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MULTIDIMENSIONAL HECKE THEOREM ON THE DISTRIBUTION OF FRACTIONAL PARTS

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Abstract. Hecke's theorem on the distribution of fractional parts on the unit circle is generalized to the tori $\mathbb{T}^D = \mathbb{R}^D/L$ of arbitrary dimension D. It is proved that $|\delta_k(i)| \le c_k\, n$ for $i = 0, 1, 2, \dots$, where $\delta_k(i) = r_k(i) - ia_k$ is the deviation of the number $r_k(i)$ of returns in i steps into $\mathbb{T}^D_k \subset \mathbb{T}^D$ for the points of an S_β-orbit from its mean value $a_k = \mathrm{vol}(\mathbb{T}^D_k)/\mathrm{vol}(\mathbb{T}^D)$, where $\mathrm{vol}(\mathbb{T}^D_k)$ and $\mathrm{vol}(\mathbb{T}^D)$ denote the volumes of the tile $\mathbb{T}^D_k$ and of the torus $\mathbb{T}^D$. The tiles $\mathbb{T}^D_k$ in question have the following property: for the torus $\mathbb{T}^D$ there exists a development $T^D \subset \mathbb{R}^D$ such that a shift S_α of the torus $\mathbb{T}^D$ is equivalent to some exchange transformation of the corresponding tiles T^D_k in a partition of the development $T^D = T^D_0 \sqcup T^D_1 \sqcup \dots \sqcup T^D_D$. The torus shift vectors S_α, S_β satisfy the condition $\alpha \equiv n\beta \bmod L$, where n is any natural number, and the constants c_k in the inequalities are expressed in terms of the diameter of the development T^D.

Introduction

Let α be a real number, and let $\beta = \frac{1}{n}(\alpha + b)$, where $n = 1, 2, 3, \dots$ and b is an arbitrary integer. We define counting functions by

$$\begin{aligned} r_0(i) &= \sharp\{j; \{j\beta\} < 1 - \alpha, 0 \le j < i\}, \\ r_1(i) &= \sharp\{j; \{j\beta\} \ge 1 - \alpha, 0 \le j < i\}, \end{aligned} \tag{0.1}$$

where $\{x\}$ denotes the fractional part of a number x. Consider the deviations

$$\begin{aligned} \delta_0(i) &= r_0(i) - i(1 - \alpha), \\ \delta_1(i) &= r_1(i) - i\alpha \end{aligned}$$

of $r_0(i)$ and $r_1(i)$ from their mean values, where $1 - \alpha$ and α are the lengths of the intervals within which the fractional parts $\{j\beta\}$ fall. Hecke proved the following theorem concerning $\delta_0(i)$, $\delta_1(i)$.

Theorem 1 (Hecke [3]). *Suppose α is irrational. Then the inequalities*

$$|\delta_0(i)| \le n, \quad |\delta_1(i)| \le n \tag{0.2}$$

hold true for $i = 0, 1, 2, \dots$

Our goal is to generalize the Hecke theorem to the tori $\mathbb{T}^D$ of arbitrary dimension D. First, observe the following analogy.

The functions $r_0(i)$ and $r_1(i)$ can be related to rotations of the circle. Let

$$S_\alpha \,:\, x \mapsto x + \alpha \bmod 1 \tag{0.3}$$

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be a rotation of $C = \mathbb{T}^1$ by α. Then $r_0(i)$ and $r_1(i)$ can be viewed as the numbers of points

$$S_\beta^0(0) = 0, \quad S_\beta^1(0), \ \ldots, \ S_\beta^{i-1}(0)$$

(here S_β^j is the j-fold iteration of the rotation S_β) that lie in the semiintervals $[0, 1-\alpha)$ and $[1-\alpha, 1)$, respectively. At the same time, $\delta_0(i)$, $\delta_1(i)$ can be viewed as the deviations in the problem on of the fractional parts distribution. Cutting at some point, we identify the circle with the unit semiinterval $T^1 = [0, 1)$, and then write $T^1 = [0, 1)$ as the union

$$T^1 = T_0^1 \sqcup T_1^1 \tag{0.4}$$

of two semiintervals $T_0^1 = [0, 1-\alpha)$, $T_1^1 = [1-\alpha, 1)$. A rotation of the circle (0.3) is equivalent to exchanging the semiintervals T_0^1 and T_1^1:

$$T^1 \xrightarrow{S} T^1 \ : \ S(x) = x + v_k$$

for $x \in T_k^1$, $k = 0, 1$, where the shift vectors v_0, v_1 are equal to α, $\alpha - 1$, respectively.

In the case where $D \geq 2$, consider a shift of the torus $\mathbb{T}^D \simeq \mathbb{R}^D/L$,

$$\mathbb{T}^D \xrightarrow{S_\alpha} \mathbb{T}^D \ : \ x \mapsto S_\alpha(x) \equiv x + \alpha \bmod L, \tag{0.5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_D)$ is a shift vector and L is a nonsingular lattice in $\mathbb{R}^D$. Cut the torus $\mathbb{T}^D$ to place its development T^D in $\mathbb{R}^D$. Now if we identify the torus $\mathbb{T}^D$ with its development T^D, then the shift (0.5) will correspond to an exchange transformation of the tiles T_k belonging to the tiling

$$T^D = T_0 \sqcup T_1 \sqcup \cdots \sqcup T_s \tag{0.6}$$

of this development. In nonsingular cases, the minimal number of regions in a partition of the form (0.6) is equal to $s_{\min} = D + 1$. For example, in the one-dimensional case we get the two semiintervals (0.4); for $D = 2$ we can take a planar projection of a three-dimensional cube consisting of three exchanging parallelograms.

In order to obtain a multidimensional generalization of the Hecke theorem, we shall move in the opposite direction compared to (0.2): we start with minimal tilings of the torus

$$T^D = T_0 \sqcup T_1 \sqcup \cdots \sqcup T_D \tag{0.7}$$

that admit an exchange transformation

$$T^D \xrightarrow{S} T^D \ : \ x \mapsto S(x) = x + v_k \tag{0.8}$$

for $x \in T_k$. We need to impose the following condition on a tiling (0.7) and on shift vectors $v_0, v_1, \ldots, v_D$: upon identifying the torus $\mathbb{T}^D$ with its development T^D, the exchange transformation (0.8) corresponds to a torus shift S_α as in (0.3) for some vector α uniquely determined by the transformation (0.8).

As in (0.1), we define the counting functions

$$r_k(i) = \sharp\{j;\, S_\beta^j(0) \in T_k;\, 0 \leq j < i\}$$

for $k = 0, 1, \ldots, D$; here S_β is the shift (0.3) of the torus $\mathbb{T}^D$ by the vector

$$\beta = \frac{1}{n}(\alpha + b_1 l_1 + \cdots + b_D l_D),$$

where b_k are arbitrary integers and $l_k = v_k - v_0$ for $k = 1, \ldots, D$. Suppose the vectors $l_1, \ldots, l_D$ are linearly independent over $\mathbb{R}$. For every tile T_k, consider the deviation

$$\delta_k(i) = r_k(i) - i a_k \tag{0.9}$$

for $k = 0, 1, \dots, D$, where $a_1, \dots, a_D$ are the coordinates of the vector $-\alpha$ with respect to the basis $l_1, \dots, l_D$, and $a_0 = 1 - a_1 - \dots - a_D$. In Theorem 5.1, we shall prove the following statement concerning the deviations $\delta_k(i)$.

