MONODROMY ZETA-FUNCTION OF A POLYNOMIAL ON A COMPLETE INTERSECTION, AND NEWTON POLYHEDRA

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Abstract. For a generic (polynomial) one-parameter deformation of a complete intersection, its monodromy zeta-function is defined. Explicit formulas for this zeta-function in terms of the corresponding Newton polyhedra are obtained in the case where the deformation is nondegenerate with respect to its Newton polyhedra. This result is employed to obtain a formula for the monodromy zeta-function at the origin of a polynomial on a complete intersection, which is an analog of the Libgober–Sperber theorem.

§1. Introduction

Let \( F_0, F_1, \ldots, F_k \) be a set of functions on \( \mathbb{C}^n \) defined as polynomials in \( n \) complex variables \( z = (z_1, z_2, \ldots, z_n) \). Consider the family of varieties

\[ V_c = \{ z \in \mathbb{C}^n \mid F_0(z) = c, F_i(z) = 0, \ i = 1, 2, \ldots, k \}, \]

where \( c \in \mathbb{C} \) is a complex parameter. This family provides a fibration over the punctured neighborhood of the origin in the parameter space with the fiber \( V_c \) over a point \( c \) (see below). In this paper we obtain a formula for the monodromy zeta-function of the above fibration in terms of the Newton polyhedra of the polynomials \( F_0, F_1, \ldots, F_k \). This result can be viewed as a global analog of [3, Theorem 2.2] and an analog of [5, Theorem 5.5], where the monodromy zeta-function at infinity is calculated. In §2 we consider the case where \( F_0(z) = z^n \), so that the fibration corresponds to a polynomial deformation of a set of polynomials in \( n-1 \) variables \( z_1, z_2, \ldots, z_{n-1} \). The general case is deduced from this special one in §3. The study is partially motivated by the results of D. Siersma and M. Tibar (6).

Let \( A = \mathbb{C}^n \setminus Y \) be the complement to an arbitrary algebraic hypersurface \( Y \subset \mathbb{C}^n \). Let \( Z = \{ z \in \mathbb{C}^n \mid F_i = 0, \ i = 1, 2, \ldots, k \} \cap A \). We denote by \( \mathbb{D}_r \) and \( \mathbb{D}_r^* \) the closed disk in \( \mathbb{C} \) of radius \( r \) centered at the origin and the punctured disk \( \mathbb{D}_r^* := \mathbb{D}_r \setminus \{0\} \), respectively. From [11, Theorem 5.1] it follows that there exists a finite set \( B \subset \mathbb{C} \) such that the restriction \( F = F_0|_Z \) of the function \( F_0 \) is a topological fibration over \( \mathbb{C} \setminus B \). In particular, the map \( F|_{F^{-1}(\mathbb{D}_r^*)} (F|_{F^{-1}(\mathbb{C}\setminus\mathbb{D}_d)}) \) is a fibration for any sufficiently small \( \delta \) (for any sufficiently large \( d \)). Consider the restriction of this fibration to the cycle \( \{ c \cdot \exp(2\pi it) \mid t \in [0, 1] \} \), where \( |c| \) is sufficiently small (large, respectively). Consider the monodromy transformation \( h_{F,0} : Z_c \to Z_c \) (\( h_{F,\infty} : Z_c \to Z_c \)) of the fiber \( Z_c \) over the point \( c \) of the resulting fibration.

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The *zeta-function* of an arbitrary transformation \( h : X \to X \) of a topological space \( X \) is the rational function
\[
\zeta_h(t) = \prod_{i \geq 0} \left( \det(\text{Id} - th_i |_{H^i_c(X;\mathbb{C})}) \right)^{(-1)^i},
\]
where \( H^i_c(X;\mathbb{C}) \) denotes the \( i \)th homology group with closed support.

**Definition 1.** The monodromy zeta-function (at the origin) of the function \( F_0 \) on the set \( Z \) is the zeta-function of the transformation \( h_{F,0}, \zeta_{F,0}(t) := \zeta_{h_{F,0}}(t) \). The monodromy zeta-function at infinity of the function \( F_0 \) on the set \( Z \) is the zeta-function of the transformation \( h_{F,\infty}, \zeta_{F,\infty}(t) := \zeta_{h_{F,\infty}}(t) \).

Let \( S_1, S_2, \ldots, S_n \subset \mathbb{R}^n \) be a collection of convex bodies. We denote by \( S_1 S_2 \ldots S_n \) their Minkovski mixed volume (see, e.g., [8]). If \( S_j = \emptyset \) for some \( j \), we put \( S_1 S_2 \ldots S_n = 0 \). For a homogeneous polynomial \( T(x_1, x_2, \ldots, x_k) = \sum \alpha_{i_1i_2 \ldots i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \) of degree \( n \), we define \( T(S_1, S_2, \ldots, S_k) \) as \( \sum \alpha_{i_1i_2 \ldots i_n} S_{i_1} S_{i_2} \ldots S_{i_n} \).

Let \( S_1, S_2, \ldots, S_l \subset L \subset \mathbb{R}^n \) be a collection of convex bodies that lie in an \( l \)-dimensional rational affine subspace \( L \). We define \( S_1 S_2 \ldots S_l \) as the \( l \)-dimensional integer mixed volume, that is, the Minkovskian mixed volume in the affine subspace \( L \) normalized in such way that the \( l \)-dimensional volume of the minimal parallelepiped with integer vertices equals one.

In this paper we obtain a formula for the zeta-function \( \zeta_{F_0,V}(t) \), \( V = \{ z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0 \} \) for a generic set of polynomials \( F_0, F_1, \ldots, F_k \) in terms of the integer mixed volumes of the faces of their Newton polyhedra \( \Delta_0, \Delta_1, \ldots, \Delta_k \).

