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AVERAGE NUMBER OF LOCAL MINIMA FOR THREE-DIMENSIONAL INTEGRAL LATTICES

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Abstract. An asymptotic formula is found for the average number of local minima of three-dimensional complete integral lattices with determinant in the interval $[1, N]$. This is a generalization to the two-dimensional case of the classical result about the average length of a finite continued fraction with denominator belonging to $[1, N]$.

Notation

$\mathbb{R}_+$ is the set of positive real numbers;

$X^{s\times s}$ is the set of $(s \times s)$-matrices with entries in X;

$\mathrm{GL}_s(\mathbb{R})$ is the set of nonsingular matrices in $\mathbb{R}^{s\times s}$;

$\mathrm{GL}_s(X; N) = \{M \in X^{s\times s} : \det M = N\}$;

for $m^{(1)}, \dots, m^{(s)} \in \mathbb{R}^s$, the symbol $[m^{(1)}, \dots, m^{(s)}]$ denotes the matrix with the columns ${m^{(1)}}^T, \dots, {m^{(s)}}^T$;

$\partial\Omega$ is the boundary of a set Ω, and $\bar{\Omega}$ is the closure of Ω;

$\#X$ is the cardinality of a finite set X.

We write $f(x) \ll g(x)$ (or $f(x) = O(g(x))$) for $x \in X$ if there exists a constant $C > 0$ such that $|f(x)| \le C \cdot g(x)$ for all $x \in X$. If C depends on a parameter θ, we write $f(x) \ll_\theta g(x)$ (or $f(x) = O_\theta(g(x))$). We write $f \asymp g$ if $f \ll g \ll f$.

Introduction

Two generalizations of continued fractions to the multidimensional case emerged at the end of the 19th century. One of them was due to F. Klein [1], the other was suggested by Voronoĭ [2], and, independently, by Minkowski [3]. Each one is related to isolation and the study of certain sets of nodes of an s-dimensional lattice Γ, specifically, the Klein polytops and (accordingly) the set $\mathfrak{M}(\Gamma)$ of all local minima for Γ.

We recall the definitions. *A complete s-dimensional integral lattice is a set of the form*

$$\Gamma = \left\{ \sum_{i=1}^{s} k_i m^{(i)} \,:\, k_i \in \mathbb{Z} \right\},$$

where the $m^{(i)}$ $(i = 1, \dots, s)$ are linearly independent vectors in $\mathbb{Z}^s$ (a basis of Γ).

The quantity $\det \Gamma = |\det((m_j^{(i)}))|$ is called the *determinant* of Γ.

A nonzero node $\gamma \in \Gamma$ is called a *local minimum* of Γ if there is no other nonzero node $\gamma' \in \Gamma$ with

$$|\gamma_i'| \le |\gamma_i| \quad i = 1, \dots, s, \quad \sum_{i=1}^{s} |\gamma_i'| < \sum_{i=1}^{s} |\gamma_i'|.$$

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We use the following notation:

$\mathcal{L}_s(\mathbb{Z}; N)$ is the set of all complete s-dimensional integral lattices;

$\mathfrak{M}(\Gamma)$ is the set of local minima of a lattice Γ.

The construction of Voronoĭ and Minkowski was motivated by the Lagrange theorem on best approximation with the help of continued fractions. For instance, if $\alpha \in (0, 1/2)$, then for the lattice Γ_α with the basis $(1, \alpha)$, $(0, 1)$ we have

$$\mathfrak{M}(\Gamma_\alpha) = \{\pm(Q_i, \alpha Q_i - P_i) \,:\, i = 0, 1, \dots\}, \tag{1}$$

where $Q_0 = 0$, $P_0 = 1$, and P_i/Q_i is the ith convergent for α if $i \geq 1$.

The notion of a local minimum emerges in various mathematical fields. For example, it was observed in [4, 5] that the set $\mathfrak{M}(\Gamma)$ of local minima determines the discrepancy of Korobov's multidimensional quadrature formulas (see also [6]).

Despite a considerable interest, fairly little is known about the number of local minima for lattices in dimensions 3 and higher (for two-dimensional lattices, the number of local minima is determined by the length of the continued fraction expansion for the corresponding α). Only the following estimates (from above and from below) are known:

$$\#\mathfrak{M}(\Gamma) \ll_s \ln^{s-1} N \quad \forall \Gamma \in \mathcal{L}(\mathbb{Z}^s; N), \quad N \geq 2, \tag{2}$$

$$\sum_{\Gamma \in \mathcal{L}(\mathbb{Z}^s; N)} \#\mathfrak{M}(\Gamma) \gg_s N^{s-1} \ln^{s-1} N. \tag{3}$$

For complete lattices, inequality (2) was proved in [4, 5] (see [7] for an estimate of the constant), and inequality (3) was proved in [10]. The results of [10] show also that (2) is sharp, up to a constant depending on the dimension s. Similar estimates for noncomplete lattices were obtained in [8].

We introduce the quantity

$$E_s(N) = \frac{\sum_{n=1}^{N} \sum_{\Gamma \in \mathcal{L}_s(\mathbb{Z}; n)} \#\mathfrak{M}(\Gamma)}{\sum_{n=1}^{N} \#\mathcal{L}_s(\mathbb{Z}; n)},$$

i.e., the average number or local minima for s-dimensional complete integral lattices with determinant belonging to $[1, N]$.

It can easily be proved (see Lemma 2 below) that

$$\sum_{n=1}^{N} \#\mathcal{L}_s(\mathbb{Z}; n) \ll_s N^s. \tag{4}$$

Using (2), (3), and (4), it is easy to observe that

$$E_s(N) \asymp_s \ln^{s-1} N \quad \text{for} \quad N \geq 2.$$

Conjecture. *There exists a constant $C(s)$ such that*

$$E_s(N) \sim C(s) \ln^{s-1} N \quad \textit{as} \quad N \to +\infty. \tag{5}$$

For $s = 2$, the asymptotic formula (5) can easily be deduced from Heilbronn's classical result [9] about the average length of a finite continued fraction (see Lemma 3 below); moreover,

$$C(2) = \frac{4 \ln 2}{\zeta(2)}.$$

Throughout, ζ stands for the Riemann zeta-function.

Our aim in the present paper is to prove formula (5) for $s = 3$. This generalizes classical results about the average length of a finite continued fraction with denominator in $[1, N]$ (see [11, 9]) to the two-dimensional case.

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§1. Statement of the main result

If

$$\Omega = \left\{ \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{R}) : \begin{array}{l} (b_1, c_1) \in W_1 \cdot a_1, \\ (a_2, c_2) \in W_2 \cdot b_2, \\ (a_3, b_3) \in W_3 \cdot c_3, \end{array} \right\},$$

where the W_i are Jordan measurable subsets of $\mathbb{R}^2$, then

$$\mu(\Omega) = \int_{W_1 \times W_2 \times W_3} \frac{db_1\, dc_1\, da_2\, dc_2\, da_3\, db_3}{|\det M(a,b,c)|^3}, \quad M(a,b,c) = \begin{pmatrix} 1 & b_1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & b_3 & 1 \end{pmatrix}.$$

Note that μ is invariant under the action of the group $\mathrm{GL}_3(\mathbb{R})$.

We introduce the following sets of matrices:

$$\Omega_1 = \left\{ \begin{pmatrix} a_1 & -b_1 & -c_1 \\ a_2 & b_2 & -c_2 \\ a_3 & -b_3 & c_3 \end{pmatrix} : a_1 < b_1 + c_1 \right\},$$

$$\Omega_2 = \left\{ \begin{pmatrix} a_1 & -b_1 & -c_1 \\ a_2 & b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{pmatrix} : \begin{array}{l} a_2 > c_2 \\ \text{or} \\ c_1 > b_1 \end{array} \right\},$$

$$\Omega_3 = \left\{ \begin{pmatrix} a_1 & -b_1 & -c_1 \\ a_2 & b_2 & -c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \right\},$$

$$\Omega_4 = \left\{ \begin{pmatrix} a_1 & -b_1 & c_1 \\ a_2 & b_2 & -c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} : \begin{array}{l} b_2 < a_2 + c_2 \\ b_1 > c_1 \end{array} \right\},$$

where

$$\begin{aligned} &a_i, b_i, c_i \in \mathbb{R}_+, \quad i = 1, 2, 3, \\ &a_1 > b_1, c_1, \quad b_2 > a_2, c_2, \quad c_3 > a_3 > b_3. \end{aligned}$$

We state the main result of this paper.

Theorem 1. *For N running through the natural numbers, we have*

$$E_3(N) = \frac{4C}{\zeta(2)\zeta(3)} \ln^2 N + O(\ln N + 1),$$

where $C = \mu(\Omega_1) + \mu(\Omega_2) + 2\mu(\Omega_3) + 2\mu(\Omega_4)$.

Remark 1. Apparently, C cannot be expressed in terms of known constants. The calculation of the quantities $\mu(\Omega_i)$ reduces to evaluating certain 6-fold integrals, in which two integrations can be done explicitly. Approximately, we have

$$\mu(\Omega_1) \approx 0.0442, \quad \mu(\Omega_2) \approx 0.0922, \quad \mu(\Omega_3) \approx 0.1119, \quad \mu(\Omega_4) \approx 0.0268,$$
$$C \approx 0.4120, \quad C(3) = \frac{4C}{\zeta(2)\zeta(3)} \approx 0.8335.$$

Note that $C(2) \approx 1.6855$.

The next section contains some preliminaries. In §3 it is shown that finding $E_3(N)$ reduces to the calculation of the number of integral matrices of a specific form. In §4, a formula is presented for the number of integral matrices in a given domain. In the last section we finish the proof of Theorem 1.

§2. Preliminary information

A matrix is said to be basic if its columns constitute a basis of the lattice Γ.

