# FINITE REPRESENTABILITY OF $\ell_p$ -SPACES IN SYMMETRIC SPACES

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ABSTRACT. For a separable rearrangement invariant space X on the semiaxis,  $\mathcal{F}(X)$  is defined to be the set of all  $p \in [1, \infty]$  such that  $\ell_p$  is finitely representable in X in such a way that the standard basis vectors of  $\ell_p$  correspond to equimeasurable mutually disjoint functions. In the paper, a characterization of the set  $\mathcal{F}(X)$  is obtained. As a consequence, a version of Krivine's well-known theorem is proved for rearrangement invariant spaces. Next, a description of the sets  $\mathcal{F}(X)$  for certain Lorentz spaces is found.

## §1. INTRODUCTION

In 1974, Tsirel'son constructed an example of a Banach space containing no isomorphic copies of  $\ell_p$ ,  $1 \leq p < \infty$ , and  $c_0$ . Two years later, Krivine proved a theorem that showed once again the fundamental difference between the properties of infinite-dimensional subspaces of a Banach space and subspaces of finite (though large) dimension. To state and discuss it, we introduce some notions.

**Definition.** Suppose X is a Banach space,  $1 \leq p \leq \infty$ , and  $\{z_i\}_{i=1}^{\infty}$  is a bounded sequence in X. The space  $\ell_p$  is said to be *block finitely representable in*  $\{z_i\}_{i=1}^{\infty}$  if for every  $n \in \mathbb{N}$  and  $\varepsilon > 0$  there exist  $0 = p_0 < p_1 < \cdots < p_n$  and  $\alpha_i \in \mathbb{C}$  such that the vectors  $u_k = \sum_{i=p_{k-1}+1}^{p_k} \alpha_i z_i$   $(k = 1, 2, \ldots, n)$  satisfy the inequality

$$(1+\varepsilon)^{-1} \| (a_k)_{k=1}^n \|_p \le \left\| \sum_{k=1}^n a_k u_k \right\|_X \le (1+\varepsilon) \| (a_k)_{k=1}^n \|_p$$

for arbitrary  $a_1, a_2, \ldots, a_n \in \mathbb{C}$ . Here, as usual,

$$||(a_k)_{k=1}^n||_p := \left(\sum_{k=1}^n |a_k|^p\right)^{1/p}$$
 if  $p < \infty$ , and  $||(a_k)_{k=1}^n||_\infty := \max_{k=1,2,\dots,n} |a_k|.$ 

**Definition.** Let X be a Banach space, and let  $1 \le p \le \infty$ . The space  $\ell_p$  is said to be *finitely representable* in X if for every  $n \in \mathbb{N}$  and  $\varepsilon > 0$  there exist  $x_1, x_2, \ldots, x_n \in X$  such that

(1.1) 
$$(1+\varepsilon)^{-1} \| (a_k)_{k=1}^n \|_p \le \left\| \sum_{k=1}^n a_k x_k \right\|_X \le (1+\varepsilon) \| (a_k)_{k=1}^n \|_p$$

for arbitrary  $a_1, a_2, \ldots, a_n \in \mathbb{C}$ .

By the celebrated Dvoretzky theorem (see [2] or [3, Theorem 5.8]),  $\ell_2$  is finitely representable in an arbitrary infinite-dimensional Banach space X. Clearly, if  $\ell_p$  is block

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finitely representable in some sequence  $\{z_i\}_{i=1}^{\infty} \subset X$ , then  $\ell_p$  is finitely representable in X. Therefore, the following statement proved by Krivine in [4] is an important supplement to the Dvoretzky theorem.

**Theorem 1** (see [5, Theorem 11.3.9]). Let  $\{z_i\}_{i=1}^{\infty}$  be an arbitrary normalized sequence in a Banach space X such that the vectors  $z_i$  do not form a relatively compact set. Then  $\ell_p$  is block finitely representable in  $\{z_i\}_{i=1}^{\infty}$  for some  $p, 1 \leq p \leq \infty$ .

Undoubtedly, this statement is a central result in the geometric theory of Banach spaces. It has numerous applications (see, e.g., [5] and [3]). In this connection, it is natural to look for a description of all  $p \in [1, \infty]$  such that  $\ell_p$  is finitely representable in a given Banach space. It has turned out that the (Rademacher) type and cotype of the space play an important role here. Denote by  $r_n : [0,1] \to \mathbb{R}$   $(n \in \mathbb{N})$  the Rademacher functions, i.e.,  $r_n(t) = \operatorname{sign}(\sin 2^n \pi t)$ . A Banach space X is said to have type  $p, 1 \leq p \leq 2$ , if there exists a constant K > 0 such that for every  $n \in \mathbb{N}$  and arbitrary  $x_1, \ldots, x_n$  we have

$$\int_{0}^{1} \left\| \sum_{k=1}^{n} r_{k}(t) x_{k} \right\| dt \leq K \left( \sum_{k=1}^{n} \|x_{k}\|^{p} \right)^{1/p}$$

A Banach space X is said to have cotype  $q \ge 2$  if there is a constant K > 0 such that for every  $n \in \mathbb{N}$  and arbitrary  $x_1, \ldots, x_n$  we have

$$\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{1/q} \le K \int_0^1 \left\|\sum_{k=1}^{n} r_k(t) x_k\right\| dt.$$

If  $q = \infty$ , the left-hand side of the last inequality must be replaced by  $\max_{1 \le k \le n} ||x_k||$ . The space X is said to have *trivial type* (*trivial cotype*) if X only has type 1 (only has infinite cotype). Detailed information about these notions can be found in [6] and [7]. We introduce the notation

$$p_X := \sup\{p \in [1,2] : X \text{ has type } p\}$$

and

$$q_X := \inf\{q \in [2, \infty] : X \text{ is of cotype } q\}.$$

It can easily be checked that if  $\ell_p$  is finitely representable in a Banach space X, then  $p \in [p_X, q_X]$ . Moreover, Maurey and Pisier proved the following refinement of Theorem 1 (see [8]).

**Theorem 2** (Maurey–Pisier). For every infinite-dimensional Banach space X, the spaces  $\ell_{p_X}$  and  $\ell_{q_X}$  are finitely representable in X.

Suppose now that a Banach space X is endowed with a partial order making X a Banach lattice. If  $\ell_p$  is finitely representable in X and, moreover, the images  $x_1, x_2, \ldots, x_n$  in X of the basis vectors of  $\ell_p$  can be chosen mutually disjoint, we say that  $\ell_p$  is *lattice finitely representable* in the Banach lattice X. In this case, the role of the type and cotype is played by the upper and lower estimates for the lattice.

We recall that a Banach lattice X admits an *upper p-estimate* if there exists a constant M such that for every  $n \in \mathbb{N}$  and arbitrary mutually disjoint vectors  $x_1, \ldots, x_n \in X$  we have

$$\left\|\sum_{k=1}^{n} x_k\right\| \le M\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p} \quad (p < \infty)$$

 $\left\|\sum_{k=1}^{n} x_k\right\| \le M \max_{1 \le k \le n} \|x_k\| \quad (p = \infty).$ 

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A Banach lattice X is said to admit a *lower q-estimate* if there exists a constant M > 0such that for every  $n \in \mathbb{N}$  and arbitrary mutually disjoint vectors  $x_1, \ldots, x_n \in X$  we have

$$\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{1/q} \le M \left\|\sum_{k=1}^{n} x_k\right\| \quad (q < \infty)$$

and

$$\max_{1 \le k \le n} \|x_k\| \le M \left\| \sum_{k=1}^n x_k \right\| \quad (q = \infty)$$

Let

 $s(X) := \sup\{p \ge 1 : X \text{ admits an upper } p\text{-estimate}\}$ 

and

$$\sigma(X) := \inf\{q \ge 1 : X \text{ admits a lower } q \text{-estimate}\}.$$

If  $\ell_p$  is lattice finitely representable in a Banach lattice X, then  $p \in [s(X), \sigma(X)]$ . Furthermore, the following lattice version of the Maurey–Pisier and Krivine theorems was proved in [9].

**Theorem 3** (Schep). If X is an infinite-dimensional Banach lattice, then  $\ell_{s(X)}$  and  $\ell_{\sigma(X)}$  are finitely representable in X.

In what follows, we shall consider a special class of Banach function lattices, specifically, the symmetric (in other terminology, rearrangement invariant) spaces. See the next section for the definition. If x(t) is a measurable function on  $[0, \alpha)$  ( $0 < \alpha \le \infty$ ), we denote  $n_x(\tau) := m(\{s \in [0, \alpha) : |x(s)| > \tau\})$ . Here and in the sequel *m* denotes the Lebesgue measure.

**Definition.** Let X be a symmetric space on  $[0, \infty)$ . We denote by  $\mathcal{F}(X)$  the set of  $p \in [1, \infty]$  with the property that for every  $n \in \mathbb{N}$  and every  $\varepsilon > 0$  there exist  $x_k \in X$  (k = 1, 2, ..., n) such that  $\operatorname{supp} x_i \cap \operatorname{supp} x_j = \emptyset$  for  $(i \neq j)$ ,  $n_{x_k}(\tau) = n_{x_1}(\tau)$   $(k = 1, 2, ..., n; \tau > 0)$ , and for every  $a_k \in \mathbb{C}$  we have

(1.2) 
$$(1+\varepsilon)^{-1} \| (a_k)_{k=1}^n \|_p \le \left\| \sum_{k=1}^n a_k x_k \right\|_X \le (1+\varepsilon) \| (a_k)_{k=1}^n \|_p$$

(with a natural modification for  $p = \infty$ ).