### §2. Zeta-function of a polynomial deformation

In this section we study the case where \( F_0(z) = z_n \). Consider the set of deformations \( f_{i,\sigma}(z_1, \ldots, z_{n-1}) := F_i(z_1, \ldots, z_{n-1}, \sigma) \) of the functions \( f_i := f_{i,0} \) on the set \( \mathbb{C}^{n-1}, \ i = 1, 2, \ldots, k, \) where \( \sigma \in \mathbb{C} \) is the deformation parameter. The fiber over the point \( c \) of the vibration provided by the function \( F_0 \) on the set \( \{ F_1 = F_2 = \cdots = F_k = 0 \} \) is \( \{ f_{1,c} = f_{2,c} = \cdots = f_{k,c} = 0 \} \times \{ c \} \). This fact motivates the following definition.

**Definition 2.** Consider \( V = \{ z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0 \} \). The zeta-function \( \zeta_{z_n,V}(t) \) \( (\zeta_{z_n,V}(t)) \) will be called the monodromy zeta-function (at infinity) of the deformation \( \{ f_{i,\sigma} \mid i = 1, 2, \ldots, k \} \).

### 2.1. Formulas for the zeta-function of a deformation

Consider the representation \( F_i = \sum_{k \in \mathbb{Z}^n} F_{i,k} z^k \), where the \( F_{i,k} \in \mathbb{C}, k \in \mathbb{Z}^n \), are the coefficients of the polynomial \( F_i \) and \( k = (k_1, k_2, \ldots, k_n) \) are the coordinates in the space \( \mathbb{R}^n \) that correspond to the variables \( (z_1, z_2, \ldots, z_n) \). Let \( \Delta_i = \Delta(F_i) \) denote the Newton polyhedron of the polynomial \( F_i, i = 1, 2, \ldots, k, \) i.e., the convex hull of the set \( \{ k \in \mathbb{Z}^n \mid F_{i,k} \neq 0 \} \). A subset \( I \) of the set \( \{ 1, 2, \ldots, n \} \) will be called an index set. Denote \( \mathbb{R}^I = \{ k \in \mathbb{R}^n \mid k_i = 0, i \notin I \} \). Let \( j_1^I < j_2^I < \cdots < j_{k(I)}^I \) be the elements of the set \( \{ j \in \{ 1, 2, \ldots, k \} \mid \Delta_j \cap \mathbb{R}^I \neq \emptyset \} \). We put \( \Delta_I^j = \Delta_j^I \cap \mathbb{R}^I, i = 1, 2, \ldots, k(I) \) and \( F_I^j = \sum_{k \in \Delta_I^j} F_{j,k} z^k \).

An integer covector is said to be primitive if it is not a multiple of another integer covector. We denote by \( \mathbb{Z}^I \) the set of primitive covectors in the dual space \( (\mathbb{R}^I)^* \). For a convex set \( S \subset \mathbb{R}^I \) and a covector \( \alpha \in \mathbb{Z}^I \), let \( S^\alpha \) be the subset of \( S \) formed by the points where the function \( \alpha |_S \) attains its minimal value: \( S^\alpha = \{ x \in S \mid \alpha(x) = \min(\alpha(S)) \} \).

For an arbitrary polynomial \( P = \sum_{k \in \Delta} P_k z^k \), with the Newton polyhedron \( \Delta \subset \mathbb{R}^I \) and a covector \( \alpha \in \mathbb{Z}^I \), we denote by \( P^\alpha \) the polynomial \( \sum_{k \in \Delta^\alpha} P_k z^k \). For an index set \( I \) containing \( n \), let \( \mathbb{Z}^I_+ \subset \mathbb{Z}^I \) \( (\mathbb{Z}^I_- \subset \mathbb{Z}^I) \) be the subset of covectors \( \alpha = \cdots + \alpha_n \) with strictly positive last component: \( \alpha_n > 0 \) (strictly negative last component: \( \alpha_n < 0 \)).
Definition 3. Consider a covector \( \alpha \in \mathbb{Z}^{(1,2,\ldots,m)} \). We say that a system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \alpha \)-nondegenerate with respect to its Newton polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \) if the 1-forms \( dF_i^\alpha \), \( i = 1, 2, \ldots, k \), are linearly independent at all the points of the set \( \{ z \in (\mathbb{C}^*)^m \mid F_1^\alpha(z) = F_2^\alpha(z) = \cdots = F_k^\alpha(z) = 0 \} \).

We say that a system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \sigma \)-nondegenerate (at infinity) with respect to its Newton polyhedra if for each index set \( I \) containing \( n \) and each covector \( \alpha \in \mathbb{Z}_+^I \) \( (\alpha \in \mathbb{Z}_-^I) \), the system of polynomials \( F_1^\alpha, F_2^\alpha, \ldots, F_k^\alpha \) is \( \alpha \)-nondegenerate with respect to its Newton polyhedra.

Finally, a system of polynomials \( F_1, F_2, \ldots, F_k \) is said to be nondegenerate with respect to its Newton polyhedra if for each index set \( I \) and each \( \alpha \in \mathbb{Z}_+^I \) the system of polynomials \( F_1^\alpha, F_2^\alpha, \ldots, F_k^\alpha \) is \( \alpha \)-nondegenerate.

For each index set \( I \subset \{1,2,\ldots,n\} \) containing \( n \), we define the following rational functions:

\[
\zeta_{\Delta_1,\Delta_2,\ldots,\Delta_k}^I(t) = \prod_{\alpha \in \mathbb{Z}_+^I} \left( 1 - t^{\alpha(\frac{\partial}{\partial z_n})} \right) Q^I_{\alpha}(\Delta_{1,\Delta_2,\ldots,\Delta_k}^I),
\]

\[
\zeta_{\Delta_1,\Delta_2,\ldots,\Delta_k}^{\infty}(t) = \prod_{\alpha \in \mathbb{Z}_+^I} \left( 1 - t^{-\alpha(\frac{\partial}{\partial z_n})} \right) Q^I_{\alpha}(\Delta_{1,\Delta_2,\ldots,\Delta_k}^\infty),
\]

where \( l = |I|-1 \), \( \frac{\partial}{\partial z_n} \) is the vector in \( \mathbb{R}^l \) whose only nonzero coordinate is \( k_n = 1 \), and \( Q^I_{\alpha}(x_1, x_2, \ldots, x_k) := \prod_{i=1}^k \frac{x_i}{1+x_i}^\alpha \), where \( [\cdot]_l \) denotes the degree \( l \) homogeneous part of the power series under consideration. In particular, \( Q^I_{\alpha} \equiv 0 \) for \( l > 0 \) and \( Q^I_{\alpha} \equiv 1 \).

