FORMATIONS OF FINITE $C_\pi$-GROUPS

E. P. VDOVIN, D. O. REVIN, AND L. A. SHEMETKOV

ABSTRACT. It is proved that the class of all finite $C_\pi$-groups is closed under subdirect products. Conditions for a given formation of $C_\pi$-groups to be saturated or partially saturated are found.

§1. INTRODUCTION

By a group we always mean a finite group. Throughout, a (fixed) set of primes is denoted by $\pi$ and $p$ is a prime. The set of primes not belonging to $\pi$ is denoted by $\pi'$. Given a natural number $n$, we denote by $\pi(n)$ the set of all prime divisors of $n$, and $\pi(G)$ equals $\pi(|G|)$ for a group $G$.

Recall that a group $G$ with $\pi(G) \subseteq \pi$ is called a $\pi$-group. A subgroup $H$ of $G$ is called a $\pi$-Hall subgroup if $\pi(H) \subseteq \pi$ and $\pi(|G:H|) \subseteq \pi'$.

In accordance with [1], we say that $G$ satisfies $E_\pi$ if $G$ possesses a $\pi$-Hall subgroup. If $G$ satisfies $E_\pi$ and every two $\pi$-Hall subgroups are conjugate, then we say that $G$ satisfies $C_\pi$. If $G$ satisfies $C_\pi$ and each $\pi$-subgroup of $G$ is included in a $\pi$-Hall subgroup, then we say that $G$ satisfies $D_\pi$. A group satisfying $E_\pi$ ($C_\pi$, $D_\pi$) is also called an $E_\pi$-group (respectively, $C_\pi$-group, $D_\pi$-group). The symbols $E_\pi$, $C_\pi$, and $D_\pi$ will be used also for the classes of all $E_\pi$-, $C_\pi$-, and $D_\pi$-groups, respectively. If $X$ is a class of groups, then by $E_\pi X$ we denote the class of all groups possessing a $\pi$-Hall subgroup from $X$, while $C_\pi X$ and $D_\pi X$ are $C_\pi \cap (E_\pi X)$ and $D_\pi \cap (E_\pi X)$, respectively.

Recall that a class of groups $\mathfrak{F}$ is called a formation if it is closed under homomorphic images and finite subdirect products (in other words, for each $G$ and its normal subgroups $M$ and $N$, $G \in \mathfrak{F}$ imply $G/M \in \mathfrak{F}$, while $G/M \in \mathfrak{F}$ and $G/N \in \mathfrak{F}$ imply $G/(M \cap N) \in \mathfrak{F}$).

A formation $\mathfrak{F}$ is said to be:

1) $p$-saturated if $G/N \in \mathfrak{F}$ and $N \leq O_p(\Phi(G))$ imply $G \in \mathfrak{F}$ (here and below $O_p(\Phi(G))$ is the largest $p$-subgroup of $G$ for a fixed set of primes $\pi$, while $\Phi(G)$ is the Frattini subgroup of $G$, i.e., the intersection of all maximal subgroups);

2) saturated if $\mathfrak{F}$ is $p$-saturated for each $p$;

3) $p$-solubly saturated if the conditions $G/N \in \mathfrak{F}$ and $N \leq O_p(\Phi(G_{p-\pi}))$, where $G_{p-\pi}$ is a $p$-solvable radical of $G$, always imply $G \in \mathfrak{F}$;

4) solubly saturated if the conditions $G/N \in \mathfrak{F}$ and $N \leq \Phi(G_{\pi})$, where $G_{\pi}$ is a solvable radical of $G$, always imply $G \in \mathfrak{F}$.

In [2] Corollary 8], the classification of finite simple group was used to show that the class $E_\pi$ is a formation (notice that the proof of this statement in [19], which appeared

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earlier than [2, had a gap). In the series of papers [5-15], by using the classification of finite simple groups, it was proved that a group satisfies \(D \pi\) if and only if each its composition factor satisfies \(D \pi\), see [5, Theorem 7.7]. It follows that \(D \pi\) is a formation.

The two formations \(E \pi\) and \(D \pi\) are saturated because the Frattini subgroup is nilpotent.

In [4] it was proved that if \(F\) is a saturated formation, then the class of all solvable \(C \pi F\)-groups is also a formation. In 1978, L. A. Shemetkov proposed the following conjecture.

