# FINITE REPRESENTABILITY OF $\ell_{p}$-SPACES IN SYMMETRIC SPACES 

S. V. ASTASHKIN


#### Abstract

For a separable rearrangement invariant space $X$ on the semiaxis, $\mathcal{F}(X)$ is defined to be the set of all $p \in[1, \infty]$ such that $\ell_{p}$ is finitely representable in $X$ in such a way that the standard basis vectors of $\ell_{p}$ correspond to equimeasurable mutually disjoint functions. In the paper, a characterization of the set $\mathcal{F}(X)$ is obtained. As a consequence, a version of Krivine's well-known theorem is proved for rearrangement invariant spaces. Next, a description of the sets $\mathcal{F}(X)$ for certain Lorentz spaces is found.


## §1. Introduction

In 1974, Tsirel'son constructed an example of a Banach space containing no isomorphic copies of $\ell_{p}, 1 \leq p<\infty$, and $c_{0}$. Two years later, Krivine proved a theorem that showed once again the fundamental difference between the properties of infinite-dimensional subspaces of a Banach space and subspaces of finite (though large) dimension. To state and discuss it, we introduce some notions.
Definition. Suppose $X$ is a Banach space, $1 \leq p \leq \infty$, and $\left\{z_{i}\right\}_{i=1}^{\infty}$ is a bounded sequence in $X$. The space $\ell_{p}$ is said to be block finitely representable in $\left\{z_{i}\right\}_{i=1}^{\infty}$ if for every $n \in \mathbb{N}$ and $\varepsilon>0$ there exist $0=p_{0}<p_{1}<\cdots<p_{n}$ and $\alpha_{i} \in \mathbb{C}$ such that the vectors $u_{k}=\sum_{i=p_{k-1}+1}^{p_{k}} \alpha_{i} z_{i}(k=1,2, \ldots, n)$ satisfy the inequality

$$
(1+\varepsilon)^{-1}\left\|\left(a_{k}\right)_{k=1}^{n}\right\|_{p} \leq\left\|\sum_{k=1}^{n} a_{k} u_{k}\right\|_{X} \leq(1+\varepsilon)\left\|\left(a_{k}\right)_{k=1}^{n}\right\|_{p}
$$

for arbitrary $a_{1}, a_{2}, \ldots, a_{n} \in \mathbb{C}$. Here, as usual,

$$
\left\|\left(a_{k}\right)_{k=1}^{n}\right\|_{p}:=\left(\sum_{k=1}^{n}\left|a_{k}\right|^{p}\right)^{1 / p} \text { if } p<\infty, \text { and }\left\|\left(a_{k}\right)_{k=1}^{n}\right\|_{\infty}:=\max _{k=1,2, \ldots, n}\left|a_{k}\right| .
$$

Definition. Let $X$ be a Banach space, and let $1 \leq p \leq \infty$. The space $\ell_{p}$ is said to be finitely representable in $X$ if for every $n \in \mathbb{N}$ and $\varepsilon>0$ there exist $x_{1}, x_{2}, \ldots, x_{n} \in X$ such that

$$
\begin{equation*}
(1+\varepsilon)^{-1}\left\|\left(a_{k}\right)_{k=1}^{n}\right\|_{p} \leq\left\|\sum_{k=1}^{n} a_{k} x_{k}\right\|_{X} \leq(1+\varepsilon)\left\|\left(a_{k}\right)_{k=1}^{n}\right\|_{p} \tag{1.1}
\end{equation*}
$$

for arbitrary $a_{1}, a_{2}, \ldots, a_{n} \in \mathbb{C}$.
By the celebrated Dvoretzky theorem (see [2] or [3, Theorem 5.8]), $\ell_{2}$ is finitely representable in an arbitrary infinite-dimensional Banach space $X$. Clearly, if $\ell_{p}$ is block

[^0]finitely representable in some sequence $\left\{z_{i}\right\}_{i=1}^{\infty} \subset X$, then $\ell_{p}$ is finitely representable in $X$. Therefore, the following statement proved by Krivine in [4] is an important supplement to the Dvoretzky theorem.

Theorem 1 (see [5, Theorem 11.3.9]). Let $\left\{z_{i}\right\}_{i=1}^{\infty}$ be an arbitrary normalized sequence in a Banach space $X$ such that the vectors $z_{i}$ do not form a relatively compact set. Then $\ell_{p}$ is block finitely representable in $\left\{z_{i}\right\}_{i=1}^{\infty}$ for some $p, 1 \leq p \leq \infty$.

Undoubtedly, this statement is a central result in the geometric theory of Banach spaces. It has numerous applications (see, e.g., 5 and 3]). In this connection, it is natural to look for a description of all $p \in[1, \infty]$ such that $\ell_{p}$ is finitely representable in a given Banach space. It has turned out that the (Rademacher) type and cotype of the space play an important role here. Denote by $r_{n}:[0,1] \rightarrow \mathbb{R}(n \in \mathbb{N})$ the Rademacher functions, i.e., $r_{n}(t)=\operatorname{sign}\left(\sin 2^{n} \pi t\right)$. A Banach space $X$ is said to have type $p, 1 \leq p \leq 2$, if there exists a constant $K>0$ such that for every $n \in \mathbb{N}$ and arbitrary $x_{1}, \ldots, x_{n}$ we have

$$
\int_{0}^{1}\left\|\sum_{k=1}^{n} r_{k}(t) x_{k}\right\| d t \leq K\left(\sum_{k=1}^{n}\left\|x_{k}\right\|^{p}\right)^{1 / p}
$$

A Banach space $X$ is said to have cotype $q \geq 2$ if there is a constant $K>0$ such that for every $n \in \mathbb{N}$ and arbitrary $x_{1}, \ldots, x_{n}$ we have

$$
\left(\sum_{k=1}^{n}\left\|x_{k}\right\|^{q}\right)^{1 / q} \leq K \int_{0}^{1}\left\|\sum_{k=1}^{n} r_{k}(t) x_{k}\right\| d t
$$

If $q=\infty$, the left-hand side of the last inequality must be replaced by $\max _{1 \leq k \leq n}\left\|x_{k}\right\|$. The space $X$ is said to have trivial type (trivial cotype) if $X$ only has type 1 (only has infinite cotype). Detailed information about these notions can be found in [6] and [7]. We introduce the notation

$$
p_{X}:=\sup \{p \in[1,2]: X \text { has type } p\}
$$

and

$$
q_{X}:=\inf \{q \in[2, \infty]: X \text { is of cotype } q\}
$$

It can easily be checked that if $\ell_{p}$ is finitely representable in a Banach space $X$, then $p \in\left[p_{X}, q_{X}\right]$. Moreover, Maurey and Pisier proved the following refinement of Theorem 1 (see [8]).
Theorem 2 (Maurey-Pisier). For every infinite-dimensional Banach space $X$, the spaces $\ell_{p_{X}}$ and $\ell_{q_{X}}$ are finitely representable in $X$.

Suppose now that a Banach space $X$ is endowed with a partial order making $X$ a Banach lattice. If $\ell_{p}$ is finitely representable in $X$ and, moreover, the images $x_{1}, x_{2}, \ldots, x_{n}$ in $X$ of the basis vectors of $\ell_{p}$ can be chosen mutually disjoint, we say that $\ell_{p}$ is lattice finitely representable in the Banach lattice $X$. In this case, the role of the type and cotype is played by the upper and lower estimates for the lattice.

We recall that a Banach lattice $X$ admits an upper p-estimate if there exists a constant $M$ such that for every $n \in \mathbb{N}$ and arbitrary mutually disjoint vectors $x_{1}, \ldots, x_{n} \in X$ we have

$$
\left\|\sum_{k=1}^{n} x_{k}\right\| \leq M\left(\sum_{k=1}^{n}\left\|x_{k}\right\|^{p}\right)^{1 / p} \quad(p<\infty)
$$

and

$$
\left\|\sum_{k=1}^{n} x_{k}\right\| \leq M \max _{1 \leq k \leq n}\left\|x_{k}\right\| \quad(p=\infty)
$$

A Banach lattice $X$ is said to admit a lower $q$-estimate if there exists a constant $M>0$ such that for every $n \in \mathbb{N}$ and arbitrary mutually disjoint vectors $x_{1}, \ldots, x_{n} \in X$ we have

$$
\left(\sum_{k=1}^{n}\left\|x_{k}\right\|^{q}\right)^{1 / q} \leq M\left\|\sum_{k=1}^{n} x_{k}\right\| \quad(q<\infty)
$$

and

$$
\max _{1 \leq k \leq n}\left\|x_{k}\right\| \leq M\left\|\sum_{k=1}^{n} x_{k}\right\| \quad(q=\infty)
$$

Let

$$
s(X):=\sup \{p \geq 1: X \text { admits an upper } p \text {-estimate }\}
$$

and

$$
\sigma(X):=\inf \{q \geq 1: X \text { admits a lower } q \text {-estimate }\} .
$$

If $\ell_{p}$ is lattice finitely representable in a Banach lattice $X$, then $p \in[s(X), \sigma(X)]$. Furthermore, the following lattice version of the Maurey-Pisier and Krivine theorems was proved in 9 .

Theorem 3 (Schep). If $X$ is an infinite-dimensional Banach lattice, then $\ell_{s(X)}$ and $\ell_{\sigma(X)}$ are finitely representable in $X$.

In what follows, we shall consider a special class of Banach function lattices, specifically, the symmetric (in other terminology, rearrangement invariant) spaces. See the next section for the definition. If $x(t)$ is a measurable function on $[0, \alpha)(0<\alpha \leq \infty)$, we denote $n_{x}(\tau):=m(\{s \in[0, \alpha):|x(s)|>\tau\})$. Here and in the sequel $m$ denotes the Lebesgue measure.

Definition. Let $X$ be a symmetric space on $[0, \infty)$. We denote by $\mathcal{F}(X)$ the set of $p \in[1, \infty]$ with the property that for every $n \in \mathbb{N}$ and every $\varepsilon>0$ there exist $x_{k} \in X$ $(k=1,2, \ldots, n)$ such that $\operatorname{supp} x_{i} \cap \operatorname{supp} x_{j}=\varnothing$ for $(i \neq j), n_{x_{k}}(\tau)=n_{x_{1}}(\tau)(k=$ $1,2, \ldots, n ; \tau>0$ ), and for every $a_{k} \in \mathbb{C}$ we have

$$
\begin{equation*}
(1+\varepsilon)^{-1}\left\|\left(a_{k}\right)_{k=1}^{n}\right\|_{p} \leq\left\|\sum_{k=1}^{n} a_{k} x_{k}\right\|_{X} \leq(1+\varepsilon)\left\|\left(a_{k}\right)_{k=1}^{n}\right\|_{p} \tag{1.2}
\end{equation*}
$$

(with a natural modification for $p=\infty$ ).
If $\alpha_{X}$ and $\beta_{X}$ are the Boyd indices of a symmetric space $X$, it can easily be shown that $\mathcal{F}(X) \subset\left[1 / \beta_{X}, 1 / \alpha_{X}\right]$. Next, in the monograph [7] the following version of Krivine's theorem was stated without proof (see also the remark after Theorem 3.3 in [10]).