Theorem 1. Suppose a system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \sigma \)-nondegenerate with respect to its Newton polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \). Then

\[
(1) \quad \zeta_{z_n, V \cap (\mathbb{C}^*)^n}^\alpha(t) = \zeta_{\Delta_1,\Delta_2,\ldots,\Delta_k}^I(t),
\]

\[
(2) \quad \zeta_{z_n, V}(t) = \prod_{I : n \in I \subset \{1,2,\ldots,n\}} \zeta_{\Delta_1,\Delta_2,\ldots,\Delta_k}^I(t),
\]

where \( V = \{ z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0 \} \).

Theorem 2. Suppose a system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \sigma \)-nondegenerate at infinity with respect to its Newton polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \). Then

\[
(3) \quad \zeta_{z_n, V \cap (\mathbb{C}^*)^n}^{\infty}(t) = \zeta_{\Delta_1,\Delta_2,\ldots,\Delta_k}^{\infty} \zeta_{\Delta_1,\Delta_2,\ldots,\Delta_k}^I(t),
\]

\[
(4) \quad \zeta_{z_n, V}^{\infty}(t) = \prod_{I : n \in I \subset \{1,2,\ldots,n\}} \zeta_{\Delta_1,\Delta_2,\ldots,\Delta_k}^{\infty}(t),
\]

where \( V = \{ z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0 \} \).

Remark 1. For \( k = 1 \), equation (11) implies

\[
(5) \quad \zeta_{z_n, V \cap (\mathbb{C}^*)^n}^\alpha(t) = \prod_{\alpha \in \mathbb{Z}_+^0} \left( 1 - t^{\alpha(\frac{\partial}{\partial z_n})} \right)^{-\alpha(n-1)!} \text{Vol}_{n-1}(\Delta_1^\alpha),
\]

where \( \text{Vol}_l(\cdot) \) denotes the \( l \)-dimensional integer volume, \( I_0 = \{1,2,\ldots,n\} \). This relation is similar to formula (1) in [3] Theorem 2.2 for the zeta-function of a singularity deformation. In fact, let \( f_\sigma \) denote the germ at the origin of the deformation defined by \( f_\sigma(z_1, \ldots, z_{n-1}) = F_1(z_1, \ldots, z_{n-1}, \sigma) \). Using the equation in [3], we obtain

\[
\zeta_{f_\sigma|\cap (\mathbb{C}^*)^{n-1}}(t) = \prod_{\alpha \in \mathbb{Z}_+^0} \left( 1 - t^{\alpha(\frac{\partial}{\partial z_n})} \right)^{-\alpha(n-1)!} \text{Vol}_{n-1}(\Delta_1^{I_0 \alpha}),
\]

where \( \mathbb{Z}_+^0 \) is the subset of covectors in \( \mathbb{Z}_+^{I_0} \) whose components are all strictly positive. Hence, the local zeta-function
\[ \zeta_{f^*|_{\mathbb{C}^*}^{n-1}}(t) \] is a “natural” factor of the global one \( \zeta_{\mathbb{C}^*}^{n} (t) \). The same observation follows from the localization principle (see below and [2]).

**Example 1.** Assume that \( n = 2 \) and \( k = 1 \). Consider the polynomial \( F_1(z_1, z_2) = z_1 + z_2(1 + z_1^2) \). Identities (1), (2) and the corresponding relations in [3] Theorem 2.2 imply that \( \zeta_{f^*|_{\mathbb{C}^*}^{n}}(t) = (1 - t) \), \( \zeta_{f^*|_{\mathbb{C}^*}^{2}}(t) = (1 - t)^2 \). The same results can be obtained by the following arguments. The global fiber is \( \mathbb{V}^*_t = \{ z_1 \mid F_1(z_1, \sigma) = 0 \} = \{ \frac{-1 + \sqrt{1 - 4\sigma^2}}{2\sigma} \} = \{ x_1(\sigma), x_2(\sigma) \} \), where \( x_1(\sigma) \approx -\sigma, \ x_2(\sigma) \approx -\sigma^{-1} \) for \( |\sigma| \ll 1 \). Thus, it consists of two points, one of them close to the origin and the other close to infinity. The monodromy transformation is the identical map of the fiber itself, whence \( \zeta_{f^*|_{\mathbb{C}^*}^{2}}(t) = \det((1 - t)\text{Id}) = (1 - t)^2 \). Since the local fiber \( \{ f_\sigma(z_1) = 0 \} = \{ x_1(\sigma) \} \) consists of one point, we have \( \zeta_{f^*|_{\mathbb{C}^*}^{n}}(t) = (1 - t) \).

### 2.2. Proofs of the theorems.

We reduce the calculation of the zeta-function to integration with respect to the Euler characteristic (see., e.g., [7]), using the following localization principle.

We recall the notion of the zeta-function as applied to a family of sections of a line bundle over a variety, introduced by S. M. Gusein-Zade and D. Siersma in [2]. Let \( W \) be a compact complex analytic variety, and let \( W_1 \) be the complement to a compact subvariety of \( W \). Let \( L \) be a line bundle over \( W \), and let \( q_\sigma \) be a family of sections of \( L \) analytic in \( \sigma \in \mathbb{C}_\sigma \). Let \( U \) be the subset of \( W_1 \times \mathbb{C}_\sigma \) given by \( q_\sigma(x) = 0 \). The restriction to \( U \) of the projection \( W_1 \times \mathbb{C}_\sigma \to \mathbb{C}_\sigma \) is a fibration over the punctured disk \( \mathbb{D}_\sigma^* \subset \mathbb{C}_\sigma \) for \( |\sigma| \ll 1 \). The zeta-function of a family of sections \( q_\sigma \) restricted to the set \( W_1 \) is the zeta-function of the monodromy transformation of the above fibration. We denote it by \( \zeta_{q_\sigma|_{W_1}}(t) \).