Lemma 1. *For every lattice $\Gamma \in \mathcal{L}_s(\mathbb{Z}; n)$, there exists a unique basic matrix of the form*

$$M = \begin{pmatrix} m_1 & m_{12} & m_{13} & \dots & m_{1s} \\ 0 & m_2 & m_{23} & \dots & m_{2s} \\ 0 & 0 & m_3 & \dots & m_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & m_s \end{pmatrix}, \tag{6}$$

where $0 < m_{ij} \le m_i$, $i = i+1, \dots, s$, $j = 1, \dots, s$, $m_1 m_2 \cdots m_s = n$.

Proof. The existence is well known (see, e.g., [17, Chapter 1]). We prove uniqueness. Suppose M and M' are two matrices of this sort. Then there is a unimodular integral matrix S with $M = M' \cdot S$, and it can easily be proved consecutively that

$$\begin{aligned} s_{ij} = 0 \ \text{ for } \ i > j \ &\Longrightarrow \ s_{ii} = 1 \ \Longrightarrow \ m_i = m'_i, \ i = 1, \dots, s \\ &\Longrightarrow m_{ij} \equiv m'_{ij} \pmod{m_i} \ \forall j \ne i \ \Longrightarrow \ m_{ij} = m'_{ij}, \ i, j = 1, \dots, s. \end{aligned}$$ □

Lemma 2. *For $s \ge 2$ and $N \in \mathbb{N}$, we have*

$$\sum_{n=1}^{N} \#\mathcal{L}_s(\mathbb{Z}; n) = \frac{\zeta(2)\zeta(3)\cdots\zeta(s)}{s} N^s + O_s\left(N^{s-1} \ln N + 1\right). \tag{7}$$

Proof. By Lemma 1, the number of lattices in $\mathcal{L}_s(\mathbb{Z}; n)$ is equal to the number of integral matrices of the form (6). Therefore,

$$\begin{aligned} \#\mathcal{L}_s(\mathbb{Z}; n) &= \sum_{\substack{m_1, \dots, m_s \in \mathbb{N}, \\ m_1 \cdots m_s = n}} m_1^{s-1} m_2^{s-2} \cdots m_{s-1}, \\ \sum_{n=1}^{N} \#\mathcal{L}_s(\mathbb{Z}; n) &= \sum_{\substack{m_1, \dots, m_s \in \mathbb{N}, \\ m_1 \cdots m_s \le N}} m_1^{s-1} m_2^{s-2} \cdots m_{s-1} \\ &= \frac{N^s}{s} \sum_{\substack{m_2, \dots, m_s \in \mathbb{N}, \\ m_2 \cdots m_s \le N}} \frac{1}{m_2^2 m_3^3 \cdots m_s^s} + O_s\left(\sum_{\substack{m_2, \dots, m_s \in \mathbb{N}, \\ m_2 \cdots m_s \le N}} \frac{N^{s-1}}{m_2 m_3^2 \cdots m_s^{s-1}} \right) \\ &= \frac{N^s}{s} \zeta(2)\zeta(3)\cdots\zeta(s) + O_s\left(N^{s-1} \ln N + 1\right). \end{aligned}$$ □

For completeness, we show how to deduce the asymptotic expression for the number of relative minima of two-dimensional lattices (though this result will not be used in the sequel). We put

$$\sigma(N) = \sum_{d|N} d, \quad \Lambda(N) = \begin{cases} \ln p & \text{if } N = p^k, \quad p \text{ is prime}, \\ 0 & \text{otherwise}; \end{cases}$$

these are the sum of divisors of N and the Mangoldt function.

Lemma 3. *For every $N \geq 2$, we have*

$$\sum_{\Gamma \in \mathcal{L}_2(\mathbb{Z};N)} \#\mathfrak{M}(\Gamma) = \frac{4\ln 2}{\zeta(2)} \sum_{d \backslash N} d\left(\ln d - \sum_{r \backslash d} \frac{\Lambda(r)}{r}\right) + O\left(\sigma(N)\right), \tag{8}$$

$$\sum_{n=1}^{N} \sum_{\Gamma \in \mathcal{L}_2(\mathbb{Z};n)} \#\mathfrak{M}(\Gamma) = 2\ln(2) \cdot N^2 \ln N + O(N^2), \tag{9}$$

$$E_2(N) = \frac{4\ln 2}{\zeta(2)} \ln N + O(1). \tag{10}$$

Proof. We denote by $l_d(a)$ the length of the continued fraction expansion of $\frac{a}{d}$, and by $\Gamma(a,d)$ the lattice with the basis $(d,0)$, $(a,1)$ $(a,d \in \mathbb{N})$. Formula (1) shows that

$$\#\mathfrak{M}(\Gamma(a,d)) = 2l_d(a) + 2 \quad \text{for} \quad 0 < a < d, \quad a \neq \frac{d}{2}.$$

Since $\#\mathfrak{M}(\Gamma(a,d)) = O(1)$ and $l_d(a) = O(1)$ for $d \in \{a, 2a\}$, we obtain

$$\sum_{a=1}^{d} \#\mathfrak{M}(\Gamma(a,d)) = 2\sum_{a=1}^{d} l_d(a) + O(d). \tag{11}$$

A well-known result by Porter [12] implies the following formula (see [13, § 4.5.3]):

$$\sum_{a=1}^{d} l_d(a) = \frac{2\ln 2}{\zeta(2)} d\left(\ln d - \sum_{r \backslash d} \frac{\Lambda(r)}{r}\right) + O\left(d\right). \tag{12}$$

By Lemma 1, any lattice in $\mathcal{L}_2(\mathbb{Z};N)$ has a unique basis of the form

$$(d,0), \quad (a,d'), \tag{13}$$

where $dd' = N$, $1 \leq a \leq d$. The transformation of nodes given by $(\gamma_1, \gamma_2) \to (\gamma_1, \gamma_2/d')$ transforms the lattice with the basis (13) to the lattice $\Gamma(a,d)$. Consequently,

$$\sum_{\Gamma \in \mathcal{L}_2(\mathbb{Z};N)} \#\mathfrak{M}(\Gamma) = \sum_{d \backslash N} \sum_{a=1}^{d} \#\mathfrak{M}(\Gamma(a,d)).$$

Applying (11) and (12), we arrive at (8). Now, (9) follows from (8) by summation. It suffices to observe that

$$\begin{aligned}
\sum_{n=1}^{N} \sum_{d \backslash n} d \ln d &= \sum_{n=1}^{N} \sum_{d \backslash n} \frac{n}{d} \cdot \ln(n/d) = \sum_{d=1}^{N} \sum_{\substack{d \leq n \leq N, \\ n \equiv 0 \pmod{d}}} \frac{n}{d} \cdot \ln(n/d) \\
&= \sum_{d=1}^{N} \left(\frac{1}{d} \int_{d}^{N} \frac{t}{d} \cdot \ln(t/d)\, dt + O\left(\frac{N}{d} \ln(N/d)\right)\right) \\
&= \sum_{d=1}^{N} \left(\frac{N^2}{2d^2} \ln(N/d) + O\left(\frac{N^2}{d^2} + \frac{N}{d} \ln(N/d)\right)\right) = \frac{N^2 \ln N}{2} \zeta(2) + O(N^2),
\end{aligned}$$

$$\sum_{n=1}^{N} \sum_{d \backslash n} d \sum_{r \backslash d} \frac{\Lambda(r)}{r} = \sum_{d=1}^{N} \sum_{r \backslash d} \sum_{\substack{1 \leq n \leq N, \\ n \equiv 0 \pmod{d}}} d \cdot \frac{\Lambda(r)}{r} \ll N \sum_{d=1}^{N} \sum_{r \backslash d} \frac{\Lambda(r)}{r} = O(N^2),$$

$$\sum_{n=1}^{N} \sigma(n) = O(N^2).$$

This leads to (9). Formula (10) is a consequence of (9) and (7). □

We pass to three-dimensional lattices. Put

$$\mathcal{U}(N) = \{(\gamma, \Gamma) \,:\, \Gamma \in \mathcal{L}_3(\mathbb{Z}; N),\ \gamma \in \mathfrak{M}(\Gamma)\}, \quad N \in \mathbb{N},$$
$$\mathcal{U}_+(N) = \{(\gamma, \Gamma) \in \mathcal{U}(N) \,:\, \gamma_{1,2,3} > 0\}.$$

By using (2), it is easy to show (see [8]) for more details) that, for every lattice in $\mathcal{L}_3(\mathbb{Z}; n)$ with $n \geq 2$, the number of relative minima with at least one zero coordinate is $O(\ln n)$. So, (4) implies

$$\sum_{\Gamma \in \mathcal{L}_3(\mathbb{Z}; n)} \#\mathfrak{M}(\Gamma) \equiv \#\mathcal{U}(n) = 2^3 \cdot \#\mathcal{U}_+(n) + O\left(\ln n \cdot \#\mathcal{L}_3(\mathbb{Z}; n)\right).$$

Taking (7) into account, we obtain

$$E_3(N) = \frac{24}{N^3 \zeta(2)\zeta(3)} \sum_{n=1}^{\infty} \mathcal{U}_+(n) + O(\ln N). \tag{14}$$

For two-dimensional lattices, the calculation of the cardinality of the set

$$\mathcal{U}_+^{(2)}(N) = \{(\gamma, \Gamma) \,:\, \Gamma \in \mathcal{L}_2(\mathbb{Z}; N),\ \gamma \in \mathfrak{M}(\Gamma),\ \gamma_{1,2} > 0\}$$

is based on the following observation. Let $(\gamma, \Gamma) \in \mathcal{U}_+^{(2)}(N)$. We choose a node $b \in \mathfrak{M}(\Gamma) \setminus \{a\}$ in accordance with the conditions

$$|b_1| \leq a_1, \quad 0 \leq b_2 \to \min. \tag{15}$$

Then $b_1 \leq 0$, $b_2 \geq a_2$, and the vectors a, b form a basis of Γ. Consequently, any pair $(a, \Gamma) \in \mathcal{U}_+^{(2)}(N)$ gives rise to a matrix

$$A \in \omega(N) = \left\{ \begin{pmatrix} a_1 & -b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}; N) \,:\, 0 \leq b_1 \leq a_1,\ 0 < a_2 \leq b_2 \right\}.$$

The converse is also true. If $[a, b] \in \omega(N)$ and Γ is the lattice with the basis a, b, then $a \in \mathfrak{M}(\Gamma)$ and (15) holds true. Therefore, the sets $\mathcal{U}_+^{(2)}(N)$ and $\omega(N)$ are in one-to-one correspondence. Consequently, $\#\mathcal{U}_+^{(2)}(N) = \#\omega(N)$.