Theorem 4 (see [7, Theorem 2.b.6]). If $X$ is an arbitrary symmetric space, then $\max \mathcal{F}(X)=1 / \alpha_{X}$ and $\min \mathcal{F}(X)=1 / \beta_{X}$.

The last theorem and, in general, the structure of $\mathcal{F}(X)$ play an important role in the study of geometric properties of symmetric spaces (see, e.g., [11) and also in the study of normal solvability and invertibility of operators between function spaces, which, in its turn, is important for the theory of functional-differential equations, the theory of dynamical systems, etc. (see [12] and references therein).

The following notion will be important in what follows.
Definition. Let $A: X \rightarrow X$ be a bounded linear operator, $X$ being a Banach space over $\mathbb{C}$. A sequence $\left\{u_{n}\right\}_{n=1}^{\infty} \subset X,\left\|u_{n}\right\|=1(n=1,2, \ldots)$ is called an approximate eigenvector corresponding to an approximate eigenvalue $\lambda \in \mathbb{C}$ for $A$ if $\left\|A u_{n}-\lambda u_{n}\right\| \rightarrow 0$.

It can easily be shown (see [3, 12.1]) that every bounded linear operator has at least one approximate eigenvalue. The main result of this paper is the following characterization of the set $\mathcal{F}(X)$ for an arbitrary separable symmetric space on $[0, \infty)$. Theorem 4 will turn out to be an easy consequence of it.

Theorem 5. Let $X$ be an arbitrary separable symmetric space on $[0, \infty)$. Then $p \in$ $\mathcal{F}(X)$ if and only if $\lambda:=2^{1 / p}$ is an approximate eigenvalue for the dilation operator $T x(t):=x(t / 2)$.

The proof of this theorem is the main topic of 43 (in $\$ 2$ we introduce the necessary definitions and notation). As a consequence, we shall prove Theorem 4 (we remind the reader that it was stated without proof in [7]). In the second part of the paper ( $\$ 5{ }_{5}$ ), we describe $\mathcal{F}(X)$ completely in the case where $X$ is a Lorentz space. It will be shown that, depending on the function that generates this space, either $\mathcal{F}(X)$ is the entire interval $\left[1 / \beta_{X}, 1 / \alpha_{X}\right]$ or $\mathcal{F}(X)$ is the union of two intervals. In the proof, the results of [13] are used substantially; that paper contains a thorough description of the image of $S_{\lambda}=S-\lambda I$ ( $S$ is the shift operator, $I$ is the identity, and $\lambda \in \mathbb{C}$ ) in a weighted $\ell_{p}$-space. This description is presented in detail in $\S 4$. Finally, $\sqrt[6]{6}$ contains examples and remarks.

## §2. Definitions and notation

A Banach space $\left(X,\|\cdot\|_{X}\right)$ of complex-valued and Lebesgue measurable functions on the interval $[0, \alpha)(0<\alpha \leq \infty)$ is said to be symmetric (or rearrangement invariant) if, whenever $y \in X$ and $x^{*}(t) \leq y^{*}(t)(t \in[0, \alpha))$, we have $x \in X$ and $\|x\|_{X} \leq\|y\|_{X}$. Here and below, $x^{*}(t)$ is the right continuous nondecreasing rearrangement of $|x(s)|$, i.e., $x^{*}(t)=\inf \left\{\tau \geq 0: n_{x}(\tau) \leq t\right\}(t>0)$.

A symmetric space (s.s. for short) $X$ is said to be maximal (or to have the Fatou property) if the conditions $\left\{f_{n}\right\}_{n=1}^{\infty} \subseteq X, f_{n} \rightarrow f$ a.e. on $[0, \alpha)$ and $\sup _{n}\left\|f_{n}\right\|_{X}<\infty$ imply $f \in X$ and $\|f\|_{X} \leq \liminf _{n \rightarrow \infty}\left\|f_{n}\right\|_{X}$. As in [7], in what follows we assume that $X$ is either separable or maximal.

For every s.s. $X$ on $[0, \infty)$ we have the following continuous embeddings:

$$
L_{1} \cap L_{\infty} \subseteq X \subseteq L_{1}+L_{\infty}
$$

We denote by $X_{0}$ the closure of $L_{1} \cap L_{\infty}$ in $X$; this set is called the separable part of $X$. If $X \neq L_{1} \cap L_{\infty}$, then $X_{0}$ is separable.

Let $X$ be a symmetric space. The function $\varphi_{X}(t):=\left\|\chi_{A}\right\|_{X}$, where $A \subset(0, \infty)$, $\lambda(A)=t$, and $\chi_{A}$ is the characteristic function of $A$, is called the fundamental function of $X$. Another important characteristic of a symmetric space $X$ is its Boyd indices. We recall their definition in the case of a space on $[0, \infty)$. For any $\tau>0$, the dilation operator $\sigma_{\tau} x(t):=x(t / \tau)$ is bounded in any s.s. $X$ and $\left\|\sigma_{\tau}\right\|_{X \rightarrow X} \leq \max (1, \tau)(\tau>0)$; see 14, Theorem 2.4.4]. The numbers

$$
\alpha_{X}:=\lim _{\tau \rightarrow 0} \frac{\ln \left\|\sigma_{\tau}\right\|_{X \rightarrow X}}{\ln \tau} \quad \text { and } \quad \beta_{X}:=\lim _{\tau \rightarrow \infty} \frac{\ln \left\|\sigma_{\tau}\right\|_{X \rightarrow X}}{\ln \tau}
$$

are called the Boyd indices of $X$; we always have $0 \leq \alpha_{X} \leq \beta_{X} \leq 1$.
The dilation function of a positive function $\psi(t), t \in(0, \infty)$, is defined by the formula

$$
M_{\psi}(s)=\sup _{t>0} \frac{\psi(s t)}{\psi(t)}, \quad 0<s<\infty
$$

Next, the numbers

$$
\gamma_{\psi}=\lim _{s \rightarrow 0+} \frac{\ln M_{\psi}(s)}{\ln s} \quad \text { and } \quad \delta_{\psi}=\lim _{s \rightarrow \infty} \frac{\ln M_{\psi}(s)}{\ln s}
$$

are called the lower and the upper dilation indices of $\psi(t)$. If $\psi$ is quasiconcave, that is, $\psi(t)$ is monotone increasing and $\psi(t) / t$ is monotone decreasing for $t>0$, then $0 \leq \gamma_{\psi} \leq$ $\delta_{\psi} \leq 1$ (see [14, §2.1.2]).

Important examples of symmetric spaces are the $L_{p}$-spaces $(1 \leq p \leq \infty)$ and their generalization, the Orlicz spaces. Let $\Phi$ be an Orlicz function on $[0, \infty)$, i.e., $\Phi$ is a convex continuous monotone increasing function on $[0, \infty)$ with $\Phi(0)=0$ and $\Phi(\infty)=\infty$. The Orlicz space $L_{\Phi}$ on the semiaxis consists of all measurable functions $f$ on $[0, \infty)$ for which the norm $\|f\|_{L_{\Phi}}=\inf \left\{\rho>0: \int_{0}^{\infty} \Phi(|f(t)| / \rho) d t \leq 1\right\}$ is finite.

The Lorentz spaces, considered in the second part of the paper, constitute another class of symmetric spaces. Let $1 \leq q<\infty$, and let $\psi$ be a positive function on $(0, \infty)$ satisfying the following conditions:
(a) the dilation indices of $\psi$ are nontrivial, i.e., $0<\gamma_{\psi} \leq \delta_{\psi}<1$;
(b) the function $\psi(t)^{q} / t$ is monotone decreasing.

The Lorentz space $\Lambda_{q}(\psi)$ consists of all functions $x(t)$ measurable on $(0, \infty)$ and satisfying

$$
\begin{equation*}
\|x\|_{\Lambda_{q}(\psi)}:=\left(\int_{0}^{\infty} x^{*}(t)^{q} \psi(t)^{q} \frac{d t}{t}\right)^{1 / q}<\infty \tag{2.1}
\end{equation*}
$$

This a separable s.s., and it can easily be checked that its Boyd indices coincide with the corresponding dilation indices for $\psi$, i.e., $\alpha_{\Lambda_{q}(\psi)}=\gamma_{\psi}, \beta_{\Lambda_{q}(\psi)}=\delta_{\psi}$. Next, since the dilation indices are nontrivial, the function $\psi$ is equivalent to its least concave majorant (see [14, Corolary 2.1.2]) and, consequently, we may assume that $\psi$ is quasiconcave. See the monographs [7, 14, 15] for more details.

Finally, we present the definitions and notation to be used in the proof of Theorem 5 , Let $E$ be a space of complex sequences in which the standard unit vectors $e_{n}(n \in \mathbb{N})$ form a symmetric basis. Throughout, we denote by $c_{0,0}$ the set of all finitely supported sequences, i.e., $x=\left(x_{n}\right)_{n=1}^{\infty} \in c_{0,0}$ if $x_{n}=0$ for all sufficiently large $n$. For arbitrary $x=\left(x_{n}\right), y=\left(y_{n}\right) \in c_{0,0}$, we denote by $x \oplus y$ their disjoint sum. This is an arbitrary vector in $c_{0,0}$ whose nonzero coordinates coincide with all nonzero coordinates of $x$ and $y$. For instance, if $n_{0}=\max \left\{n \in \mathbb{N}: x_{n} \neq 0\right\}$, then for the disjoint sum of $x$ and $y$ we can take the vector:

$$
x \oplus y=\sum_{n=1}^{n_{0}} x_{n} e_{n}+\sum_{n=n_{0}+1}^{\infty} y_{n-n_{0}} e_{n} .
$$

Since the basis $\left\{e_{n}\right\}_{n=1}^{\infty}$ is symmetric in $E$, the norm $\|x \oplus y\|_{E}$ does not depend on a specific choice for $x \oplus y$.

We say that a vector $x$ is replaceable ( $\varepsilon$-replaceable) by a vector $y$ if for arbitrary $u$, $v \in c_{0,0}$ we have

$$
\|u \oplus x \oplus v\|_{E}=\|u \oplus y \oplus v\|_{E}
$$

(respectively,

$$
\left.\left|\|u \oplus x \oplus v\|_{E}-\|u \oplus y \oplus v\|_{E}\right|<\varepsilon\right) .
$$

All Banach spaces are assumed to be complex. We write $f \asymp g$ if $c f \leq g \leq C f$ for some constants $c>0$ and $C>0$ that do not depend on the values of all (or some) of the arguments of $f$ and $g$.

## §3. Characterization of $\mathcal{F}(X)$ for s.s. on the semiaxis

Proof of Theorem [5. Observe that the operator $T$ is bounded on $X$ and $1 \leq\|T\|_{X \rightarrow X} \leq$ 2 (see the preceding section).