The fibration \( L \) is trivial over a neighborhood of a point \( x \in W \). Therefore, using a fixed coordinate system, we can view the family of germs at the point \( x \) of sections \( q_\sigma \) as a deformation in the parameter \( \sigma \) of a function germ. We denote by \( \zeta_{q_\sigma|_{W_1}, x}(t) \) the zeta-function of the germ at the point \( \sigma = 0 \) of the above deformation restricted to the set \( W_1 \) (see, e.g., [3]).

**Theorem 3** ([2], “localization principle”). We have

\[ \zeta_{q_\sigma|_{W_1}}(t) = \int_W \zeta_{q_\sigma|_{W_1}, x}(t) \, d\chi. \]

Using the Newton polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \) of the polynomials \( F_1, F_2, \ldots, F_k \), we construct a unimodular simplicial partition \( \Lambda \) of the dual space \( \mathbb{R}^n \); we assume that this partition is sufficiently fine for the system \( \{ \Delta_i \} \) in the sense of [9]. Consider the toroidal compactification \( X_\Lambda \) of the torus \( (\mathbb{C}^*)^n \) that corresponds to the partition \( \Lambda \). Recall that the standard action of the torus \( (\mathbb{C}^*)^n \) on itself uniquely extends to an action of the torus on the variety \( X_\Lambda \). The cones \( \lambda \in \Lambda \) of the partition are in one-to-one correspondence with the orbits \( T_\lambda \subset X_\Lambda \) of this action and the orbit \( T_\lambda \) is isomorphic to \( (\mathbb{C}^*)^{n - \dim \lambda} \). Denote by \( X'_\Lambda \) the complement in \( X_\Lambda \) to the torus \( T_{\{0\}} \cong (\mathbb{C}^*)^n \). Let \( \overline{V} \) be the closure of the set \( V \cap T_{\{0\}} \subset X_\Lambda \), and let \( V' = \overline{V} \cap X'_\Lambda \). We prove the following statement.

**Lemma 1.** For a sufficiently fine partition \( \Lambda \), we have

\[ \zeta_{\mathbb{C}^*|_{\mathbb{C}^*}^{n}}(t) = \int_{V'} \zeta_{\mathbb{C}^*|_{\mathbb{C}^*}^{n}, x}(t) \, d\chi, \]

\[ \zeta_{\mathbb{C}^*|_{\mathbb{C}^*}^{\infty}}(t) = \int_{V'} \zeta_{\mathbb{C}^*|_{\mathbb{C}^*}^{\infty}, x}(t) \, d\chi. \]
where, for a germ at \( x \in V' \) of a meromorphic function \( f \) on the set \( \tilde{V} \) and for an open subset \( A \subset \tilde{V} \), the expression \( \zeta_{f|A,x}(t) (\zeta_{f|A,x}^\infty(t)) \) denotes the local zeta-function (at infinity) of the germ at \( x \) of the function \( f \) restricted to \( A \).

**Proof.** We may assume the partition \( \Lambda \) to be a subdivision of the standard partition \( \Pi \) of the space \( (\mathbb{R}^n)^* \) corresponding to the \( n \)-dimensional projective space: \( X_\Pi = \mathbb{CP}^n \supset (\mathbb{C}^*)^n \). Let \( p : X_\Lambda \to \mathbb{CP}^n \) be the map of the toric varieties induced by the refinement \( \Lambda \prec \Pi \). Consider the family of global sections \( s_\sigma, \sigma \in \mathbb{C}, \) of the fibration \( \mathcal{O}(1) \) over \( \mathbb{CP}^n \) that is defined by the condition \( s_\sigma|_\mathbb{C}^n = z_n - \sigma \). Denote \( \pi = p \circ \text{inj} \), where \( \text{inj} : \tilde{V} \hookrightarrow X_\Lambda \) is the inclusion map. Let \( S_\sigma = \pi^*(s_\sigma) \) be the family of sections of the bundle \( \pi^*(\mathcal{O}(1)) \) that is the pull-back of \( s_\sigma \). In a similar way, consider a family of sections \( s'_\sigma, \sigma \in \mathbb{C}, \) of the fibration \( \mathcal{O}(1) \) that is defined by the condition \( s'_\sigma|_\mathbb{C}^n = 1 - \sigma z_n \), and consider the pull-back \( S'_\sigma = \pi^*(s'_\sigma) \).

By simple reformulations, we can easily show that

\[
\zeta_{z_n,V \cap \mathbb{C}^n}(t) = \zeta_{s_\sigma|V \cap \mathbb{C}^n}(t), \quad \zeta_{z_n|V \cap \mathbb{C}^n}(t) = \zeta_{s'_\sigma|V \cap \mathbb{C}^n}(t), \quad \zeta_{z_n,V \cap \mathbb{C}^n, x}(t) = \zeta_{s_\sigma|V \cap \mathbb{C}^n, x}(t), \quad \zeta_{z_n|V \cap \mathbb{C}^n, x}(t) = \zeta_{s'_\sigma|V \cap \mathbb{C}^n, x}(t).
\]

Applying Theorem 3 to the families \( S_c \) and \( S'_c \), we obtain

\[
\zeta_{s_\sigma|V \cap \mathbb{C}^n}(t) = \int_{\tilde{V}} \zeta_{s_\sigma|V \cap \mathbb{C}^n, x}(t) d\chi = \int_{\tilde{V}} \zeta_{z_n|V \cap \mathbb{C}^n, x}(t) d\chi,
\]

\[
\zeta_{s'_\sigma|V \cap \mathbb{C}^n}(t) = \int_{\tilde{V}} \zeta_{s'_\sigma|V \cap \mathbb{C}^n, x}(t) d\chi = \int_{\tilde{V}} \zeta_{z_n|V \cap \mathbb{C}^n, x}(t) d\chi.
\]