**Conjecture** [5, Problem 19]. Let \(\pi\) be a set of primes and \(\mathfrak{F}\) a saturated formation. Then \(C \pi \mathfrak{F}\) is a saturated formation.

Later in [6, Theorem 2], a criterion for \(C \pi \mathfrak{F}\) to be saturated was obtained, where \(\mathfrak{F}\) is a formation such that \(C \pi \mathfrak{F}\) is a formation. As a corollary, it was proved that \(C \pi \mathfrak{R}\) may fail to be saturated, \(\mathfrak{R}\) being the formation of all nilpotent groups, so that the original Shemetkovs' conjecture was disproved.

In this paper we prove that \(C \pi \mathfrak{F}\) is a formation for every formation \(\mathfrak{F}\) (see Theorem 1 below). We use the results of [7], where a criterion for a group to satisfy \(C \pi\) was obtained. This criterion was based on the classification of finite simple groups, and it was formulated in terms of the normal structure of the group. Moreover, it was proved that the \(C \pi\) property is inherited by some subgroups of index not divisible by primes from \(\pi\), and is inherited by the factor groups. We complete the analysis started in [6] by finding the cases where \(C \pi \mathfrak{F}\) turns out to be \(p\)-saturated or \(p\)-solubly saturated.

We formulate the main results of the paper.

**Theorem 1.**

1) \(C \pi\) is a formation;

2) \(C \pi \mathfrak{F}\) is a formation for each formation \(\mathfrak{F}\).

**Corollary 1.1.** Each finite group admits a \(C \pi\)-coradical, i.e., the smallest normal subgroup such that the factor group by this subgroup is a \(C \pi\)-group.

**Theorem 2.** If \(\mathfrak{F}\) is a \(p\)-solubly saturated formation, then \(C \pi \mathfrak{F}\) is also a \(p\)-solubly saturated formation.

**Corollary 2.1.** Let \(\mathfrak{F}\) be a \(p\)-solubly saturated formation including all \(p\)-groups. Assume that \(G\) possesses a normal subgroup \(A\) and a normal \(p\)-subgroup \(B\) such that \(B \leq A\) and \(A/B \in C \pi \mathfrak{F}\). If \(B\) is included in the Frattini subgroup of the \(p\)-solvable radical of \(G\), then \(A \in C \pi \mathfrak{F}\).

**Corollary 2.2.** If \(\mathfrak{F}\) is a solubly saturated formation, then \(C \pi \mathfrak{F}\) is also a solubly saturated formation.

**Corollary 2.3.** Let \(\mathfrak{F}\) be a solubly saturated formation. Assume that \(G\) possesses normal subgroups \(A\) and \(B\) such that \(B \leq A\) and \(A/B \in C \pi \mathfrak{F}\). If \(B\) is included in the Frattini subgroup of the solvable radical of \(G\), then \(A \in C \pi \mathfrak{F}\).

In two last claims given below we use the following notation and definitions. By \(O'_{p'}(G)\) we denote the \(p\)-nilpotent radical of \(G\) (i.e., the largest normal subgroup possessing a normal \(p'\)-Hall subgroup). If \(\mathfrak{X}\) is a class of groups, then the smallest formation containing \(\mathfrak{X}\) (the intersection of all formations containing \(\mathfrak{X}\)) is denoted by \(\text{form}(\mathfrak{X})\). A group is said to be monolitic if it has a unique minimal normal subgroup. The class of all \(p\)-groups is denoted by \(\mathfrak{N}_p\). If \(\mathfrak{X}\) is a formation, then \(\mathfrak{N}_p \mathfrak{X}\) is the class of all extensions of \(p\)-groups by \(\mathfrak{X}\)-groups.
Theorem 3. Let $\mathfrak{F}$ be a $p$-saturated formation, and let $p \in \pi$. Then $C_\pi \mathfrak{F}$ is $p$-saturated if and only if any $\pi$-Hall subgroup of each monolitic $C_\pi \mathfrak{F}$-group with nonabelian socle of order divisible by $p$ belongs to $\mathfrak{N}_p \text{form}(G/O_{p',p}(G) \mid G \in \mathfrak{F})$.