First, let $p \in \mathcal{F}(X)$. Then for every $n \in \mathbb{N}$ there exists a collection $\left\{x_{k}\right\}_{k=1}^{2^{n}}$ of equimeasurable and mutually disjoint functions (i.e., $\operatorname{supp} x_{i} \cap \operatorname{supp} x_{j}=\varnothing$ for $i \neq j$ and
$n_{x_{k}}(\tau)=n_{x_{1}}(\tau)$ for $\tau>0$ and $\left.k=1,2, \ldots, 2^{n}\right)$ such that

$$
\begin{equation*}
\frac{1}{2}\left(\sum_{k=1}^{2^{n}}\left|a_{k}\right|^{p}\right)^{1 / p} \leq\left\|\sum_{k=1}^{2^{n}} a_{k} x_{k}\right\|_{X} \leq 2\left(\sum_{k=1}^{2^{n}}\left|a_{k}\right|^{p}\right)^{1 / p} \tag{3.1}
\end{equation*}
$$

(we assume that $p<\infty$; the case of $p=\infty$ is treated similarly). Consider the functions

$$
y_{0}:=x_{1}, \quad y_{s}:=\sum_{i=2^{s-1}+1}^{2^{s}} x_{i} \quad(s=1,2, \ldots, n) .
$$

They are also disjoint; moreover,

$$
\begin{equation*}
n_{y_{s}}(\tau)=2^{s-1} n_{y_{0}}(\tau) \quad(\tau>0), \quad s=1,2, \ldots, n . \tag{3.2}
\end{equation*}
$$

Since $X$ is separable and symmetric, we always can ensure by approximation that $\operatorname{supp} y_{0}=\operatorname{supp} x_{1} \subset[0, A]$ for some $A>0$. Putting

$$
z_{0}(t):=y_{0}^{*}(t), \quad z_{s}(t):=z_{0}\left(2^{1-s} t-A\right) \quad(s=1,2, \ldots, n),
$$

we have supp $z_{s} \subset\left[2^{s-1} A, 2^{s} A\right]$ for $s=1,2, \ldots, n$ (in particular, the $z_{s}$ are mutually disjoint). Next, by (3.2), $n_{z_{s}}(\tau)=n_{y_{s}}(\tau)=2^{s-1} n_{y_{0}}(\tau)(\tau>0)$. Therefore, since $X$ is symmetric, we obtain

$$
\begin{equation*}
\left\|\sum_{s=1}^{n} b_{s} z_{s}\right\|_{X}=\left\|\sum_{s=1}^{n} b_{s} y_{s}\right\|_{X} \tag{3.3}
\end{equation*}
$$

for arbitrary $b_{s} \in \mathbb{C}$. From the definition of the functions $y_{s}$ and relation (3.1), we deduce that

$$
\frac{1}{2} 2^{(s-1) / p} \leq\left\|y_{s}\right\|_{X} \leq 2 \cdot 2^{(s-1) / p} \quad \text { and } \quad \frac{1}{2} n^{1 / p} \leq\left\|\sum_{s=1}^{n} 2^{(1-s) / p} y_{s}\right\|_{X} \leq 2 n^{1 / p}
$$

By (3.3), it follows that the functions $\bar{z}_{s}(t):=2^{(1-s) / p} z_{s}(t)$ satisfy

$$
\begin{equation*}
\frac{1}{2} \leq\left\|\bar{z}_{s}\right\|_{X} \leq 2 \quad \text { and } \quad \frac{1}{2} n^{1 / p} \leq\left\|\sum_{s=1}^{n} \bar{z}_{s}\right\|_{X} \leq 2 n^{1 / p} \tag{3.4}
\end{equation*}
$$

Thus, putting

$$
v_{n}(t):=n^{-1 / p} \sum_{s=1}^{n} \bar{z}_{s}(t) \quad(n=1,2, \ldots),
$$

we have

$$
\begin{equation*}
\frac{1}{2} \leq\left\|v_{n}\right\|_{X} \leq 2 \quad(n=1,2, \ldots) \tag{3.5}
\end{equation*}
$$

Moreover, since $\bar{z}_{s}(t / 2)=2^{1 / p} \bar{z}_{s+1}(t)(s=1,2, \ldots, n-1)$ by the definition of $\bar{z}_{s}$, we see that for $\lambda=2^{1 / p}$ the operator $T_{\lambda}:=T-\lambda I$ ( $I$ is the identity) satisfies

$$
\begin{aligned}
T_{\lambda} v_{n}(t) & =n^{-1 / p}\left(\sum_{s=1}^{n} \bar{z}_{s}(t / 2)-\lambda \sum_{s=1}^{n} \bar{z}_{s}(t)\right) \\
& =n^{-1 / p}\left(2^{1 / p} \sum_{s=1}^{n-1} \bar{z}_{s+1}(t)+\bar{z}_{n}(t / 2)-2^{1 / p} \sum_{s=1}^{n} \bar{z}_{s}(t)\right) \\
& =n^{-1 / p}\left(\bar{z}_{n}(t / 2)-2^{1 / p} \bar{z}_{1}(t)\right) .
\end{aligned}
$$

Therefore, taking the first inequality in (3.4) into account, we obtain $\left\|T_{\lambda} v_{n}\right\|_{X} \leq 8 n^{-1 / p}$, whence $\left\|T_{\lambda} v_{n}\right\|_{X} \rightarrow 0$ as $n \rightarrow \infty$. Putting $\bar{v}_{n}:=v_{n} /\left\|v_{n}\right\|$, by (3.5) we see that also $\left\|T_{\lambda} \bar{v}_{n}\right\|_{X} \rightarrow 0$. So, $\lambda=2^{1 / p}$ is an approximate eigenvalue for $T$.

Now, we prove the converse. We start with some notation. For $n=0,1,2, \ldots$ and $k=1,2, \ldots$, we introduce the intervals $\Delta_{k}^{n}=\left[2^{-n}(k-1), 2^{-n} k\right)$ and the functions

$$
f_{k}^{n}:=\frac{1}{\varphi_{X}\left(2^{-n}\right)} \cdot \chi_{\Delta_{k}^{n}}, \quad \text { where } \quad \varphi_{X}\left(2^{-n}\right)=\left\|\chi_{\Delta_{k}^{n}}\right\|_{X}
$$

Next, let $X_{n}$ be the linear hull of the sequence $\left\{f_{k}^{n}\right\}_{k=1}^{\infty}$, and let $U$ denote the linear mapping of $c_{00}$ to $X_{n}$ determined by the condition $U e_{k}=f_{k}^{n}(k=1,2, \ldots)$. Given a symmetric space $X$, for $n \in \mathbb{N}$ we introduce the norm $\|a\|_{E_{n}}:=\|U a\|_{X}$ on $c_{00}$. The completion of $c_{00}$ under this norm is a symmetric sequence space $E_{n}$, and $\left\{e_{k}\right\}_{k=1}^{\infty}$ is a normalized symmetric basis in it.

We observe that

$$
T f_{k}^{n}=\frac{1}{\varphi_{X}\left(2^{-n}\right)} \cdot\left(\chi_{\Delta_{2 k-1}^{n}}+\chi_{\Delta_{2 k}^{n}}\right)=f_{2 k-1}^{n}+f_{2 k}^{n}
$$

for all $n=0,1,2, \ldots$ and $k=1,2, \ldots$ Therefore, whenever $a=\sum a_{k} e_{k} \in c_{00}$, for every $b \in c_{00}$ and $d \in c_{00}$ we have

$$
\begin{align*}
\|b \oplus a \oplus a \oplus d\|_{E_{n}} & =\|U b \oplus U a \oplus U a \oplus U d\|_{X} \\
& =\left\|U b \oplus \sum a_{k}\left(f_{2 k-1}^{n}+f_{2 k}^{n}\right) \oplus U d\right\|_{X}  \tag{3.6}\\
& =\|U b \oplus T U a \oplus U d\|_{X},
\end{align*}
$$

because $\left\{f_{k}^{n}\right\}_{k=1}^{\infty}$ is a symmetric basic sequence in $X$. Let $\lambda$ be an approximate eigenvalue of $T$, and let $\left\{g_{l}\right\}_{l=1}^{\infty} \subset X,\left\|g_{l}\right\|=1(l=1,2, \ldots)$, be the corresponding approximate eigenvector. Since $X$ is separable, there is no loss of generality in assuming that $g_{l} \in X_{n_{l}}$ for some $n_{1}<n_{2}<\cdots$ and

$$
\left\|T g_{l}-\lambda g_{l}\right\|_{X} \leq \frac{1}{l} \quad(l=1,2, \ldots)
$$

We show that the vector $U^{-1} g_{l} \oplus U^{-1} g_{l}$ is $1 / l$-replaceable by $\lambda U^{-1} g_{l}$ in $E_{n_{l}}$. Indeed, let $b, d \in c_{00}$. Choosing due representatives for disjoint sums, by (3.6) and the preceding inequality we obtain

$$
\begin{align*}
\mid \| b \oplus U^{-1} g_{l} & \oplus U^{-1} g_{l} \oplus d\left\|_{E_{n_{l}}}-\right\| b \oplus \lambda U^{-1} g_{l} \oplus d \|_{E_{n_{l}}} \mid \\
& =\left|\left\|U b \oplus T g_{l} \oplus U d\right\|_{X}-\left\|U b \oplus \lambda g_{l} \oplus U d\right\|_{X}\right|  \tag{3.7}\\
& \leq\left\|T g_{l}-\lambda g_{l}\right\|_{X} \leq \frac{1}{l}
\end{align*}
$$

For every $l \in \mathbb{N}$, we introduce a new sequence space $F_{l}$, namely, the completion of $c_{00}$ with respect to the norm

$$
\begin{equation*}
\left\|\sum_{j=1}^{m} a_{j} e_{j}\right\|_{F_{l}}:=\left\|a_{1} U^{-1} g_{l} \oplus a_{2} U^{-1} g_{l} \oplus \cdots \oplus a_{m} U^{-1} g_{l}\right\|_{E_{n_{l}}} \quad(m \in \mathbb{N}) \tag{3.8}
\end{equation*}
$$

Clearly, the sequence $\left\{e_{k}\right\}_{k=1}^{\infty}$ of unit vectors in $F_{l}$ is isometric to a sequence of disjoint functions in $X_{n_{l}}$ that are equimeasurable with $g_{l}$.