If  $\alpha_X$  and  $\beta_X$  are the Boyd indices of a symmetric space X, it can easily be shown that  $\mathcal{F}(X) \subset [1/\beta_X, 1/\alpha_X]$ . Next, in the monograph [7] the following version of Krivine's theorem was stated without proof (see also the remark after Theorem 3.3 in [10]).

**Theorem 4** (see [7, Theorem 2.b.6]). If X is an arbitrary symmetric space, then  $\max \mathcal{F}(X) = 1/\alpha_X$  and  $\min \mathcal{F}(X) = 1/\beta_X$ .

The last theorem and, in general, the structure of  $\mathcal{F}(X)$  play an important role in the study of geometric properties of symmetric spaces (see, e.g., [11]) and also in the study of normal solvability and invertibility of operators between function spaces, which, in its turn, is important for the theory of functional-differential equations, the theory of dynamical systems, etc. (see [12] and references therein).

The following notion will be important in what follows.

**Definition.** Let  $A : X \to X$  be a bounded linear operator, X being a Banach space over  $\mathbb{C}$ . A sequence  $\{u_n\}_{n=1}^{\infty} \subset X$ ,  $||u_n|| = 1$  (n = 1, 2, ...) is called an *approximate eigenvector* corresponding to an *approximate eigenvalue*  $\lambda \in \mathbb{C}$  for A if  $||Au_n - \lambda u_n|| \to 0$ . It can easily be shown (see [3, 12.1]) that every bounded linear operator has at least one approximate eigenvalue. The main result of this paper is the following characterization of the set  $\mathcal{F}(X)$  for an arbitrary separable symmetric space on  $[0, \infty)$ . Theorem 4 will turn out to be an easy consequence of it.

**Theorem 5.** Let X be an arbitrary separable symmetric space on  $[0, \infty)$ . Then  $p \in \mathcal{F}(X)$  if and only if  $\lambda := 2^{1/p}$  is an approximate eigenvalue for the dilation operator Tx(t) := x(t/2).

The proof of this theorem is the main topic of §3 (in §2 we introduce the necessary definitions and notation). As a consequence, we shall prove Theorem 4 (we remind the reader that it was stated without proof in [7]). In the second part of the paper (§5), we describe  $\mathcal{F}(X)$  completely in the case where X is a Lorentz space. It will be shown that, depending on the function that generates this space, either  $\mathcal{F}(X)$  is the entire interval  $[1/\beta_X, 1/\alpha_X]$  or  $\mathcal{F}(X)$  is the union of two intervals. In the proof, the results of [13] are used substantially; that paper contains a thorough description of the image of  $S_{\lambda} = S - \lambda I$  (S is the shift operator, I is the identity, and  $\lambda \in \mathbb{C}$ ) in a weighted  $\ell_p$ -space. This description is presented in detail in §4. Finally, §6 contains examples and remarks.

### §2. Definitions and notation

A Banach space  $(X, \|\cdot\|_x)$  of complex-valued and Lebesgue measurable functions on the interval  $[0, \alpha)$   $(0 < \alpha \leq \infty)$  is said to be symmetric (or rearrangement invariant) if, whenever  $y \in X$  and  $x^*(t) \leq y^*(t)$   $(t \in [0, \alpha))$ , we have  $x \in X$  and  $\|x\|_x \leq \|y\|_x$ . Here and below,  $x^*(t)$  is the right continuous nondecreasing rearrangement of |x(s)|, i.e.,  $x^*(t) = \inf\{\tau \geq 0 : n_x(\tau) \leq t\}$  (t > 0).

A symmetric space (s.s. for short) X is said to be maximal (or to have the Fatou property) if the conditions  $\{f_n\}_{n=1}^{\infty} \subseteq X$ ,  $f_n \to f$  a.e. on  $[0, \alpha)$  and  $\sup_n ||f_n||_X < \infty$  imply  $f \in X$  and  $||f||_X \leq \liminf_{n\to\infty} ||f_n||_X$ . As in [7], in what follows we assume that X is either separable or maximal.

For every s.s. X on  $[0, \infty)$  we have the following continuous embeddings:

$$L_1 \cap L_\infty \subseteq X \subseteq L_1 + L_\infty.$$

We denote by  $X_0$  the closure of  $L_1 \cap L_\infty$  in X; this set is called the *separable part* of X. If  $X \neq L_1 \cap L_\infty$ , then  $X_0$  is separable.

Let X be a symmetric space. The function  $\varphi_X(t) := \|\chi_A\|_X$ , where  $A \subset (0, \infty)$ ,  $\lambda(A) = t$ , and  $\chi_A$  is the characteristic function of A, is called the *fundamental function* of X. Another important characteristic of a symmetric space X is its Boyd indices. We recall their definition in the case of a space on  $[0, \infty)$ . For any  $\tau > 0$ , the dilation operator  $\sigma_\tau x(t) := x(t/\tau)$  is bounded in any s.s. X and  $\|\sigma_\tau\|_{X\to X} \leq \max(1, \tau)$  ( $\tau > 0$ ); see [14, Theorem 2.4.4]. The numbers

$$\alpha_X := \lim_{\tau \to 0} \frac{\ln \|\sigma_\tau\|_{X \to X}}{\ln \tau} \quad \text{and} \quad \beta_X := \lim_{\tau \to \infty} \frac{\ln \|\sigma_\tau\|_{X \to X}}{\ln \tau}$$

are called the *Boyd indices* of X; we always have  $0 \le \alpha_X \le \beta_X \le 1$ .

The dilation function of a positive function  $\psi(t), t \in (0, \infty)$ , is defined by the formula

$$M_{\psi}(s) = \sup_{t>0} \frac{\psi(st)}{\psi(t)}, \quad 0 < s < \infty.$$

Next, the numbers

$$\gamma_{\psi} = \lim_{s \to 0+} \frac{\ln M_{\psi}(s)}{\ln s}$$
 and  $\delta_{\psi} = \lim_{s \to \infty} \frac{\ln M_{\psi}(s)}{\ln s}$ 

are called the *lower* and the *upper dilation indices* of  $\psi(t)$ . If  $\psi$  is quasiconcave, that is,  $\psi(t)$  is monotone increasing and  $\psi(t)/t$  is monotone decreasing for t > 0, then  $0 \le \gamma_{\psi} \le \delta_{\psi} \le 1$  (see [14, §2.1.2]).

Important examples of symmetric spaces are the  $L_p$ -spaces  $(1 \le p \le \infty)$  and their generalization, the Orlicz spaces. Let  $\Phi$  be an Orlicz function on  $[0, \infty)$ , i.e.,  $\Phi$  is a convex continuous monotone increasing function on  $[0, \infty)$  with  $\Phi(0) = 0$  and  $\Phi(\infty) = \infty$ . The *Orlicz space*  $L_{\Phi}$  on the semiaxis consists of all measurable functions f on  $[0, \infty)$  for which the norm  $||f||_{L_{\Phi}} = \inf\{\rho > 0 : \int_{0}^{\infty} \Phi(|f(t)|/\rho) dt \le 1\}$  is finite. The Lorentz spaces, considered in the second part of the paper, constitute another

The Lorentz spaces, considered in the second part of the paper, constitute another class of symmetric spaces. Let  $1 \le q < \infty$ , and let  $\psi$  be a positive function on  $(0, \infty)$  satisfying the following conditions:

(a) the dilation indices of  $\psi$  are nontrivial, i.e.,  $0 < \gamma_{\psi} \leq \delta_{\psi} < 1$ ;

(b) the function  $\psi(t)^q/t$  is monotone decreasing.

The Lorentz space  $\Lambda_q(\psi)$  consists of all functions x(t) measurable on  $(0, \infty)$  and satisfying

(2.1) 
$$\|x\|_{\Lambda_q(\psi)} := \left(\int_0^\infty x^*(t)^q \psi(t)^q \frac{dt}{t}\right)^{1/q} < \infty.$$

This a separable s.s., and it can easily be checked that its Boyd indices coincide with the corresponding dilation indices for  $\psi$ , i.e.,  $\alpha_{\Lambda_q(\psi)} = \gamma_{\psi}$ ,  $\beta_{\Lambda_q(\psi)} = \delta_{\psi}$ . Next, since the dilation indices are nontrivial, the function  $\psi$  is equivalent to its least concave majorant (see [14, Corolary 2.1.2]) and, consequently, we may assume that  $\psi$  is quasiconcave. See the monographs [7, 14, 15] for more details.

Finally, we present the definitions and notation to be used in the proof of Theorem 5. Let E be a space of complex sequences in which the standard unit vectors  $e_n$   $(n \in \mathbb{N})$  form a symmetric basis. Throughout, we denote by  $c_{0,0}$  the set of all finitely supported sequences, i.e.,  $x = (x_n)_{n=1}^{\infty} \in c_{0,0}$  if  $x_n = 0$  for all sufficiently large n. For arbitrary  $x = (x_n), y = (y_n) \in c_{0,0}$ , we denote by  $x \oplus y$  their disjoint sum. This is an arbitrary vector in  $c_{0,0}$  whose nonzero coordinates coincide with all nonzero coordinates of x and y. For instance, if  $n_0 = \max\{n \in \mathbb{N} : x_n \neq 0\}$ , then for the disjoint sum of x and y we can take the vector:

$$x \oplus y = \sum_{n=1}^{n_0} x_n e_n + \sum_{n=n_0+1}^{\infty} y_{n-n_0} e_n.$$

Since the basis  $\{e_n\}_{n=1}^{\infty}$  is symmetric in E, the norm  $||x \oplus y||_E$  does not depend on a specific choice for  $x \oplus y$ .