For the first time, similar arguments were used in Heilbronn's classical paper [9] for calculation of the average length of a continued fraction. In the next section, we shall obtain a similar correspondence for three-dimensional lattices.

§3. An analog of Heilbronn's correspondence

Along with the sets Ω_i $(i = 1, 2, 3, 4)$ of matrices introduced above, we also employ the sets $\partial\Omega_i$ (the matrices belonging to the boundary of Ω_i) and $\bar{\Omega}_i = \Omega_i \cup \partial\Omega_i$.

If $\Omega \subset \mathbb{R}^{3\times 3}$, then $\omega(N) = \Omega \cap \mathrm{GL}_3(\mathbb{Z}; N)$. In particular,

$$\bar{\omega}_i(N) = \bar{\Omega}_i \cap \mathrm{GL}_3(\mathbb{Z}; N), \quad \partial\omega_i(N) = \partial\Omega_i \cap \mathrm{GL}_3(\mathbb{Z}; N).$$

Our aim in this section is the proof of the following lemma.

Lemma 4. *For every natural number N, we have*

$$\begin{aligned} \#\mathcal{U}_+(N) &= \#\omega_1(N) + \#\omega_2(N) + 2 \cdot \#\omega_3(N) + 2 \cdot \#\omega_4(N) \\ &\quad + O\left(\sum_{i=1}^{4} \#\partial\omega_i(N)\right) + O_\epsilon(N^{1+\epsilon}) \quad \forall \epsilon > 0. \end{aligned} \tag{16}$$

This statement will be verified at the end of the section. The arguments are based on the considerations described below. Let $(a,\Gamma)\in\mathcal{U}_+(N)$. We choose nodes $b\in\mathfrak{M}(\Gamma)\setminus\{a\}$ and $c\in\mathfrak{M}(\Gamma)\setminus\{a,b\}$ in accordance with the following conditions:

$$|b_i|\le a_i,\ i=1,3,\quad 0\le b_2\to\min,$$
$$|c_1|\le a_1,\quad |c_2|\le b_2,\quad 0\le c_3\to\min.$$

If we discard certain "bad" cases (whose number is estimated by the remainder term in (16)) and eliminate certain "symmetric" variants (the sets $V_2(N)$ and $V_3(N)$ below), then a,b,c is a basis of Γ, and moreover, the matrix $[a,b,c]$ belongs to the union of the sets $\omega_i(N)$.

We shall use the so-called minimal sets. Recall the definition.

Definition 1. A set $M\subset\Gamma\in\mathcal{L}_s(\mathbb{R})$ is said to be minimal if $M\subset\mathfrak{M}(\Gamma)$ and there is no nonzero node $\gamma\in\Gamma$ such that

$$|\gamma_i|<\max_{(\eta_1,\dots,\eta_s)\in M}|\eta_i|,\quad i=1,\dots,s.$$

The notion of a minimal set first appeared in the work of Voronoĭ and of Minkowski (see [2, 14]) in connection with methods of constructing units in number fields. They analyzed various properties of minimal systems of two- and three-dimensional latices in general position (without proofs in the 3-dimensional case). A fairly exhaustive study of three-dimensional lattices in general position was done in [15]. Minimal systems of lattices in $\mathcal{L}_3(\mathbb{Z})$ were investigated in [16].

In the present paper, we shall content ourselves with the following result. Denote by $\widetilde{\mathcal{L}}_3(\mathbb{Z};N)$ the set of lattices $\Gamma\in\mathcal{L}_3(\mathbb{Z};N)$ with the following property: if a set $\{a,b,c\}\subset\Gamma$ is minimal and consists of linearly independent nodes, then a,b,c is a basis in Γ. The statement below shows that there are not many lattices without this property.

Lemma 5. *For every natural N we have*

$$\#(\mathcal{L}_3(\mathbb{Z};N)\setminus\widetilde{\mathcal{L}}_3(\mathbb{Z};N))\ll_\epsilon N^{1+\epsilon}\quad\forall\epsilon>0. \tag{17}$$

Proof. Let $\Gamma\in\mathcal{L}_3(\mathbb{Z};N)\setminus\widetilde{\mathcal{L}}_3(\mathbb{Z};N)$. By [16, Theorem 2], there exists a basis of Γ which, up to the order of coordinates and up to signs, has the form

$$(a_1,a_2,a_3),\quad(0,a_2,c_3),\quad(0,0,2c_3), \tag{18}$$

where $a_1,a_2,c_3>0$, $a_3\ge0$. We denote by $L(N)$ the set of lattices in $\mathcal{L}_3(\mathbb{Z};N)$ generated by vectors of the form (18). By using Lemma 1, it is easy to observe that every lattice in $L(N)$ admits a unique basis of the form (18) that obeys additionally the requirement $a_3\in[1,2c_3]$. Consequently,

$$\#L(N)=\sum_{2c_3a_2|N}2c_3\ll_\epsilon N^{1+\epsilon}. \qquad\square$$

It can easily be checked that

$$a_1b_2c_3\le\det A\quad\forall A=\begin{pmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{pmatrix}\in\bigcup_{i=1}^4\bar\Omega_i. \tag{19}$$

Put

$$\widetilde{\mathcal{U}}_+(N)=\{(\gamma,\Gamma)\in\mathcal{U}_+(N)\,:\,\Gamma\in\widetilde{\mathcal{L}}_3(\mathbb{Z};N)\}.$$

From (2) and (17) it follows that

$$\#\big(\mathcal{U}_+(N)\setminus\widetilde{\mathcal{U}}_+(N)\big)\ll\ln^2N\cdot\#\big(\mathcal{L}_3(\mathbb{Z};N)\setminus\widetilde{\mathcal{L}}_3(\mathbb{Z};N)\big)\ll_\epsilon N^{1+\epsilon} \tag{20}$$

for $N \geq 2$ and every $\epsilon > 0$. Thus, it suffices to obtain a formula for the number $\#\widetilde{\mathcal{U}}_+(N)$. We shall use the following result to estimate the cardinality of subsets of $\widetilde{\mathcal{U}}_+(N)$.

Lemma 6. *Suppose the following conditions are fulfilled*:

(a) $U \subset \widetilde{\mathcal{U}}_+(N)$, $\Omega \subset \bigcup_{i=1}^4 \bar{\Omega}_i$, $\omega(N) = \Omega \cap \mathrm{GL}_3(\mathbb{Z}; N)$;

(b) *there exists an injection* $F : \mathbb{Z}^{3\times 3} \to \mathbb{Z}^{3\times 3}$ *that preserves the modulus of the determinant of any matrix*;

(c) *for every pair* $(a, \Gamma) \in U$ *there exist nodes* $b, c \in \Gamma$ *such that the set* $\{a, b, c\}$ *is minimal and* $F([a, b, c]) \in \Omega$.

Then

$$\#U \leq \#\omega(N). \tag{21}$$

Proof. We introduce a map $\Phi : U \to \Omega$ that takes each pair $(a, \Gamma) \in U$ to the matrix $F([a, b, c])$. By (19) and the condition

$$F([a, b, c]) \in \bigcup_{i=1}^4 \widetilde{\Omega}_i,$$

we obtain $\det F([a, b, c]) > 0$. Consequently, $\det[a, b, c] \neq 0$, i.e., the nodes a, b, c are linearly independent and, therefore, they constitute a basis of Γ (because $\Gamma \in \widetilde{\mathcal{L}}_3(\mathbb{Z}; N)$). Then $|\det[a, b, c]| = N$, $\det \Phi(a, \Gamma) = N$, whence $\Phi(a, \Gamma) \in \omega(N)$. It can easily be checked that

$$\Phi(a, \Gamma) = \Phi(a', \Gamma') \quad \Longleftrightarrow \quad (a, \Gamma) = (a', \Gamma').$$

Thus, the map $\Phi : U \to \omega(N)$ is an injection, and (21) is fulfilled. □

Let $(a, \Gamma) \in \widetilde{\mathcal{U}}_+(N)$. We denote

$$H_2(a, \Gamma) = \{\gamma \in \mathfrak{M}(\Gamma) \setminus \{\pm a\} \,:\, |\gamma_i| \leq a_i, \;\; i = 1, 3\}.$$

We observe immediately that the set $H_2(a, \Gamma)$ is nonempty because it contains a node of the form $(0, n, 0)$. Moreover,

$$|\gamma_2| \geq a_2 \quad \forall \gamma \in H_2(a, \Gamma),$$

because otherwise a is not minimal.

Also, for every pair $(a, \Gamma) \in \widetilde{\mathcal{U}}_+(N)$ we define

$$\widetilde{H}_2(a, \Gamma) = \{\gamma \in \Gamma \setminus \{0, \pm a\} \,:\, |\gamma_i| \leq a_i,\, i = 1, 3\} \setminus \{\gamma \in \Gamma \,:\, |\gamma_i| = a_i,\, i = 1, 3,\, |\gamma_2| > a_2\}.$$

Lemma 7. *Suppose that* $(a, \Gamma) \in \widetilde{\mathcal{U}}_+(N)$ *and a node* $b \in H_2(a, \Gamma)$ *satisfies the inequalities*

$$0 < b_2 \leq |\gamma_2| \quad \forall \gamma \in H_2(a, \Gamma). \tag{22}$$

Then

$$b_2 \leq |\gamma_2| \quad \forall \gamma \in \widetilde{H}_2(a, \Gamma), \tag{23}$$

and the set $\{a, b\}$ *is minimal*; *moreover, either* $b_1 \leq 0$ *or* $b_3 \leq 0$.