Corollary 3.1. Let $\mathfrak{F}$ be a saturated formation. Then $C_\pi \mathfrak{F}$ is saturated if and only if any $\pi$-Hall subgroup of each monolitic $C_\pi \mathfrak{F}$-group $H$ with nonabelian socle $N$ belongs to $\mathfrak{N}_p \text{form}(G/O_{p',p}(G) \mid G \in \mathfrak{F})$ for every $p \in \pi(N) \cap \pi$.

§2. Preliminary results

A natural number divisible by $p$ (not divisible by $p$) is called a $pd$-number ($p'$-number, respectively). A group is called a $pd$-group if its order is a $pd$-number. We denote by $\mathcal{K}(X)$ the class of all simple groups isomorphic to composition factors of $X$.

The following statement is well known (see, e.g., [1, Lemma 1] or [8, Chapter IV, (5.11)]).

Lemma 1. Let $G$ be a group and $A$ its normal subgroup. If $H$ is a $\pi$-Hall subgroup of $G$, then $H \cap A$ is a $\pi$-Hall subgroup of $A$, while the factor group $HA/A$ is a $\pi$-Hall subgroup of $G/A$.

Lemma 2 (S. A. Chunikhin, see [9]).

1) Let $G$ be a group and $A$ its normal subgroup. If $G/A$ and $A$ satisfy $C_\pi$, then $G$ satisfies $C_\pi$.

2) If all composition factors of a subnormal series of $G$ are either $\pi$-groups or $\pi'$-groups, then $G$ satisfies $D_\pi$ (see also [8, Chapter V, Theorem 3.7]).

Lemma 3 [2, Corollary 7]. Let $G$ be a group, and let $M$ and $N$ be its normal subgroups. If $G/M$ and $G/N$ are $E_\pi$-groups, then $G/(M \cap N) \in E_\pi$.

Lemma 4 [7, Lemma 9]. Let $G$ be a $C_\pi$-group and $A$ its normal subgroup. Then $G/A$ satisfies $C_\pi$.

Lemma 5 [7, Lemma 7]. Let $G$ be a $C_\pi$-group, $A$ its normal subgroup, and $H$ a $\pi$-Hall subgroup of $G$. Then $N_G(HA)$ and $N_G(H \cap A)$ satisfy $C_\pi$.

Lemma 6 [7, Theorem 1]. Let $G$ be a $C_\pi$-group, $A$ its normal subgroup, and $H$ a $\pi$-Hall subgroup of $G$. Then $HA$ satisfies $C_\pi$.

Following [10–12], we recall the terminology and some results on partially saturated formations.

Definition 1. Let $H/K$ be a normal section of $G$. The small centralizer $c_G(H/K)$ of $H/K$ is the subgroup generated by all normal subgroups $N$ of $G$ such that $K(NK/K) \cap K(H/K) = \varnothing$.

Definition 2. Let $f$ be a function that maps each $pd$-number to one and the same formation $f(p)$ and each $p'$-number to one and the same formation $f(p')$. Such $f$ is called a $p$-local satellite.

Definition 3. Let $\mathfrak{F}$ be a formation and $f$ a $p$-local satellite. We say that $f$ is a $p$-local satellite of $\mathfrak{F}$ if $\mathfrak{F}$ coincides with the class of all groups $G$ such that each composition factor $H/K$ of $G$ has the following properties:

1) if $H/K$ is a $pd$-group, then $G/c_G(H/K) \in f(p)$;

2) if $H/K$ is a $p'$-group, then $G/c_G(H/K) \in f(p')$.

Definition 4. Let $f$ be a $p$-local satellite of $\mathfrak{F}$. We say that $f$ is a canonical $p$-local satellite of $\mathfrak{F}$ if $f(p) = \mathfrak{N}_p f(p) \subseteq \mathfrak{F}$ and $f(p') = \mathfrak{F}$.

1See also [1, Theorems C1 and C1] or [8, Chapter V, (3.12)].
Lemma 7 [10] Theorem 1]. A nonempty formation is \( p \)-saturated if and only if it possesses a \( p \)-local satellite. Each \( p \)-saturated formation \( \mathcal{F} \) containing \( \mathcal{N}_p \) possesses a canonical \( p \)-local satellite \( f \), and \( f(p) = \mathcal{N}_p \text{form}(G/O_{p'}(G) \mid G \in \mathcal{F}) \).