Theorem 2. *Suppose that vectors $l_1, \dots, l_D$ are linearly independent over $\mathbb{R}$ and that a shift vector α of the torus $\mathbb{T}^D = \mathbb{R}^D/L$ is irrational, namely, its coordinates $\alpha'_1, \dots, \alpha'_D$ in some basis of the lattice L have the following property: the numbers*

$$\alpha'_1, \dots, \alpha'_D, 1 \tag{0.10}$$

are linearly independent over $\mathbb{Z}$. Then for any $k = 0, 1, \dots, D$ we have

$$|\delta_k(i)| \le c_k n \tag{0.11}$$

for every $i = 0, 1, 2, \dots$. Here the $c_k = c_{T,k}$ are the constants (5.13) *independent of n and i, determined only by the size of a development of the torus T^D.*

Let A be the affine map that takes the vectors $l_1, \dots, l_D$ to the standard basis $e_1 = (1, 0, \dots, 0), \dots, e_D = (0, 0, \dots, 1)$. The image AT of a torus development T is again a development of some torus of the same dimension D. Passing from T to AT preserves all the constructions above. Moreover, the constants in the inequalities (0.11) are invariant under affine maps A:

$$c_{AT,k} = c_{T,k}.$$

Hence, without loss of generality we may assume that

$$l_1 = e_1, \ \dots, \ l_D = e_D. \tag{0.12}$$

Under these assumptions we prove the following theorem.

Theorem 3 (Multidimensional Hecke theorem)**.** *Under the assumptions of Theorem* 2 *together with* (0.12)*, we have*

$$|\delta_k(i)| \le d_T\, n \ \text{ for } \ k = 1, \dots, D,$$
$$|\delta_k(i)| \le D d_T\, n \ \text{ for } \ k = 0,$$

where d_T is the diameter of the development T^D. Moreover, if every tile T_k of the tiling (0.7) *is Jordan measurable, the deviations* (0.9) *can be written in the form*

$$\delta_k(i) = r_k(i) - i \operatorname{vol}(T_k)$$

for $k = 0, 1, \dots, D$, where $\operatorname{vol}(T_k)$ is the volume of the tile T_k.

It is informative to state the claim of this theorem analytically (rather than geometrically) in terms of the function $\{x\}$, as in the Hecke Theorem (0.2). Consider the following example in the case where $D = 2$. We introduce the counting function

$$r_1(i) = \sharp\big\{0 \le j < i;\ \{\{j(\beta_1 - \alpha_1\beta_2)\} + \{j\beta_2\}\alpha_1\} \ge 1 - \alpha_1,\ \{j\beta_2\} < 1 - \alpha_2\big\}, \tag{0.13}$$

for any vector $\alpha = (\alpha_1, \alpha_2)$ and

$$\beta_1 = \frac{1}{n}(\alpha_1 - b_1 - b_2\alpha_1), \quad \beta_2 = \frac{1}{n}(\alpha_2 - b_2).$$

In the standard basis $e_1 = (1, 0), e_2 = (0, 1)$, the counting function (0.13) can be written in an equivalent form

$$r_1(i) = \sharp\big\{0 \le j < i;\ 1 - \alpha_1 \le \{j\widetilde{\beta}_1\} + \alpha_1\{j\beta_2\} < 1,\ \{j\beta_2\} < 1 - \alpha_2\big\}, \tag{0.14}$$

where $\widetilde{\beta}_1 = \frac{1}{n}(\alpha_1 - b_1 - \alpha_1\alpha_2)$.

Theorem 4. *Suppose that the numbers $\alpha_1 - \alpha_1\alpha_2, \alpha_2, 1$ are linearly independent over $\mathbb{Z}$. Then the deviation*

$$\delta_1(i) = r_1(i) - i(\alpha_1 - \alpha_1\alpha_2)$$

satisfies the inequality

$$|\delta_1(i)| \le (1 + \alpha_1)n \tag{0.15}$$

for all $i = 0, 1, 2, \dots$

The tiles T_k in the tilings (0.7) are bounded remainder sets on the torus $\mathbb{T}^D$. In particular, by (0.15), the sets T_1 defined by the inequalities occurring in (0.14) are bounded remainder sets. In fact, they are parallelograms determined by the five independent variables α_1, α_2, b_1, b_2, n. The family T_1 of parallelograms contains a subfamily of parallelograms [7] that have fixed parameters $b_1 = 0$, $b_2 = 0$. All the parallelograms in T_1 have acute angles, and this is not a coincidence: Liardet [4] proved that no nontrivial rectangle on a torus can be a bounded remainder set. Bounded remainder sets with fractal boundaries in the two-dimensional case were studied previously in [8].

Our method is geometrical. It resembles that of French mathematicians (Rauzy [6], Ferenczi [2], etc.).

This technique allows us 1) to construct bounded remainder sets on the torus $\mathbb{T}^D$ for any dimension D; 2) to find explicit estimates for the deviations $\delta_k(i)$ on these sets, namely, approximate bounds as in (0.15) (see Theorem 5.1) and sharp bounds for deviations (see Theorems 6.1 and 7.1); 3) to compute the mean values $\langle\delta_k\rangle$ of the deviations $\delta_k(i)$ (see Theorem 8.1).

§1. Exchange transformations for toric developments

Let

$$\mathbb{T}^D \simeq \mathbb{R}^D/L \tag{1.1}$$

be a D-dimensional torus, where L is a nonsingular lattice in the real coordinate space $\mathbb{R}^D$. In other words, the lattice L has dimension D over $\mathbb{R}$. Suppose we are given a shift

$$\mathbb{T}^D \xrightarrow{S_\alpha} \mathbb{T}^D \,:\, x \mapsto S_\alpha(x) \equiv x + \alpha \bmod L \tag{1.2}$$

of the torus. A subset T of $\mathbb{R}^D$ is called a development of the torus $\mathbb{T}^D$ if the restriction of the canonical projection

$$\mathbb{R}^D \xrightarrow{\bmod L} \mathbb{T}^D \,:\, x \mapsto x \bmod L$$

onto $T \subset \mathbb{R}^D$ gives a bijection

$$T \overset{\bmod L}{\underset{\sim}{\longrightarrow}} \mathbb{T}^D. \tag{1.3}$$

Suppose T is a toric development satisfying the following condition: there exists a partition

$$T = T_0 \sqcup T_1 \sqcup \dots \sqcup T_D \tag{1.4}$$

of T such that after identifying the torus $\mathbb{T}^D$ with its development T via (1.3), the map

$$T \xrightarrow{S_\alpha} T \tag{1.5}$$

induced by the shift (1.2) is equivalent to a $(D+1)$-exchange transformation of the development T:

$$S_\alpha(x) = x + v(x), \tag{1.6}$$

where a shift vector $v(x)$ depends on $x \in T$ and satisfies

$$v(x) = v_k \quad \text{if} \quad x \in T_k, \tag{1.7}$$

where $v_0, v_1, \dots, v_D$ are some fixed vectors.