Moreover, it is easily seen that \( \zeta_{z_n|V \cap \mathbb{C}^n, x}(t) = \zeta_{z_n|V \cap \mathbb{C}^n}(t) = 1 \) for \( x \notin V' \). Therefore, using the multiplicative property of the integration, we get

\[
\int_{\tilde{V}} \zeta_{z_n|V \cap \mathbb{C}^n, x}(t) d\chi = \int_{\tilde{V}} \zeta_{z_n|V \cap \mathbb{C}^n, x}(t) d\chi = \int_{\tilde{V}} \zeta_{z_n|V \cap \mathbb{C}^n, x}(t) d\chi.
\]

\( \square \)

Let \( \Lambda_+ \subset \Lambda \) and \( \Lambda_\pm \subset \Lambda \) be the subset of cones \( \lambda \in \Lambda \) generated by a set of primitive covectors \( \alpha_1, \alpha_2, \ldots, \alpha_l \) lying in \( \mathbb{Z}^{1,2,\ldots,n} \setminus \mathbb{Z}^+_{1,2,\ldots,n} \) in \( \mathbb{Z}^{1,2,\ldots,n} \setminus \mathbb{Z}^+_{1,2,\ldots,n} \), respectively. We may assume that \( \Lambda = \pm \Lambda_+ \pm \Lambda_+ = \Lambda \).

Consider an arbitrary point \( x_0 \in V' \). It is contained in the torus \( T_\lambda \) that corresponds to some \( l \)-dimensional cone \( \lambda \in \Lambda, l < n \). This cone lies on the border of an \( n \)-dimensional cone \( \lambda' \in \Lambda \). Denote by \( \alpha_1, \alpha_2, \ldots, \alpha_l \) the primitive integer covectors that generate the cone \( \lambda \). The cone \( \lambda' \) is generated by the covectors \( \alpha_1, \alpha_2, \ldots, \alpha_l \) and some covectors \( \alpha_{l+1}, \alpha_{l+2}, \ldots, \alpha_n \). Consider the coordinate system \( u = (u_1, u_2, \ldots, u_n) \) corresponding to the set of covectors \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \). We have \( u_i(x_0) = 0, i \leq l, u_i(x_0) \neq 0, i > l \). We express the monomial \( z_n \) as a function \( F \) of the variables \( u \):

\[
F(u) = b \cdot u_1^{\alpha_1 (\partial / \partial k_n)} u_2^{\alpha_2 (\partial / \partial k_n)} \cdots u_l^{\alpha_l (\partial / \partial k_n)},
\]

where \( b(u) = \prod_{j=l+1}^n u_j^{\alpha_j (\partial / \partial k_n)} \), \( b(x_0) \in \mathbb{C}^* \). Now we are ready to calculate the values of the integrands \( \zeta_{z_n|V \cap \mathbb{C}^n, x_0}(t) \) and \( \zeta_{z_n|V \cap \mathbb{C}^n, x_0}(t) \) in the following two cases.

1. If \( \lambda \in \Lambda_\pm \), then the value of the function \( F \) at the point \( x \) is not zero, so that \( \zeta_{z_n|V \cap \mathbb{C}^n, x_0}(t) = 1 \). Accordingly, assume that \( \lambda \notin \Lambda_\pm \). Then the point \( x_0 \) is not a pole of the function \( F \) and therefore \( \zeta_{z_n|V \cap \mathbb{C}^n, x_0}(t) = 1 \).

2. Assume that \( \lambda \in \Lambda \setminus \Lambda_\pm \). The system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \sigma \)-nondegenerate (at infinity) with respect to its polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \).
Therefore, \( l + k \leq n \), and there is a coordinate system \((u_1, \ldots, u_l, w_{l+1}, \ldots, w_n)\) in a neighborhood of \( x_0 \) such that \( u_i(x_0) = 0 \), \( i = l + 1, \ldots, n \), and

\[
(7) \quad F_i = a_i u_1^{m_{i,1}} \cdots u_l^{m_{i,l}} \cdot w_{n-i+1}, \quad i = 1, 2, \ldots, k,
\]

where \( m_{i,j} = \min(\alpha_j|\Delta_i) \) and \( a_i \) is a germ of an analytic function such that \( a_i(x_0) \neq 0 \). Denote \( V_{x_0} = V \cap (C^*)^n \cap U \). By (7), we have

\[
V_{x_0} = \{ u_i \neq 0, \ i \leq l; \ w_i = 0, \ i > n - k \} \subset U.
\]

Hence,

\[
(8) \quad \zeta_{\varphi|_{V \cap (C^*)^n}, x_0}(t) = \zeta_{g|_{(u_i \neq 0, i \leq l)}, 0}(t) \quad (\zeta_{\varphi|_{V \cap (C^*)^n}, x_0}(t) = \zeta_{g|_{(u_i \neq 0, i \leq l)}, 0}(t)),
\]

where \( g \) is the germ of the function in the variables \((u_1, \ldots, u_l, w_{l+1}, \ldots, w_{n-k})\) that is given by the relation

\[
g = \prod_{j=1}^l u_j^{\alpha_j} (\partial/\partial k_n) \cdot b(u_1, \ldots, u_l, w_{l+1}, \ldots, w_{n-k}, 0, \ldots, 0).
\]

Using the Varchenko-type formula for meromorphic functions (see \( \Pi \)), we calculate the right-hand side of (8). For \( l = 1 \), we obtain

\[
(9) \quad \zeta_{\varphi|_{V \cap (C^*)^n}, x_0}(t) = 1 - t^{\alpha_1}(\partial/\partial k_n) \quad \left( \zeta_{\varphi|_{V \cap (C^*)^n}, x_0}(t) = 1 - t^{\alpha_1}(\partial/\partial k_n) \right).
\]

Finally, the two zeta-functions in question are trivial if \( l > 1 \).