Definition 5. Let \( f \) be a function that maps each nonzero power of \( p \) to one and the same formation \( f(p) \), while each number that is not equal to a nonzero power of \( p \) is mapped to one and same formation \( f(p') \). Such \( f \) is called a \( p \)-composite satellite.

Definition 6. Let \( \mathcal{F} \) be a formation and \( f \) a \( p \)-composite satellite. We say that \( f \) is a \( p \)-composite satellite of \( \mathcal{F} \) if \( \mathcal{F} \) coincides with the class of all groups \( G \) such that each chief composition factor \( H/K \) of \( G \) has the following properties:

1) if \( H/K \) is a \( p \)-group, then \( G/C_G(H/K) \in f(p) \);
2) if \( H/K \) is not a \( p \)-group, then \( G/c_G(H/K) \in f(p') \).

We note that the class of all groups \( G \) satisfying conditions 1) and 2) of Definition 6 is a formation (see [10] Example 6).

Definition 7. Let \( f \) be a \( p \)-composite satellite of a formation \( \mathcal{F} \). We say that \( f \) is a canonical \( p \)-composite satellite of \( \mathcal{F} \) if \( f(p) = \mathcal{N}_p \text{form}(G/O_{p'}(G) \mid G \in \mathcal{F}) \) and \( f(p') = \mathcal{F} \).

Lemma 8 (see [10] Lemma 7; 12, Theorem 1]). A nonempty formation is \( p \)-solubly saturated if and only if it possesses a \( p \)-composite satellite. Each \( p \)-solubly saturated formation possesses a canonical \( p \)-composite satellite.

Lemma 9 (see [10] Theorem 1]). Let \( \mathcal{F} \) be a formation containing all \( p \)-groups. Assume that \( G \) possesses a normal subgroup \( A \) and a normal \( p \)-subgroup \( B \) such that \( B \leq A \) and \( A/B \in \mathcal{F} \). Then \( A \in \mathcal{F} \) whenever one of the following condition is fulfilled:

1) \( \mathcal{F} \) is \( p \)-saturated and \( B \leq \Phi(G) \);
2) \( \mathcal{F} \) is \( p \)-solubly saturated and \( B \) is included in the Frattini subgroup of the \( p \)-solvable radical of \( G \).

The next lemma follows from the properties of the Frattini \( p \)-elementary extension of a \( pd \)-group (see [14], p. 853).

Lemma 10. Let \( G \) be a monolithic group with a nonabelian socle whose order is divisible by \( p \). Then there exists a group \( E \) and an elementary Abelian normal \( p \)-subgroup \( A \neq 1 \) of \( E \) such that \( E/A \cong G \), \( A \leq \Phi(E) \), and \( O_{p'}(H) = 1 \) for each subgroup \( H \) with \( A \leq H \leq E \).

Lemma 11. Let \( f \) be a canonical \( p \)-local satellite of a formation \( \mathcal{F} \). If a \( pd \)-group \( G \) lies in \( \mathcal{F} \) and \( O_{p'}(G) = 1 \), then \( G \in f(p) \).

Proof. If \( H/K \) is a chief \( pd \)-factor of \( G \), then \( G/C_G(H/K) \in f(p) \) by Definition 3. Since \( f(p) \) is a formation, it follows that \( G/R \in f(p) \), where \( R \) is the intersection of all centralizers of all chief \( pd \)-factors. It is known that \( R = O_{p'}(G) \). Since \( O_{p'}(G) = 1 \), we obtain \( G \in \mathcal{N}_p f(p) \). Consequently, \( G \in f(p) \), because for a canonical \( p \)-local satellite \( f \) we have \( \mathcal{N}_p f(p) = f(p) \).

Lemma 12 (P. Shmid, see. [4] Theorem 7.11] and [20] Lemma 4)). If \( \Phi(G) = 1 \) and \( S \) is the socle of \( G \), then \( C_G(S) \leq S \).

Lemma 13. Let \( f \) be a canonical \( p \)-local satellite of a formation \( \mathcal{F} \). Let \( K \) be a normal Abelian \( p \)-subgroup of \( G \) such that \( G/K \in \mathcal{F} \) and \( G/C_G(K) \in f(p) \). Then \( G \in \mathcal{F} \).