Examples of toric developments.

1. The one-dimensional torus $\mathbb{T}^1$ is the same as the unit circle C; a torus shift S_α is a rotation of C by the angle α. If we identify the circle C with the unit interval $I = [0, 1)$, the rotation S_α becomes equivalent to the exchange of the two semiintervals $[0, 1-\alpha)$, $[1-\alpha, 1)$ inside I.

2. For $D = 2$, we can choose a development T of the torus $\mathbb{T}^2$ to be, for instance, a hexagon divided into three parallelograms, and their exchange corresponds to some rotation of the torus $\mathbb{T}^2$.

3. For the general D, the construction of exchange transformations for a development T of the torus $\mathbb{T}^D$ will be discussed in §9.

Note that (1.2), (1.6), and (1.7) imply

$$v_k \equiv \alpha \bmod L \quad \text{for} \quad k = 0, 1, \dots, D, \tag{1.8}$$

and the vectors

$$l_k = v_k - v_0, \quad k = 1, \dots, D, \tag{1.9}$$

lie in the period lattice L of the torus (1.1). These vectors generate a sublattice

$$L' = \mathbb{Z}[l_1, \dots, l_D] \tag{1.10}$$

of the lattice L, where $l_1, \dots, l_D$ are generators of L' over $\mathbb{Z}$.

A shift S_α of the torus (1.2) is said to be *irrational* if the numbers

$$\alpha'_1, \dots, \alpha'_D, 1 \tag{1.11}$$

are linearly independent over $\mathbb{Z}$, where $\alpha'_1, \dots, \alpha'_D$ are the coordinates of the shift vector α with respect to some basis of the period lattice L

Proposition 1.1. *Suppose that a shift S_α (1.2) of the torus $\mathbb{T}^D$ (1.1) and its development T satisfy the following conditions*:

1) *the shift S_α is irrational*;

2) *the closure $\bar{T}$ of the net T has dimension $\dim \bar{T} = D$, and therefore, $\bar{T}$ is not contained in any subspace $\mathbb{R}^{D'} \subset \mathbb{R}^D$ of dimension $D' < D$*;

3) *the development T has finite diameter $d_T = \sup_{x,x' \in \bar{T}} |x - x'|$.*

Then

$$\operatorname{rank} L' = \operatorname{rank} L = D,$$

where $\operatorname{rank} M$ *denotes the rank of a lattice $M \subset \mathbb{R}^D$, which is equal to the dimension of the lattice M over $\mathbb{R}$.*

Proof. Suppose the contrary. Assume that for the lattice L' we have

$$L' \subset \mathbb{R}^{D'}, \quad \text{where} \quad D' < D. \tag{1.12}$$

Case 1. Suppose $\alpha \in \mathbb{R}^{D'}$. Then (1.12) together with the definition (1.6) implies the inclusion $\operatorname{Orb}_{S_\alpha}(0) \subset \mathbb{R}^{D'}$. Therefore, we have $\overline{\operatorname{Orb}}_{S_\alpha}(0) \subset \mathbb{R}^{D'}$ for the closure $\overline{\operatorname{Orb}}(0)_{S_\alpha}$ of the orbit $\operatorname{Orb}(0)_{S_\alpha}$. Now assumption 1), the theorem about the ergodicity of a torus shift [10, p. 66], and (1.5) imply

$$\overline{\operatorname{Orb}}(0) = \bar{T}.$$

It follows that the closure $\bar{T}$ of the development is contained in the subspace $\mathbb{R}^{D'}$, which contradicts 2).

Case 2. Now suppose $\alpha \notin \mathbb{R}^{D'}$. Then

$$S_\alpha^i(0) \subset i\alpha + \mathbb{R}^{D'} \tag{1.13}$$

for every $i = 0, 1, 2, \ldots$. Since $\alpha \notin \mathbb{R}^{D'}$, the inclusion (1.13) shows that the point $S_\alpha^i(0)$ tends to infinity as $i \to +\infty$. By (1.6), this contradicts the fact that

$$S_\alpha^i(0) \subset T$$

and the assumption $d_T < \infty$ about the diameter of the development T. This completes the proof. □

§2. Vector T-fractional part

From now on we assume that L' is a spanning lattice, namely, that

$$\text{the family } \ l_1, \ldots, l_D \ \text{ has rank } \ D \ \text{ over } \ \mathbb{R}. \tag{2.1}$$

For any $x \in \mathbb{R}^D$, we denote by $\mathrm{Fr}(x)$ its vector T-fractional part

$$\mathrm{Fr}(x) = x', \tag{2.2}$$

where $x' \equiv x \bmod L$ and $x' \in T$. This notion is well defined because there is a partition

$$\mathbb{R}^D = \coprod_{l \in L} T[l] \tag{2.3}$$

of $\mathbb{R}^D$ into the sets

$$T[l] = T + l = \{x + l; x \in T\}$$

obtained by shifts of T by the vectors l.

We often write $\mathrm{Fr}_T(x)$ instead of $\mathrm{Fr}(x)$ to emphasize its dependence on the development T. The point (or the vector) $\mathrm{Fr}_T(x)$ is called the *T-fractional part* of $x \in \mathbb{R}^D$.

Proposition 2.1. *Let*

$$\Delta\,\mathrm{Fr}(x) = \mathrm{Fr}(x + \alpha) - \mathrm{Fr}(x)$$

be a vector-valued difference function with step α, where α is a shift vector of the torus $\mathbb{T}^D$. Then

$$\Delta\,\mathrm{Fr}(x) = v(x) \tag{2.4}$$

for any $x \in \mathbb{R}^D$, where

$$v(x) = \alpha + l(x), \tag{2.5}$$

and $l(x) = l_k$ for $x \in T_k$, $k = 1, \ldots, D$, while $l(x) = l_0 = 0$ for $x \in T_0$. Here the l_k are the vectors as in (1.9).