We arrange all vectors in $c_{00}$ with rational coordinates in a sequence $\left(a_{1}^{(k)}, \ldots, a_{r_{k}}^{(k)}\right)_{k=1}^{\infty}$ and construct a decreasing family $\left(l_{i}^{k}\right)_{i=1}^{\infty}(k=1,2, \ldots)$ of infinite sequences of natural numbers such that for every $k=1,2, \ldots$ the limit

$$
\lim _{i \rightarrow \infty}\left\|\sum_{j} a_{j}^{(m)} e_{j}\right\|_{F_{l_{i}^{k}}}
$$

exists for all $1 \leq m \leq k$. The standard diagonal procedure yields a sequence $\left(l_{s}\right)_{s=1}^{\infty}$ included in each subsequence $\left(l_{i}^{k}\right)_{i=1}^{\infty}$ up to finitely many terms. The usual arguments
based on the density of the rationals in the set of reals show that the limit

$$
\lim _{s \rightarrow \infty}\left\|\sum_{j} a_{j} e_{j}\right\|_{F_{l_{s}}}
$$

exists for arbitrary $a=\left(a_{j}\right) \in c_{00}$. Therefore, we can introduce a new norm $\|a\|_{F}$ on $c_{00}$ which is equal to this limit. We denote by $F$ the completion of $c_{00}$ in this norm. By the definition and [5, Lemma 11.1.11], it is clear that for every $\varepsilon>0$ and $m \in \mathbb{N}$ there exists $s_{0} \in \mathbb{N}$ such that for all $s \geq s_{0}$ and arbitrary $a_{j} \in \mathbb{C}$ we have

$$
\begin{equation*}
(1+\varepsilon)^{-1}\left\|\sum_{j=1}^{m} a_{j} e_{j}\right\|_{F_{l_{s}}} \leq\left\|\sum_{j=1}^{m} a_{j} e_{j}\right\|_{F} \leq(1+\varepsilon)\left\|\sum_{j=1}^{m} a_{j} e_{j}\right\|_{F_{l_{s}}} . \tag{3.9}
\end{equation*}
$$

We prove that the sum $e_{1}+e_{2}$ is replaceable in $F$ by $\lambda e_{1}$. Indeed, let $b, d \in c_{00}$ be arbitrary. Relations (3.7) and (3.8) show that

$$
\begin{aligned}
& \left|\left\|b \oplus e_{1}+e_{2} \oplus d\right\|_{F_{l_{s}}}-\left\|b \oplus \lambda e_{1} \oplus d\right\|_{F_{l_{s}}}\right| \\
& \quad=\left|\left\|b^{\prime} \oplus U^{-1} g_{l} \oplus U^{-1} g_{l} \oplus d^{\prime}\right\|_{E_{n_{l_{s}}}}-\left\|b^{\prime} \oplus \lambda U^{-1} g_{l} \oplus d^{\prime}\right\|_{E_{n_{l_{s}}}}\right| \leq \frac{1}{l_{s}}
\end{aligned}
$$

for all $s \in \mathbb{N}$. Thus, by (3.9), for every $\delta>0$ and all $s \in \mathbb{N}$ sufficiently large, we have

$$
\begin{aligned}
& \left|\left\|b \oplus e_{1}+e_{2} \oplus d\right\|_{F}-\left\|b \oplus \lambda e_{1} \oplus d\right\|_{F}\right| \\
& \quad \leq\left|\left\|b \oplus e_{1}+e_{2} \oplus d\right\|_{F_{l_{s}}}-\left\|b \oplus \lambda e_{1} \oplus d\right\|_{F_{l_{s}}}\right|+\delta\left(\left\|b \oplus e_{1}+e_{2} \oplus d\right\|_{F}+\left\|b \oplus \lambda e_{1} \oplus d\right\|_{F}\right) \\
& \quad \leq \frac{1}{l_{s}}+\delta\left(\left\|b \oplus e_{1}+e_{2} \oplus d\right\|_{F}+\left\|b \oplus \lambda e_{1} \oplus d\right\|_{F}\right) .
\end{aligned}
$$

Since the right-hand side in the last inequality can be made arbitrarily small, we see that

$$
\left\|u \oplus e_{1}+e_{2} \oplus v\right\|_{F}=\left\|u \oplus \lambda e_{1} \oplus v\right\|_{F} .
$$

Now, if $\lambda=1$, then $\left\|e_{1}+e_{2}+\cdots+e_{n}\right\|_{F}=1(n \in \mathbb{N})$, and $F$ is isometric to $c_{0}$. Furthermore, (3.6), (3.8), and (3.9) show that for arbitrary $\varepsilon>0$ and $n \in \mathbb{N}$ there exists a collection of mutually disjoint and equimeasurable functions $w_{1}, w_{2}, \ldots, w_{n}$ in $X$ such that

$$
\begin{equation*}
(1+\varepsilon)^{-1}\left\|\sum_{k=1}^{n} a_{k} e_{k}\right\|_{F} \leq\left\|\sum_{k=1}^{n} a_{k} w_{k}\right\|_{X} \leq(1+\varepsilon)\left\|\sum_{k=1}^{n} a_{k} e_{k}\right\|_{F} \tag{3.10}
\end{equation*}
$$

for arbitrary $a_{1}, a_{2}, \ldots, a_{n} \in \mathbb{C}$. Consequently, $p=\infty \in \mathcal{F}(X)$.
Assume that $1<\lambda \leq 2$, i.e., $\lambda:=2^{1 / p}$ with $1 \leq p<\infty$. Then, by Lemmas 11.3.11 and 11.3.12(ii) in [5, there exists a sequence $\left\{b_{n}\right\}_{n=1}^{\infty} \subset c_{00}$ with $\left\|b_{n}\right\|_{F}=1(n=1,2, \ldots)$ such that the vectors $b_{n} \oplus b_{n}$ and $b_{n} \oplus b_{n} \oplus b_{n}$ are $1 / n$-replaceable by $2^{1 / p} b_{n}$ and $3^{1 / p} b_{n}$, respectively. Acting in the same way as in the passage from $E_{n_{l}}$ to $F$, we obtain a space $G$ in which $e_{1}+e_{2}$ and $e_{1}+e_{2}+e_{3}$ will be replacable by $2^{1 / p} e_{1}$ and $3^{1 / p} e_{1}$, respectively. But then $G$ is isometric to $\ell_{p}$, see [5, Lemma 11.3.11]. As before, for every $\varepsilon>0$ and $n \in \mathbb{N}$, we can find a collection of mutually disjoint equimeasurable functions in $X$ such that a formula similar to (3.10) is valid for the unit vectors $\left\{e_{k}\right\}_{k=1}^{n}$ in $G$ and arbitrary $a_{1}, a_{2}, \ldots, a_{n} \in \mathbb{C}$. This means that $p \in \mathcal{F}(X)$, and the theorem is proved.

A statement similar to the following one (but for the inverse of $T_{\lambda}$ ) can be found in [16] (Theorem 3.2).

Proposition 1. Let $X$ be a symmetric space on $[0, \infty)$ with Boyd indices $\alpha_{X}$ and $\beta_{X}$. Then the operator $T_{\lambda}=T-\lambda I$ (as before, $T x(t):=x(t / 2)$ ) is an isomorphism of $X$ for $|\lambda| \notin\left[2^{\alpha_{X}}, 2^{\beta_{X}}\right]$.

Proof. Let $|\lambda|>2^{\beta_{X}}$. Suppose that the equation $T_{\lambda} x=y$ or, equivalently,

$$
\begin{equation*}
x(t / 2)-\lambda x(t)=y(t) \quad(t>0) \tag{3.11}
\end{equation*}
$$

has a solution $x=x(t) \in X$ for arbitrary $y=y(t) \in X$. Then

$$
\lambda^{-1} x\left(t / 2^{2}\right)-x(t / 2)=\lambda^{-1} y(t / 2)
$$

Adding the last two identities, we obtain

$$
\lambda^{-1} x\left(t / 2^{2}\right)-\lambda x(t)=y(t)+\lambda^{-1} y(t / 2)
$$

whence

$$
x(t)=-\lambda^{-1}\left(y(t)+\lambda^{-1} y(t / 2)\right)+\lambda^{-2} x\left(t / 2^{2}\right)
$$

Proceeding in the same way, we arrive at the relation

$$
\begin{equation*}
x(t)=-\sum_{k=1}^{n} \lambda^{-k} y\left(2^{-k+1} t\right)+\lambda^{-n} x\left(2^{-n} t\right) \quad(t>0) \tag{3.12}
\end{equation*}
$$

which is valid for arbitrary $n$. Take $\varepsilon>0$ with $2^{\beta_{X}+\varepsilon}<|\lambda|$. By the definition of the Boyd indices,

$$
\left\|\sigma_{\tau}\right\|_{X \rightarrow X} \leq C \tau^{\beta_{X}+\varepsilon} \quad(\tau \geq 1)
$$

therefore,

$$
\left\|x\left(2^{-n} t\right)\right\|_{X} \leq C 2^{n\left(\beta_{X}+\varepsilon\right)}\|x\|_{X}
$$

Consequently,

$$
\left\|\lambda^{-n} x\left(2^{-n} t\right)\right\|_{X} \leq C\left(|\lambda|^{-1} 2^{\beta_{X}+\varepsilon}\right)^{n}\|x\|_{X} \rightarrow 0 \text { as } n \rightarrow \infty
$$

and, by (3.12),

$$
\begin{equation*}
x(t)=-\sum_{k=1}^{\infty} \lambda^{-k} y\left(2^{-k+1} t\right) \quad(t>0) \tag{3.13}
\end{equation*}
$$

Thus, if a solution $x \in X$ of equation (3.11) exists, it must have the form (3.13) (the series on the right in (3.13) converges absolutely because $|\lambda|>2^{\beta_{X}}$ ). On the other hand, it is straightforward that the function (3.13) is a solution of equation (3.11). To summarize, this equation has a unique solution $x \in X$ for an arbitrary $y \in X$ on the right. Thus, the operator $T_{\lambda}: X \rightarrow X$ is an isomorphism.

The case where $|\lambda|<2^{\alpha_{X}}$ is treated similarly.
We present a consequence of the above statement and Theorem 5.
Corollary 1. The spectrum $\sigma(T)$ of $T$ lies inside the annulus $\left\{\lambda \in \mathbb{C}:|\lambda| \in\left[2^{\alpha_{X}}, 2^{\beta_{X}}\right]\right\}$, and the set $\mathcal{F}(X)$ lies inside the interval $\left[1 / \beta_{X}, 1 / \alpha_{X}\right]$.

We show that all boundary points of this annulus are approximate eigenvalues for $T$.
Theorem 6. Let $\alpha_{X}$ and $\beta_{X}$ be the Boyd indices of a symmetric space $X$. Then any $\lambda \in \mathbb{C}$ with $|\lambda|=2^{\beta_{X}}$ or $|\lambda|=2^{\alpha_{X}}$ is an approximate eigenvalue for $T$. In particular, the spectral radius $r(T)$ is $2^{\beta_{X}}$.
Proof. First, let $\lambda=2^{\beta_{X}}$. The definition of the Boyd indices (see [14, $\S \S 2.1 .1$ and 2.4.3]) shows that

$$
\beta_{X}=\inf _{\tau \geq 1} \frac{\ln \left\|\sigma_{\tau}\right\|_{X \rightarrow X}}{\ln \tau}
$$

Therefore,

$$
\begin{equation*}
\left\|T^{n}\right\|_{X \rightarrow X}=\left\|\sigma_{2^{n}}\right\|_{X \rightarrow X} \geq 2^{n \beta_{X}} \quad(n \in \mathbb{N}) \tag{3.14}
\end{equation*}
$$

and we can argue as in the proof of Theorem 11.3.12 in [5]. We give the details for completeness.