We say that a vector x is replaceable ( $\varepsilon$ -replaceable) by a vector y if for arbitrary u,  $v \in c_{0,0}$  we have

$$||u \oplus x \oplus v||_E = ||u \oplus y \oplus v||_E$$

(respectively,

$$\left| \|u \oplus x \oplus v\|_E - \|u \oplus y \oplus v\|_E \right| < \varepsilon ).$$

All Banach spaces are assumed to be complex. We write  $f \simeq g$  if  $cf \leq g \leq Cf$  for some constants c > 0 and C > 0 that do not depend on the values of all (or some) of the arguments of f and g.

#### §3. CHARACTERIZATION OF $\mathcal{F}(X)$ FOR S.S. ON THE SEMIAXIS

Proof of Theorem 5. Observe that the operator T is bounded on X and  $1 \leq ||T||_{X \to X} \leq 2$  (see the preceding section).

First, let  $p \in \mathcal{F}(X)$ . Then for every  $n \in \mathbb{N}$  there exists a collection  $\{x_k\}_{k=1}^{2^n}$  of equimeasurable and mutually disjoint functions (i.e.,  $\operatorname{supp} x_i \cap \operatorname{supp} x_j = \emptyset$  for  $i \neq j$  and

 $n_{x_k}(\tau) = n_{x_1}(\tau)$  for  $\tau > 0$  and  $k = 1, 2, \ldots, 2^n$  such that

(3.1) 
$$\frac{1}{2} \left( \sum_{k=1}^{2^n} |a_k|^p \right)^{1/p} \le \left\| \sum_{k=1}^{2^n} a_k x_k \right\|_X \le 2 \left( \sum_{k=1}^{2^n} |a_k|^p \right)^{1/p}$$

(we assume that  $p < \infty$ ; the case of  $p = \infty$  is treated similarly). Consider the functions

$$y_0 := x_1, \quad y_s := \sum_{i=2^{s-1}+1}^{2^s} x_i \quad (s = 1, 2, \dots, n).$$

They are also disjoint; moreover,

(3.2) 
$$n_{y_s}(\tau) = 2^{s-1} n_{y_0}(\tau) \quad (\tau > 0), \quad s = 1, 2, \dots, n.$$

Since X is separable and symmetric, we always can ensure by approximation that  $\operatorname{supp} y_0 = \operatorname{supp} x_1 \subset [0, A]$  for some A > 0. Putting

$$z_0(t) := y_0^*(t), \quad z_s(t) := z_0 \left( 2^{1-s}t - A \right) \quad (s = 1, 2, \dots, n),$$

we have supp  $z_s \subset [2^{s-1}A, 2^sA]$  for  $s = 1, 2, \ldots, n$  (in particular, the  $z_s$  are mutually disjoint). Next, by (3.2),  $n_{z_s}(\tau) = n_{y_s}(\tau) = 2^{s-1}n_{y_0}(\tau)$  ( $\tau > 0$ ). Therefore, since X is symmetric, we obtain

(3.3) 
$$\left\|\sum_{s=1}^{n} b_s z_s\right\|_X = \left\|\sum_{s=1}^{n} b_s y_s\right\|_X$$

for arbitrary  $b_s \in \mathbb{C}$ . From the definition of the functions  $y_s$  and relation (3.1), we deduce that

$$\frac{1}{2}2^{(s-1)/p} \le \|y_s\|_X \le 2 \cdot 2^{(s-1)/p} \quad \text{and} \quad \frac{1}{2}n^{1/p} \le \left\|\sum_{s=1}^n 2^{(1-s)/p} y_s\right\|_X \le 2n^{1/p}.$$

By (3.3), it follows that the functions  $\bar{z}_s(t) := 2^{(1-s)/p} z_s(t)$  satisfy

(3.4) 
$$\frac{1}{2} \le \|\bar{z}_s\|_X \le 2 \text{ and } \frac{1}{2}n^{1/p} \le \left\|\sum_{s=1}^n \bar{z}_s\right\|_X \le 2n^{1/p}$$

Thus, putting

$$v_n(t) := n^{-1/p} \sum_{s=1}^n \overline{z}_s(t) \quad (n = 1, 2, ...),$$

we have

(3.5) 
$$\frac{1}{2} \le \|v_n\|_X \le 2 \quad (n = 1, 2, \dots)$$

Moreover, since  $\bar{z}_s(t/2) = 2^{1/p} \bar{z}_{s+1}(t)$  (s = 1, 2, ..., n-1) by the definition of  $\bar{z}_s$ , we see that for  $\lambda = 2^{1/p}$  the operator  $T_{\lambda} := T - \lambda I$  (*I* is the identity) satisfies

$$T_{\lambda}v_{n}(t) = n^{-1/p} \left( \sum_{s=1}^{n} \overline{z}_{s}(t/2) - \lambda \sum_{s=1}^{n} \overline{z}_{s}(t) \right)$$
  
=  $n^{-1/p} \left( 2^{1/p} \sum_{s=1}^{n-1} \overline{z}_{s+1}(t) + \overline{z}_{n}(t/2) - 2^{1/p} \sum_{s=1}^{n} \overline{z}_{s}(t) \right)$   
=  $n^{-1/p} \left( \overline{z}_{n}(t/2) - 2^{1/p} \overline{z}_{1}(t) \right).$ 

Therefore, taking the first inequality in (3.4) into account, we obtain  $||T_{\lambda}v_n||_X \leq 8n^{-1/p}$ , whence  $||T_{\lambda}v_n||_X \to 0$  as  $n \to \infty$ . Putting  $\bar{v}_n := v_n/||v_n||$ , by (3.5) we see that also  $||T_{\lambda}\bar{v}_n||_X \to 0$ . So,  $\lambda = 2^{1/p}$  is an approximate eigenvalue for T. Now, we prove the converse. We start with some notation. For n = 0, 1, 2, ... and k = 1, 2, ..., we introduce the intervals  $\Delta_k^n = [2^{-n}(k-1), 2^{-n}k)$  and the functions

$$f_k^n := \frac{1}{\varphi_X(2^{-n})} \cdot \chi_{\Delta_k^n}, \quad \text{where} \quad \varphi_X(2^{-n}) = \|\chi_{\Delta_k^n}\|_X.$$

Next, let  $X_n$  be the linear hull of the sequence  $\{f_k^n\}_{k=1}^\infty$ , and let U denote the linear mapping of  $c_{00}$  to  $X_n$  determined by the condition  $Ue_k = f_k^n$  (k = 1, 2, ...). Given a symmetric space X, for  $n \in \mathbb{N}$  we introduce the norm  $||a||_{E_n} := ||Ua||_X$  on  $c_{00}$ . The completion of  $c_{00}$  under this norm is a symmetric sequence space  $E_n$ , and  $\{e_k\}_{k=1}^\infty$  is a normalized symmetric basis in it.

We observe that

$$Tf_k^n = \frac{1}{\varphi_X(2^{-n})} \cdot \left(\chi_{\Delta_{2k-1}^n} + \chi_{\Delta_{2k}^n}\right) = f_{2k-1}^n + f_{2k}^n$$

for all n = 0, 1, 2, ... and k = 1, 2, ... Therefore, whenever  $a = \sum a_k e_k \in c_{00}$ , for every  $b \in c_{00}$  and  $d \in c_{00}$  we have

(3.6)  
$$\|b \oplus a \oplus a \oplus d\|_{E_n} = \|Ub \oplus Ua \oplus Ua \oplus Ud\|_X$$
$$= \|Ub \oplus \sum_{k=1}^{n} a_k(f_{2k-1}^n + f_{2k}^n) \oplus Ud\|_X$$
$$= \|Ub \oplus TUa \oplus Ud\|_X,$$

because  $\{f_k^n\}_{k=1}^{\infty}$  is a symmetric basic sequence in X. Let  $\lambda$  be an approximate eigenvalue of T, and let  $\{g_l\}_{l=1}^{\infty} \subset X$ ,  $||g_l|| = 1$  (l = 1, 2, ...), be the corresponding approximate eigenvector. Since X is separable, there is no loss of generality in assuming that  $g_l \in X_{n_l}$ for some  $n_1 < n_2 < \cdots$  and

$$||Tg_l - \lambda g_l||_X \le \frac{1}{l} \quad (l = 1, 2, \dots).$$

We show that the vector  $U^{-1}g_l \oplus U^{-1}g_l$  is 1/l-replaceable by  $\lambda U^{-1}g_l$  in  $E_{n_l}$ . Indeed, let  $b, d \in c_{00}$ . Choosing due representatives for disjoint sums, by (3.6) and the preceding inequality we obtain

$$(3.7) \qquad \begin{aligned} \| b \oplus U^{-1}g_l \oplus U^{-1}g_l \oplus d\|_{E_{n_l}} - \| b \oplus \lambda U^{-1}g_l \oplus d\|_{E_{n_l}} \| \\ &= \left\| \| Ub \oplus Tg_l \oplus Ud \|_X - \| Ub \oplus \lambda g_l \oplus Ud \|_X \right\| \\ &\leq \| Tg_l - \lambda g_l \|_X \leq \frac{1}{l}. \end{aligned}$$

For every  $l \in \mathbb{N}$ , we introduce a new sequence space  $F_l$ , namely, the completion of  $c_{00}$  with respect to the norm

(3.8) 
$$\left\|\sum_{j=1}^{m} a_j e_j\right\|_{F_l} := \|a_1 U^{-1} g_l \oplus a_2 U^{-1} g_l \oplus \dots \oplus a_m U^{-1} g_l\|_{E_{n_l}} \quad (m \in \mathbb{N}).$$

Clearly, the sequence  $\{e_k\}_{k=1}^{\infty}$  of unit vectors in  $F_l$  is isometric to a sequence of disjoint functions in  $X_{n_l}$  that are equimeasurable with  $g_l$ .

