{S}$ we mean the transformed rate-of-strain tensor: $\widetilde{S}(\boldsymbol{u}) = (\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^T$, $\Pi\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{f})$. The normals $\boldsymbol{n}(e_\rho)$ and $\boldsymbol{N}$ are related by the formula

$$
\boldsymbol{n}(e_\rho(y)) = \frac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|}.
$$

The expression $\mathfrak{L}\rho = -\Delta_{\mathcal{G}}\rho + (\mathcal{H}^2-2\mathcal{K})\rho$ is computed as the first variation of

$$
H - \mathcal{H} = -\nabla_x\cdot\boldsymbol{n}(x)|_{x=e_\rho} + \nabla_y\cdot\boldsymbol{N}(y)
$$

with respect to ρ.

Theorem 5.1. *If* $\boldsymbol{u}_0 \in W_2^{l+1}(\mathcal{F})$, $\rho_0\in W_2^{l+2}(\mathcal{G})$, *and the compatibility and smallness conditions* $\widetilde{\nabla}\cdot\boldsymbol{u}_0 = 0$, $\Pi\widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n} = 0$, $t=0$, $\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \le \epsilon \ll 1$ *are satisfied, then problem* (5.3) *has a unique solution with a finite norm* $\widehat{Y}_T(\boldsymbol{u},q,\rho)$ (*see* (4.21)) *defined on a certain* (*small*) *time interval* $(0,T)$.

The solvability of problem (5.3) can be established by the method of successive approximations, in accordance with the usual pattern:

$$
(5.5)\quad \begin{cases}
\boldsymbol{u}_{m+1,t}(y,t) - \nu\nabla^2\boldsymbol{u}_{m+1} + \nabla q_{m+1} = \boldsymbol{l}_1(\boldsymbol{u}_m,q_m,\rho_m),\\
\nabla\cdot\boldsymbol{u}_{m+1} = l_2(\boldsymbol{u}_m,\rho_m), \quad y\in\mathcal{F}, t>0,\\
\Pi_{\mathcal{G}}S(\boldsymbol{u}_{m+1})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}_m,\rho_m),\\
-q_{m+1} + \nu\boldsymbol{N}\cdot S(\boldsymbol{u}_{m+1})\boldsymbol{N}(y) + \sigma\mathfrak{L}\rho_{m+1} = l_4(\boldsymbol{u}_m,\rho_m) + l_5(\rho_m) + \sigma\mathcal{H}(y),\\
\rho_{m+1,t} + \boldsymbol{V}(y)\cdot\nabla_\tau\rho_{m+1} - \boldsymbol{u}_{m+1}\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u}_m,\rho_m), \ \ y\in\mathcal{G},\\
\boldsymbol{u}_{m+1}(y,0) = \boldsymbol{u}_0(y), \ \ y\in\mathcal{F}, \quad \rho_{m+1}(y,0) = \rho_0(y), \ \ y\in\mathcal{G},
\end{cases}
$$

$m = 1,2,\dots$. As the first approximation, we take the functions $(\boldsymbol{u}_1,\rho_1)$ satisfying the conditions $\boldsymbol{u}_1(y,0) = \boldsymbol{u}_0(y)$, $\rho_1(y,0) = \rho_0(y)$, and we set $q_1 = 0$. We require that

$$
(5.6)\quad \begin{aligned}
&\|\boldsymbol{u}_1\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F})},\\
&\|\rho_1\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_{1,t}\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \le c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}.
\end{aligned}
$$

Then the compatibility conditions (1.3) in the linear problems (5.5) are satisfied for all $m\ge 1$. Moreover, estimates of nonlinear terms (we omit them) enable us to show, by using Theorem 2.1, that

$$
(5.7)\quad \begin{aligned}
\widehat{Y}_T(\boldsymbol{u}_{m+1},q_{m+1},\rho_{m+1}) &\le \delta_1\sum_{j=1}^3 \widehat{Y}_T^j(\boldsymbol{u}_m,q_m,\rho_m)\\
&\quad + c\Big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{q}_0\|_{W_2^l(\mathcal{F})} + \|\rho_0\|_{W_2^{l+2}(\mathcal{F})} + \|\mathcal{H}\|_{W_2^l(\mathcal{G})}\Big),
\end{aligned}
$$

where δ_1 is a number depending on T and $\|\rho_0\|_{W_2^{l+3/2}(\mathcal{F})}$ and going to zero as T and the

$W_2^{l+3/2}(\mathcal{F})$-norm of ρ_0 tend to zero. If δ_1 is sufficiently small, then inequalities (5.7) guarantee a uniform estimate for $Y_T(\boldsymbol{u}_{m+1}, q_{m+1}, \rho_{m+1})$. The convergence of $(\boldsymbol{u}_m, q_m, \rho_m)$ to the solution of (2.2) is proved by similar arguments.

Estimates (5.7) hold true if the vector field $\boldsymbol{V}(x)$ is chosen properly. In accordance with our calculations, it should belong to $W_2^{l+3/2}(\mathcal{G})$ and satisfy the condition

$$\sup_{\mathcal{G}} |\boldsymbol{V}(x) - \boldsymbol{u}_0| + \|\boldsymbol{V} - \boldsymbol{u}_0\|_{W_2^1(\mathcal{G})} \leq \delta_2 \ll 1. \tag{5.8}$$

The proof of Theorem 5.1 is given in [14].

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St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia

E-mail address: solonnik@pdmi.ras.ru

Received 8/JUL/2010

Originally published in English