By (3.14) we have

$$
\lim _{n \rightarrow \infty}\left\|(n+1) 2^{-n \beta_{X}} T^{n}\right\|_{X \rightarrow X}=\infty
$$

and the uniform boundedness principle implies the existence of $f_{0} \in X,\left\|f_{0}\right\|_{X}=1$, such that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\|(n+1) 2^{-n \beta_{X}} T^{n} f_{0}\right\|_{X}=\infty \tag{3.15}
\end{equation*}
$$

Clearly, the function $f_{0}$ may be assumed to be nonnegative.
By Corollary [1, the operator $T-r I$ is invertible in $X$ if $r>2^{\beta_{X}}$. Consequently, $(T-r I)^{-2}$ can be represented as follows (the series converges):

$$
(T-r I)^{-2}=\frac{1}{r^{2}} \sum_{n=0}^{\infty}(n+1) r^{-n} T^{n}
$$

Since $f_{0} \geq 0$ and $T \geq 0$, we conclude that

$$
\left\|(T-r I)^{-2} f_{0}\right\|_{X} \geq r^{-2}\left\|(n+1) r^{-n} T^{n} f_{0}\right\|_{X}
$$

for every $r>2^{\beta_{X}}$ and every $n \in \mathbb{N}$. By (3.15),

$$
\lim _{r \rightarrow 2^{B} X}\left\|(T-r I)^{-2} f_{0}\right\|_{X}=\infty
$$

Therefore, there exists a sequence $\left\{r_{n}\right\}$ with $r_{n} \rightarrow 2^{\beta_{X}}$ such that either

$$
\lim _{n \rightarrow \infty}\left\|\left(T-r_{n} I\right)^{-1} f_{0}\right\|_{X}=\infty
$$

or

$$
\lim _{n \rightarrow \infty} \frac{\left\|\left(T-r_{n} I\right)^{-2} f_{0}\right\|_{X}}{\left\|\left(T-r_{n} I\right)^{-1} f_{0}\right\|_{X}}=\infty
$$

In either case, it is easy to find a sequence $\left\{g_{n}\right\}_{n=1}^{\infty} \subset X,\left\|g_{n}\right\|_{X}=1$, such that

$$
\lim _{n \rightarrow \infty}\left\|\left(T-r_{n} I\right) g_{n}\right\|_{X}=0
$$

Surely, this implies that $2^{\beta_{X}}$ is an approximate eigenvalue for $T$.
If $\lambda=2^{\alpha_{X}}$, then, again by [14], we have

$$
\alpha_{X}=\sup _{0<\tau \leq 1} \frac{\ln \left\|\sigma_{\tau}\right\|_{X \rightarrow X}}{\ln \tau},
$$

implying that the operator $T^{-1} f(t)=f(2 t)$ satisfies

$$
\left\|\left(T^{-1}\right)^{n}\right\|_{X \rightarrow X}=\left\|\sigma_{2^{-n}}\right\|_{X \rightarrow X} \geq 2^{-n \alpha_{X}} \quad(n \in \mathbb{N})
$$

Arguing as in the preceding case, we deduce that $2^{-\alpha_{X}}$ is an approximate eigenvalue for $T^{-1}=\sigma_{1 / 2}$, i.e.,

$$
\lim _{n \rightarrow \infty}\left\|\left(T^{-1}-2^{-\alpha_{X}} I\right) h_{n}\right\|_{X}=0
$$

for some sequence $\left\{h_{n}\right\}_{n=1}^{\infty} \subset X$ with $\left\|h_{n}\right\|_{X}=1$. Since $1 / 2 \leq\left\|T^{-1} h_{n}\right\|_{X} \leq 1$ for all $n \in \mathbb{N}$ (see §21), and for $g_{n}:=T^{-1} h_{n} /\left\|T^{-1} h_{n}\right\|$ we have

$$
\left\|\left(T-2^{\alpha_{X}} I\right) g_{n}\right\|_{X}=\frac{2^{\alpha_{X}}}{\left\|T^{-1} h_{n}\right\|}\left\|\left(T^{-1}-2^{-\alpha_{X}} I\right) h_{n}\right\|_{X}
$$

we see that $2^{\alpha_{X}}$ is an approximate eigenvalue for $T$.
Now, let $\lambda \in \mathbb{C}$, and let, for instance, $|\lambda|=2^{\beta_{X}}$. Then $\lambda=2^{\beta_{X}} e^{i \theta}$ for some $0 \leq \theta \leq$ $2 \pi$. If $\left\{g_{n}\right\}_{n=1}^{\infty} \subset X$ is an approximate eigenvector corresponding to the approximate eigenvalue $2^{\beta_{X}}$, then it can easily be checked that the functions $f_{n}(t):=t^{-i \theta \log _{2} e} g_{n}(t)$ $(n \in \mathbb{N})$ satisfy the formula

$$
(T-\lambda I) f_{n}=e^{i \theta}\left(T-2^{\beta_{X}} I\right) g_{n} .
$$

Since $\left\|f_{n}\right\|_{X}=\left\|g_{n}\right\|_{X}=1$, we see that $\lambda$ is an approximate eigenvalue for $T$. The case where $|\lambda|=2^{\alpha_{X}}$ is treated similarly.

Since $2^{\beta_{X}}$ is an approximate eigenvalue for $T$, we have $2^{\beta_{X}} \in \sigma(T)$, and the second statement of the theorem follows from Corollary 1 and the definition of the spectral radius.

We show that, in the case of symmetric spaces on the semiaxis, Theorem 4 is an immediate consequence of the results obtained (we recall that Theorem 4 was stated in [7, theorem 2.b.6] without proof).
Proof of Theorem 4. If $X$ is separable, the claim follows from Theorems 5, 6 and Corollary 1 Otherwise, $X$ is maximal. It is easily seen that then $\left\|\sigma_{\tau}\right\|_{X \rightarrow X}=\left\|\sigma_{\tau}\right\|_{X_{0} \rightarrow X_{0}}$ for every $\tau>0$, where $X_{0}$ is the separable part of $X$ (see $\S(2)$. Thus, $\alpha_{X}=\alpha_{X_{0}}$ and $\beta_{X}=\beta_{X_{0}}$. If $X \neq L_{1} \cap L_{\infty}$, then $X_{0}$ is separable and, consequently, $\max \mathcal{F}\left(X_{0}\right)=1 / \alpha_{X}$ and $\min \mathcal{F}\left(X_{0}\right)=1 / \beta_{X}$. Since $X_{0}$ is a subspace of $X$, this implies the claim by Corollary 11 If $X=L_{1} \cap L_{\infty}$, the result is obvious.

Theorems 5 and 6 allow us to completely describe the sets $\mathcal{F}(X)$ if $X$ is a Lorentz space. In doing this, we shall crucially need the results of [13], so first we summarize them.
§4. The closedness of the operator $S_{\lambda}$ In a weighted $\ell_{q}$-Space
For a numerical sequence $\mu=\left(\mu_{k}\right)_{k=-\infty}^{\infty}$ satisfying the conditions

$$
\begin{equation*}
0<\mu_{k} \leq \mu_{k+1} \leq 2 \mu_{k} \quad(k=0, \pm 1, \pm 2, \ldots) \tag{4.1}
\end{equation*}
$$

we introduce the weighted space $\ell_{q}(\mu)$ with the norm

$$
\left\|\left(a_{k}\right)\right\|_{\ell_{q}(\mu)}:=\left(\sum_{k=-\infty}^{\infty}\left|a_{k}\right|^{q} \mu_{k}^{q}\right)^{1 / q} \quad(1 \leq q<\infty)
$$

Next, for every $\lambda \in \mathbb{C}$ we put $S_{\lambda}=S-\lambda I$, where $S\left(a_{k}\right):=\left(a_{k-1}\right)$ is the shift operator and $I$ is the identity. The operator $S_{\lambda}$ is linear and bounded on $\ell_{q}(\mu)$.

Basically, the paper [13] is devoted to the real interpolation method for subcouples of codimension 1 generated by a linear functional bounded on the intersection of the spaces of the initial couple. As an application, the following problem was resolved completely in [13]: if the weight sequence $\mu=\left(\mu_{k}\right)_{k=-\infty}^{\infty}$ satisfies (4.1), determine when the range of $S_{\lambda}$ is closed in $\ell_{q}(\mu)$. (In what follows, we say that an operator is closed if its image is closed.) To state the result, we need some definitions and notation.

Let $\psi(t)$ be a quasiconcave function on $(0, \infty)$. We introduce three dilation functions:

$$
M(t)=\sup _{s>0} \frac{\psi(t s)}{\psi(s)}, \quad M_{0}(t)=\sup _{0<s \leq \min (1,1 / t)} \frac{\psi(t s)}{\psi(s)}, \quad M_{\infty}(t)=\sup _{s \geq \max (1,1 / t)} \frac{\psi(t s)}{\psi(s)}
$$

They are submultiplicative on $(0, \infty)$ and, consequently, we can introduce the following six numbers:

$$
\begin{array}{lll}
\alpha=\lim _{t \rightarrow 0} \frac{\log _{2} M(t)}{\log _{2} t}, & \alpha_{0}=\lim _{t \rightarrow 0} \frac{\log _{2} M_{0}(t)}{\log _{2} t}, & \alpha_{\infty}=\lim _{t \rightarrow 0} \frac{\log _{2} M_{\infty}(t)}{\log _{2} t} \\
\beta=\lim _{t \rightarrow \infty} \frac{\log _{2} M(t)}{\log _{2} t}, & \beta_{0}=\lim _{t \rightarrow \infty} \frac{\log _{2} M_{0}(t)}{\log _{2} t}, & \beta_{\infty}=\lim _{t \rightarrow \infty} \frac{\log _{2} M_{\infty}(t)}{\log _{2} t}
\end{array}
$$

which are called the dilation indices for $\psi$ (observe that $\alpha$ and $\beta$ coincide with $\alpha_{0}$ and $\beta_{0}$ defined in (12). It is easily seen that $0 \leq \alpha \leq \alpha_{0} \leq \beta_{0} \leq \beta \leq 1$ and $0 \leq \alpha \leq \alpha_{\infty} \leq$ $\beta_{\infty} \leq \beta \leq 1$. Moreover, $\alpha=\min \left(\alpha_{0}, \alpha_{\infty}\right)$ and $\beta=\max \left(\beta_{0}, \beta_{\infty}\right)$, see [13, Lemma 1]. Let
$\mu_{k}:=\psi\left(2^{k}\right)(k=0, \pm 1, \pm 2, \ldots)$. Since $\psi$ is quasiconcave, the numbers $\mu_{k}$ satisfy (4.1). Next, the above dilation indices can also be calculated by the following formulas:

$$
\begin{aligned}
\alpha & =-\lim _{n \rightarrow \infty} \frac{1}{n} \log _{2} \sup _{k \in \mathbb{Z}} \frac{\mu_{k}}{\mu_{n+k}}, & \beta & =\lim _{n \rightarrow \infty} \frac{1}{n} \log _{2} \sup _{k \in \mathbb{Z}} \frac{\mu_{k}}{\mu_{k-n}}, \\
\alpha_{0} & =-\lim _{n \rightarrow \infty} \frac{1}{n} \log _{2} \sup _{k \leq 0} \frac{\mu_{k-n}}{\mu_{k}}, & & \beta_{0}=\lim _{n \rightarrow \infty} \frac{1}{n} \log _{2} \sup _{k \leq 0} \frac{\mu_{k}}{\mu_{k-n}}, \\
\alpha_{\infty} & =-\lim _{n \rightarrow \infty} \frac{1}{n} \log _{2} \sup _{k \geq 0} \frac{\mu_{k}}{\mu_{n+k}}, & \beta_{\infty} & =\lim _{n \rightarrow \infty} \frac{1}{n} \log _{2} \sup _{k \geq 0} \frac{\mu_{k+n}}{\mu_{k}} .
\end{aligned}
$$

Therefore, applying [13, Theorem 5 and Proposition 2], we obtain the following statement (we formulate it in a slightly more general but equivalent form).