Now we specify the only case where the function \( \zeta_{\varphi|_{V \cap (C^*)^n}, x_0}(t) \) is not trivial. Namely, we assume that \( x_0 \in T_\Lambda, \lambda \in \Lambda \setminus \Delta_\Lambda \) (\( \lambda \in \Lambda \setminus \Delta_\Lambda \)) and \( \dim \lambda = 1 \). Denote \( \alpha = \alpha_1 \). The set \( T_\Lambda \cap V' \) can be defined in the coordinates \((u_2, \ldots, u_{n+1})\) on the torus \( T_\Lambda = \{ u_1 = 0 \} \) by the system of equations \( \{ Q_1^\alpha = Q_2^\alpha = \cdots = Q_k^\alpha = 0 \} \), where

\[
Q_i^\alpha = \sum_{k \in \Delta_i^\alpha} F_{i,k} u_2^{\alpha_2(k)} u_3^{\alpha_3(k)} \cdots u_n^{\alpha_n(k)}.
\]

Using the main results of (9) and (10), we obtain

\[
(10) \quad \chi(T_\Lambda \cap V') = (n - 1)! Q_k^{n-1}(\Delta(Q_1^\alpha), \Delta(Q_2^\alpha), \ldots, \Delta(Q_k^\alpha)),
\]

where \( \Delta(\cdot) \) denotes the Newton polyhedron of the Laurent polynomial under consideration. The covectors \( \alpha_2, \alpha_3, \ldots, \alpha_n \) determine an isomorphism of the integer lattices of the hyperplane \( \{ \alpha = 0 \} \subset \mathbb{R}^n \) and the space \( \mathbb{R}^{n-1} \), which contains the polyhedra \( \Delta(Q_i^\alpha) \). Under this isomorphism, the polyhedra \( \Delta(Q_i^\alpha) \) correspond to parallel shifts of the polyhedra \( \Delta_i \). Therefore, the corresponding mixed integer volumes coincide and

\[
(11) \quad Q_k^{n-1}(\Delta(Q_1^\alpha), \Delta(Q_2^\alpha), \ldots, \Delta(Q_k^\alpha)) = Q_k^{n-1}(\Delta_1^\alpha, \Delta_2^\alpha, \ldots, \Delta_k^\alpha).
\]

Relations (9), (10), (11) imply the following answers:

\[
\int_{T_\Lambda \cap V'} \zeta_{\varphi|_{V \cap (C^*)^n}, x}(t) \, d\chi = \left( 1 - t^{\alpha_1(\partial/\partial k_n)} \right)^{(n-1)!} Q_k^{n-1}(\Delta_1^\alpha, \Delta_2^\alpha, \ldots, \Delta_k^\alpha),
\]

\[
\int_{T_\Lambda \cap V'} \zeta_{\varphi|_{V \cap (C^*)^n}, x}(t) \, d\chi = \left( 1 - t^{\alpha_1(\partial/\partial k_n)} \right)^{(n-1)!} Q_k^{n-1}(\Delta_1^\alpha, \Delta_2^\alpha, \ldots, \Delta_k^\alpha).
\]

We can multiply identities (12) over all strata \( T_\Lambda \subset X'_{\Lambda} \) of dimension \( n - 1 \) corresponding to the tori \( \lambda \in \Lambda \setminus \Delta_\Lambda \) (\( \lambda \in \Lambda \setminus \Delta_\Lambda \)), and apply (9), obtaining the required formulas (1) and (3). Formulas (2) and (4) follow from (1) and (3) (respectively) by the multiplicative property of zeta-functions.
§3. ZETA-FUNCTION OF A POLYNOMIAL ON A COMPLETE INTERSECTION

In this section we obtain the general formula for the zeta-function at the origin of a polynomial \( F_0 = \sum_{k \in \mathbb{Z}^n} F_{0,k} z_k^k \) on the set of common zeros of a set of polynomials \( F_1, F_2, \ldots, F_k \). We use the notation and definitions introduced in [2]. Let \( \Delta_0 \) be the Newton polyhedron of \( F_0 \). For an index set \( I \), we denote \( \Delta^I_0 = \Delta_0 \cap \mathbb{R}^I \), \( F^I_0 = \sum_{k \in \Delta^I_0} F_{0,k} z_k^k \).

For each index set \( I \subset \{1, 2, \ldots, n\} \), consider the following rational function:

\[
\tilde{\zeta}^I_{\Delta^I_0; \Delta^I_1, \ldots, \Delta^I_k}(t) := \prod_{\alpha \in \mathbb{Z}^I_{\Delta^I_0}} \left( 1 - t^{m_{\Delta^I_0}(\alpha)} \right)^{!} \tilde{Q}^I_{k(t)+1}(\Delta^I_0; \Delta^I_1, \ldots, \Delta^I_k(t)) \tag{13}
\]

where \( m_{\Delta^I_0}(\alpha) = \min(\alpha | \Delta^I_0) \) is the minimal value of the covector \( \alpha \) on the set \( \Delta^I_0 \), the symbol \( \mathbb{Z}^I_{\Delta^I_0} \) stands for the set of covectors \( \alpha \in \mathbb{Z}^I \) such that \( \min(\alpha | \Delta^I_0) > 0 \) (for \( \Delta^I_0 = \emptyset \), we put \( \mathbb{Z}^I_{\Delta^I_0} = \emptyset \)), and

\[
\tilde{Q}^I_{k+1}(x_0, x_1, \ldots, x_k) = Q^I_k(x_1, x_2, \ldots, x_k) - Q^I_{k+1}(x_0, x_1, \ldots, x_k) \tag{14}
\]

is a homogeneous polynomial of degree \( l := |I| - 1 \). The following statement is a consequence of Theorem 1 and some observations concerning the formula for the Euler characteristic of a nondegenerate complete intersection that was obtained in [10].