Proof. For any x in the development T, we have

$$S_\alpha(x) = x + v(x) \tag{2.6}$$

and $v(x) = v_k$ for $x \in T_k$, $k = 0, 1, \ldots, D$. By (1.8), we have $v(x) = \alpha + l(x)$, where $l(x) = l_k$ for $x \in T_k$ and $k = 0, 1, \ldots, D$. Therefore, (2.6) implies

$$S_\alpha(x) = x + \alpha + l(x).$$

Moreover, for any x in T its image $x + \alpha + l(x)$ also lies in the toric development T. It follows that

$$\mathrm{Fr}(x + \alpha) = x + \alpha + l(x) = x + v(x) \tag{2.7}$$

for any $x \in T$. In order to prove (2.4), note that

$$x + \alpha \equiv x + \alpha + l(x) \bmod L, \tag{2.8}$$

where $l(x) \in L$, and that (1.6) implies

$$x + \alpha + l(x) \in T. \tag{2.9}$$

From (2.7) it follows that

$$\begin{aligned}\Delta \operatorname{Fr}(x) &= \operatorname{Fr}(x+\alpha) - \operatorname{Fr}(x)\\ &= x + \alpha + l(x) - x = \alpha + l(x) = v(x)\end{aligned}$$

for any $x \in T$.

Now consider the general case: $x \in \mathbb{R}^D$. Since we have the partition (2.3), every x can be written in the form $x = x' + l$ for some $x' \in T$, $l \in L$. Hence,

$$\operatorname{Fr}(x) = x'. \tag{2.10}$$

Now (2.8) and (2.9) imply

$$x + \alpha \equiv x' + \alpha + l \equiv x' + \alpha + l(x) + l \bmod L, \tag{2.11}$$

where $x' + \alpha + l(x)$ lies in the development T. By (2.11),

$$\operatorname{Fr}(x+\alpha) = x' + \alpha + l(x) \tag{2.12}$$

for any $x \in \mathbb{R}^D$. Using (2.10) and (2.12), we get

$$\begin{aligned}\Delta \operatorname{Fr}(x) &= \operatorname{Fr}(x+\alpha) - \operatorname{Fr}(x)\\ &= x' + \alpha + l(x) - x' = \alpha + l(x) = v(x);\end{aligned}$$

therefore, once again we obtain (2.4):

$$\Delta \operatorname{Fr}(x) = v(x),$$

where $v(x) = \alpha + l(x)$, for any $x \in T$.

This concludes the proof. □

§3. Total vector deviation

We define a vector-valued function $\delta(i)$ as follows:

$$\delta(i) = \sum_{0 \le j < i} \Delta \operatorname{Fr}(j\beta) \tag{3.1}$$

for $i = 0, 1, 2, \dots$, where

$$\beta = \alpha_n(b) = \frac{1}{n}(\alpha + b \circ l), \tag{3.2}$$

n is any natural number,

$$b \circ l = b_1 l_1 + \dots + b_D l_D, \tag{3.3}$$

and $b = (b_1, \dots, b_d) \in \mathbb{Z}^D$ is a vector with integral coordinates. Formula (3.3) shows that $b \circ l$ lies in the lattice L'.

Using (2.4), we can write (3.1) in the form

$$\delta(i) = \sum_{0 \le j < i} (\alpha + l(j\beta)) = i\alpha + \sum_{0 \le j < i} l(j\beta) = i\alpha + \sum_{0 \le k \le D} \sum_{\substack{0 \le j < i \\ \operatorname{Fr}(j\beta) \in T_k}} l(j\beta). \tag{3.4}$$

Since $\operatorname{Fr}(j\beta) \in T_k$ implies $l(j\beta) = l_k$, and $l_0 = 0$ by the definition (2.5), we can use (3.4) for $\delta(i)$ to obtain

$$\delta(i) = i\alpha + \sum_{0 \le k \le D} l_k \sum_{\substack{0 \le j < i \\ \operatorname{Fr}(j\beta) \in T_k}} 1,$$

or equivalently,

$$\delta(i) = i\alpha + r_1(i)l_1 + \cdots + r_D(i)l_D, \tag{3.5}$$

where

$$r_k(i) = \sharp\{\mathrm{Fr}(j\beta) \in T_k; 0 \le j < i\}. \tag{3.6}$$

In terms of shifts of the torus, we can rewrite the counting function (3.6) in the form

$$r_k(i) = \sharp\{j; S_\beta^j(0) \in T_k; 0 \le j < i\}, \tag{3.7}$$

where S_β is the shift

$$\mathbb{T}^D \xrightarrow{S_\beta} \mathbb{T}^D \,:\, x \mapsto S_\beta(x) \equiv x + \beta \bmod L \tag{3.8}$$

of the torus by the vector β defined by (3.2).

Here is another way to obtain the sum (3.1). Using (3.1), we can write $\delta(i)$ as the difference of two sums:

$$\delta(i) = \sum_{0\le j<i} \mathrm{Fr}(j\beta + \alpha) - \sum_{0\le j<i} \mathrm{Fr}(j\beta). \tag{3.9}$$

Relation (3.2) implies that

$$\alpha \equiv n\beta \bmod L. \tag{3.10}$$

Combining (3.10) and the definition (2.2) of the vector T-fractional part $\mathrm{Fr}(x)$, we get

$$\mathrm{Fr}(j\beta + \alpha) = \mathrm{Fr}(j\beta + n\beta). \tag{3.11}$$

Now (3.11) implies

$$\sum_{0\le j<i} \mathrm{Fr}(j\beta + \alpha) = \sum_{0\le j<i} \mathrm{Fr}(j\beta + n\beta) = \sum_{n\le j<i+n} \mathrm{Fr}(j\beta). \tag{3.12}$$

Using (3.9) and (3.12), we get another expression for $\delta(i)$:

$$\delta(i) = \sum_{n\le j<i+n} \mathrm{Fr}(j\beta) - \sum_{0\le j<i} \mathrm{Fr}(j\beta). \tag{3.13}$$

Suppose $i > n$. Then, by (3.13),

$$\delta(i) = \sum_{n\le j<i+n} \mathrm{Fr}(j\beta) - \sum_{0\le j<n} \mathrm{Fr}(j\beta) = \sum_{0\le j\le n-1} \big(\mathrm{Fr}(j\beta + i\beta) - \mathrm{Fr}(j\beta)\big). \tag{3.14}$$

Proposition 3.1. *Suppose a shift vector β of S_β as in* (3.8) *is given by* (3.2)*, and $\delta(i)$ is the sum* (3.1)*. Then*

$$\delta(i) = i\alpha + r_1(i)l_1 + \cdots + r_D(i)l_D, \tag{3.15}$$

where $r_k(i)$ is the number of returns of the initial point $x = 0$ into the tile T_k of the development T under the action of S_β.

Apart from (3.15)*, we can write $\delta(i)$ in one of the following two forms*:

$$\delta(i) = \sum_{0\le j<i} (\mathrm{Fr}(j\beta + n\beta) - \mathrm{Fr}(j\beta)) \tag{3.16}$$

for $0 \le i \le n$ and

$$\delta(i) = \sum_{0\le j\le n-1} (\mathrm{Fr}(j\beta + i\beta) - \mathrm{Fr}(j\beta)) \tag{3.17}$$

for $i > n$. Here $\mathrm{Fr}(x) = \mathrm{Fr}_T(x)$ is given by (2.2) *and is equal to the vector T-fractional part of $x \in \mathbb{R}^D$.*

Proof. Identity (3.16) follows immediately from (3.1) and (3.10), while (3.17) follows from (3.14). □

Remark 3.1. In §5 we shall show that under some assumptions we have the following bounds on the absolute value $|\delta(i)|$:

$$|\delta(i)| = o(i) \quad \text{as} \quad i \to +\infty.$$

Therefore, using (3.15), we can view $\delta(i)$ as the total vector deviation from $i\alpha$ of the distribution of points in the orbit

$$\mathrm{Orb}_{S_\beta}(0) = \{S_\beta^i(0) \equiv i\beta \bmod L;\ i = 0, 1, 2, \dots\} \tag{3.18}$$

over all tiles $T_1, \dots, T_D$ of the development T.