Theorem 7. Suppose $\lambda \in \mathbb{C}$ and $1 \leq q<\infty$. Then the operator $S_{\lambda}$ is closed in $\ell_{q}(\mu)$ if and only if

$$
|\lambda| \in\left[0,2^{\alpha}\right) \cup\left(2^{\beta_{0}}, 2^{\alpha_{\infty}}\right) \cup\left(2^{\beta_{\infty}}, 2^{\alpha_{0}}\right) \cup\left(2^{\beta}, \infty\right) .
$$

Moreover, if $|\lambda| \in\left[0,2^{\alpha}\right) \cup\left(2^{\beta \infty}, 2^{\alpha_{0}}\right) \cup\left(2^{\beta}, \infty\right)$, then $\operatorname{Im} S_{\lambda}=\ell_{q}(\mu)$; if $|\lambda| \in\left(2^{\beta_{0}}, 2^{\alpha_{\infty}}\right)$, then $\operatorname{Im} S_{\lambda}$ is a closed subspace of codimension 1 in $\ell_{q}(\mu)$. The operator $S_{\lambda}$ is invertible if and only if $|\lambda| \in\left[0,2^{\alpha}\right) \cup\left(2^{\beta}, \infty\right)$. If $|\lambda| \in\left(2^{\beta_{0}}, 2^{\alpha \infty}\right)$, this operator is injective, but if $|\lambda| \in\left(2^{\beta_{\infty}}, 2^{\alpha_{0}}\right)$, it is not.

It should be noted that a weaker result, involving only four indices, was proved in [17].
In the next section, we shall apply Theorem 7 in order to deduce a similar result for the dilation operator defined on a Lorentz space.

## §5. Description of the set $\mathcal{F}(X)$ for Lorentz spaces

Let $1 \leq q<\infty, \psi$ a positive function on $(0, \infty)$ satisfying conditions (a) and (b) in §2, and $\Lambda_{q}(\psi)$ the Lorentz space whose norm in defined by (2.1). As has been mentioned, this is a separable s.s. whose Boyd indices coincide with the corresponding dilation indices of $\psi$, i.e., $\alpha_{\Lambda_{q}(\psi)}=\gamma_{\psi}, \beta_{\Lambda_{q}(\psi)}=\delta_{\psi}$. Moreover, by condition (a), we may assume that $\psi$ is quasiconcave.

We put $\Delta_{k}:=\left[2^{k}, 2^{k+1}\right)(k=0, \pm 1, \pm 2, \ldots)$ and for an arbitrary sequence $a=\left(a_{k}\right)_{k=1}^{\infty}$ of complex numbers introduce the function

$$
h_{a}(t):=\sum_{k=-\infty}^{\infty} a_{k} \chi_{\Delta_{k}}(t) .
$$

By [18, Proposition 5.1(2)], we have

$$
\left\|h_{a}\right\|_{\Lambda_{q}(\psi)} \asymp\left(\sum_{k=-\infty}^{\infty}\left|a_{k}\right|^{q} \int_{\Delta_{k}} \psi(t)^{q} \frac{d t}{t}\right)^{1 / q}
$$

with constants independent of $\left(a_{k}\right)$. Since $\psi$ is quasiconcave, we obtain

$$
\int_{\Delta_{k}} \psi(t)^{q} \frac{d t}{t} \asymp \psi\left(2^{k}\right)^{q} \quad(k=0, \pm 1, \pm 2, \ldots),
$$

whence

$$
\left\|h_{a}\right\|_{\Lambda_{q}(\psi)} \asymp\left(\sum_{k=-\infty}^{\infty}\left|a_{k}\right|^{q} \psi\left(2^{k}\right)^{q}\right)^{1 / q}
$$

with constants depending only on $\psi$ and $q$. Thus, putting $\mu_{k}=\psi\left(2^{k}\right)(k=0, \pm 1, \pm 2, \ldots)$, in the notation of the preceding section we obtain

$$
\begin{equation*}
\left\|h_{a}\right\|_{\Lambda_{q}(\psi)} \asymp\left\|\left(a_{k}\right)\right\|_{\ell_{q}(\mu)}, \tag{5.1}
\end{equation*}
$$

i.e., $\ell_{q}(\mu)$ is isomorphic to the subspace $\left[\chi_{\Delta_{k}}\right]$ spanned in $\Lambda_{q}(\psi)$ by the system of the characteristic functions of dyadic intervals.

As before, let $T x(t)=x(t / 2)$ and $T_{\lambda}:=T-\lambda I(\lambda \in \mathbb{C})$, where $I$ is the identity operator on $\Lambda_{q}(\psi)$. We show that $T_{\lambda}$ and $S_{\lambda}=S-\lambda I$ (see $\mathbb{4} 4$ ) are related in a simple way. First, for an arbitrary sequence $\left(a_{k}\right)$ we have

$$
\begin{align*}
T_{\lambda} h_{a}(t) & =\sum_{k=-\infty}^{\infty} a_{k} \chi_{\Delta_{k}}(t / 2)-\lambda \sum_{k=-\infty}^{\infty} a_{k} \chi_{\Delta_{k}}(t) \\
& =\sum_{k=-\infty}^{\infty} a_{k} \chi_{\Delta_{k+1}}(t)-\lambda \sum_{k=-\infty}^{\infty} a_{k} \chi_{\Delta_{k}}(t)  \tag{5.2}\\
& =\sum_{k=-\infty}^{\infty}\left(S_{\lambda} a\right)_{k} \chi_{\Delta_{k}}(t)
\end{align*}
$$

In particular, by (5.1) it follows that

$$
\begin{equation*}
\left\|T_{\lambda} h_{a}\right\|_{\Lambda_{q}(\psi)} \asymp\left\|S_{\lambda} a\right\|_{\ell_{q}(\mu)} \tag{5.3}
\end{equation*}
$$

In the sequel, we shall also need the following statement.
Proposition 2. For arbitrary $\lambda \in \mathbb{C}$, the following assertions are true:
(i) $T_{\lambda}$ is injective if and only if $S_{\lambda}$ is injective;
(ii) if $T_{\lambda}$ is closed, then $S_{\lambda}$ is closed;
(iii) if $S_{\lambda}$ is injective and closed, then $T_{\lambda}$ is closed.

Proof. First, we verify that it suffices to prove the proposition for $\lambda \geq 0$. Indeed, let $\lambda=|\lambda| \cdot e^{i \theta}$, where $\theta \in[0,2 \pi]$. For every $x \in \Lambda_{q}(\psi)$, we introduce the function $y(t)=$ $t^{-i \theta \log _{2} e} \cdot x(t)$. Then $y \in \Lambda_{q}(\psi),\|y\|_{\Lambda_{q}(\psi)}=\|x\|_{\Lambda_{q}(\psi)}$, and

$$
T_{\lambda} y(t)=t^{-i \theta \log _{2} e} 2^{i \theta \log _{2} e} x(t / 2)-t^{-i \theta \log _{2} e} \lambda x(t)=t^{-i \theta \log _{2} e} e^{i \theta} T_{|\lambda|} x(t)
$$

Consequently, $\left\|T_{\lambda} y\right\|_{\Lambda_{q}(\psi)}=\left\|T_{|\lambda|} x\right\|_{\Lambda_{q}(\psi)}$, and we see that $T_{\lambda}$ is injective (closed) if and only if $T_{|\lambda|}$ is injective (closed).

A similar statement is true for $S_{\lambda}$ and $S_{|\lambda|}$. In this case, if $a=\left(a_{k}\right) \in \ell_{q}(\mu)$ and $b=\left(b_{k}\right), b_{k}:=a_{k} e^{-i \theta k}$, again we have $b \in \ell_{q}(\mu)$ and $\|b\|_{\ell_{q}(\mu)}=\|a\|_{\ell_{q}(\mu)}$. Furthermore,

$$
\left(S_{\lambda} b\right)_{k}=a_{k-1} e^{-i \theta k} e^{i \theta}-\lambda a_{k} e^{-i \theta k}=e^{-i \theta(k-1)}\left(S_{|\lambda|} a\right)_{k},
$$

whence $\left\|S_{\lambda} b\right\|_{\ell_{q}(\mu)}=\left\|S_{|\lambda|} a\right\|_{\ell_{q}(\mu)}$, and the claim follows.
(i) The fact that the injectivity of $T_{\lambda}$ implies the injectivity of $S_{\lambda}$ is a direct consequence of (5.3). Since $T$ and $S$ are injective, it suffices to prove the converse for $\lambda>0$.

Suppose $x \in \Lambda_{q}(\psi), x \neq 0$, and $T_{\lambda} x=0$. Since $\left|T_{\lambda} x\right| \geq||x(t / 2)|-\lambda| x(t)| |$, we may assume that $x(t)$ is nonnegative. Outside a set of zero measure, we have

$$
\begin{equation*}
x(t / 2)=\lambda \cdot x(t) \tag{5.4}
\end{equation*}
$$

for $t>0$, whence it follows that

$$
\int_{\Delta_{k}} x(t) d t=\frac{1}{\lambda} \int_{\Delta_{k}} x(t / 2) d t=\frac{2}{\lambda} \int_{\Delta_{k-1}} x(t) d t
$$

or

$$
\int_{\Delta_{k}} x(t) d t=\left(\frac{2}{\lambda}\right)^{k} \int_{\Delta_{0}} x(t) d t \quad(k=0, \pm 1, \pm 2, \ldots)
$$

Since the space $\Lambda_{q}(\psi)$ is separable, it is an interpolation space with respect to the couple $\left(L_{1}, L_{\infty}\right)$ (see [14, the corollary to Theorem 2.4.10]. Therefore, the averaging operator

$$
\begin{equation*}
Q y(t):=\sum_{k=-\infty}^{\infty} 2^{-k} \int_{\Delta_{k}} y(s) d s \cdot \chi_{\Delta_{k}}(t) \tag{5.5}
\end{equation*}
$$

is bounded on $\Lambda_{q}(\psi)$ (see [14, $\left.\S 2.3 .2\right]$ ). Hence, by (5.1), the sequence

$$
a(x):=\left(2^{-k} \int_{\Delta_{k}} x(s) d s\right)_{k=-\infty}^{\infty}
$$

belongs to $\ell_{q}(\mu)$. Thus, by the preceding formula, the sequence $a=\left(a_{k}\right)$ with $a_{k}:=$ $\lambda^{-k} \int_{\Delta_{0}} x(s) d s$ also belongs to this space. At the same time, it is easily seen that $S_{\lambda} a=0$. Since $x \geq 0$ and $x \neq 0$, by (5.4) we deduce that $a \neq 0$. Therefore, $S_{\lambda}$ is not injective, and statement (i) is proved.
(ii) If $a^{n}=\left(a_{k}^{n}\right)_{k=-\infty}^{\infty} \in \ell_{q}(\mu)(n=1,2, \ldots)$ and $S_{\lambda} a^{n} \rightarrow b=\left(b_{k}\right)$ in $\ell_{q}(\mu)$, then, by (5.3), the functions $\left\{T_{\lambda} h_{a^{n}}\right\}$ form a Cauchy sequence in $\Lambda_{q}(\psi)$. By assumption, $T_{\lambda} h_{a^{n}} \rightarrow y:=T_{\lambda} x$, where $x \in \Lambda_{q}(\psi)$. Formula (5.2) shows that $y=h_{b}$.