**Theorem 4.** Suppose that systems of polynomials \( F_0, F_1, \ldots, F_k \) and \( F_1, F_2, \ldots, F_k \) are nondegenerate with respect to their Newton polyhedra \( \Delta_0, \Delta_1, \ldots, \Delta_k \) and \( \Delta_1, \Delta_2, \ldots, \Delta_k \), respectively. Then

\[
\zeta_{F_0, V \cap (C^*)^n}(t) = \zeta^{(1, \ldots, n)}_{\Delta^0_0; \Delta^1_1, \ldots, \Delta^k_k}(t), \quad \zeta_{F_0, V}(t) = \prod_{I \subset \{1, \ldots, n\} : I \neq \emptyset} \tilde{\zeta}^I_{\Delta^I_0; \Delta^I_1, \ldots, \Delta^I_k}(t), \tag{15, 16}
\]

where \( V = \{ z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0 \} \) is the set of common zeros of the system \( F_1, F_2, \ldots, F_k \).

**Remark 2.** Consider the case where \( k = 0 \). Using (16) and (13), we can obtain the relation

\[
\zeta_{F_0, \mathbb{C}^n}(t) = \prod_{I \neq \emptyset, 2^I_0} \prod_{\alpha \in \mathbb{Z}^I_{\Delta^I_0}} \left( 1 - t^{m_{\Delta^I_0}(\alpha)} \right)^{!} \Vol_{0}(\Delta^I_0; \alpha) \tag{17}
\]

(here we put \( \Vol_{0}(pt) = 1 \)). This is an analog of the Libgober–Sperber theorem (see [4]) and (in a slightly different form) was obtained by Y. Matsui and K. Takeuchi ([5, §4]).

**Proof of the theorem.** Note that formula (16) follows from (15) by the multiplicative property of zeta-functions. We prove (15).

Consider the system of polynomials \( G_1, G_2, \ldots, G_{k+1} \) in \( n + 1 \) variables \( (z, z_{n+1}) = (z_1, z_2, \ldots, z_{n+1}) \) given by

\[
G_i(z_1, z_2, \ldots, z_{n+1}) = F_i(z_1, z_2, \ldots, z_n), \quad i = 1, 2, \ldots, k;
\]

\[
G_{k+1}(z_1, z_2, \ldots, z_{n+1}) = F_0(z_1, z_2, \ldots, z_n) - z_{n+1}. \tag{18}
\]

Consider the set \( W = \{ (z, z_{n+1}) \in \mathbb{C}^{n+1} \mid G_1(z) = G_2(z) = \cdots = G_{k+1}(z) = 0 \} \). Since, obviously, the fibrations defined by the maps

\[
V \cap (\mathbb{C}^*)^n \cap F_0^{-1}(D^\delta_0) \to D^\delta_0 \quad \text{and} \quad W \cap (\mathbb{C}^*)^{n+1} \cap \{ 0 < |z_{n+1}| \leq \delta \} \to D^\delta_0
\]

are isomorphic, we have

\[
\zeta_{F_0, V \cap (\mathbb{C}^*)^n}(t) = \zeta_{z_{n+1}, W \cap (\mathbb{C}^*)^{n+1}}(t). \tag{19}
\]
The space \( \mathbb{R}^n \) with the coordinates \((k_1, k_2, \ldots, k_n)\) is enclosed in a standard manner in the space \( \mathbb{R}^{n+1} \) with the additional coordinate \( k_{n+1} \) that corresponds to the variable \( z_{n+1} \). For \( i \leq k \), the Newton polyhedra of the polynomials \( F_i \) and \( G_i \) coincide: \( \Delta(G_i) = \Delta_i \).

The Newton polyhedron of the polynomial \( G_{k+1} \) is a cone of integer height 1 over the Newton polyhedron of the polynomial \( F_0, \Delta(G_{k+1}) = C \Delta_0 \).

**Proposition 1.** For a system of polynomials \( F_0, F_1, \ldots, F_k \) such that both this system itself and the system \( F_1, F_2, \ldots, F_k \) are nondegenerate with respect to their Newton polyhedra, the system of polynomials \( G_1, G_2, \ldots, G_{k+1} \) is also nondegenerate with respect to its Newton polyhedra.

**Proof.** Consider an arbitrary subset \( I \subset \{1, 2, \ldots, (n+1)\} \) and an arbitrary covector \( \alpha \in \mathbb{Z}^I \). For \( n+1 \not\in I \), obviously, the conditions of \( \alpha \)-nondegeneracy as applied to the system \( \{ G_i^I \} \) and to the system \( \{ F_i^I \} \) are equivalent. Assume that \( n+1 \in I \). Denote \( I' = I \setminus \{ n+1 \} \), \( \alpha' = \alpha|_{\mathbb{R}^I} \). Extending the notation of Subsection 2.1 to the system of polynomials \( G_1, G_2, \ldots, G_{k+1} \), we see that \( k(I) = k(I') + 1, G_i^{I', \alpha}(z, z_{n+1}) = F_i^{I', \alpha'}(z) \) for \( i \leq k(I') \). Three cases are possible.

1. \( \alpha(\frac{\partial}{\partial k_{n+1}}) > \min(\alpha'|_{\Delta_i' \setminus \Delta_i}) \). Then \( (C \Delta_0 \cap \mathbb{R}^I)^\alpha = \Delta_0^{I', \alpha'} \), \( G_i^{I, \alpha}(z, z_{n+1}) = F_i^{I', \alpha'}(z) \).
2. \( \alpha(\frac{\partial}{\partial k_{n+1}}) < \min(\alpha'|_{\Delta_i' \setminus \Delta_i}) \). Then \( (C \Delta_0 \cap \mathbb{R}^I)^\alpha = \{ \frac{\partial}{\partial k_{n+1}} \}, G_i^{I, \alpha} = -z_{n+1} \).
3. \( \alpha(\frac{\partial}{\partial k_{n+1}}) = \min(\alpha'|_{\Delta_i' \setminus \Delta_i}) \). Then \( (C \Delta_0 \cap \mathbb{R}^I)^\alpha = C \Delta_0^{I', \alpha'} \) is a cone of integer height 1 over \( \Delta_0^{I', \alpha'} \), \( G_i^{I, \alpha}(z, z_{n+1}) = F_i^{I', \alpha'}(z) - z_{n+1} \).