Proof. We verify (23). Suppose $\gamma \in \Gamma \setminus \{0, \pm a\}$,

$$|\gamma_1| \leq a_1, \;\; |\gamma_2| < b_2, \;\; |\gamma_3| \leq a_3, \;\; \gamma \neq \pm a.$$

Since $\gamma \neq \pm a$, we have $\gamma \notin \mathfrak{M}(\Gamma)$ (otherwise (22) fails); therefore, there is a node $\eta \in \mathfrak{M}(\Gamma)$ with

$$|\eta_i| \leq |\gamma_i|, \quad i = 1, 2, 3, \quad \sum_{i=1}^3 |\eta_i| < \sum_{i=1}^3 |\gamma_i|$$
$$\Longrightarrow \quad |\eta_1| \leq a_1, \;\; |\eta_2| < b_2, \;\; |\eta_3| \leq a_3.$$

By (22), this is possible only if $\eta = \pm a$. Consequently,

$$a_i = |\eta_i| \le |\gamma_i| \le a_i \implies |\gamma_i| = a_i, \quad i = 1, 3.$$

Thus, $a_2 < |\gamma_2|$. This proves (23). The minimality of the set $\{a, b\}$ follows from (23). If $b_{1,3} > 0$, then the node $\gamma = a - b$ violates the minimality of $\{a, b\}$. □

We denote by $V(N)$ the set of $(a, \Gamma) \in \widetilde{\mathcal{U}}_+(N)$ for which a node $b \in H_2(a, \Gamma)$ satisfying (22) is unique. We show that the number of elements not belonging to $V(N)$ is fairly small.

Lemma 8. *We have*

$$\#\widetilde{\mathcal{U}}_+(N) = \#V(N) + O(\#\partial\omega_2(N) + \#\partial\omega_4(N)). \tag{24}$$

Proof. Suppose that $(a, \Gamma) \in \widetilde{\mathcal{U}}_+(N) \setminus V(N)$. Then there are $b, b' \in H_2(a, \Gamma)$ satisfying (22). Clearly, $b_2 = b'_2 > 0$. If $b_1 b'_1 \ge 0$ and $b_3 b'_3 \ge 0$, then the node $b - b'$ violates the minimality of a. Thus, either $b_1 b'_1 < 0$, or $b_3 b'_3 < 0$. Consequently, up to the interchangement of b and b', only the following variants are possible:

$$b = (-b_1, b_2, -b_3), \qquad b' = (-b'_1, b_2, b'_3), \tag{25}$$

$$b = (-b_1, b_2, -b_3), \qquad b' = (b'_1, b_2, -b'_3), \tag{26}$$

$$b = (-b_1, b_2, b_3), \qquad b' = (b'_1, b_2, -b'_3), \tag{27}$$

where $b_{1,3}$, $b'_{1,3} \ge 0$. We introduce the sets X_1, X_2, and X_3 that consist of $(a, \Gamma) \in (\mathcal{U}_+(N) \setminus V(N))$ such that b and b' have the form (25), (26), or (27) (respectively). Clearly,

$$\#X_1 = \#X_2,$$

because the interchangement of the 1st and 3d coordinates of nodes generates a bijection between X_1 and X_2. It remains to prove that

$$\#X_1 \ll \#\partial\omega_2(N), \quad \#X_3 \ll \#\partial\omega_4(N). \tag{28}$$

Observe at once that the set $\{a, b, b'\}$ is minimal and

$$b_1 + b'_1 > a_1 \quad \text{for } (a, \Gamma) \in X_1, \tag{29}$$

$$b_1 + b'_1 > a_1 \text{ or } b_3 + b'_3 > a_3 \quad \text{for } (a, \Gamma) \in X_3. \tag{30}$$

(Otherwise the node $b - b'$ violates the minimality of a.) We introduce the set $X_{31} \subset X_3$ ($X_{32} \subset X_3$) consisting of the pairs (a, Γ) for which the nodes b, b' satisfy $b_1 + b'_1 > a_1$ ($b_3 + b'_3 > a_3$). Then

$$\#X_{31} = \#X_{32}, \quad \#X_3 \le 2 \cdot \#X_{31}.$$

Let $(a, \Gamma) \in X_1$. We prove that $b'_3 \ge b_3$. Put

$$\gamma = a - b' + b = (a_1 - b'_1 - b_1,\ a_2,\ a_3 + b'_3 - b_3).$$

Since $a_1 \le b_1 + b'_1 \le 2a_1$, we have $|\gamma_1| \le a_1$. Since $b_3 \le a_3$, for $b'_3 < b_3$ the node γ violates the minimality of a. Therefore, $b'_3 \ge b_3$. By (29) and the conditions on b and b', the matrix $[a, b, b' - b]$ belongs to $\partial\Omega_2$. Consequently, the first inequality in (28) holds true (see Lemma 6).

Let $(a, \Gamma) \in X_{31}$. Then $[a, b, (a - b')] \in \partial\Omega_4$ and, by Lemma 6, the second inequality in (28) holds true. □

On the set $V(N)$, we can introduce a mapping Φ_2 that takes any pair $(a, \Gamma) \in V(N)$ to the node $b \in H_2(a, \Gamma)$ that satisfies (22). Note that

$$b_2 < |\gamma_2| \quad \forall \gamma \in \widetilde{H}_2(a, \Gamma),\ \gamma \ne \pm b, \tag{31}$$

for $b = \Phi_2(a, \Gamma)$, $(a, \Gamma) \in V(N)$. This can be checked much as in the proof of (23).

We introduce the set

$$H_3(a,\Gamma)=\{\gamma\in\mathfrak{M}(\Gamma)\setminus\{\pm a,\pm b\}\,:\,|\gamma_1|\le a_1,\ |\gamma_2|\le b_2\}.$$

It is nonempty (for example, it contains a node of the form $(0,0,n)$). Moreover,

$$|\gamma_3|>a_3\quad\forall\gamma\in H_3(a,\Gamma),$$

because otherwise γ violates (31).

We shall also employ the set $C(a,\Gamma)$ consisting of the nodes c that satisfy the conditions

$$c\in H_3(a,\Gamma),\quad 0<c_3\le|\gamma_3|\quad\forall\gamma\in H_3(a,\Gamma),\tag{32}$$

and

$$\begin{aligned}\Pi_1(a)&=\{x\in\Gamma\,:\,|x_1|=a_1,\ |a_i|\le|x_1|,\ i=2,3,\ |a_2|+|a_3|<|x_2|+|x_3|\}\,,\\ \Pi_2(b)&=\{x\in\Gamma\,:\,|x_2|=b_1,\ |b_i|\le|x_i|,\ i=1,3,\ |b_1|+|b_3|<|x_1|+|x_3|\}\,,\\ \widetilde{H}_3(a,\Gamma)&=\{\gamma\in\Gamma\setminus\{0,\pm a,\pm b\}\,:\,|\gamma_1|\le a_1,\ |\gamma_2|\le b_2,\ \ \gamma\notin\Pi_1(a)\cup\Pi_2(b)\}\,.\end{aligned}$$

Lemma 9. *Suppose* $(a,\Gamma)\in V(N)$, $b=\Phi_1(a,\Gamma)$, $c\in C(a,\Gamma)$. *Then*

$$c_3\le|\gamma_3|\quad\forall\gamma\in\widetilde{H}_3(a,\Gamma),\tag{33}$$

the set $\{a,b,c\}$ *is minimal, and either* $c_1\le 0$ *or* $c_2\le 0$.

The proof is similar to that of Lemma 7.

We show that the cases where $b_1=0$ or $b_3=0$ are fairly rare. Put

$$V_0(N)=\{(a,\Gamma)\,:\,b_1=0\text{ or }b_3=0\}\,,$$

where $b=\Phi_1(a,\Gamma)$.

Lemma 10. *We have*

$$\#V_0(N)\ll\sum_{i=2}^{4}\#\partial\omega_i(N).\tag{34}$$

Proof. Since $b_1\le 0$ or $b_3\le 0$ (see Lemma 7), the set $V_0(N)$ can be represented in the form

$$\begin{aligned}V_0(N)&=V_{01}\cup V_{01}'\cup V_{02}\cup V_{02}',\\ V_{01}&=\{(a,\Gamma)\in V(N)\,:\,b=(0,b_2,b_3),\ b_3\ge 0\},\\ V_{01}'&=\{(a,\Gamma)\in V(N)\,:\,b=(b_1,b_2,0),\ b_1\ge 0\},\\ V_{02}&=\{(a,\Gamma)\in V(N)\,:\,b=(-b_1,b_2,0),\ b_1>0\},\\ V_{02}'&=\{(a,\Gamma)\in V(N)\,:\,b=(0,b_2,-b_3),\ b_3>0\},\end{aligned}$$

where $b=\Phi_1(a,\Gamma)$. We have $b_2>0$ in all cases. Observe that

$$\#V_{01}=\#V_{01}',\quad\#V_{02}=\#V_{02}'$$

(the transformation interchanging the first and the third coordinates of nodes is a bijection between V_{01} and V_{01}', and also between V_{02} and V_{02}'). Thus, it suffices to estimate the cardinalities of V_{01} and V_{02}.