Next, arguing in the same way as we did to deduce (5.2), we obtain

$$
\begin{equation*}
x(t)=T_{\lambda}^{-1} h_{b}(t)=\sum_{k=-\infty}^{\infty}\left(S_{\lambda}^{-1} b\right)_{k} \chi_{\Delta_{k}}(t) \tag{5.6}
\end{equation*}
$$

where $S_{\lambda}^{-1} b=\left(b_{k+1}-\lambda^{-1} b_{k}\right)_{k}$. Since $x \in \Lambda_{q}(\psi)$, relation (5.1) shows that $c:=\left(S_{\lambda}^{-1} b\right)_{k} \in$ $\ell_{q}(\mu)$. Thus, $b=S_{\lambda} c$, i.e., $b \in \operatorname{Im} S_{\lambda}$. Consequently, $S_{\lambda}$ is a closed operator.
(iii) Suppose that $T_{\lambda}$ is not closed. Then there exists a sequence $\left\{x_{n}\right\} \subset \Lambda_{q}(\psi)$ with the following properties:

$$
\begin{equation*}
\left\|x_{n}\right\|_{\Lambda_{q}(\psi)}=1 \quad(n=1,2, \ldots) \quad \text { and } \quad\left\|T_{\lambda} x_{n}\right\|_{\Lambda_{q}(\psi)} \rightarrow 0 . \tag{5.7}
\end{equation*}
$$

Since $\Lambda_{q}(\psi)$ is an interpolation space with respect to the couple $\left(L_{1}, L_{\infty}\right)$, by [14, Lemma 2.4.6] we obtain

$$
\left\|T_{\lambda} x_{n}\right\| \geq\left\|x_{n}^{*}(t / 2)-\lambda x_{n}^{*}(t)\right\|=\left\|T_{\lambda} x_{n}^{*}\right\| .
$$

Consequently, we may assume that every function $x_{n}$ satisfies (5.7) and is nonnegative and monotone nonincreasing. Next, if $Q$ is the averaging operator defined by (5.5), then for every $x \in \Lambda_{q}(\psi)$ we have

$$
\begin{aligned}
Q T_{\lambda} x & =\sum_{k=-\infty}^{\infty} 2^{-k} \int_{\Delta_{k}} T_{\lambda} x(s) d s \cdot \chi_{\Delta_{k}} \\
& =\sum_{k=-\infty}^{\infty} 2^{-k}\left(\int_{\Delta_{k}} x(s / 2) d s-\lambda \int_{\Delta_{k}} x(s) d s\right) \cdot \chi_{\Delta_{k}} \\
& =\sum_{k=-\infty}^{\infty} 2^{-k}\left(2 \int_{\Delta_{k-1}} x(s) d s-\lambda \int_{\Delta_{k}} x(s) d s\right) \cdot \chi_{\Delta_{k}} \\
& =\sum_{k=-\infty}^{\infty}\left(a_{k-1}(x)-\lambda a_{k}(x)\right) \cdot \chi_{\Delta_{k}}=\sum_{k=-\infty}^{\infty}\left(S_{\lambda} a(x)\right)_{k} \cdot \chi_{\Delta_{k}},
\end{aligned}
$$

where, as before, $a(x)=\left(2^{-k} \int_{\Delta_{k}} x(s) d s\right)_{k=-\infty}^{\infty}$. Since

$$
\|Q x\|_{\Lambda_{q}(\psi)} \leq C\|x\|_{\Lambda_{q}(\psi)}
$$

for some $C>0$, from (5.1) and (5.7) it follows that $\left\|S_{\lambda} a\left(x_{n}\right)\right\|_{\ell_{q}(\mu)} \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, the same relation and the monotonicity of $x_{n}$ imply the inequalities

$$
\begin{aligned}
\left\|a\left(x_{n}\right)\right\|_{\ell_{q}(\mu)} & \geq c\left\|\sum_{k=-\infty}^{\infty} 2^{-k} \int_{\Delta_{k}} x_{n}(s) d s \cdot \chi_{\Delta_{k}}\right\|_{\Lambda_{q}(\psi)} \\
& \geq c\left\|\sum_{k=-\infty}^{\infty} x_{n}\left(2^{k+1}\right) \cdot \chi_{\Delta_{k}}\right\|_{\Lambda_{q}(\psi)} \geq c\left\|\sigma_{1 / 2} x_{n}\right\| \geq \frac{c}{2}\left\|x_{n}\right\|=\frac{c}{2} .
\end{aligned}
$$

Since, by assumption, $S_{\lambda}$ is injective, we see that $S_{\lambda}$ is not closed, which contradicts the assumptions.

The next statement is a direct consequence of Theorem 7 and Proposition 2.
Corollary 2. Suppose $\lambda \in \mathbb{C}$ and $1 \leq q<\infty$. Then $T_{\lambda}$ is injective and closed on $\Lambda_{q}(\psi)$ if and only if

$$
|\lambda| \in\left[0,2^{\alpha}\right) \cup\left(2^{\beta_{0}}, 2^{\alpha_{\infty}}\right) \cup\left(2^{\beta}, \infty\right)
$$

Moreover, it is invertible on $\Lambda_{q}(\psi)$ if and only if $|\lambda| \in\left[0,2^{\alpha}\right) \cup\left(2^{\beta}, \infty\right)$.
Now we are in a position to completely describe the set $\mathcal{F}(X)$ in the case where $X$ is a Lorentz space $\Lambda_{q}(\psi)$.
Theorem 8. Let $1 \leq q<\infty$, and let $\psi$ be a positive function on $(0, \infty)$ satisfying conditions (a) and (b) in 42 . If $\alpha_{\infty} \leq \beta_{0}$, then $\mathcal{F}\left(\Lambda_{q}(\psi)\right)=[1 / \beta, 1 / \alpha]$; if $\alpha_{\infty}>\beta_{0}$, then $\mathcal{F}\left(\Lambda_{q}(\psi)\right)=\left[1 / \beta, 1 / \alpha_{\infty}\right] \cup\left[1 / \beta_{0}, 1 / \alpha\right]$.
Proof. First, if $p \notin[1 / \beta, 1 / \alpha]$, then $\lambda:=2^{1 / p} \notin\left[2^{\alpha}, 2^{\beta}\right]$. By Proposition $1, T_{\lambda}$ is invertible in this case, so $\lambda$ is not an approximate eigenvalue of $T$. Consequently, by Theorem [5, $p \notin \mathcal{F}\left(\Lambda_{q}(\psi)\right)$, and $\mathcal{F}\left(\Lambda_{q}(\psi)\right) \subset[1 / \beta, 1 / \alpha]$.

Suppose that $\alpha_{\infty} \leq \beta_{0}$ and $p \in[1 / \beta, 1 / \alpha]$. Then $\lambda=2^{1 / p} \in\left[2^{\alpha}, 2^{\beta}\right]$, and, by Corollary 2 $T_{\lambda}$ is either noninjective or nonclosed. In both cases, $\lambda$ is an approximate eigenvalue of $T$. Thus, $p \in \mathcal{F}\left(\Lambda_{q}(\psi)\right)$ by Theorem 5. So, $[1 / \beta, 1 / \alpha] \subset \mathcal{F}\left(\Lambda_{q}(\psi)\right)$ in the case in question.

Now, suppose that $\alpha_{\infty}>\beta_{0}$ and $p \in\left(1 / \alpha_{\infty}, 1 / \beta_{0}\right)$. Then $\lambda=2^{1 / p} \in\left(2^{\beta_{0}}, 2^{\alpha_{\infty}}\right)$ and, again by Corollary 2 $T_{\lambda}$ is closed and injective. Therefore, there exist $c>0$ with

$$
\left\|T_{\lambda} x\right\|_{\Lambda_{q}(\psi)} \geq c\|x\|_{\Lambda_{q}(\psi)} \quad\left(x \in \Lambda_{q}(\psi)\right)
$$

Consequently, $\lambda$ is not an approximate eigenvalue for $T$, and $p \notin \mathcal{F}\left(\Lambda_{q}(\psi)\right)$ by Theorem 5 .

Remark 1. It can easily be shown (see [13]) that for arbitrary four numbers $a, b, c$, and $d$ with $0<a \leq \min (b, c) \leq \max (b, c) \leq d<1$ there exists a function $\psi$ quasiconcave on $(0, \infty)$ and such that $\alpha(\psi)=a, \beta_{0}(\psi)=b, \alpha_{\infty}(\psi)=c$, and $\beta(\psi)=d$. Thus, by Theorem 8 , for every $1 \leq q<\infty$ we have $\mathcal{F}\left(\Lambda_{q}(\psi)\right)=[1 / d, 1 / a]$ if $c \leq b$ and $\mathcal{F}\left(\Lambda_{q}(\psi)\right)=[1 / d, 1 / c] \cup[1 / b, 1 / a]$ if $c>b$.