Using the \( \alpha \)-nondegeneracy of the systems \( F_0, F_1, \ldots, F_k \) and \( F_1, F_2, \ldots, F_k \), we verify easily that the 1-forms \( dG_i^{I', \alpha}, i = 1, 2, \ldots, k(I) \), are linearly independent at the points of the set

\[
\{(z, z_{n+1}) \in (\mathbb{C}^*)^{n+1} | G_1(z, z_{n+1}) = G_2(z, z_{n+1}) = \cdots = G_{k+1}(z, z_{n+1}) = 0 \}.
\]

**Proposition 1** shows that Theorem 1 is applicable to the polynomials \( G_1, G_2, \ldots, G_{k+1} \):

\[
(19) \quad \zeta_{z_{n+1}, W \cap (\mathbb{C}^*)^{n+1}}(t) = \prod_{\alpha \in \mathbb{Z}^I_0} (1 - t^\alpha(\frac{\partial}{\partial k_{n+1}}))^n Q_{k+1}^n((C \Delta_0)^\alpha, \Delta_1^\alpha, \ldots, \Delta_k^\alpha),
\]

where \( I_0 = \{1, 2, \ldots, n+1\} \). It is easily seen that in the above cases 1 and 2 the exponent \( Q_{k+1}^n((C \Delta_0)^\alpha, \Delta_1^\alpha, \ldots, \Delta_k^\alpha) \) equals 0. Therefore,

\[
(20) \quad \zeta_{z_{n+1}, W \cap (\mathbb{C}^*)^{n+1}}(t) = \prod_{\alpha \in \mathbb{Z}^I_0} (1 - t^{n \Delta_0(\alpha)})^n Q_{k+1}^n(C(\Delta_0^\alpha, \Delta_1^\alpha, \ldots, \Delta_k^\alpha)),
\]

where \( I_0 = \{1, 2, \ldots, n\} \). Now, formula (18) follows from (18), (20), and the identity

\[
n! Q_{k+1}^n((C(\Delta_0^\alpha, \Delta_1^\alpha, \ldots, \Delta_k^\alpha)) = (n-1)! \tilde{Q}_{k+1}^{n-1}(\Delta_0^\alpha, \Delta_1^\alpha, \ldots, \Delta_k^\alpha),
\]

which is a consequence of the following statement.

**Proposition 2.** Let \( \Delta_0, \Delta_1, \ldots, \Delta_k \) be a set of integer polyhedra lying in a rational affine hyperplane \( L \subset \mathbb{R}^{n+1} \). Let \( C \Delta_0 \) be the cone over \( \Delta_0 \) with vertex at some point \( v \in \mathbb{R}^{n+1} \) that lies at the integer distance 1 from the hyperplane \( L \). Then

\[
(21) \quad (n+1)! Q_{k+1}^n(C \Delta_0, \Delta_1, \ldots, \Delta_k) = \tilde{Q}_{k+1}^n(\Delta_0, \Delta_1, \ldots, \Delta_k).
\]

**Proof.** We choose an affine integer coordinate system \( k = (k_1, k_2, \ldots, k_{n+1}) \) in the space \( \mathbb{R}^{n+1} \) in such a way that \( L = \{ k \in \mathbb{R}^n | k_{n+1} = 0 \} \) and \( v = (0, 0, \ldots, 1) \). Choose Laurent polynomials \( F_0, F_1, \ldots, F_k \) in the variables \( z = (z_1, z_2, \ldots, z_n) \) with fixed Newton polyhedra \( \Delta_0, \Delta_1, \ldots, \Delta_k \) in such a way that the systems \( F_0, F_1, \ldots, F_k \) and \( F_1, F_2, \ldots, F_k \)
are nondegenerate in the sense of [9] with respect to their Newton polyhedra. One can easily show (see Proposition 1) that the system of Laurent polynomials \(G_1, G_2, \ldots, G_{k+1}\) in \(n + 1\) variables defined in terms of the polynomials \(\{F_i\}\) by formulas (17) is also non-degenerate in the sense of [9] with respect to its Newton polyhedra \(\Delta_1, \ldots, \Delta_k, C(\Delta_0)\).

We put
\[
V = \{F_0 = F_1 = \cdots = F_k = 0\} \subset (\mathbb{C}^*)^n,\\
V_1 = \{F_1 = F_2 = \cdots = F_k = 0\} \subset (\mathbb{C}^*)^n,\\
W = \{G_1 = G_2 = \cdots = G_{k+1} = 0\} \subset (\mathbb{C}^*)^{n+1}.
\]

Applying the results of [10], we find the Euler characteristics of the sets \(V, V_1, W:\)
\[
(22) \quad \chi(V) = n! Q_{k+1}^n(\Delta_0, \Delta_1, \ldots, \Delta_k), \quad \chi(V_1) = n! Q_k^n(\Delta_1, \Delta_2, \ldots, \Delta_k),
\]
\[
(23) \quad \chi(W) = (n + 1)! Q_{k+1}^{n+1}(C\Delta_0, \Delta_1, \ldots, \Delta_k).
\]

Consider the projection \(p: (\mathbb{C}^*)^{n+1} \to (\mathbb{C}^*)^n\) to the coordinate hyperplane with the coordinates \((z_1, z_2, \ldots, z_n)\). Its restriction \(p|_W\) provides an isomorphism between \(W\) and \(V_1 \setminus V\). Therefore,
\[
\chi(W) = \chi(V_1) - \chi(V).
\]

Applying this relation and using (22), (23), and [14], we get (21). \(\square\)

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