We arrange all vectors in  $c_{00}$  with rational coordinates in a sequence  $(a_1^{(k)}, \ldots, a_{r_k}^{(k)})_{k=1}^{\infty}$ and construct a decreasing family  $(l_i^k)_{i=1}^{\infty}$   $(k = 1, 2, \ldots)$  of infinite sequences of natural numbers such that for every  $k = 1, 2, \ldots$  the limit

$$\lim_{i\to\infty} \left\|\sum_j a_j^{(m)} e_j\right\|_{F_{l_i^k}}$$

exists for all  $1 \leq m \leq k$ . The standard diagonal procedure yields a sequence  $(l_s)_{s=1}^{\infty}$  included in each subsequence  $(l_i^k)_{i=1}^{\infty}$  up to finitely many terms. The usual arguments

based on the density of the rationals in the set of reals show that the limit

$$\lim_{s \to \infty} \left\| \sum_{j} a_{j} e_{j} \right\|_{F_{l_{s}}}$$

exists for arbitrary  $a = (a_j) \in c_{00}$ . Therefore, we can introduce a new norm  $||a||_F$  on  $c_{00}$  which is equal to this limit. We denote by F the completion of  $c_{00}$  in this norm. By the definition and [5, Lemma 11.1.11], it is clear that for every  $\varepsilon > 0$  and  $m \in \mathbb{N}$  there exists  $s_0 \in \mathbb{N}$  such that for all  $s \geq s_0$  and arbitrary  $a_j \in \mathbb{C}$  we have

(3.9) 
$$(1+\varepsilon)^{-1} \left\| \sum_{j=1}^{m} a_j e_j \right\|_{F_{l_s}} \le \left\| \sum_{j=1}^{m} a_j e_j \right\|_F \le (1+\varepsilon) \left\| \sum_{j=1}^{m} a_j e_j \right\|_{F_{l_s}}$$

We prove that the sum  $e_1 + e_2$  is replaceable in F by  $\lambda e_1$ . Indeed, let  $b, d \in c_{00}$  be arbitrary. Relations (3.7) and (3.8) show that

$$\begin{split} \left| \, \| b \oplus e_1 + e_2 \oplus d \|_{F_{l_s}} - \| b \oplus \lambda e_1 \oplus d \|_{F_{l_s}} \, \right| \\ &= \left| \, \| b' \oplus U^{-1} g_l \oplus U^{-1} g_l \oplus d' \|_{E_{n_{l_s}}} - \| b' \oplus \lambda U^{-1} g_l \oplus d' \|_{E_{n_{l_s}}} \, \right| \le \frac{1}{l_s} \end{split}$$

for all  $s \in \mathbb{N}$ . Thus, by (3.9), for every  $\delta > 0$  and all  $s \in \mathbb{N}$  sufficiently large, we have  $|\|b \oplus e_1 + e_2 \oplus d\|_F - \|b \oplus \lambda e_1 \oplus d\|_F|$ 

$$\leq \left| \|b \oplus e_1 + e_2 \oplus d\|_{F_{l_s}} - \|b \oplus \lambda e_1 \oplus d\|_{F_{l_s}} \right| + \delta \left( \|b \oplus e_1 + e_2 \oplus d\|_F + \|b \oplus \lambda e_1 \oplus d\|_F \right)$$
  
 
$$\leq \frac{1}{l_s} + \delta \left( \|b \oplus e_1 + e_2 \oplus d\|_F + \|b \oplus \lambda e_1 \oplus d\|_F \right).$$

Since the right-hand side in the last inequality can be made arbitrarily small, we see that

$$\|u \oplus e_1 + e_2 \oplus v\|_F = \|u \oplus \lambda e_1 \oplus v\|_F.$$

Now, if  $\lambda = 1$ , then  $||e_1 + e_2 + \cdots + e_n||_F = 1$   $(n \in \mathbb{N})$ , and F is isometric to  $c_0$ . Furthermore, (3.6), (3.8), and (3.9) show that for arbitrary  $\varepsilon > 0$  and  $n \in \mathbb{N}$  there exists a collection of mutually disjoint and equimeasurable functions  $w_1, w_2, \ldots, w_n$  in X such that

(3.10) 
$$(1+\varepsilon)^{-1} \left\| \sum_{k=1}^{n} a_k e_k \right\|_F \le \left\| \sum_{k=1}^{n} a_k w_k \right\|_X \le (1+\varepsilon) \left\| \sum_{k=1}^{n} a_k e_k \right\|_F$$

for arbitrary  $a_1, a_2, \ldots, a_n \in \mathbb{C}$ . Consequently,  $p = \infty \in \mathcal{F}(X)$ .

Assume that  $1 < \lambda \leq 2$ , i.e.,  $\lambda := 2^{1/p}$  with  $1 \leq p < \infty$ . Then, by Lemmas 11.3.11 and 11.3.12(ii) in [5], there exists a sequence  $\{b_n\}_{n=1}^{\infty} \subset c_{00}$  with  $||b_n||_F = 1$  (n = 1, 2, ...) such that the vectors  $b_n \oplus b_n$  and  $b_n \oplus b_n \oplus b_n$  are 1/n-replaceable by  $2^{1/p}b_n$  and  $3^{1/p}b_n$ , respectively. Acting in the same way as in the passage from  $E_{n_l}$  to F, we obtain a space G in which  $e_1 + e_2$  and  $e_1 + e_2 + e_3$  will be replacable by  $2^{1/p}e_1$  and  $3^{1/p}e_1$ , respectively. But then G is isometric to  $\ell_p$ , see [5, Lemma 11.3.11]. As before, for every  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , we can find a collection of mutually disjoint equimeasurable functions in X such that a formula similar to (3.10) is valid for the unit vectors  $\{e_k\}_{k=1}^n$  in G and arbitrary  $a_1, a_2, \ldots, a_n \in \mathbb{C}$ . This means that  $p \in \mathcal{F}(X)$ , and the theorem is proved.

A statement similar to the following one (but for the inverse of  $T_{\lambda}$ ) can be found in [16] (Theorem 3.2).

**Proposition 1.** Let X be a symmetric space on  $[0, \infty)$  with Boyd indices  $\alpha_X$  and  $\beta_X$ . Then the operator  $T_{\lambda} = T - \lambda I$  (as before, Tx(t) := x(t/2)) is an isomorphism of X for  $|\lambda| \notin [2^{\alpha_X}, 2^{\beta_X}]$ . *Proof.* Let  $|\lambda| > 2^{\beta_X}$ . Suppose that the equation  $T_{\lambda}x = y$  or, equivalently,

(3.11) 
$$x(t/2) - \lambda x(t) = y(t)$$
  $(t > 0)$   
has a solution  $x = x(t) \in X$  for arbitrary  $y = y(t) \in X$ . Then

$$\lambda^{-1}x(t/2^2) - x(t/2) = \lambda^{-1}y(t/2).$$

Adding the last two identities, we obtain

$$\lambda^{-1}x(t/2^{2}) - \lambda x(t) = y(t) + \lambda^{-1}y(t/2),$$

whence

$$x(t) = -\lambda^{-1} (y(t) + \lambda^{-1} y(t/2)) + \lambda^{-2} x(t/2^2).$$

Proceeding in the same way, we arrive at the relation

(3.12) 
$$x(t) = -\sum_{k=1}^{n} \lambda^{-k} y(2^{-k+1}t) + \lambda^{-n} x(2^{-n}t) \quad (t > 0),$$

which is valid for arbitrary n. Take  $\varepsilon > 0$  with  $2^{\beta_X + \varepsilon} < |\lambda|$ . By the definition of the Boyd indices,

$$\|\sigma_{\tau}\|_{X \to X} \le C\tau^{\beta_X + \varepsilon} \quad (\tau \ge 1);$$

therefore,

$$||x(2^{-n}t)||_X \le C2^{n(\beta_X+\varepsilon)} ||x||_X$$

Consequently,

$$\|\lambda^{-n}x(2^{-n}t)\|_X \le C\left(|\lambda|^{-1}2^{\beta_X+\varepsilon}\right)^n \|x\|_X \to 0 \text{ as } n \to \infty,$$

and, by (3.12),

(3.13) 
$$x(t) = -\sum_{k=1}^{\infty} \lambda^{-k} y(2^{-k+1}t) \quad (t > 0).$$

Thus, if a solution  $x \in X$  of equation (3.11) exists, it must have the form (3.13) (the series on the right in (3.13) converges absolutely because  $|\lambda| > 2^{\beta_X}$ ). On the other hand, it is straightforward that the function (3.13) is a solution of equation (3.11). To summarize, this equation has a unique solution  $x \in X$  for an arbitrary  $y \in X$  on the right. Thus, the operator  $T_{\lambda} : X \to X$  is an isomorphism.