## §6. Concluding examples and remarks

Example 1 (see also [9). For arbitrary $1<p<r<\infty$, consider the quasiconcave functions $\psi_{1}(t)=\max \left(t^{1 / p}, t^{1 / r}\right)$ and $\psi_{2}(t)=\min \left(t^{1 / p}, t^{1 / r}\right)$. It is easy to verify that $M\left(\psi_{i}\right)=\psi_{i}(i=1,2), M_{0}\left(\psi_{1}\right)(t)=M_{\infty}\left(\psi_{2}\right)(t)=t^{1 / r}$, and $M_{\infty}\left(\psi_{1}\right)(t)=M_{0}\left(\psi_{2}\right)(t)=$ $t^{1 / p}$. Therefore, $\alpha\left(\psi_{i}\right)=1 / r, \beta\left(\psi_{i}\right)=1 / p(i=1,2), \alpha_{0}\left(\psi_{1}\right)=\beta_{0}\left(\psi_{1}\right)=\alpha_{\infty}\left(\psi_{2}\right)=$ $\beta_{\infty}\left(\psi_{2}\right)=1 / r$, and $\alpha_{\infty}\left(\psi_{1}\right)=\beta_{\infty}\left(\psi_{1}\right)=\alpha_{0}\left(\psi_{2}\right)=\beta_{0}\left(\psi_{2}\right)=1 / p$. Thus, by Theorem 8 , $\mathcal{F}\left(\Lambda_{q}\left(\psi_{1}\right)\right)=\{p, r\}$ and $\mathcal{F}\left(\Lambda_{q}\left(\psi_{2}\right)\right)=[p, r]$ for every $1 \leq q<\infty$. Observe that

$$
\Lambda_{q}\left(\psi_{1}\right)=\Lambda_{q}\left(t^{1 / p}\right) \cap \Lambda_{q}\left(t^{1 / r}\right) \quad \text { and } \quad \Lambda_{q}\left(\psi_{2}\right)=\Lambda_{q}\left(t^{1 / p}\right)+\Lambda_{q}\left(t^{1 / r}\right)
$$

Example 2. Now, let $1 \leq p<r \leq \infty$, and let $X=L_{p}(0, \infty) \cap L_{r}(0, \infty), Y=L_{p}(0, \infty)+$ $L_{r}(0, \infty)$ with the usual norms:

$$
\begin{gathered}
\|f\|_{X}:=\max \left(\|f\|_{L_{p}},\|f\|_{L_{r}}\right), \\
\mid f \|_{Y}=\inf \left\{\|g\|_{L_{p}}+\|h\|_{L_{r}}: f=g+h, g \in L_{p}, h \in L_{r}\right\} .
\end{gathered}
$$

Then $\psi_{1}(t)=\max \left(t^{1 / p}, t^{1 / r}\right)$ and $\psi_{2}(t)=\min \left(t^{1 / p}, t^{1 / r}\right)$ are the fundamental functions for $X$ and $Y$, respectively; so, as in the preceding example, we have $\alpha_{X}=\alpha_{Y}=1 / r$ and $\beta_{X}=\beta_{Y}=1 / p$. By Proposition $T_{\lambda}=T-\lambda I$ is an isomorphism in $L_{s}(0, \infty)$ provided that $\lambda \neq 2^{1 / s}$. Consequently, for every $\lambda$ different from $2^{1 / p}$ and $2^{1 / r}$ there exists $c>0$ such that

$$
\left\|T_{\lambda} f\right\|_{L_{p}} \geq c\|f\|_{L_{p}} \quad\left(f \in L_{p}\right) \quad \text { and } \quad\left\|T_{\lambda} f\right\|_{L_{r}} \geq c\|f\|_{L_{r}} \quad\left(f \in L_{r}\right) .
$$

Thus, $\left\|T_{\lambda} f\right\|_{X} \geq c\|f\|_{X}$ and we see that no such $\lambda$ is an approximate eigenvalue for $T$ on $X$. By Theorem 4 and 5 , $\mathcal{F}(X)=\{p, r\}$.

In order to find $\mathcal{F}(Y)$, we show that $T_{\lambda}$ is not injective on $Y$ for $2^{1 / r}<\lambda<2^{1 / p}$. Indeed, let $f_{0}$ be an arbitrary positive function belonging to $L_{\infty}[1,2]$. It can easily be verified that $T_{\lambda} f=0$ if

$$
f(t)=\sum_{n=-\infty}^{\infty} \lambda^{-n} f_{0}\left(2^{-n} t\right) \chi_{\left[2^{n}, 2^{n+1}\right)}(t) \quad(t>0)
$$

Next, by the definition of the norm on a sum of spaces, we have

$$
\begin{aligned}
\|f\|_{Y} & \leq\left\|\sum_{n=0}^{-\infty} \lambda^{-n} f_{0}\left(2^{-n} t\right) \chi_{\left[2^{n} .2^{n+1}\right)}(t)\right\|_{L_{p}}+\left\|\sum_{n=1}^{\infty} \lambda^{-n} f_{0}\left(2^{-n} t\right) \chi_{\left[2^{n} .2^{n+1}\right)}(t)\right\|_{L_{r}} \\
& =\left(\sum_{n=0}^{-\infty} \lambda^{-n p} \int_{2^{n}}^{2^{n+1}}\left|f_{0}\left(2^{-n} t\right)\right|^{p} d t\right)^{1 / p}+\left(\sum_{n=1}^{\infty} \lambda^{-n r} \int_{2^{n}}^{2^{n+1}}\left|f_{0}\left(2^{-n} t\right)\right|^{r} d t\right)^{1 / r} \\
& \leq\left\{\left(\sum_{n=0}^{-\infty} \lambda^{-n p} 2^{n}\right)^{1 / p}+\left(\sum_{n=1}^{\infty} \lambda^{-n r} 2^{n}\right)^{1 / r}\right\}\left\|f_{0}\right\|_{L_{\infty}[1,2]}
\end{aligned}
$$

(we consider the case where $r<\infty$; for $r=\infty$ the arguments are similar). Since $2^{1 / r}<\lambda<2^{1 / p}$, the two series on the right in the last inequality converge. Consequently, $f \in Y$ and $T_{\lambda}$ is not injective on $Y$. Thus, an arbitrary $\lambda \in\left(2^{1 / r}, 2^{1 / p}\right)$ is an eigenvalue of $T$. Applying Theorems 5and 4 , we deduce that $\mathcal{F}(Y)=[p, r]$.

Remark 2. If we denote by $\alpha, \alpha_{0}, \alpha_{\infty}, \beta, \beta_{0}, \beta_{\infty}$ the dilation indices of the fundamental function of a symmetric space, then, formally, the results of the preceding example fit into the pattern of Theorem 8 (though it is not applicable because $X$ and $Y$ are not Lorentz spaces). It seems quite natural to conjecture that the domain of applicability of Theorem 8 is wider than the class of Lorentz spaces. In particular, it would be interesting to prove a similar result for Orlicz spaces on the semiaxis (the spaces of Example 2belong to this class). Note that for the Orlicz spaces $X=L_{M}[0,1]$ on an interval we always have $\mathcal{F}(X)=\left[1 / \beta_{X}, 1 / \alpha_{X}\right]$ (see [19, Theorem 4.a.9] and [7, remark on pp. 140-141]).

## References

[1] B. S. Tsirel'son, It is impossible to imbed $\ell_{p}$ or $c_{0}$ into an arbitrary Banach space, Funktsional. Anal. i Prilozhen. 8 (1974), no. 2, 57-60; English transl., Funct. Anal. Appl. 8 (1974), $138-141$. MR0350378 (50:2871)
[2] A. Dvoretzky, Some results on convex bodies and Banach spaces, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Acad. Press, Jerusalem, 1961, pp. 123-160. MR0139079 (25:2518)
[3] V. D. Milman and G. Schechtman, Asymptotic theory of finite-dimensional normed spaces, Lecture Notes in Math., vol. 1200, Springer-Verlag, Berlin, 1986. MR 0856576 ( $87 \mathrm{~m}: 46038$ )
[4] J. L. Krivine, Sous-espaces de dimension finie des espaces de Banach réticulés, Ann. of Math. (2) 104 (1976), 1-29. MR0407568(53:11341)
[5] F. Albiac and N. J. Kalton, Topics in Banach space theory, Grad. Texts in Math., vol. 233, Springer, New York, 2006. MR2192298 (2006h:46005)
[6] J. Diestel, H. Jarchow, and A. Tonge, Absolutely summing operators, Cambridge Stud. Adv. Math., vol. 43, Cambridge Univ. Press, Cambridge, 1995. MR1342297 (96i:46001)
[7] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. II. Function spaces. Ergeb. Math. Grenzgeb., Bd. 97, Springer-Verlag, Berlin-New York, 1979. MR0540367|(81c:46001)
[8] B. Maurey and G. Pisier, Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, Studia Math. 58 (1976), 45-90. MR0443015 (56:1388)
[9] A. R. Schep, Krivine's theorem and the indices of a Banach lattice, Positive Operators and Semigroups on Banach Lattices (Curaçao, 1990), Acta Appl. Math. 27 (1992), 111-121. MR1184883 (93j:46025)
[10] H. P. Rosenthal, On a theorem of J. L. Krivine concerning block finite representability of $\ell^{p}$ in general Banach spaces, J. Funct. Anal. 28 (1978), 197-225. MR0493384 (81d:46020)
[11] S. V. Astashkin, Tensor product in symmetric function spaces, Function Spaces (Zielona Góra, Poland, 1995), Collect. Math. 48 (1997), no. 4-6, 375-391. MR1602623 (99a:46043)
[12] A. B. Antonevich, Linear functional equations. The operator approach, Belorus. Univ., Minsk, 1988; English transl., Oper. Theory Adv. Appl., vol. 83, Birkhäuser Verlag, Basel, 1996. MR0993293 (90f:39001) MR1382652 (97i:47001)
[13] S. V. Astashkin and P. Sunehag, Real method of interpolation on subcouples of codimension one, Studia Math. 185 (2008), no. 2, 151-168. MR2379965 (2009c:46032)
[14] S. G. Kreĭn, Yu. I. Petunin, and E. M. Semenov, Interpolation of linear operators, Nauka, Moscow, 1978; English transl., Transl. Math. Monogr., vol. 54, Amer. Math. Soc., Providence, RI, 1982. MR0506343 (81f:46086) MR0649411 (84j:46103)
[15] C. Bennett and R. Sharpley, Interpolation of operators, Pure Appl. Math., vol. 129, Acad. Press, Boston, 1988. MR0928802 (89e:46001)
[16] S. V. Astashkin, Images of operators in rearrangement invariant spaces and interpolation, Function Spaces (Wroclaw, Poland, 2001), World Sci. Publ., River Edge, NJ, 2003, pp. 49-64. MR2082317 (2005g:46059)
[17] , Interpolation of intersections by the real method, Algebra i Analiz 17 (2005), no. 2, 33-69; English transl., St. Petersburg Math. J. 17 (2006), no. 2, 239-265. MR2159583|(2006d:46021)
[18] N. J. Kalton, Calderón couples of rearrangement invariant spaces, Studia Math. 106 (1993), no. 3, 233-277. MR1239419 (94k:46152)
[19] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. I. Sequence spaces, Ergeb. Math. Grenzgeb., Bd. 92, Springer-Verlag, Berlin-New York, 1977. MR 0500056 (58:17766)

Samara State University, ul. Akademika Pavlova 1, Samara 443011, Russia
E-mail address: astashkn@ssu.samara.ru
Received 7/OCT/2009
Translated by S. KISLYAKOV


[^0]:    2010 Mathematics Subject Classification. Primary 46E30.
    Key words and phrases. Finite representability of $\ell_{p}$-spaces, symmetric spaces, Boyd indices, Lorentz space, spectrum, weighted spaces.

    Supported in part by RFBR, grant no. 07-01-96603.