We estimate $\#V_{01}$. Let $(a,\Gamma)\in V_{01}$. Then $b=(0,b_2,b_3)$, $a_3\ge b_3\ge 0$, $b_2\ge a_2>0$. The node $\gamma=a-b$ satisfies the relations

$$|\gamma_1|=a_1,\quad|\gamma_2|=b_2-a_2<b_2,\quad|\gamma_3|=a_3-b_3\le a_3.$$

By (31), it follows that either $\gamma\in\Pi_1(a)$ or $\gamma=\pm a$. Therefore,

$$\gamma_2\ge a_2,\ |\gamma_3|\ge a_3\implies b_2\ge 2a_2,\ b_3=0.$$

Thus, $b = (0, b_2, 0)$, $b_2 \geq 2a_2$. We choose $c \in C(a, \Gamma)$ arbitrarily. Then either $c_1 \leq 0$ or $c_2 \leq 0$ (see Lemma 9). We show that the case where $c = (c_1, -c_2, c_3)$, $c_{1,3} > 0$, $c_2 \geq 0$ is impossible. Indeed, in this case the node $\gamma = a - c - b$ satisfies the relations

$$|\gamma_1| = a_1 - c_1 < a_1, \quad |\gamma_2| = |a_2 + c_2 - b_2| < b_2, \quad |\gamma_3| = c_3 - a_3 < c_3$$

(the second inequality follows from the estimates $a_2 > 0$; $2a_2, c_2 \leq b_2$). By the minimality of $\{a, b, c\}$, this is possible only if $\gamma = 0$. However, in this case the node $-c = b - a$ contradicts the choice of b. Thus, only the following two cases may occur:

1) $c = (-c_1, c_2, c_3)$, $c_{1,2,3} \geq 0$, then $[a, b, c] \in \partial\Omega_2$;

2) $c = (-c_1, -c_2, c_3)$, $c_{1,2,3} \geq 0$, then $[a, b, c] \in \partial\Omega_3$.

So, we have proved that

$$\#V_{01} = \#V'_{01} \leq \#\partial\omega_2(N) + \#\partial\omega_3(N)$$

(see Lemma 6).

We estimate $\#V_{02}$. Let $(a, \Gamma) \in V_{02}$. Then $b = (-b_1, b_2, 0)$, $b_{1,2} > 0$. Take an arbitrary node $c \in C(a, b)$. The following cases are possible.

If $c = (-c_1, -c_2, c_3)$, $c_{1,2} \geq 0$, then $[a, b, c] \in \partial\Omega_3$.

Let $c = (-c_1, c_2, c_3)$, $c_{1,2} \geq 0$. Then the node $\gamma = a + b - c$ satisfies the relations

$$|\gamma_1| = a_1 - b_1 + c_1, \ |\gamma_2| = b_2 + a_2 - c_2, \ |\gamma_3| = c_3 - a_3 < c_3.$$

Therefore, either $c_1 \geq b_1$ or $a_2 \geq c_2$ (otherwise $0 < |\gamma_1| < a_1$, $|\gamma_2| < b_2$, and the minimality of $\{a, b, c\}$ is violated). Thus, $[a, b, c] \in \partial\Omega_2$.

Let $c = (c_1, -c_2, c_3)$, $c_1 > 0$, $c_2 \geq 0$. Then the node $\gamma = a - c$ satisfies the relations

$$|\gamma_1| = a_1 - c_1 < a_1, \ |\gamma_2| = a_2 + c_2, \ |\gamma_3| = c_3 - a_3 < c_3$$

and $a_2 + c_2 = b_2$ by the minimality of $\{a, b, c\}$. We put $\eta = a - b - c$. Then

$$|\eta_1| = a_1 + b_1 - c_1 > 0, \ |\eta_2| = a_2 + c_2 - b_2 = 0, \ |\eta_3| = c_3 - a_3 < c_3.$$

Thus, $b_1 \geq c_1$ (otherwise $\{a, b, c\}$) is not minimal), $[a, b, c] \in \partial\Omega_4$.

We have proved that

$$\#V_{02} = \#V'_{02} \ll \sum_{i=2}^{4} \#\partial\omega_i(N). \qquad \square$$

We split the set $V(N) \setminus V_0(N)$ into mutually disjoint parts:

$$V_1(N) = \{(a, \Gamma) \in V(N) \,:\, \Phi_1(a, \Gamma) = (-b_1, b_2, -b_3)\},$$
$$V_2(N) = \{(a, \Gamma) \in V(N) \,:\, \Phi_1(a, \Gamma) = (-b_1, b_2, b_3)\},$$
$$V_3(N) = \{(a, \Gamma) \in V(N) \,:\, \Phi_1(a, \Gamma) = (b_1, b_2 - b_3)\},$$

where $b_{1,2,3} > 0$. Then

$$V(N) = \bigcup_{i=0}^{3} V_i(N).$$

Since $\#V_2(N) = \#V_3(N)$ (the map interchanging the first and the third coordinates of nodes is a bijection between $V_2(N)$ and $V_3(N)$), from (24) and (34) it follows that

$$\#\widetilde{\mathcal{U}}_+(N) = \#V_1(N) + 2\#V_2(N) + O\Big(\sum_{i=2}^{4} \#\partial\omega_i(N)\Big). \tag{35}$$

We put

$$\widetilde{V}_1(N) = \{(a, \Gamma) \in V_1(N) \,:\, c_1 < 0 \ \forall c \in C(a, \Gamma)\},$$
$$\widetilde{V}_2(N) = \{(a, \Gamma) \in V_2(N) \,:\, c_2 < 0 \ \forall c \in C(a, \Gamma)\}$$

and prove that the collection of elements not belonging to these sets has a "relatively small" cardinality.

Lemma 11. *We have*

$$\#V_1(N) = \#\widetilde{V}_1(N) + O\big(\#\partial\omega_1(N) + \#\partial\omega_2(N)\big), \tag{36}$$

$$\#V_2(N) = \#\widetilde{V}_1(N) + O\big(\#\partial\omega_3(N)\big). \tag{37}$$

Proof. Suppose that $(a,\Gamma) \in V_1(N) \setminus \widetilde{V}_1(N)$, and let $b = \Phi_1(a,\Gamma)$. Then

$$b = (-b_1, b_2, -b_3), \quad b_{1,2,3} > 0$$

and there exists a node $c \in C(a,\Gamma)$ with $c_1 \geq 0$; by Lemma 9, either $c_1 = 0$ or $c_2 \leq 0$. Consequently, the following 5 cases are possible.

1. If $c = (0,0,c_3)$, then $[a,b,c] \in \partial\Omega_2$.
2. If $c = (0,-c_2,c_3)$ and $c_2 > 0$, then the node $\gamma = b + c$ satisfies the relations

$$|\gamma_1| = b_1 \leq a_1, \quad |\gamma_2| = b_2 - c_2 < b_2, \quad |\gamma_3| = c_3 - b_3 < c_3;$$

therefore, $a_1 = b_1$ (otherwise the minimality of $\{a,b,c\}$ is violated) and $[a,b,c] \in \partial\Omega_1$.

3. Let $c = (0,c_2,c_3)$ with $c_{2,3} > 0$, then the node $\gamma = a - c$ satisfies the relations

$$|\gamma_1| = a_1, \quad |\gamma_2| = |a_2 - c_2| < b_2, \quad |\gamma_3| = c_3 - a_3 < c_3,$$

and formula (33) implies that

$$a_2 \leq |\gamma_2|,\ a_3 \leq |\gamma_3| \quad \Longrightarrow \quad c_2 \geq 2a_2,\ c_3 > 2a_3.$$

Put $\eta = a + b - c$. Then

$$|\eta_1| = a_1 - b_1 < a_1, \quad |\eta_2| = a_2 + b_2 - c_2 < b_2, \quad |\eta_3| = c_3 + b_3 - a_3 \leq c_3.$$

Therefore, $b_3 = a_3$ and the matrix $[a,b,c-a]$ belongs to $\partial\Omega_2$.

4. Let $c = (c_1,0,c_3)$ with $c_1 > 0$. The node $\gamma = a - c$ satisfies the relations

$$|\gamma_1| = a_1 - c_1 < a_1, \quad |\gamma_2| = a_2, \quad |\gamma_3| = c_3 - a_3 < c_3.$$

By (33), this is possible only if $\gamma \in \Pi_2(b)$, i.e.,

$$b_i \leq |\gamma_i|,\ i = 1,3,\ |\gamma_2| = b_2 \quad \Longrightarrow \quad a_2 = b_2, \quad a_3 + b_3 \leq c_3.$$

Next, consider the node $\theta = a - b - c$. It satisfies the relations

$$0 < |\theta_1| = a_1 + b_1 - c_1,\ |\theta_2| = 0,\ |\theta_3| = c_3 - (a_3 + b_3) < c_3.$$

Consequently, $b_1 > c_1$ (otherwise (33) fails). Then the node $\eta = b + c$ contradicts (33). We see that this case is impossible.

5. The case where $c = (c_1,-c_2,c_3)$, $c_{1,2} > 0$, is also impossible, because then the node $b + c$ violates the minimality of $\{a,b,c\}$.

Applying Lemma 6, we arrive at

$$\#(V_1(N) \setminus \widetilde{V}_1(N)) \ll \#\partial\omega_1(N) + \#\partial\omega_2(N).$$

This proves (36).

Now, we prove (37). Suppose $(a,\Gamma) \in V_2(N) \setminus \widetilde{V}_2(N)$, and let $b = \Phi_1(a,\Gamma)$, $b = (-b_1,b_2,b_3)$ with $b_{1,2,3} > 0$. Then there exists $c \in C(a,\Gamma)$ with the property that $c_2 \geq 0$, and moreover, either $c_2 = 0$ or $c_1 \leq 0$ (by Lemma 9). The following 4 cases are possible.

1. If $c = (-c_1,0,c_3)$, $c_1 \geq 0$, then $[a,b,c] \in \partial\Omega_3$.
2. If $c = (-c_1,c_2,c_3)$, $c_{1,2} > 0$, then the node $b - c$ violates the minimality of $\{a,b,c\}$.
3. Let $c = (c_1,0,c_3)$ with $c_1 > 0$. Then the node $\gamma = a - c$ satisfies the relations

$$|\gamma_1| = a_1 - c_1 < a_1, \quad |\gamma_2| = a_2, \quad |\gamma_3| = c_3 - a_3 < c_3.$$

Thus, $c_3 \geq 2a_3$ (otherwise $a \notin \mathfrak{M}(\Gamma)$) and $[a,b,c-b] \in \partial\Omega_3$.