The case where  $|\lambda| < 2^{\alpha_X}$  is treated similarly.

We present a consequence of the above statement and Theorem 5.

**Corollary 1.** The spectrum  $\sigma(T)$  of T lies inside the annulus  $\{\lambda \in \mathbb{C} : |\lambda| \in [2^{\alpha_X}, 2^{\beta_X}]\}$ , and the set  $\mathcal{F}(X)$  lies inside the interval  $[1/\beta_X, 1/\alpha_X]$ .

We show that all boundary points of this annulus are approximate eigenvalues for T.

**Theorem 6.** Let  $\alpha_X$  and  $\beta_X$  be the Boyd indices of a symmetric space X. Then any  $\lambda \in \mathbb{C}$  with  $|\lambda| = 2^{\beta_X}$  or  $|\lambda| = 2^{\alpha_X}$  is an approximate eigenvalue for T. In particular, the spectral radius r(T) is  $2^{\beta_X}$ .

*Proof.* First, let  $\lambda = 2^{\beta_X}$ . The definition of the Boyd indices (see [14, §§2.1.1 and 2.4.3]) shows that

$$\beta_X = \inf_{\tau \ge 1} \frac{\ln \|\sigma_\tau\|_{X \to X}}{\ln \tau}$$

Therefore,

(3.14) 
$$||T^n||_{X \to X} = ||\sigma_{2^n}||_{X \to X} \ge 2^{n\beta_X} \quad (n \in \mathbb{N}),$$

and we can argue as in the proof of Theorem 11.3.12 in [5]. We give the details for completeness.

By (3.14) we have

$$\lim_{n \to \infty} \|(n+1)2^{-n\beta_X}T^n\|_{X \to X} = \infty,$$

and the uniform boundedness principle implies the existence of  $f_0 \in X$ ,  $||f_0||_X = 1$ , such that

(3.15) 
$$\limsup_{n \to \infty} \| (n+1)2^{-n\beta_X} T^n f_0 \|_X = \infty.$$

Clearly, the function  $f_0$  may be assumed to be nonnegative.

By Corollary 1, the operator T - rI is invertible in X if  $r > 2^{\beta_X}$ . Consequently,  $(T - rI)^{-2}$  can be represented as follows (the series converges):

$$(T - rI)^{-2} = \frac{1}{r^2} \sum_{n=0}^{\infty} (n+1)r^{-n}T^n.$$

Since  $f_0 \ge 0$  and  $T \ge 0$ , we conclude that

$$||(T - rI)^{-2} f_0||_X \ge r^{-2} ||(n+1)r^{-n}T^n f_0||_X$$

for every  $r > 2^{\beta_X}$  and every  $n \in \mathbb{N}$ . By (3.15),

$$\lim_{x \to 2^{\beta_X}} \| (T - rI)^{-2} f_0 \|_X = \infty.$$

Therefore, there exists a sequence  $\{r_n\}$  with  $r_n \to 2^{\beta_X}$  such that either

$$\lim_{n \to \infty} \|(T - r_n I)^{-1} f_0\|_X = \infty,$$

or

$$\lim_{n \to \infty} \frac{\|(T - r_n I)^{-2} f_0\|_X}{\|(T - r_n I)^{-1} f_0\|_X} = \infty.$$

In either case, it is easy to find a sequence  $\{g_n\}_{n=1}^{\infty} \subset X$ ,  $\|g_n\|_X = 1$ , such that

$$\lim_{n \to \infty} \|(T - r_n I)g_n\|_X = 0$$

Surely, this implies that  $2^{\beta_X}$  is an approximate eigenvalue for T.

If  $\lambda = 2^{\alpha_X}$ , then, again by [14], we have

$$\alpha_X = \sup_{0 < \tau \le 1} \frac{\ln \|\sigma_\tau\|_{X \to X}}{\ln \tau},$$

implying that the operator  $T^{-1}f(t) = f(2t)$  satisfies

$$||(T^{-1})^n||_{X \to X} = ||\sigma_{2^{-n}}||_{X \to X} \ge 2^{-n\alpha_X} \quad (n \in \mathbb{N}).$$

Arguing as in the preceding case, we deduce that  $2^{-\alpha_X}$  is an approximate eigenvalue for  $T^{-1} = \sigma_{1/2}$ , i.e.,

$$\lim_{n \to \infty} \| (T^{-1} - 2^{-\alpha_X} I) h_n \|_X = 0$$

for some sequence  ${h_n}_{n=1}^{\infty} \subset X$  with  $||h_n||_X = 1$ . Since  $1/2 \leq ||T^{-1}h_n||_X \leq 1$  for all  $n \in \mathbb{N}$  (see §2), and for  $g_n := T^{-1}h_n/||T^{-1}h_n||$  we have

$$\|(T - 2^{\alpha_X}I)g_n\|_X = \frac{2^{\alpha_X}}{\|T^{-1}h_n\|} \|(T^{-1} - 2^{-\alpha_X}I)h_n\|_X,$$

we see that  $2^{\alpha_X}$  is an approximate eigenvalue for T.

Now, let  $\lambda \in \mathbb{C}$ , and let, for instance,  $|\lambda| = 2^{\beta_X}$ . Then  $\lambda = 2^{\beta_X} e^{i\theta}$  for some  $0 \leq \theta \leq 2\pi$ . If  $\{g_n\}_{n=1}^{\infty} \subset X$  is an approximate eigenvector corresponding to the approximate eigenvalue  $2^{\beta_X}$ , then it can easily be checked that the functions  $f_n(t) := t^{-i\theta \log_2 e} g_n(t)$   $(n \in \mathbb{N})$  satisfy the formula

$$(T - \lambda I)f_n = e^{i\theta}(T - 2^{\beta_X}I)g_n.$$

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Since  $||f_n||_X = ||g_n||_X = 1$ , we see that  $\lambda$  is an approximate eigenvalue for T. The case where  $|\lambda| = 2^{\alpha_X}$  is treated similarly.

Since  $2^{\beta_X}$  is an approximate eigenvalue for T, we have  $2^{\beta_X} \in \sigma(T)$ , and the second statement of the theorem follows from Corollary 1 and the definition of the spectral radius.

We show that, in the case of symmetric spaces on the semiaxis, Theorem 4 is an immediate consequence of the results obtained (we recall that Theorem 4 was stated in [7, theorem 2.b.6] without proof).

Proof of Theorem 4. If X is separable, the claim follows from Theorems 5, 6 and Corollary 1. Otherwise, X is maximal. It is easily seen that then  $\|\sigma_{\tau}\|_{X \to X} = \|\sigma_{\tau}\|_{X_0 \to X_0}$ for every  $\tau > 0$ , where  $X_0$  is the separable part of X (see § 2). Thus,  $\alpha_X = \alpha_{X_0}$  and  $\beta_X = \beta_{X_0}$ . If  $X \neq L_1 \cap L_\infty$ , then  $X_0$  is separable and, consequently, max  $\mathcal{F}(X_0) = 1/\alpha_X$ and min  $\mathcal{F}(X_0) = 1/\beta_X$ . Since  $X_0$  is a subspace of X, this implies the claim by Corollary 1. If  $X = L_1 \cap L_\infty$ , the result is obvious.

Theorems 5 and 6 allow us to completely describe the sets  $\mathcal{F}(X)$  if X is a Lorentz space. In doing this, we shall crucially need the results of [13], so first we summarize them.

# §4. The closedness of the operator $S_{\lambda}$ in a weighted $\ell_q$ -space

For a numerical sequence  $\mu = (\mu_k)_{k=-\infty}^{\infty}$  satisfying the conditions

(4.1) 
$$0 < \mu_k \le \mu_{k+1} \le 2\mu_k \quad (k = 0, \pm 1, \pm 2, \dots),$$

we introduce the weighted space  $\ell_q(\mu)$  with the norm

$$\|(a_k)\|_{\ell_q(\mu)} := \left(\sum_{k=-\infty}^{\infty} |a_k|^q \mu_k^q\right)^{1/q} \quad (1 \le q < \infty).$$

Next, for every  $\lambda \in \mathbb{C}$  we put  $S_{\lambda} = S - \lambda I$ , where  $S(a_k) := (a_{k-1})$  is the shift operator and I is the identity. The operator  $S_{\lambda}$  is linear and bounded on  $\ell_q(\mu)$ .

Basically, the paper [13] is devoted to the real interpolation method for subcouples of codimension 1 generated by a linear functional bounded on the intersection of the spaces of the initial couple. As an application, the following problem was resolved completely in [13]: if the weight sequence  $\mu = (\mu_k)_{k=-\infty}^{\infty}$  satisfies (4.1), determine when the range of  $S_{\lambda}$  is closed in  $\ell_q(\mu)$ . (In what follows, we say that an operator is closed if its image is closed.) To state the result, we need some definitions and notation.