§4. Colored k-deviations

Suppose L' is a spanning lattice (2.1). Then its basis $l_1, \dots, l_D$ has a dual basis $l_1^*, \dots, l_D^*$ defined by

$$l_k^* \cdot l_m = \delta_{k,m}, \tag{4.1}$$

where $x = (x_1, \dots x_D)$ and $y = (y_1, \dots y_D)$ are in $\mathbb{R}^D$, and $x \cdot y$ denotes the scalar product

$$x \cdot y = x_1 y_1 + \dots + x_D y_D.$$

Using (3.15) and the dual basis (4.1), we get

$$l_k^* \cdot \delta(i) = r_k(i) - i l_k^* \cdot \alpha \quad \text{for} \quad k = 1, \dots, D. \tag{4.2}$$

Note that (4.2) can be written in the form

$$\delta_k(i) = r_k(i) - i a_k, \quad k = 1, \dots, D. \tag{4.3}$$

Here the $\delta_k(i)$ are given by

$$\delta_k(i) = l_k^* \cdot \delta(i). \tag{4.4}$$

Condition (2.1) on the vectors $l_1, \dots, l_D$ together with (4.1) implies that the coefficients a_k are uniquely determined by the equation

$$-\alpha = a_1 l_1 + \dots + a_D l_D. \tag{4.5}$$

The number $\delta_k(i)$ is called the deviation of the distribution of points in the orbit $\mathrm{Orb}_{S_\beta}(0)$ for the tile $T_k \subset T$. Short version: *k-deviation*, or *colored deviation*.

From (4.1) and (4.4) it follows that the vector-valued function $\delta(i)$ is related to the colored deviations $\delta_k(i)$ by the identity

$$\delta(i) = \delta_1(i) l_1 + \dots + \delta_D(i) l_D. \tag{4.6}$$

Therefore, it is natural to call $\delta(i)$ *the total vector deviation* of the distribution of points in the orbit $\mathrm{Orb}_{S_\beta}(0)$ with respect to the tiles $T_1, \dots, T_D$ of the development T.

It remains to compute the deviation $\delta_0(i)$ with respect to the tile T_0. Note that the counting functions $r_0(i), r_1(i), \dots, r_D(i)$ are not independent, because the definitions (3.6) and (3.7) show that

$$r_0(i) + r_1(i) + \dots + r_D(i) = i \tag{4.7}$$

for $i = 0, 1, 2, \dots$. Let a_0 be a number such that

$$a_0 + a_1 + \dots + a_D = 1. \tag{4.8}$$

Using (4.7) and (4.8), we get

$$[r_0(i) - i a_0] + [r_1(i) - i a_1] + \dots + [r_D(i) - i a_D] = 0. \tag{4.9}$$

As in (4.3) for the deviations $\delta_1(i), \dots, \delta_D(i)$, we can define the zeroth colored deviation as follows:

$$\delta_0(i) = r_0(i) - ia_0. \tag{4.10}$$

By (4.3) and (4.9), the colored deviations satisfy $\delta_0(i) + \delta_1(i) + \cdots + \delta_D(i) = 0$ for all $i = 0, 1, 2, \dots$. Therefore,

$$\delta_0(i) = -\delta_1(i) - \cdots - \delta_D(i). \tag{4.11}$$

On the other hand, relations (4.4) and (4.11) show that

$$\delta_0(i) = -l_1^* \cdot \delta(i) - \cdots - l_D^* \cdot \delta(i) = -(l_1^* + \cdots + l_D^*) \cdot \delta(i). \tag{4.12}$$

Now, put

$$l_0^* = -(l_1^* + \cdots + l_D^*). \tag{4.13}$$

Using (4.12), we get

$$\delta_0(i) = l_0^* \cdot \delta(i). \tag{4.14}$$

Comparing this with (4.4), we see that symmetry is restored for the colored deviations with respect to all tiles $T_0, T_1, \dots, T_D$ in the development T. This symmetry has been broken because we distinguished the vector v_0 in order to define the basis (1.9) of the lattice L', see (1.10).

Proposition 4.1. *Suppose L' is a complete lattice* (1.10) *and*

$$\delta_k(i) = r_k(i) - ia_k \quad \text{for} \quad k = 0, 1, \dots, D, \tag{4.15}$$

where the numbers $a_1, \dots, a_D$ are the coordinates of $-\alpha$ with respect to the basis $l_1, \dots, l_D$, and $a_0 = 1 - a_1 - \cdots - a_D$. Then for the k-deviations (4.15) *we have*

$$\delta_0(i) + \delta_1(i) + \cdots + \delta_D(i) = 0 \tag{4.16}$$

for all $i = 0, 1, 2, \dots$;

$$\delta_k(i) = l_k^* \cdot \delta(i) \quad \text{for} \quad k = 0, 1, \dots, D, \tag{4.17}$$

where $l_1^*, \dots, l_D^*$ is the basis dual to the basis $l_1, \dots, l_D$ of L' (see (4.1)*), the vector l_0^* is given by* (4.13)*, and the vector deviation $\delta(i)$ is given by* (3.1)*. Moreover, the deviations $\delta(i)$ and $\delta_k(i)$ are related to each other by the formula*

$$\delta(i) = \delta_1(i) l_1 + \cdots + \delta_D(i) l_D \tag{4.18}$$

for all $i = 0, 1, 2, \dots$.

Proof. Using (4.3), (4.10), and (4.4), we get (4.16) and (4.17) for the k-deviations $\delta_k(i)$; (4.6) implies formula (4.18). □

Remark 4.1. In the next section we shall prove the following asymptotic bound for the k-deviations $\delta_k(i)$ (see Theorem 5.1):

$$\delta_k(i) = o(i) \quad \text{as} \quad i \to +\infty, \tag{4.19}$$

where $k = 0, 1, \dots, D$. From (4.15) and (4.19) it follows that

$$\lim_{i \to +\infty} \frac{r_k(i)}{i} = a_k. \tag{4.20}$$

Combining this with the definition of the counting functions (3.6), we get

$$a_k \geq 0 \quad \text{for all} \quad k = 0, 1, \dots, D.$$

Therefore, a_k is the mean value of the number of the points in $\mathrm{Orb}_{S_\beta}(0)$ that lie in the tile T_k.