Proof. If \( O_{p'}(G) \neq 1 \), then \( G/O_{p'}(G) \in \mathcal{F} \) by induction. Since \( G/K \in \mathcal{F} \), we obtain \( G \in \mathcal{F} \). If \( O_{p'}(G) = 1 \), then \( G/O_p(G) \in f(p) \), and we use the condition \( \mathcal{N}_p f(p) = f(p) \).
Lemma 14. Let $M$ and $N$ be normal subgroups of $G$ and let $M \cap N = 1$. Then for any two subgroups $A$ and $B$ of $M$ the following is true:

1. $N \leq C_G(A)$;
2. if $AN = BN$, then $A = B$;

Proof. Since the product of $M$ and $N$ is direct, we have (1). Applying the coordinate projection from $M \times N$ onto $M$ to the subgroup $AN = BN$, we obtain $A = B$, and (2) follows. Now we prove (3). The inclusion $N_G(A)/N \leq N_G/(AN/N)$ is obvious. If $Ng \in N_G/(AN/N)$ for some $g \in G$, then

$$A^gN = (AN)^g = AN$$

and $A^g \leq M$ because $M$ is normal. Applying (2), we obtain $A^g = A$ and $g \in N_G(A)$. Statement (3) is proved. $\square$

§3. Proof of Theorem 1 and its corollary

First, we prove statement 1) of Theorem 1. By Lemma 4, $C_\pi$ is closed under factor groups. Assume that it is not closed under subdirect products. Then there exists a group $G$ and its subgroups $M$ and $N$ such that $G/M$ and $G/N$ satisfy $C_\pi$, while $G/(M \cap N) \notin C_\pi$. Assume that $G$ has a minimal possible order. We show that the conjecture about the existence of such a group leads to a contradiction. We split the proof into a series of steps.

By Lemma 3, we have

1. $G/(M \cap N) \in E_\pi$.  

The minimality of the order of $G$ implies that

2. $M \cap N = 1$ and $G \simeq G/(M \cap N)$. In particular, $[M, N] = 1$.

Therefore,

3. $G \in E_\pi$ and $G \notin C_\pi$.

It follows that

4. $G$ possesses two nonconjugate $\pi$-Hall subgroups $H$ and $K$.

Since $G/N \in C_\pi$, and $NH/N$ and $NK/N$ are $\pi$-Hall subgroups of $G/N$, there is no loss of generality in assuming that

5. $NH = NK$;

Indeed, suppose $MNH < G$. We have

$$MNH/M = (MN/M)(MH/M),$$

while $MN/M \leq G/M$ and $HM/M$ is a $\pi$-Hall subgroup of $G/M \in C_\pi$. Applying Lemmas 1, 2, and 6, we see that $MNH/M \in C_\pi$. The same arguments imply $MNH/N \in C_\pi$. By the minimality of the order of $G$, we have

$$MNH \simeq MNH/(M \cap N) \in C_\pi.$$  

But $H$ and $K$ are $\pi$-Hall subgroups of $MNH$ by (5), so that they are conjugate in contrast to (4), a contradiction. The fact that $MNH = MNK$ follows from (5).

7. $(M \cap H)N/N = (HN/N) \cap (MN/N)$, 
   $(M \cap K)N/N = (KN/N) \cap (MN/N)$.  

Indeed, $M \cap H$ is a $\pi$-Hall subgroup of $M$ by Lemma 1, whence $(M \cap H)N/N$ is a $\pi$-Hall subgroup of $MN/N$. Since $HN/N$ is a $\pi$-Hall subgroup of $G/N$ and $MN/N \leq G/N$, Lemma 1 implies that $(HN/N) \cap (MN/N)$ is a $\pi$-Hall subgroup of $MN/N$. Since, clearly,

$$(M \cap H)N/N \leq (HN/N) \cap (MN/N),$$

the coincidence of orders on the right and on the left shows that

$$(M \cap H)N/N = (HN/N) \cap (MN/N).$$

The relation

$$(M \cap K)N/N = (KN/N) \cap (MN/N)$$

is checked in the same way. Statement (7) is proved.

From (5) and (7) it follows that

$$(M \cap H)N/N = (HN/N) \cap (MN/N) = (KN/N) \cap (MN/N) = (M \cap K)N/N.$$ 

Hence, $(M \cap H)N = (M \cap K)N$, and by Lemma 14(2) we obtain

(8) $M \cap H = M \cap K$;

(9) $M \cap H = M \cap K \leq G$.