Let  $\psi(t)$  be a quasiconcave function on  $(0,\infty)$ . We introduce three dilation functions:

$$M(t) = \sup_{s>0} \frac{\psi(ts)}{\psi(s)}, \quad M_0(t) = \sup_{0 < s \le \min(1,1/t)} \frac{\psi(ts)}{\psi(s)}, \quad M_\infty(t) = \sup_{s \ge \max(1,1/t)} \frac{\psi(ts)}{\psi(s)}.$$

They are submultiplicative on  $(0, \infty)$  and, consequently, we can introduce the following six numbers:

$$\alpha = \lim_{t \to 0} \frac{\log_2 M(t)}{\log_2 t}, \qquad \alpha_0 = \lim_{t \to 0} \frac{\log_2 M_0(t)}{\log_2 t}, \qquad \alpha_\infty = \lim_{t \to 0} \frac{\log_2 M_\infty(t)}{\log_2 t},$$
$$\beta = \lim_{t \to \infty} \frac{\log_2 M(t)}{\log_2 t}, \qquad \beta_0 = \lim_{t \to \infty} \frac{\log_2 M_0(t)}{\log_2 t}, \qquad \beta_\infty = \lim_{t \to \infty} \frac{\log_2 M_\infty(t)}{\log_2 t},$$

which are called the *dilation indices* for  $\psi$  (observe that  $\alpha$  and  $\beta$  coincide with  $\alpha_0$  and  $\beta_0$  defined in §2). It is easily seen that  $0 \leq \alpha \leq \alpha_0 \leq \beta_0 \leq \beta \leq 1$  and  $0 \leq \alpha \leq \alpha_\infty \leq \beta_\infty \leq \beta \leq 1$ . Moreover,  $\alpha = \min(\alpha_0, \alpha_\infty)$  and  $\beta = \max(\beta_0, \beta_\infty)$ , see [13, Lemma 1]. Let

 $\mu_k := \psi(2^k) \ (k = 0, \pm 1, \pm 2, ...)$ . Since  $\psi$  is quasiconcave, the numbers  $\mu_k$  satisfy (4.1). Next, the above dilation indices can also be calculated by the following formulas:

$$\begin{aligned} \alpha &= -\lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \in \mathbb{Z}} \frac{\mu_k}{\mu_{n+k}}, \qquad \beta &= \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \in \mathbb{Z}} \frac{\mu_k}{\mu_{k-n}}, \\ \alpha_0 &= -\lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \le 0} \frac{\mu_{k-n}}{\mu_k}, \qquad \beta_0 &= \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \le 0} \frac{\mu_k}{\mu_{k-n}}, \\ \alpha_\infty &= -\lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \ge 0} \frac{\mu_k}{\mu_{n+k}}, \qquad \beta_\infty &= \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \ge 0} \frac{\mu_{k+n}}{\mu_k}. \end{aligned}$$

Therefore, applying [13, Theorem 5 and Proposition 2], we obtain the following statement (we formulate it in a slightly more general but equivalent form).

**Theorem 7.** Suppose  $\lambda \in \mathbb{C}$  and  $1 \leq q < \infty$ . Then the operator  $S_{\lambda}$  is closed in  $\ell_q(\mu)$  if and only if

$$|\lambda| \in [0, 2^{\alpha}) \cup (2^{\beta_0}, 2^{\alpha_{\infty}}) \cup (2^{\beta_{\infty}}, 2^{\alpha_0}) \cup (2^{\beta}, \infty)$$

Moreover, if  $|\lambda| \in [0, 2^{\alpha}) \cup (2^{\beta_{\infty}}, 2^{\alpha_0}) \cup (2^{\beta}, \infty)$ , then  $\operatorname{Im} S_{\lambda} = \ell_q(\mu)$ ; if  $|\lambda| \in (2^{\beta_0}, 2^{\alpha_{\infty}})$ , then  $\operatorname{Im} S_{\lambda}$  is a closed subspace of codimension 1 in  $\ell_q(\mu)$ . The operator  $S_{\lambda}$  is invertible if and only if  $|\lambda| \in [0, 2^{\alpha}) \cup (2^{\beta}, \infty)$ . If  $|\lambda| \in (2^{\beta_0}, 2^{\alpha_{\infty}})$ , this operator is injective, but if  $|\lambda| \in (2^{\beta_{\infty}}, 2^{\alpha_0})$ , it is not.

It should be noted that a weaker result, involving only four indices, was proved in [17].

In the next section, we shall apply Theorem 7 in order to deduce a similar result for the dilation operator defined on a Lorentz space.

# §5. Description of the set $\mathcal{F}(X)$ for Lorentz spaces

Let  $1 \leq q < \infty$ ,  $\psi$  a positive function on  $(0, \infty)$  satisfying conditions (a) and (b) in §2, and  $\Lambda_q(\psi)$  the Lorentz space whose norm in defined by (2.1). As has been mentioned, this is a separable s.s. whose Boyd indices coincide with the corresponding dilation indices of  $\psi$ , i.e.,  $\alpha_{\Lambda_q(\psi)} = \gamma_{\psi}$ ,  $\beta_{\Lambda_q(\psi)} = \delta_{\psi}$ . Moreover, by condition (a), we may assume that  $\psi$ is quasiconcave.

We put  $\Delta_k := [2^k, 2^{k+1})$   $(k = 0, \pm 1, \pm 2, ...)$  and for an arbitrary sequence  $a = (a_k)_{k=1}^{\infty}$  of complex numbers introduce the function

$$h_a(t) := \sum_{k=-\infty}^{\infty} a_k \chi_{\Delta_k}(t).$$

By [18, Proposition 5.1(2)], we have

$$\|h_a\|_{\Lambda_q(\psi)} \asymp \left(\sum_{k=-\infty}^{\infty} |a_k|^q \int_{\Delta_k} \psi(t)^q \frac{dt}{t}\right)^{1/q}$$

with constants independent of  $(a_k)$ . Since  $\psi$  is quasiconcave, we obtain

$$\int_{\Delta_k} \psi(t)^q \, \frac{dt}{t} \asymp \psi(2^k)^q \quad (k = 0, \pm 1, \pm 2, \dots),$$

whence

$$\|h_a\|_{\Lambda_q(\psi)} \asymp \left(\sum_{k=-\infty}^{\infty} |a_k|^q \psi(2^k)^q\right)^{1/q}$$

with constants depending only on  $\psi$  and q. Thus, putting  $\mu_k = \psi(2^k)$   $(k = 0, \pm 1, \pm 2, ...)$ , in the notation of the preceding section we obtain

(5.1) 
$$\|h_a\|_{\Lambda_q(\psi)} \asymp \|(a_k)\|_{\ell_q(\mu)},$$

i.e.,  $\ell_q(\mu)$  is isomorphic to the subspace  $[\chi_{\Delta_k}]$  spanned in  $\Lambda_q(\psi)$  by the system of the characteristic functions of dyadic intervals.

As before, let Tx(t) = x(t/2) and  $T_{\lambda} := T - \lambda I$  ( $\lambda \in \mathbb{C}$ ), where I is the identity operator on  $\Lambda_q(\psi)$ . We show that  $T_{\lambda}$  and  $S_{\lambda} = S - \lambda I$  (see §4) are related in a simple way. First, for an arbitrary sequence  $(a_k)$  we have

(5.2)  

$$T_{\lambda}h_{a}(t) = \sum_{k=-\infty}^{\infty} a_{k}\chi_{\Delta_{k}}(t/2) - \lambda \sum_{k=-\infty}^{\infty} a_{k}\chi_{\Delta_{k}}(t)$$

$$= \sum_{k=-\infty}^{\infty} a_{k}\chi_{\Delta_{k+1}}(t) - \lambda \sum_{k=-\infty}^{\infty} a_{k}\chi_{\Delta_{k}}(t)$$

$$= \sum_{k=-\infty}^{\infty} (S_{\lambda}a)_{k}\chi_{\Delta_{k}}(t).$$

In particular, by (5.1) it follows that

(5.3) 
$$||T_{\lambda}h_a||_{\Lambda_q(\psi)} \asymp ||S_{\lambda}a||_{\ell_q(\mu)}$$

In the sequel, we shall also need the following statement.

**Proposition 2.** For arbitrary  $\lambda \in \mathbb{C}$ , the following assertions are true:

- (i)  $T_{\lambda}$  is injective if and only if  $S_{\lambda}$  is injective;
- (ii) if  $T_{\lambda}$  is closed, then  $S_{\lambda}$  is closed;
- (iii) if  $S_{\lambda}$  is injective and closed, then  $T_{\lambda}$  is closed.

*Proof.* First, we verify that it suffices to prove the proposition for  $\lambda \geq 0$ . Indeed, let  $\lambda = |\lambda| \cdot e^{i\theta}$ , where  $\theta \in [0, 2\pi]$ . For every  $x \in \Lambda_q(\psi)$ , we introduce the function  $y(t) = t^{-i\theta \log_2 e} \cdot x(t)$ . Then  $y \in \Lambda_q(\psi)$ ,  $\|y\|_{\Lambda_q(\psi)} = \|x\|_{\Lambda_q(\psi)}$ , and

$$T_{\lambda}y(t) = t^{-i\theta \log_2 e} 2^{i\theta \log_2 e} x(t/2) - t^{-i\theta \log_2 e} \lambda x(t) = t^{-i\theta \log_2 e} e^{i\theta} T_{|\lambda|} x(t).$$

Consequently,  $||T_{\lambda}y||_{\Lambda_q(\psi)} = ||T_{|\lambda|}x||_{\Lambda_q(\psi)}$ , and we see that  $T_{\lambda}$  is injective (closed) if and only if  $T_{|\lambda|}$  is injective (closed).