4. It only remains to consider the case where $c = (0, c_2, c_3)$, $c_2 > 0$. The nodes $\gamma = a - c$ and $\eta = b - c$ satisfy the relations

$$|\gamma_1| = a_1, \quad |\gamma_2| = |a_2 - c_2| < b_2, \quad |\gamma_3| = c_3 - a_3 < c_3,$$
$$|\eta_1| = |b_1|, \quad |\eta_2| = b_2 - c_2 < b_2, \quad |\eta_3| = c_3 - b_3 < c_3.$$

By (33), this is possible only if η, $\gamma \in \Pi_1(a)$,

$$a_i \le |\gamma_i|, |\eta_i|, \ i = 2, 3, \ |\eta_1| = |\gamma_1| = a_1$$
$$\Longrightarrow a_2 \ge 2c_2, \ c_3 \ge 2a_3, \ a_1 = b_1, \ b_2 \ge a_2 + c_2, \ c_3 \ge a_3 + b_3,$$

and the matrix $[a, b, c - a]$ belongs to $\partial\Omega_3$.

Applying Lemma 6, we obtain

$$\#(V_2(N) \setminus \widetilde{V}_2(N)) \ll \#\partial\omega_3(N).$$

This proves formula (37). □

Let $(a, \Gamma) \in \widetilde{V}_1(N) \cup \widetilde{V}_2(N)$. If there exist nodes $c, c' \in C(a, \Gamma)$ with $c_i c'_i \ge 0$, $i = 1, 2$, then $\gamma = c - c'$ satisfies the relations

$$|\gamma_1| \le a_1, \quad |\gamma_2| \le b_2, \quad \gamma_3 = 0.$$

This contradicts (31). Therefore, either $C(a, \Gamma)$ is a singleton or it consists of two elements one of which is a node of the form $c = (-c_1, -c_2, c_3)$, $c_{1,2} > 0$. Consequently, we can introduce a mapping $\Phi_2 : \widetilde{V}_1(N) \cup \widetilde{V}_2(N) \to \mathbb{Z}^3$ acting in accordance with the rule $\Phi_2(a, \Gamma) = c$, where $c \in C(a, \Gamma)$ is as above. If there are two elements with these property, we choose c to be of the form $c = (-c_1, c_2, -c_3)$ with $c_{1,2,3} > 0$.

Lemma 12. *We have*

$$\#\widetilde{V}_1(N) \le \#\bar{\omega}_1(N) + \#\bar{\omega}_2(N), \quad \#\widetilde{V}_2(N) \le \#\bar{\omega}_3(N) + \#\bar{\omega}_4(N). \tag{38}$$

Proof. Suppose that $(a, \Gamma) \in \widetilde{V}_1(N)$, let $b = \Phi_1(a, \Gamma)$, and let $c = \Phi_2(a, \Gamma)$. Then $b = (-b_1, b_2, -b_3)$ with $b_{1,2,3} > 0$. Two cases are possible.

1. Let $c = (-c_1, -c_2, c_3)$, $c_{1,2,3} > 0$. Put $\gamma = b + c$. Then

$$|\gamma_1| = |b_1 + c_1|, \quad |\gamma_2| = |b_2 - c_2| < b_2, \quad |\gamma_3| = c_3 - b_3 < c_3.$$

Consequently, $a_1 \le b_1 + c_1$ (otherwise γ violates the minimality of $\{a, b, c\}$) and $[a, b, c] \in \bar{\Omega}_1$.

2. Suppose $c = (-c_1, c_2, c_3)$, $c_{1,3} > 0$, $c_2 \ge 0$. In this case $C(a, \Gamma)$ is a singleton, and, as in Lemma 7, we prove that

$$c_3 < |\gamma_3| \ \forall \gamma \in \widetilde{H}_3(a, \Gamma), \ \gamma \ne \pm c. \tag{39}$$

Put $\gamma = -b - a + c$. Then

$$|\gamma_1| = |a_1 + (c_1 - b_1)|, \quad |\gamma_2| = |b_2 + (a_2 - c_2)|, \quad 0 < |\gamma_3| = c_3 + (b_3 - a_3) \le c_3.$$

Consequently, either $c_1 \ge b_1$ or $a_2 \ge c_2$ (otherwise γ violates (39)) and $[a, b, c] \in \bar{\Omega}_2$. By Lemma 6, the first estimate in (38) is true.

Let $(a, \Gamma) \in \widetilde{V}_2(N)$, let $b = \Phi_1(a, \Gamma)$, and let $c = \Phi_2(a, \Gamma)$. Then $b = (-b_1, b_2, b_3)$, $b_{1,2,3} > 0$. Two cases are possible.

1. Suppose $c = (-c_1, -c_2, c_3)$, $c_{1,2,3} > 0$. Then $[a, b, c] \in \bar{\Omega}_3$.

2. Suppose $c = (-c_1, c_2, c_3)$, $c_{1,3} > 0$, $c_2 \ge 0$. Then the set $C(a, \Gamma)$ is a singleton and (39) is fulfilled. Put $\gamma = -a + c$, then

$$|\gamma_1| = |a_1 - c_1| < a_1, \quad |\gamma_2| = |a_2 + c_2|, \quad |\gamma_3| = c_3 - a_3 < c_3.$$

Therefore, $a_2 + c_2 \geq b_2$ (otherwise γ violates the minimality of $\{a, b, c\}$). We prove that $b_1 \geq c_1$. For this, consider the node $\eta = -a + b + c$. We have

$$|\eta_1| = |a_1 + (b_1 - c_1)|, \quad |\eta_2| = a_2 + c_2 - b_2 \leq b_2 \quad (\text{because } a_2 + c_2 \leq 2b_2),$$
$$|\eta_3| = c_3 + (b_3 - a_3) \leq c_3 \quad (\text{because } b_3 \leq a_3).$$

If $a_2 + c_2 < 2b_2$, then $|\eta_2| < b_2$, and, therefore, $b_1 \geq c_1$ (otherwise η violates condition (39)). Let $a_2 + c_2 = 2b_2$. Then $a_2 = b_2 = c_2$, consequently, $|a_1| = |b_1|$, $|a_3| = |b_3|$ (otherwise b violates the minimality of a), and $b_1 = a_1 \geq c_1$. Hence, $[a, b, c] \in \overline{\Omega}_4$. Applying Lemma 6, we obtain the second inequality in (38). □

Now, we prove the existence of injections

$$\omega_1(N) \cup \omega_2(N) \to \widetilde{V}_1(N), \quad \omega_3(N) \cup \omega_4(N) \to \widetilde{V}_2(N). \tag{40}$$

Lemma 13. *Let a lattice Γ be generated by vectors a, b, c. Consider the matrix $A = [a, b, c]$. Then:*

if $A \in \omega_1(N) \cup \omega_2(N)$, then $(a, \Gamma) \in \widetilde{V}_1(N)$, and
if $A \in \omega_3(N) \cup \omega_4(N)$, then $(a, \Gamma) \in \widetilde{V}_2(N)$;

furthermore, $b = \Phi_1(a, \Gamma)$ and $c = \Phi_2(a, \Gamma)$ in both cases.

Proof. It suffices to show that there are no integers m, n, and k with the property that the node

$$\gamma = ma + nb + kc$$

satisfies the conditions

$$|\gamma_1| \leq a_1, \quad |\gamma_2| \leq b_2, \quad |\gamma_3| \leq c_3, \quad \gamma \notin \{0, \pm a, \pm b, \pm c\}.$$

The numbers m, n, and k will be called coefficients.

If two coefficients are equal to zero and the third is of modulus greater than 1, then either $|\gamma_1| > b_1$, or $|\gamma_2| > b_2$, or $|\gamma_3| > c_3$.

Suppose that precisely one of the coefficients is equal to zero. Then:

if $m \cdot n > 0$, $k = 0$, then $|\gamma_2| = |ma_2| + |nb_2| > |b_2|$;
if $m \cdot n < 0$, $k = 0$, then $|\gamma_1| = |ma_1| + |nb_1| > |a_1|$;
if $m \cdot k > 0$, $n = 0$, then $|\gamma_3| = |ma_3| + |kc_3| > |c_3|$;
if $m \cdot k < 0$, $n = 0$, then

$$\begin{aligned} |\gamma_1| &= |ma_1| + |kc_1| > |a_1| && \text{for } A \in \omega_1(N) \cup \omega_2(N) \cup \omega_3(N), \\ |\gamma_2| &= |ma_2| + |kc_2| \geq |a_2| + |c_2| > |b_2| && \text{for } A \in \omega_4(N); \end{aligned}$$

if $m = 0$, $n \cdot k > 0$, then

$$\begin{aligned} |\gamma_1| &= |nb_1| + |kc_1| \geq |b_1| + |c_1| > a_1 && \text{for} \quad A \in \omega_1(N), \\ |\gamma_2| &= |nb_2| + |kc_2| > b_2 && \text{for} \quad A \in \omega_2(N), \\ |\gamma_3| &= |nb_3| + |kc_3| > c_3 && \text{for} \quad A \in \omega_3(N) \cup \omega_4(N), \end{aligned}$$

if $m = 0$, $n \cdot k < 0$, then

$$\begin{aligned} |\gamma_3| &= |nb_3| + |kc_3| > c_3 && \text{for } A \in \omega_1(N) \cup \omega_2(N), \\ |\gamma_2| &= |nb_2| + |kc_2| > b_2 && \text{for } A \in \omega_3(N) \cup \omega_4(N). \end{aligned}$$

It remains to consider the cases where m, n, $k \neq 0$. There is no loss of generality in assuming that $m > 0$. Four cases are possible.