Suppose that (9) fails. Then $N_G(M \cap H) < G$. By (2) and Lemma 14(1) it follows that $N \leq N_G(M \cap H)$. By (7) and Lemma 14(3),

$$N_G(M \cap H)/N = N_G((HN/N) \cap (MN/N)),$$

and this group satisfies $C_\pi$ by Lemma 5. Since $H, N \leq N_G(M \cap H)$, (6) implies $G = MN_G(M \cap H)$. Now $N_M(M \cap H) \leq N_G(M \cap H)$ and

$$N_G(M \cap H)/N_M(M \cap H) \simeq MN_G(M \cap H)/M = G/M \in C_\pi.$$

Thus, $N_G(M \cap H)$ possesses two normal subgroups $N_M(M \cap H)$ and $N$ with trivial intersection, and the factors by these subgroups satisfy $C_\pi$. The minimality of $G$ implies that $N_G(M \cap H) \leq C_\pi$, and since $H, K \leq N_G(M \cap H)$ by (8), the subgroups $H$ and $K$ are conjugate in contrast to (4), a contradiction.

Now $M \cap H \leq H$ follows from (9), so that $M \cap H$ is a unique $\pi$-Hall subgroup of $M$. In particular,

(10) $M \in C_\pi$.

We conclude that $G$ is an extension of the $C_\pi$-group $M$ by the $C_\pi$-group $G/M$; then $G \in C_\pi$ by Lemma 2, which contradicts (3).

Statement 1) of Theorem 1 is proved.

We prove statement 2) of Theorem 1. By Lemmas 1–4, $C_\pi \mathfrak{F}$ is closed under homomorphic images. We show that it is closed under subdirect products. Let $G$ be a group such that $M, N \leq G$ and $G/M, G/N \in C_\pi \mathfrak{F}$. We may assume that $M \cap N = 1$. Since statement 1) of Theorem 1 shows that $G \in C_\pi$, it remains to show that a $\pi$-Hall subgroup $H$ of $G$ is an $\mathfrak{F}$-group. Since $H/(H \cap M) \simeq MH/M, H/(H \cap N) \simeq NH/N$, and $MH/M$ and $NH/N$ are $\pi$-Hall subgroups of $G/M$ and $G/N$, it follows that $H/(H \cap M)$ and $H/(H \cap N)$ are $\mathfrak{F}$-groups. Since $\mathfrak{F}$ is a formation, $H \simeq H/(H \cap M \cap N) \in \mathfrak{F}$.

Theorem 1 is proved.

Corollary 1.1 follows from item 1) of Theorem 1 and [14, Chapter II, Lemma (2.4)].
§4. Proof of Theorem 2 and its corollaries

For applications, it is not sufficient to know that a formation is p-soluble saturated; the form of its p-composite satellite is also important. For this reason, we prove a refined version of Theorem 2.

Theorem 2*. Let \( p \in \pi \) and let \( f \) be a canonical p-composite satellite of \( \mathcal{F} \). Then \( C_\pi \mathcal{F} \) is a p-solubly saturated formation with a p-composite satellite \( h \) such that

\[
h(x) = \begin{cases} 
C_\pi f(p) & \text{if } x = p, \\
C_\pi \mathcal{F} & \text{if } x = p'. 
\end{cases}
\]

Proof. This is obvious for \( f(p) = \emptyset \). Assume that \( f(p) \neq \emptyset \), i.e., \( \mathcal{F} \) contains all p-groups. Using Definition 6, we construct a formation \( \mathcal{H} \) with a p-composite satellite \( h \) as in the theorem. We prove that \( C_\pi \mathcal{F} \subseteq \mathcal{H} \).