§5. Approximate bounds for k-deviations

For $x \in \mathbb{R}^D$, define a vector-valued function $\mathcal{D}$ by

$$\mathcal{D}(x) = \sum_{0 \le j \le n-1} (\mathrm{Fr}(j\beta + x) - \mathrm{Fr}(j\beta)).$$

The definition (2.2) of the T-fractional part $\mathrm{Fr}(x)$ implies that

$$\mathcal{D}(x + l) = \mathcal{D}(x)$$

for any vector l in the lattice L, which means that $\mathcal{D}(x)$ is a periodic function on the vector space $\mathbb{R}^D$ with the spanning period lattice L. We can rewrite equation (3.17) in terms of the function $\mathcal{D}(x)$:

$$\delta(i) = \mathcal{D} \circ \mathrm{Fr}(i\beta) = \mathcal{D}(\mathrm{Fr}(i\beta)), \tag{5.1}$$

i.e., $\delta(i)$ is the composition of $\mathrm{Fr}(x)$ and $\mathcal{D}(x)$.

Remark 5.1. Suppose S_α is an irrational shift (1.2) of the torus $\mathbb{T}^D$. The definition (1.11) shows that the shift S_β, where β is as in (3.2), is also irrational. It is known [9] that in this case the points of the orbit (3.18) are dense and equidistributed on the development of the torus T. Hence, in order to determine the behavior of the vector-valued deviation $\delta(i)$ for $i = 0, 1, 2, \dots$, it suffices to write $\delta(i)$ as the composition (5.1) and estimate the value of $\mathcal{D}(x)$, where x runs over the development of T

Let $i > n$. We introduce the difference function

$$\Delta(x, y) = \mathrm{Fr}(x + y) - \mathrm{Fr}(y); \tag{5.2}$$

by (2.2), it has the periodicity property

$$\Delta(x + l, y + l') = \Delta(x, y)$$

with respect to the periods l, l' in the lattice L. Using (5.2), we can rewrite the function $\mathcal{D}(x)$ in the form

$$\mathcal{D}(x) = \sum_{0 \le j \le n-1} \Delta(x, j\beta). \tag{5.3}$$

By (5.2), for any $x, y \in \mathbb{R}^D$ we have

$$\Delta(x, y) \in T_\Delta, \tag{5.4}$$

where

$$T_\Delta = T - T = \{t - t'; t, t' \in T\} \tag{5.5}$$

is the difference subset of the development T. The definition (5.5) implies that the difference set T_Δ is symmetric,

$$T_\Delta \overset{-}{\rightleftarrows} T_\Delta \,:\, t \mapsto -t,$$

with respect to the origin of $\mathbb{R}^D$. Using (5.3) and (5.4), we see that

$$\mathcal{D}(x) \in nT_\Delta \tag{5.6}$$

for any x in $\mathbb{R}^D$, where multiplication by n on the right-hand side of (5.6) means the homothety $t \mapsto nt$ with coefficient $n = 1, 2, 3, \dots$.

By (5.4), for any $x, y \in \mathbb{R}^D$ we have

$$l_k^* \cdot \Delta(x, y) \in (l_k^* \cdot T)_\Delta, \quad k = 0, 1, \dots, D, \tag{5.7}$$

because $l_k^* \cdot (T)_\Delta = (l_k^* \cdot T)_\Delta$ by the identity $l_k^* \cdot T = \{l_k^* \cdot t; t \in T\}$. We put

$$\mathcal{D}_k(x) = l_k^* \cdot \mathcal{D}(x); \tag{5.8}$$

then, using (5.6) and (5.8), for any $x \in \mathbb{R}^D$ we get

$$\mathcal{D}_k(x) \in n(l_k^* \cdot T)_\Delta, \quad k = 0, 1, \dots, D. \tag{5.9}$$

For any subset $X \subset \mathbb{R}^D$, we define the boundary values

$$\underline{\mathrm{pr}}_l X = \inf_{x \in X} \mathrm{pr}_l\, x, \quad \overline{\mathrm{pr}}_l X = \sup_{x \in X} \mathrm{pr}_l\, x \tag{5.10}$$

of the orthogonal projections

$$\mathrm{pr}_l\, x = \frac{l \cdot x}{|l|}$$

of the vectors $x \in X$ to the vector l, where $|l| = \sqrt{l \cdot l}$ is the length of l.

Relations (5.9) and (5.10) show that

$$|\mathcal{D}_k(x)| \le n|l_k^*|\big(\overline{\mathrm{pr}}_{l_k^*} T - \underline{\mathrm{pr}}_{l_k^*} T\big) \tag{5.11}$$

for any $x \in \mathbb{R}^D$ and $k = 0, 1, \dots, D$.

By Proposition 3.1, the vector-valued deviation $\delta(i)$ can be expressed as a composition:

$$\delta(i) = \mathcal{D} \circ \mathrm{Fr}(i\beta) \quad \text{for} \quad i > n.$$

Therefore, using (5.11), we obtain the inequalities

$$|\delta_k(i)| \le c_k n \quad \text{for} \quad i > n, \tag{5.12}$$

where

$$c_k = c_{T,k} = |l_k^*|\big(\overline{\mathrm{pr}}_{l_k^*} T - \underline{\mathrm{pr}}_{l_k^*} T\big) \tag{5.13}$$

are the constants independent of n, i; they depend solely on the size of the development T (see (1.4)) of the torus $\mathbb{T}^D$.

It remains to consider the case of $0 \le i \le n$, where, by (3.16), the vector-valued deviation $\delta(i)$ has the form

$$\delta(i) = \sum_{0 \le j < i} (\mathrm{Fr}(j\beta + n\beta) - \mathrm{Fr}(j\beta))$$

or, in terms of the difference function (5.2),

$$\delta(i) = \sum_{0 \le j < i} \Delta(n\beta, j\beta). \tag{5.14}$$

In this case instead of (5.12) we can use (5.7) and (5.14) to obtain the inequalities

$$|\delta_k(i)| \le c_k i \quad \text{for } 0 \le i \le n. \tag{5.15}$$

Theorem 5.1. *Suppose the vectors $l_1, \dots, l_D$ are linearly independent over $\mathbb{R}$. Then:*

1) *for any $k = 0, 1, \dots, D$, we have*

$$|\delta_k(i)| \le c_k n \tag{5.16}$$

for all $i = 0, 1, 2, \dots$, where the $c_k = c_{T,k}$ are the constants (5.13) *independent of n and i and determined by the size of the toric development T, see* (1.4);

2) *if, moreover, α is an irrational shift* (1.11)*, and every tile T_k in the tiling* (1.4) *of the development of T is Jordan measurable, then the k-deviations $\delta_k(i)$ defined by* (4.3) *and* (4.14) *have the form*

$$\delta_k(i) = r_k(i) - i a_k, \quad \textit{where} \quad a_k = \frac{\mathrm{vol}\, T_k}{\mathrm{vol}\, T}, \tag{5.17}$$

for $k = 0, 1, \dots, D$. Here $\mathrm{vol}\, T_k$ and $\mathrm{vol}\, T$ denote the volumes of the tiles T_k and of the whole development of T.