1. Let $n, k > 0$.

If $A \in \omega_1(N)$, then

$$\begin{aligned} |\gamma_3| &= |ma_3| - |nb_3| + |kc_3| \geq ||kc_3| - |(n-1)b_3| > c_3 && \text{for } k \geq n, \\ |\gamma_2| &= |ma_2| + |nb_2| - |kc_2| > b_2 && \text{for } k < n. \end{aligned}$$

If $A \in \omega_2(N)$, then $|\gamma_2| = |ma_2| + |nb_2| + |kc_2| > |b_2|$.
If $A \in \omega_3(N) \cup \omega_4(N)$, then $|\gamma_3| = |ma_3| + |nb_3| + |kc_3| > |c_3|$.
2. Let $n, k < 0$.
If $A \in \omega_1(N) \cup \omega_2(N) \cup \omega_3(N)$, then $|\gamma_1| = |ma_1| + |nb_1| + |kc_1| > |a_1|$.
If $A \in \omega_4(N)$, then

$$\begin{aligned} |\gamma_1| &= |ma_1| + |nb_1| - |kc_1| > a_1 && \text{for } m > k \text{ or } n \geq k, \\ |\gamma_3| &= -|ma_3| + |nb_3| + |kc_3| > c_3 && \text{for } m < k, \\ |\gamma_2| &= |ma_2| - |nb_2| + |mc_2| > b_2 && \text{for } k = m > n. \end{aligned}$$

3. Let $n < 0$ and $k > 0$.
If $A \in \omega_1(N) \cup \omega_2(N)$, then $|\gamma_3| = |ma_3| + |nb_3| + |kc_3| > |c_3|$.
If $A \in \omega_3(N)$, then

$$\begin{aligned} |\gamma_2| &= -|ma_2| + |nb_2| + |kc_2| > b_2 && \text{for } n > m, \\ |\gamma_3| &= |ma_3| - |nb_3| + |kc_3| > c_3 && \text{for } n \leq m. \end{aligned}$$

If $A \in \omega_4(N)$, then $|\gamma_1| = |ma_1| + |nb_1| + |kc_1| > |a_1|$.
4. Let $n > 0$ and $k < 0$.
If $A \in \omega_1(N) \cup \omega_3(N) \cup \omega_4(N)$, then $|\gamma_2| = |ma_2| + |nb_2| + |kc_2| > |b_2|$.
If $A \in \omega_2(N)$, then

$$\begin{aligned} |\gamma_1| &= |ma_1| - |nb_1| + |kc_1| > a_1 && \text{for } \ m > n, \\ |\gamma_2| &= |ma_2| + |nb_2| - |kc_2| > b_2 && \text{for } \ n > k, \\ |\gamma_3| &= -|ma_3| + |nb_3| + |kc_3| > c_3 && \text{for } \ k > m, \\ |\gamma_1| &> a_1 \ \text{ or } \ |\gamma_2| > b_2 && \text{for } \ k = m = n. \end{aligned}$$

□

Proof of Lemma 4. By (20), (35), (36), and (37), it suffices to prove that

$$\begin{aligned} \#\widetilde{V}_1(N) &= \#\omega_1(N) + \#\omega_2(N) + O\big(\#\partial\omega_1(N) + \#\partial\omega_2(N)\big), \\ \#\widetilde{V}_2(N) &= \#\omega_3(N) + \#\omega_4(N) + O\big(\#\partial\omega_3(N) + \#\partial\omega_4(N)\big). \end{aligned} \tag{41}$$

From (38) it follows that

$$\begin{aligned} \#\widetilde{V}_1(N) &\leq \#\omega_1(N) + \#\omega_2(N) + O\big(\#\partial\omega_1(N) + \#\partial\omega_2(N)\big), \\ \#\widetilde{V}_2(N) &\leq \#\omega_3(N) + \#\omega_4(N) + O\big(\#\partial\omega_3(N) + \#\partial\omega_4(N)\big). \end{aligned} \tag{42}$$

By Lemma 13, the injections (40) exist (they act by the formula $A \to (a, \Gamma)$, where a is the first column of the matrix A and the lattice Γ is generated by the columns of A). Taking into account the relations

$$\omega_1(N) \cap \omega_2(N) = \varnothing, \quad \omega_3(N) \cap \omega_4(N) = \varnothing,$$

we obtain

$$\#\omega_1(N) + \#\omega_2(N) \leq \#\widetilde{V}_1(N), \quad \#\omega_3(N) + \#\omega_4(N) \leq \#\widetilde{V}_2(N). \tag{43}$$

Now, (42) is a consequence of (43) and (41). □

§4. The number of integral matrices in a given domain

By Lemma 4, our initial problem reduces to the calculation of the number of integral matrices that lie in a certain domain Ω_i (not depending on N) with determinant belonging to $[1, N]$. It can easily be observed that this is equivalent to the problem about the number of integral points in a fixed domain. In the two-dimensional case, the following formula is well known for an arbitrary domain U with boundary ∂U:

$$\#(U \cap \mathbb{Z}^2) = \operatorname{meas} U + O(\operatorname{meas} \partial U + 1),$$

where meas is Lebesgue measure. For a convex domain, this is a consequence of the Jarnik inequality. The proof for nonconvex domains can be found, e.g., in [18]. It is easily seen that the formula fails for s-dimensional domains with $s \geq 3$. For that, it suffices to consider the cylinder

$$\{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2 < L^{-4},\ \ 0<z<L\}$$

as $L \to +\infty$. In the present paper, the following result will suffice. Put

$$\begin{aligned} \|x\|_\infty &= \max_{1\leq i\leq s} |x_i| \quad \forall x \in \mathbb{R}^s, \\ \rho_\infty(X,Y) &= \inf_{x\in X, y\in Y} \|x-y\|_\infty \quad \forall X,Y \subset \mathbb{R}^s, \\ B(S,r) &= \{x \in \mathbb{R}^s \,:\, \rho_\infty(x,S) < r\} \quad \forall S \subset \mathbb{R}^s,\ \ r>0. \end{aligned}$$

Lemma 14. *Let U be a finitely connected Lebesgue measurable subset of $\mathbb{R}^s$ $(s \geq 2)$. Then*

$$|\#(U \cap \mathbb{Z}^s) - \operatorname{meas} U| \leq 2^s \cdot \operatorname{meas} B(\partial U, 1).$$

Proof. Put

$$\begin{aligned} \Pi(x) &= \{y \in \mathbb{R}^s \,:\, y_i = x_i + t_i,\ \ t_i \in [0,1),\ \ i=1,\dots,s\} \quad \forall x \in \mathbb{Z}^s, \\ X &= \{x \in \mathbb{Z}^s \,:\, \Pi(x) \subset U\}, \quad X' = (U \cap \mathbb{Z}^s) \setminus X. \end{aligned}$$

Clearly,

$$0 \leq \operatorname{meas} U - \#X \leq \operatorname{meas}\{x \in U \,:\, \rho_\infty(x,\partial U) < 1\} \leq \operatorname{meas} B(\partial U, 1).$$

Since

$$\#X' \leq 2^s \cdot \#\{x \in \mathbb{Z}^s \,:\, \Pi(x) \cap \partial U \neq \varnothing\} \leq 2^s \operatorname{meas} B(\partial U, 1),$$

we have

$$|\operatorname{meas} U - \#(U \cap \mathbb{Z}^s)| = |(\operatorname{meas} U - \#X) - \#X'| \leq 2^s \operatorname{meas} B(\partial U, 1). \qquad \square$$

To an arbitrary matrix $A \in \mathrm{GL}_3(\mathbb{R})$ with entries a_{ij}, we assign the point

$$A = (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}) \in \mathbb{R}^9. \tag{44}$$

For $\Omega \subset \mathrm{GL}_3(\mathbb{R})$, $\operatorname{meas}\Omega$ denotes the Lebesgue measure of Ω,

$$\begin{aligned} \Omega([1;N]) &= \{A \in \Omega \,:\, 1 \leq a_{ii},\ \ i=1,2,3,\ \ \det A \in [1,N]\}, \\ \omega([1;N]) &= \Omega([1;N]) \cap \mathrm{GL}_3(\mathbb{Z}). \end{aligned}$$

A surface $L \subset \mathbb{R}^s$ is said to belong to the class $\widetilde{C}^1$ if $L \subset \widetilde{L}$, where $\widetilde{L}$ is the graph of a function $f \in C^1(\mathbb{R}^{s-1})$ whose derivatives are uniformly bounded on $\mathbb{R}^{s-1}$.

A surface $L \subset \mathbb{R}^s$ is said to be piecewise smooth if it is composed of a fixed number of surfaces belonging to $\widetilde{C}^1$.

Lemma 15. *Let Ω be a connected set in $\mathrm{GL}_3(\mathbb{R})$ (independent of N) with piecewise smooth boundary, and suppose that there exists a constant $C = C(\Omega)$ such that*

$$\begin{aligned} &|a_{ij}| < C \cdot a_{ii}, \quad i,j=1,2,3, \\ &\det A \geq C \cdot a_{11}a_{22}a_{33} \quad \forall A = ((a_{ij})) \in \Omega. \end{aligned} \tag{45}$$

Then

$$\#\omega([1;N]) = \operatorname{meas}\Omega([1;N]) + O_\Omega(N^3 \ln N) \tag{46}$$

for $N > 1$.

Proof. We denote by $D_{ij} = D_{ij}(A)$ the determinant obtained after elimination the ith row and jth column from a square matrix A; $D(A) = \det A$.

By using (45), it can easily be checked that Ω can be represented as the union of 3 mutually nonintersecting sets $\Omega^{(i)}$ $(i = 1, 2, 3)$ with piecewise smooth boundary, in such a way that

$$|D_{ii}(A)| \gg |a_{11}a_{22}a_{33}a_{ii}^{-1}| \quad \forall A \in \Omega^{(i)}.$$

Throughout this proof, the constants involved in the inequalities $\gg$ and in the notation $O(\cdot)$ depend on Ω but not on N.