Suppose \( C_\pi \mathcal{F} \not\subseteq \mathcal{H} \). Let \( G \) be a group of minimal order in \( (C_\pi \mathcal{F}) \setminus \mathcal{H} \). Clearly, \( G \) possesses a unique minimal normal subgroup \( K \), and moreover, \( G/K \in \mathcal{H} \). If \( K \) is not a p-group, then \( c_G(K) = 1 \), \( h(p') = C_\pi \mathcal{F} \), and, by the definition of the satellite of a formation, we have \( G/c_G(K) \cong G \in C_\pi \mathcal{F} \). Let \( K \) be a p-group. Clearly, \( K \) is included in a \( \pi \)-Hall subgroup \( R \) of \( G \). Since \( K \leq R \in \mathcal{F} \) and \( f \) is a canonical p-composite satellite of \( \mathcal{F} \), the group \( R \) induces a group of automorphisms on each \( R \)-chief factor of \( K \), and this group of automorphisms is included in \( f(p) \). Then \( R/C_R(K) \in \mathcal{H}_p.f(p) = f(p) \). Consequently, \( G/C_G(K) \in C_\pi f(p) \). Thus, we have \( G \in \mathcal{H} \) by Definition 6. The inclusion \( C_\pi \mathcal{F} \subseteq \mathcal{H} \) follows.

Now we assume that \( G \) is a group of minimal order in \( \mathcal{H} \setminus C_\pi \mathcal{F} \). \( G \) possesses a unique minimal normal subgroup \( N \), and moreover, \( G/N \in C_\pi \mathcal{F} \). If \( N \) is not a p-group, then \( c_G(N) = 1 \), \( h(p') = C_\pi \mathcal{F} \), and \( G/c_G(N) \cong G \in h(p') = C_\pi \mathcal{F} \). Let \( N \) be a p-group. Clearly, \( G \in C_\pi \) and \( N \) is included in a \( \pi \)-Hall subgroup \( T \) of \( G \). Since \( G/c_G(N) \in C_\pi f(p) \), we have \( Tc_G(N)/C_G(N) \cong T/C_T(N) \in f(p) \). By Lemma 13, it follows that \( T \in \mathcal{F} \). Thus, \( G \) is a \( C_\pi \)-group with a \( \pi \)-Hall subgroup \( R \in \mathcal{F} \). The inclusion \( \mathcal{H} \subseteq C_\pi \mathcal{F} \) follows. Theorem 2* and, by Lemma 8, also Theorem 2 are proved.

Corollary 2.1 follows from Theorem 2 and Lemma 9.

Proof of Corollary 2.2. A formation is said to be \( \mathcal{H}_p \)-saturated if \( G \in \mathcal{F} \) whenever \( G/N \in \mathcal{F} \). A formation is known to be a p-solubly saturated if and only if it is \( \mathcal{H}_p \)-saturated (see [12 Theorem 1; 13, Theorem 3.2]). It follows that a formation is solubly saturated if and only if it is p-solubly saturated for every prime \( p \). It remains to apply Theorem 2.

Corollary 2.3 follows from Corollary 2.2 and Lemma 9.

§5. Proof of Theorem 3 and its corollary

The conclusion of the theorem is obvious if \( \mathcal{F} \) consists of \( p' \)-groups. So, we assume that \( \mathcal{F} \) contains all p-groups. Let \( f \) denote a canonical p-local satellite of \( \mathcal{F} \) (its existence follows from Lemma 7). If the order of a socle is divisible by \( p \), we call this socle a \( pd \)-socle for brevity.

The “only if” part. Assume that \( C_\pi \mathcal{F} \) is \( p \)-saturated and \( G \) is a monolithic \( C_\pi \mathcal{F} \)-group with nonabelian \( pd \)-socle. Assume that \( E \) and \( A \) are chosen as in Lemma 10. Since \( A \leq \Phi(E) \) and \( C_\pi \mathcal{F} \) is a \( p \)-saturated formation, we have \( E \in C_\pi \mathcal{F} \). Let \( H \) be a \( \pi \)-Hall subgroup of \( E \). By Lemma 10, \( O_{p'}(H) = 1 \). Thus, \( H \in f(p) \) by Lemma 11. It remains to recall that \( f(p) = \mathcal{H}_p form(G/O_{p',p}(G) \mid G \in \mathcal{F}) \) by Lemma 7.
The “if” part. Assume that the following condition (*) is fulfilled: any \(\pi\)-Hall subgroup of each monolithic \(C_\pi\mathfrak{F}\)-group with a nonabelian \(pd\)-socle belongs to \(\mathfrak{N}_p\) form \(G/O_{p',p}(G)\mid G \in \mathfrak{F}\).