Proof. Inequality (5.16) follows from (5.12) and (5.15), while (5.17) follows from (4.3), (4.14), and (4.20). □

Now we estimate the absolute value of the total vector-valued deviation $\delta(i)$. Using (5.1) and (5.6), we get

$$\delta(i) \in nT_\Delta.$$

Therefore,

$$|\delta(i)| \leq d_T n \tag{5.18}$$

for all $i = 0, 1, 2, \dots$, where

$$d_T = \sup_{t,t' \in \bar{T}} |t - t'| \tag{5.19}$$

is the diameter of the development of T.

As above, using (5.6) and (5.8), see that

$$\delta_k(i) \in nl_k^* \cdot T_\Delta.$$

Therefore,

$$|\delta_k(i)| \leq |l_k^*| d_T n \tag{5.20}$$

for $k = 0, 1, \dots, D$.

Assume once again that the vectors $l_1, \dots, l_D$ are linearly independent over $\mathbb{R}$. Let A be a nonsingular affine map on the vector space $\mathbb{R}^D$. Consider the action of this map on the torus

$$A : \mathbb{T}_L^D \simeq \mathbb{R}^D/L \longrightarrow \mathbb{T}_{AL}^D \simeq \mathbb{R}^D/AL.$$

We obtain the following substitutions for the basic elements of the construction in §1:

$$L \to AL, \quad \alpha \to A\alpha;$$
$$T \to AT, \quad L' \to AL', \quad v_k \to Av_k, \quad v_k \to Av_k.$$

Consider the action of the affine map A on the vector-valued deviation $\delta(i) = \delta_T(i)$. The definition (3.1) shows that

$$\delta_{AT}(i) = A\delta_T(i) \tag{5.21}$$

for $i = 0, 1, 2, \dots$. Formula (5.21) means that A acts on the vector-valued deviation $\delta_T(i)$ in the same way as on the vectors $x \mapsto Ax$ of the vector space $\mathbb{R}^D$. The choice of the starting basis $l_1, \dots, l_D$ is arbitrary: we can use it to simplify the construction of exchanging toric developments (1.4). Therefore, we conclude that the vector deviation $\delta(i) = \delta_T(i)$ has no natural scale.

Now, consider the action of the affine map A on the k-deviations $\delta_k(i)$. Suppose we have started with some other basis $Al_1, \dots, Al_D$. By (4.18),

$$\delta_{AT}(i) = \delta_{AT,1}(i)Al_1 + \dots + \delta_{AT,D}(i)Al_D. \tag{5.22}$$

Now apply (4.18) again for the basis $l_1, \dots, l_D$. Considering the action of A on (4.18), we get

$$A\delta_T(i) = \delta_{T,1}(i)Al_1 + \dots + \delta_{T,D}(i)Al_D. \tag{5.23}$$

The left-hand sides of (5.22) and (5.23) are equal by (5.21). Therefore, the right-hand sides of (5.22) and (5.23) are also equal. Together with the linear independence of $Al_1, \dots, Al_D$, this implies

$$\delta_{AT,k}(i) = \delta_{T,k}(i)$$

for all $k = 0, 1, \dots, D$. As we can see, unlike the vector-valued deviation $\delta_T(i)$ (see (5.21)), the colored k-deviations $\delta_{T,k}(i)$ are invariant under the action of the affine map

A. Therefore, they do not depend on the choice of the basis $l_1, \dots, l_D$. The constants in (5.13) are also invariant,

$$c_{AT,k} = c_{T,k}, \tag{5.24}$$

and so are the coordinates (4.5),

$$a_{AT,k} = a_{T,k} \tag{5.25}$$

for any $k = 0, 1, \dots, D$.

Proof of Theorem 3 (see the Introduction). Suppose the vectors $l_1, \dots, l_D$ are linearly independent over $\mathbb{R}$. There is no loss of generality in assuming that

$$l_1 = e_1, \dots, l_D = e_D. \tag{5.26}$$

In the standard basis, inequalities (5.20) can be written as

$$|\delta_k(i)| \le d_T n \tag{5.27}$$

for $k = 1, \dots, D$. For $k = 0$ they can be written as

$$|\delta_0(i)| \le D d_T n, \tag{5.28}$$

because from (5.26) and (4.13) applied to l_0^* it follows that

$$l_0^* = -(e_1 + \dots + e_D) = (-1, \dots, -1).$$

Suppose also that the following conditions are satisfied:

1) the basic period lattice L of the torus $\mathbb{T}^D = \mathbb{R}^D/L$ has the form

$$L = L', \tag{5.29}$$

where $L' = \mathbb{Z}[e_1, \dots, e_D]$ is a cubic lattice in $\mathbb{R}^D$;

2) α is an irrational shift of $\mathbb{T}^D$;

3) the toric development T and all its tiles T_k are Jordan measurable.

Then conditions 2) and 3) imply (see [9]) that for any $k = 0, 1, \dots, D$ there is a frequency

$$\nu_k = \lim_{i \to +\infty} \frac{r_k(i)}{i} \tag{5.30}$$

of the number of returns of the orbit $\mathrm{Orb}_{S_\beta}(0)$ into the tile T_k. Moreover, we have

$$\nu_k = \operatorname{vol} T_k, \tag{5.31}$$

because 1) together with (5.29) implies that the volume of the toric development equals $\operatorname{vol} T = 1$. At the same time, by (4.20) and (5.30) we have

$$\nu_k = a_k;$$

therefore, (5.31) yields

$$a_k = \operatorname{vol} T_k. \tag{5.32}$$

Moreover, using (4.5), we obtain

$$\operatorname{vol} T_k = -e_k \cdot \alpha = -\alpha_k \tag{5.33}$$

for $k = 1, \dots, D$, where the shift vector $\alpha = (\alpha_1, \dots, \alpha_D)$ is expressed in the standard basis $e_1, \dots, e_D$.

Proposition 4.1 and (5.32) show that

$$\delta_k(i) = r_k(i) - i \operatorname{vol} T_k. \tag{5.34}$$

Combining inequalities (5.27), (5.28), and (5.34), we obtain Theorem 3. □

§6. Precise bounds for k-deviations

Let $\Delta(x,y)$ denote the difference function (5.2). It can be written as

$$\Delta(x,y) = x - l(x,y) \tag{6.1}$$

for any $x, y \in T$, where $l(x,y) = l$ is a unique vector in the lattice L such that

$$x \in T[l] - y \sim x + y \in T[l],$$

with $T[l] = T + l$. Indeed, there is a unique vector l in L such that $x + y \in T[l]$, whence $x + y - l \in T[0] = T$. Therefore,

$$\mathrm{Fr}(x+y) = x + y - l = x + \mathrm{Fr}(y) - l$$

for any $y \in T$. Thus, we have (6.1).