A similar statement is true for  $S_{\lambda}$  and  $S_{|\lambda|}$ . In this case, if  $a = (a_k) \in \ell_q(\mu)$  and  $b = (b_k), b_k := a_k e^{-i\theta k}$ , again we have  $b \in \ell_q(\mu)$  and  $\|b\|_{\ell_q(\mu)} = \|a\|_{\ell_q(\mu)}$ . Furthermore,

$$(S_{\lambda}b)_k = a_{k-1}e^{-i\theta k}e^{i\theta} - \lambda a_k e^{-i\theta k} = e^{-i\theta(k-1)}(S_{|\lambda|}a)_k,$$

whence  $||S_{\lambda}b||_{\ell_q(\mu)} = ||S_{|\lambda|}a||_{\ell_q(\mu)}$ , and the claim follows.

(i) The fact that the injectivity of  $T_{\lambda}$  implies the injectivity of  $S_{\lambda}$  is a direct consequence of (5.3). Since T and S are injective, it suffices to prove the converse for  $\lambda > 0$ .

Suppose  $x \in \Lambda_q(\psi)$ ,  $x \neq 0$ , and  $T_{\lambda}x = 0$ . Since  $|T_{\lambda}x| \geq ||x(t/2)| - \lambda |x(t)||$ , we may assume that x(t) is nonnegative. Outside a set of zero measure, we have

(5.4) 
$$x(t/2) = \lambda \cdot x(t)$$

for t > 0, whence it follows that

$$\int_{\Delta_k} x(t) dt = \frac{1}{\lambda} \int_{\Delta_k} x(t/2) dt = \frac{2}{\lambda} \int_{\Delta_{k-1}} x(t) dt$$

$$\int_{\Delta_k} x(t) dt = \left(\frac{2}{\lambda}\right)^k \int_{\Delta_0} x(t) dt \quad (k = 0, \pm 1, \pm 2, \dots).$$

Since the space  $\Lambda_q(\psi)$  is separable, it is an interpolation space with respect to the couple  $(L_1, L_\infty)$  (see [14, the corollary to Theorem 2.4.10]. Therefore, the averaging operator

(5.5) 
$$Qy(t) := \sum_{k=-\infty}^{\infty} 2^{-k} \int_{\Delta_k} y(s) \, ds \cdot \chi_{\Delta_k}(t)$$

is bounded on  $\Lambda_q(\psi)$  (see [14, §2.3.2]). Hence, by (5.1), the sequence

$$a(x) := \left(2^{-k} \int_{\Delta_k} x(s) \, ds\right)_{k=-\infty}^{\infty}$$

belongs to  $\ell_q(\mu)$ . Thus, by the preceding formula, the sequence  $a = (a_k)$  with  $a_k := \lambda^{-k} \int_{\Delta_0} x(s) ds$  also belongs to this space. At the same time, it is easily seen that  $S_{\lambda}a = 0$ . Since  $x \ge 0$  and  $x \ne 0$ , by (5.4) we deduce that  $a \ne 0$ . Therefore,  $S_{\lambda}$  is not injective, and statement (i) is proved.

(ii) If  $a^n = (a_k^n)_{k=-\infty}^{\infty} \in \ell_q(\mu)$  (n = 1, 2, ...) and  $S_{\lambda}a^n \to b = (b_k)$  in  $\ell_q(\mu)$ , then, by (5.3), the functions  $\{T_{\lambda}h_{a^n}\}$  form a Cauchy sequence in  $\Lambda_q(\psi)$ . By assumption,  $T_{\lambda}h_{a^n} \to y := T_{\lambda}x$ , where  $x \in \Lambda_q(\psi)$ . Formula (5.2) shows that  $y = h_b$ .

Next, arguing in the same way as we did to deduce (5.2), we obtain

(5.6) 
$$x(t) = T_{\lambda}^{-1} h_b(t) = \sum_{k=-\infty}^{\infty} \left( S_{\lambda}^{-1} b \right)_k \chi_{\Delta_k}(t),$$

where  $S_{\lambda}^{-1}b = (b_{k+1} - \lambda^{-1}b_k)_k$ . Since  $x \in \Lambda_q(\psi)$ , relation (5.1) shows that  $c := (S_{\lambda}^{-1}b)_k \in \ell_q(\mu)$ . Thus,  $b = S_{\lambda}c$ , i.e.,  $b \in \text{Im } S_{\lambda}$ . Consequently,  $S_{\lambda}$  is a closed operator.

(iii) Suppose that  $T_{\lambda}$  is not closed. Then there exists a sequence  $\{x_n\} \subset \Lambda_q(\psi)$  with the following properties:

(5.7) 
$$||x_n||_{\Lambda_q(\psi)} = 1 \quad (n = 1, 2, ...) \text{ and } ||T_\lambda x_n||_{\Lambda_q(\psi)} \to 0.$$

Since  $\Lambda_q(\psi)$  is an interpolation space with respect to the couple  $(L_1, L_\infty)$ , by [14, Lemma 2.4.6] we obtain

$$||T_{\lambda}x_n|| \ge ||x_n^*(t/2) - \lambda x_n^*(t)|| = ||T_{\lambda}x_n^*||.$$

Consequently, we may assume that every function  $x_n$  satisfies (5.7) and is nonnegative and monotone nonincreasing. Next, if Q is the averaging operator defined by (5.5), then for every  $x \in \Lambda_q(\psi)$  we have

$$QT_{\lambda}x = \sum_{k=-\infty}^{\infty} 2^{-k} \int_{\Delta_{k}} T_{\lambda}x(s) \, ds \cdot \chi_{\Delta_{k}}$$
$$= \sum_{k=-\infty}^{\infty} 2^{-k} \left( \int_{\Delta_{k}} x(s/2) \, ds - \lambda \int_{\Delta_{k}} x(s) \, ds \right) \cdot \chi_{\Delta_{k}}$$
$$= \sum_{k=-\infty}^{\infty} 2^{-k} \left( 2 \int_{\Delta_{k-1}} x(s) \, ds - \lambda \int_{\Delta_{k}} x(s) \, ds \right) \cdot \chi_{\Delta_{k}}$$
$$= \sum_{k=-\infty}^{\infty} \left( a_{k-1}(x) - \lambda a_{k}(x) \right) \cdot \chi_{\Delta_{k}} = \sum_{k=-\infty}^{\infty} \left( S_{\lambda}a(x) \right)_{k} \cdot \chi_{\Delta_{k}},$$

where, as before,  $a(x) = \left(2^{-k} \int_{\Delta_k} x(s) \, ds\right)_{k=-\infty}^{\infty}$ . Since

$$\|Qx\|_{\Lambda_q(\psi)} \le C \|x\|_{\Lambda_q(\psi)}$$

for some C > 0, from (5.1) and (5.7) it follows that  $||S_{\lambda}a(x_n)||_{\ell_q(\mu)} \to 0$  as  $n \to \infty$ . Furthermore, the same relation and the monotonicity of  $x_n$  imply the inequalities

$$\|a(x_n)\|_{\ell_q(\mu)} \ge c \left\| \sum_{k=-\infty}^{\infty} 2^{-k} \int_{\Delta_k} x_n(s) \, ds \cdot \chi_{\Delta_k} \right\|_{\Lambda_q(\psi)}$$
$$\ge c \left\| \sum_{k=-\infty}^{\infty} x_n(2^{k+1}) \cdot \chi_{\Delta_k} \right\|_{\Lambda_q(\psi)} \ge c \|\sigma_{1/2}x_n\| \ge \frac{c}{2} \|x_n\| = \frac{c}{2}.$$

Since, by assumption,  $S_{\lambda}$  is injective, we see that  $S_{\lambda}$  is not closed, which contradicts the assumptions.

The next statement is a direct consequence of Theorem 7 and Proposition 2.

**Corollary 2.** Suppose  $\lambda \in \mathbb{C}$  and  $1 \leq q < \infty$ . Then  $T_{\lambda}$  is injective and closed on  $\Lambda_q(\psi)$  if and only if

$$|\lambda| \in [0, 2^{\alpha}) \cup (2^{\beta_0}, 2^{\alpha_{\infty}}) \cup (2^{\beta}, \infty).$$

Moreover, it is invertible on  $\Lambda_q(\psi)$  if and only if  $|\lambda| \in [0, 2^{\alpha}) \cup (2^{\beta}, \infty)$ .

Now we are in a position to completely describe the set  $\mathcal{F}(X)$  in the case where X is a Lorentz space  $\Lambda_q(\psi)$ .

**Theorem 8.** Let  $1 \leq q < \infty$ , and let  $\psi$  be a positive function on  $(0,\infty)$  satisfying conditions (a) and (b) in §2. If  $\alpha_{\infty} \leq \beta_0$ , then  $\mathcal{F}(\Lambda_q(\psi)) = [1/\beta, 1/\alpha]$ ; if  $\alpha_{\infty} > \beta_0$ , then  $\mathcal{F}(\Lambda_q(\psi)) = [1/\beta, 1/\alpha]$ ; if  $\alpha_{\infty} > \beta_0$ , then  $\mathcal{F}(\Lambda_q(\psi)) = [1/\beta, 1/\alpha]$ ; if  $\alpha_{\infty} > \beta_0$ , then  $\mathcal{F}(\Lambda_q(\psi)) = [1/\beta, 1/\alpha]$ .