Therefore, there is no loss of generality in assuming that

$$|D_{33}(A)| \gg |a_{11}a_{22}| \quad \forall A \in \Omega. \tag{47}$$

By Lemma 14, it suffices to show that

$$\operatorname{meas} P(N) = O(N^3 \ln N), \tag{48}$$

where $P(N) = B(\partial\Omega([1; N]), 1)$. Put

$$K_s(N) = \left\{ x \in \mathbb{R}^s \,:\, 1 \le x_i,\ i = 1, \dots, s,\ \prod_{i=1}^{s} x_i \le N \right\} \quad \text{for } s \ge 2,$$
$$\underline{P}(N) = \{A \in P(N) \,:\, 1 \le a_{ii},\ i = 1, 2, 3\}.$$

Conditions (45) and (47) imply the relations

$$\begin{gathered} a_{11}a_{22}a_{33},\ a_{11}a_{22},\ a_{11}a_{33},\ a_{22}a_{33},\ a_{ii} \in [0, cN], \quad c = c(\Omega), \\ |a_{ij}| \ll a_{ii} + 1, \quad i, j = 1, 2, 3, \end{gathered} \tag{49}$$

for every $A \in P(N)$. If follows that

$$\operatorname{meas}(P(N) \setminus \underline{P}(N)) \ll 1 + \int_0^N (x+1)^2\, dx + \int_{K_2(cN)} (x+1)^2 (y+1)^2\, dx\, dy \ll N^3 \ln N$$

(the first summand is responsible for the case where all a_{ii} are smaller than 1, the second is responsible for the case where precisely two of the a_{ii} are smaller than 1, and the third for the case where precisely one a_{ii} is smaller than 1).

It suffices to estimate the measure of $\underline{P}(N)$. We denote by $\widetilde{P}(N)$ the set of $A \in \mathbb{R}^{3\times 3}$ with

$$|a_{ij}| \ll a_{ii},\ i, j = 1, 2, 3, \quad (a_{11}, a_{22}, a_{33}) \in K_3(cN). \tag{50}$$

Then $\underline{P}(N) \subset \widetilde{P}(N)$ by (49). By assumption, the set $\partial\Omega([1; N])$ is included in the union of a fixed number of $\widetilde{C}^1$-surfaces and $\mathrm{GL}_3(\mathbb{R}; N)$, $\mathrm{GL}_3(\mathbb{R}; 1)$. Thus, it suffices to estimate the measures of the sets

$$\begin{aligned} P_1(N) &= \{A \in \widetilde{P}(N) \,:\, \rho_\infty(A, L) < 1\}, \\ P_2(N) &= \{A \in \widetilde{P}(N) \,:\, \rho_\infty(A, \mathrm{GL}_3(\mathbb{R}; N)) < 1\}, \\ P_3(N) &= \{A \in \widetilde{P}(N) \,:\, \rho_\infty(A, \mathrm{GL}_3(\mathbb{R}; 1)) < 1\}, \end{aligned}$$

where the surface $L \in \widetilde{C}^1$ is independent of N.

We estimate $P_1(N)$. Let L be defined by the equation $a_{kl} = f(A')$, where A' is the collection obtained by deleting the variable a_{kl} from $(a_{11}, \dots, a_{33})$; the function $f \in C^1(\mathbb{R}^8)$ has uniformly bounded derivatives on $\mathbb{R}^8$. Without loss of generality, we assume that $k = 3$. Then it can easily be checked that the inequality $\rho_\infty(A, L) < 1$ entails

$$|a_{3l} - f(A')| \ll 1. \tag{51}$$

From (50) we deduce that

$$|a_{3j}| \ll a_{33} \ll \frac{N}{a_{11}a_{11}}, \quad j = 1,2,3$$
$$\Longrightarrow \operatorname{meas} P_1(N) \ll \int_{K_2(N)} a_{11}^2 a_{22}^2 \Big(\frac{N}{a_{11}a_{22}}\Big)^2 \, da_{11}\, da_{22} = O(N^3 \ln N).$$

We estimate $\operatorname{meas} P_2(N)$. Let $A \in P_2(N)$. Then there exists a point B satisfying $\rho_\infty(A,B) < 1$, $D(B) = N$. Expanding the function $B \to D(B)$ in the Taylor series near A, keeping in mind that the 4th order derivatives vanish, and using (50), we obtain

$$|N - D(A)| = |D(B) - D(A)| \ll \sum_{i,j=1}^{3} (|D_{ij}(A)| + |a_{ij}|) + 1$$
$$\ll a_{22}a_{33} + a_{11}a_{33} + a_{11}a_{22} \ll a_{11}a_{22} + \frac{N}{a_{11}} + \frac{N}{a_{22}}.$$

Consequently, the transformation

$$b_{33} = N - D(A),\ b_{ij} = a_{ij},\ i,j = 1,2,3,\ (i,j) \neq (3,3),$$

takes $P_2(N)$ into the set $\widetilde{P}_2(N)$ whose points B satisfy

$$(b_{11}, b_{22}) \in K_2(cN), \quad |b_{ij}| \ll b_{ii},\ j \neq i,\ i = 1,2,$$
$$|b_{3j}| \ll \frac{N}{b_{11}b_{22}},\ j = 2,3, \quad |b_{33}| \ll \left(b_{11}b_{22} + \frac{N}{b_{11}} + \frac{N}{b_{22}}\right).$$

For the absolute value of the Jacobian, we have $|D_{33}(a)| = |D_{33}(b)| \gg b_{11}b_{22}$. Therefore,

$$\operatorname{meas} P_2(N) = \int_{\widetilde{P}_2(N)} \frac{db_{11}\cdots db_{33}}{|D_{33}(b)|}$$
$$\ll \int_{K_2(cN)} b_{11}^2 b_{22}^2 \frac{N^2}{b_{11}^2 b_{22}^2} \frac{1}{b_{11}b_{22}} \Big(b_{11}b_{22} + \frac{N}{b_{11}} + \frac{N}{b_{22}}\Big)\, db_{22} db_{33}$$
$$= O(N^3 \ln N).$$

The measure of $P_3(N)$ is estimated similarly. We have proved that $\operatorname{meas} P_i(N) = O(N^3 \ln N)$; consequently, (48) is fulfilled. □

The function μ was defined in §1.

Theorem 2. *Let W_i be a finitely connected subset of $\mathbb{R}^2$ with piecewise smooth boundary ∂W_i $(i = 1,2,3)$, and let*

$$\Omega = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{GL}_3(\mathbb{R}) : \begin{array}{ll} (a_{12}, a_{13}) \in W_1 \cdot a_{11}, & a_{11} \in \mathbb{R}_+ \\ (a_{21}, a_{22}) \in W_2 \cdot a_{22}, & a_{23} \in \mathbb{R}_+ \\ (a_{31}, a_{32}) \in W_3 \cdot a_{33}, & a_{33} \in \mathbb{R}_+ \end{array} \right\}.$$

Moreover, suppose that there exists a constant $C = C(W_1, W_2, W_3)$ with

$$a_{11}a_{22}a_{33} \le C \cdot \det A \quad \forall A \in \Omega.$$

Then for $N > 1$ we have

$$\#\omega([1;N]) = N^3 \Big(\frac{\mu(\Omega)}{6} \cdot \ln^2 N + O_\Omega(\ln N)\Big).$$

Proof. It can easily be observed that Ω satisfies the assumptions of Lemma 15. Thus, it suffices to prove the relation

$$\operatorname{meas} \Omega([1;N]) = \frac{\mu(\Omega)}{6} N^3 \left(\ln^2 N + O(\ln N)\right).$$

For this, we change the variables in the integral

$$\operatorname{meas}\Omega([1;N]) = \int_{\Omega([1;N])} da_{11}\cdots da_{33},$$

putting

$$a_{ij} = b_{ij}\cdot a_{ii} \text{ for } i\neq j, \quad a_{ii} = b_{ii}, \quad i = 1,2,3.$$

Recall that $D(A) = \det A$. Since

$$\frac{D(A)}{a_{11}a_{22}a_{33}} = D'(B), \quad D'(B) = \det\begin{pmatrix} 1 & b_{12} & b_{13} \\ b_{21} & 1 & b_{23} \\ b_{31} & b_{32} & 1 \end{pmatrix},$$

the set $\Omega([1;N])$ is taken to

$$W([1;N]) = \left\{ B\in\mathbb{R}^{3\times 3} \ : \ \begin{matrix} (b_{12},b_{13})\in W_1, \\ (b_{21},b_{23})\in W_2, \\ (b_{31},b_{32})\in W_3, \end{matrix} \quad \frac{1}{D'(B)} \le b_{11}b_{22}b_{33} \le \frac{N}{D'(B)} \right\}.$$

We observe that $D'(B)\asymp 1$ for $B\in W([1;N])$ (this follows from the assumptions of the theorem). Integrating with respect to b_{ii}, we obtain

$$\begin{aligned} \operatorname{meas}\Omega([1;N]) &= \int_{W([1;N])} b_{11}^2 b_{22}^2 b_{33}^2\, db_{11}\cdots db_{33} \\ &= \int_{W_1\times W_2\times W_3} \frac{N^3}{6(D'(B))^3}(\ln^2 N + O(\ln N))\, db_{12}\, db_{13}\, db_{21}\, db_{23}\, db_{31}\, db_{32} \\ &= \frac{N^6}{6}\left(\ln^2 N + O(\ln N)\right)\cdot \mu(\Omega). \end{aligned}$$

□

§5. Proof of Theorem 1

Put

$$\omega_i([1,N]) = \bigcup_{n=1}^{N} \omega_i(n), \quad \partial\omega_i([1,N]) = \bigcup_{n=1}^{N} \partial\omega_i(n), \quad i = 1,2,3,4.$$

The sets Ω_i were defined in §1. Since Ω_i and $\partial\Omega_i$ satisfy the assumptions of Theorem 2, we have

$$\#\omega_i([1,N]) = \frac{\mu(\Omega_i)}{6} N^3 \ln^2 N + O(N^3\ln N), \quad \#\partial\omega_i([1,N]) = O(N^3\ln N).$$

The claim of Theorem 1 follows from this combined with (14) and (16).

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