We prove that \(C_\pi\mathfrak{F}\) is \(p\)-saturated. Using Definition 6, we construct a formation \(\Phi\) with \(p\)-local satellite \(g\) such that

\[
g(x) = \begin{cases} 
C_\pi f(p) & \text{if } x = p, \\
C_\pi\mathfrak{F} & \text{if } x = p'. 
\end{cases}
\]

Then \(\Phi\) is \(p\)-saturated by Lemma 7. We prove that \(C_\pi\mathfrak{F} = \Phi\).

Assume that \(C_\pi\mathfrak{F} \not\subseteq \Phi\). Let \(G\) be a group of minimal order in \(C_\pi\mathfrak{F} \setminus \Phi\). Clearly, \(G\) possesses a unique minimal normal subgroup \(K\); moreover, \(G/K \in \Phi\) and \(c_G(K) = 1\). If \(K\) is a \(p'\)-group, then \(g(p') = C_\pi\mathfrak{F}\) and \(G/c_G(K) \simeq G \in C_\pi\mathfrak{F}\). Since \(G/K \in \Phi\), we obtain \(G \in \Phi\) by Definition 3.

Now, assume that \(K\) is a nonabelian \(pd\)-group. It is clear that \(C_G(K) = 1\). By (*), any \(\pi\)-Hall subgroup of \(G\) belongs to \(f(p)\). Hence, \(G/C_G(K) \simeq G \in C_\pi f(p)\). Therefore, \(G \in g(p)\), and \(G \in \Phi\) by Definition 3.

Finally, assume that \(K\) is an Abelian \(p\)-group. Since \(\Phi\) is \(p\)-saturated (see Lemma 7), we have \(\Phi(G) = 1\), whence \(K = C_G(K)\) by Lemma 12. Hence, if \(H\) is a \(\pi\)-Hall subgroup of \(K\), then \(O_{p'}(H) = 1\) and, by Lemma 11, we have \(H \in f(p)\). Thus, \(G/C_G(K) = G/K \in g(p)\). Since \(G/K \in \Phi\), it follows that \(G \in \Phi\). The inclusion \(C_\pi\mathfrak{F} \subseteq \Phi\) is proved.

Now, let \(G\) be a group of minimal order in \(\Phi \setminus C_\pi\mathfrak{F}\). \(G\) possesses a unique minimal normal subgroup \(N\), and \(G/N \in C_\pi\mathfrak{F}\). If \(N\) is a \(p'\)-group, then \(c_G(N) = 1\) and \(G/c_G(N) \simeq G \in g(p') = C_\pi\mathfrak{F}\). If \(N\) is a nonabelian \(pd\)-group, then \(C_G(N) = 1\) and \(G/C_G(N) \simeq G \in g(p) \subseteq C_\pi\mathfrak{F}\). Assume that \(N\) is a \(p\)-group. Then \(G/C_G(N) \in C_\pi f(p)\) because \(G \in \Phi\). Obviously, \(G \in C_\pi\). It is clear that \(N\) is included in a \(\pi\)-Hall subgroup \(R\) of \(G\). Then \(G/C_G(N) \in C_\pi f(p)\), \(G/N \in C_\pi\mathfrak{F}\), whence \(R/N \in \mathfrak{F}\) and \(R/C_R(N) \in f(p)\). Lemma 13 shows that \(R \in \mathfrak{F}\). Thus, \(G\) is a \(C_\pi\)-group with a \(\pi\)-Hall subgroup \(R \in \mathfrak{F}\). The inclusion \(\Phi \subseteq C_\pi\mathfrak{F}\) follows.

So, \(C_\pi\mathfrak{F}\) coincides with a \(p\)-saturated formation \(\Phi\). Theorem 3 is proved.

Corollary 3.1 follows from the definition of a saturated formation and Theorem 3.

\section*{References}


Sobolev Institute of mathematics, pr. Academica Koptyuga 4, Novosibirsk 630090, Russia
E-mail address: vodvin@math.nsc.ru

Sobolev Institute of mathematics, pr. Academica Koptyuga 4, Novosibirsk 630090, Russia
E-mail address: revin@math.nsc.ru

Francisk Skorina Gomel State University, ul. Sovetskaya 104, Gomel 246019, Belarus
E-mail address: shemetkov@gau.by

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Translated by E. P. VDOVIN