*Proof.* First, if  $p \notin [1/\beta, 1/\alpha]$ , then  $\lambda := 2^{1/p} \notin [2^{\alpha}, 2^{\beta}]$ . By Proposition 1,  $T_{\lambda}$  is invertible in this case, so  $\lambda$  is not an approximate eigenvalue of T. Consequently, by Theorem 5,  $p \notin \mathcal{F}(\Lambda_q(\psi))$ , and  $\mathcal{F}(\Lambda_q(\psi)) \subset [1/\beta, 1/\alpha]$ .

Suppose that  $\alpha_{\infty} \leq \beta_0$  and  $p \in [1/\beta, 1/\alpha]$ . Then  $\lambda = 2^{1/p} \in [2^{\alpha}, 2^{\beta}]$ , and, by Corollary 2,  $T_{\lambda}$  is either noninjective or nonclosed. In both cases,  $\lambda$  is an approximate eigenvalue of T. Thus,  $p \in \mathcal{F}(\Lambda_q(\psi))$  by Theorem 5. So,  $[1/\beta, 1/\alpha] \subset \mathcal{F}(\Lambda_q(\psi))$  in the case in question.

Now, suppose that  $\alpha_{\infty} > \beta_0$  and  $p \in (1/\alpha_{\infty}, 1/\beta_0)$ . Then  $\lambda = 2^{1/p} \in (2^{\beta_0}, 2^{\alpha_{\infty}})$  and, again by Corollary 2,  $T_{\lambda}$  is closed and injective. Therefore, there exist c > 0 with

$$||T_{\lambda}x||_{\Lambda_q(\psi)} \ge c||x||_{\Lambda_q(\psi)} \quad (x \in \Lambda_q(\psi)).$$

Consequently,  $\lambda$  is not an approximate eigenvalue for T, and  $p \notin \mathcal{F}(\Lambda_q(\psi))$  by Theorem 5.

Remark 1. It can easily be shown (see [13]) that for arbitrary four numbers a, b, c, and d with  $0 < a \leq \min(b, c) \leq \max(b, c) \leq d < 1$  there exists a function  $\psi$  quasiconcave on  $(0, \infty)$  and such that  $\alpha(\psi) = a$ ,  $\beta_0(\psi) = b$ ,  $\alpha_\infty(\psi) = c$ , and  $\beta(\psi) = d$ . Thus, by Theorem 8, for every  $1 \leq q < \infty$  we have  $\mathcal{F}(\Lambda_q(\psi)) = [1/d, 1/a]$  if  $c \leq b$  and  $\mathcal{F}(\Lambda_q(\psi)) = [1/d, 1/c] \cup [1/b, 1/a]$  if c > b.

# §6. CONCLUDING EXAMPLES AND REMARKS

**Example 1** (see also [9]). For arbitrary  $1 , consider the quasiconcave functions <math>\psi_1(t) = \max(t^{1/p}, t^{1/r})$  and  $\psi_2(t) = \min(t^{1/p}, t^{1/r})$ . It is easy to verify that  $M(\psi_i) = \psi_i$   $(i = 1, 2), M_0(\psi_1)(t) = M_{\infty}(\psi_2)(t) = t^{1/r}$ , and  $M_{\infty}(\psi_1)(t) = M_0(\psi_2)(t) = t^{1/p}$ . Therefore,  $\alpha(\psi_i) = 1/r, \beta(\psi_i) = 1/p$   $(i = 1, 2), \alpha_0(\psi_1) = \beta_0(\psi_1) = \alpha_{\infty}(\psi_2) = \beta_{\infty}(\psi_2) = 1/r$ , and  $\alpha_{\infty}(\psi_1) = \beta_{\infty}(\psi_1) = \alpha_0(\psi_2) = \beta_0(\psi_2) = 1/p$ . Thus, by Theorem 8,  $\mathcal{F}(\Lambda_q(\psi_1)) = \{p, r\}$  and  $\mathcal{F}(\Lambda_q(\psi_2)) = [p, r]$  for every  $1 \le q < \infty$ . Observe that

$$\Lambda_q(\psi_1) = \Lambda_q(t^{1/p}) \cap \Lambda_q(t^{1/r}) \quad \text{and} \quad \Lambda_q(\psi_2) = \Lambda_q(t^{1/p}) + \Lambda_q(t^{1/r}).$$

**Example 2.** Now, let  $1 \le p < r \le \infty$ , and let  $X = L_p(0, \infty) \cap L_r(0, \infty)$ ,  $Y = L_p(0, \infty) + L_r(0, \infty)$  with the usual norms:

$$\|f\|_X := \max(\|f\|_{L_p}, \|f\|_{L_r}),$$
  
$$\|f\|_Y = \inf\{\|g\|_{L_p} + \|h\|_{L_r} : f = g + h, g \in L_p, h \in L_r\}.$$

Then  $\psi_1(t) = \max(t^{1/p}, t^{1/r})$  and  $\psi_2(t) = \min(t^{1/p}, t^{1/r})$  are the fundamental functions for X and Y, respectively; so, as in the preceding example, we have  $\alpha_X = \alpha_Y = 1/r$  and  $\beta_X = \beta_Y = 1/p$ . By Proposition 1,  $T_{\lambda} = T - \lambda I$  is an isomorphism in  $L_s(0, \infty)$  provided that  $\lambda \neq 2^{1/s}$ . Consequently, for every  $\lambda$  different from  $2^{1/p}$  and  $2^{1/r}$  there exists c > 0such that

$$||T_{\lambda}f||_{L_p} \ge c||f||_{L_p}$$
  $(f \in L_p)$  and  $||T_{\lambda}f||_{L_r} \ge c||f||_{L_r}$   $(f \in L_r).$ 

Thus,  $||T_{\lambda}f||_X \ge c||f||_X$  and we see that no such  $\lambda$  is an approximate eigenvalue for T on X. By Theorem 4 and 5,  $\mathcal{F}(X) = \{p, r\}$ .

In order to find  $\mathcal{F}(Y)$ , we show that  $T_{\lambda}$  is not injective on Y for  $2^{1/r} < \lambda < 2^{1/p}$ . Indeed, let  $f_0$  be an arbitrary positive function belonging to  $L_{\infty}[1,2]$ . It can easily be verified that  $T_{\lambda}f = 0$  if

$$f(t) = \sum_{n=-\infty}^{\infty} \lambda^{-n} f_0(2^{-n}t) \chi_{[2^n, 2^{n+1})}(t) \quad (t > 0).$$

Next, by the definition of the norm on a sum of spaces, we have

$$\begin{split} \|f\|_{Y} &\leq \left\|\sum_{n=0}^{\infty} \lambda^{-n} f_{0}(2^{-n}t) \chi_{[2^{n}.2^{n+1})}(t)\right\|_{L_{p}} + \left\|\sum_{n=1}^{\infty} \lambda^{-n} f_{0}(2^{-n}t) \chi_{[2^{n}.2^{n+1})}(t)\right\|_{L_{r}} \\ &= \left(\sum_{n=0}^{-\infty} \lambda^{-np} \int_{2^{n}}^{2^{n+1}} |f_{0}(2^{-n}t)|^{p} dt\right)^{1/p} + \left(\sum_{n=1}^{\infty} \lambda^{-nr} \int_{2^{n}}^{2^{n+1}} |f_{0}(2^{-n}t)|^{r} dt\right)^{1/r} \\ &\leq \left\{\left(\sum_{n=0}^{-\infty} \lambda^{-np} 2^{n}\right)^{1/p} + \left(\sum_{n=1}^{\infty} \lambda^{-nr} 2^{n}\right)^{1/r}\right\} \|f_{0}\|_{L_{\infty}[1,2]} \end{split}$$

(we consider the case where  $r < \infty$ ; for  $r = \infty$  the arguments are similar). Since  $2^{1/r} < \lambda < 2^{1/p}$ , the two series on the right in the last inequality converge. Consequently,  $f \in Y$  and  $T_{\lambda}$  is not injective on Y. Thus, an arbitrary  $\lambda \in (2^{1/r}, 2^{1/p})$  is an eigenvalue of T. Applying Theorems 5 and 4, we deduce that  $\mathcal{F}(Y) = [p, r]$ .

Remark 2. If we denote by  $\alpha$ ,  $\alpha_0$ ,  $\alpha_\infty$ ,  $\beta$ ,  $\beta_0$ ,  $\beta_\infty$  the dilation indices of the fundamental function of a symmetric space, then, formally, the results of the preceding example fit into the pattern of Theorem 8 (though it is not applicable because X and Y are not Lorentz spaces). It seems quite natural to conjecture that the domain of applicability of Theorem 8 is wider than the class of Lorentz spaces. In particular, it would be interesting to prove a similar result for Orlicz spaces on the semiaxis (the spaces of Example 2 belong to this class). Note that for the Orlicz spaces  $X = L_M[0, 1]$  on an interval we always have  $\mathcal{F}(X) = [1/\beta_X, 1/\alpha_X]$  (see [19, Theorem 4.a.9] and [7, remark on pp. 140–141]).

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