The case where $i > n$. By (5.3) and (6.1), we have

$$\mathcal{D}(x) = \sum_{0 \le j \le n-1} \Delta(x, j\beta) = nx - \sum_{0 \le j \le n-1} l(x, j\beta), \tag{6.2}$$

where for any $x \in T$ we denote by $l(x, j\beta) = l$ a unique vector in L such that $x + \mathrm{Fr}(j\beta) \in T[l]$. Looking at (6.2), we introduce the function

$$\mathcal{A}(x) = \sum_{0 \le j \le n-1} l(x, j\beta).$$

Remark 6.1. Our use of the vector-valued fractional part $\mathrm{Fr}(j\beta)$ instead of $j\beta$ itself corresponds to localizing the situation: we are interested only in the tiles $T[l]$ adjacent to the tile $T = T[0]$.

From now on we assume that the toric development T and all its tiles T_k are unions of finitely many arbitrary polygons.

We define a set of vectors:

$$L(j\beta) = \{l \in L; T \cap (T_l - \mathrm{Fr}(j\beta)) \neq \varnothing\}. \tag{6.3}$$

Note that there can be several vectors l satisfying the condition in (6.3).

Consider the union of the following boundaries of polygons:

$$\partial_n = \bigcup_{0 \le j \le n-1} \bigcup_{l \in L(j\beta)} \partial(T_l - \mathrm{Fr}(j\beta)),$$

where for any set $X \subset \mathbb{R}^D$ by ∂X we denote the boundary of X:

$$\partial X = \bar{X} \setminus X^{\mathrm{int}}.$$

Consider a partition

$$\mathrm{Til}_n(T) = \bigcup_p C_p \tag{6.4}$$

of T into a finite number of cells C_p, where every C_p is a closed polygon (i.e., $C_p = \overline{C_p}$), hence a simply connected set. The partition $\mathrm{Til}_n(T)$ is uniquely determined by the conditions

$$\partial(\mathrm{Til}_n(T)) = \bar{T} \cap \partial_n,$$

and any two cells C_p, $C_{p'}$ in (6.4) have no interior points in common.

By (5.3) and (5.8),

$$\mathcal{D}_k(x) = nl_k^* \cdot x - \mathcal{A}_k(x), \tag{6.5}$$

where

$$\mathcal{A}_k(x) = \sum_{0\le j\le n-1} l_k^* \cdot l(x, j\beta) \quad \text{for} \quad k = 0, 1, \dots, D. \tag{6.6}$$

Using the definitions of $l(x, j\beta)$ and $\mathrm{Til}_n(T)$, we get the following property of $\mathcal{A}_k(x)$.

Lemma 6.1. *Suppose $\mathcal{A}_k(x)$ is the function defined by* (6.6). *Then*:
1) *$\mathcal{A}_k(x)$ is constant on every set C_p^{int} in* (6.4);
2) *all jumps of $\mathcal{A}_k(x)$ are contained in the boundaries ∂C_p.*

The first term $nl_k^* \cdot x$ on the right-hand side of (6.5) is a nonzero linear form, which, when viewed as a function on C_p, has extrema points only at the vertices $v \in V(C_p)$ of the polygon C_p.

Denote by $V(\mathrm{Til}_n(T))$ the set of all vertices of all the polygons C_p in the partition $\mathrm{Til}_n(T)$. For any vertex v in $V(\mathrm{Til}_n(T))$, put

$$\mathcal{A}_{k,C_p} = \mathcal{A}_k(x_{C_p}^{\mathrm{int}}), \tag{6.7}$$

where $x_{C_p}^{\mathrm{int}}$ is any interior point of C_p. By (6.6), we have

$$\mathcal{A}_{k,C_p} = \sum_{0\le j\le n-1} l_k^* \cdot l(x_{C_p}^{\mathrm{int}}, j\beta).$$

For every vertex v in $V(\mathrm{Til}_n(T))$, consider the following two extremal quantities:

$$\min_k(v) = \min_{C_p\in \mathrm{Til}_n(T)} \mathcal{A}_{k,C_p}, \quad \max_k(v) = \max_{C_p\in \mathrm{Til}_n(T)} \mathcal{A}_{k,C_p}.$$

Let

$$m_k' = \inf_{x\in T} \mathcal{D}_k(x), \quad M_k' = \sup_{x\in T} \mathcal{D}_k(x). \tag{6.8}$$

Lemma 6.1 and relations (6.5) and (6.7) show that

$$m_k' = \min_{v\in V(\mathrm{Til}_n(T))} (nl_k^* \cdot v - \max_k(v)), \tag{6.9}$$

$$M_k' = \max_{v\in V(\mathrm{Til}_n(T))} (nl_k^* \cdot v - \min_k(v)) \tag{6.10}$$

for any $i > n$.

The case where $0 \le i \le n$. Let

$$\mathcal{D}_{k,i} = \sum_{0\le j\ <i} l_k^* \cdot \Delta(n\beta, j\beta),$$

and let

$$m_k'' = \min_{0\le i\le n} \mathcal{D}_{k,i}, \quad M_k'' = \max_{0\le i\le n} \mathcal{D}_{k,i}.$$

Proposition 3.1 and relations (5.1), (6.9), and (6.10) imply the following result.

Theorem 6.1. *Suppose the vectors $l_1, \dots, l_D$ are linearly independent over $\mathbb{R}$. Then, for any $k = 0, 1, \dots, D$, the deviations $\delta_k(i)$ satisfy the inequalities*

$$m_k \le \delta_k(i) \le M_k \tag{6.11}$$

for any $i = 0, 1, 2, \dots,$ where

$$m_k = \min\{m_k', m_k''\}, \quad M_k = \max\{M_k', M_k''\}. \tag{6.12}$$

Moreover, the bounds in (6.11) *are sharp, i.e., for any $\varepsilon > 0$ there exist i_1 and i_2 such that*

$$m_k + \varepsilon \ge \delta_k(i_1), \quad \delta_k(i_2) \ge M_k - \varepsilon.$$

Remark 6.2. The computational complexity of the sharp bounds m_k, M_k in (6.11) depends on the number $c_{n,D}(T)$ of the cells C_p in the partitions $\mathrm{Til}_n(T)$, see (6.4). Choosing the D-dimensional unit cube I^D (see §9) as a toric development T, we have the following estimates for $c_{n,D}(T)$:

$$c_{n,D}(T) = O(n^D) \quad \text{as} \quad n, D \to +\infty.$$

Remark 6.3. The sharp bounds (6.11) are useful, e.g., in solving the problem of embedding lattices into quasiperiodic partitions [11], and in constructing quasiperiodic sequences close to periodic ones, or balanced words (see, e.g., [1, 5]).

§7. The case where $n = 1$

For $n = 1$, the deviations $\delta(i)$ have the simplest behavior, which is completely determined by the geometry of the toric development T as in (1.4). When $n > 1$, the deviation $\delta(i)$ is seriously affected by the metric properties of the shift vector α, see (1.2).

So, let $n = 1$. Then, for $i \geq 2$, from (3.17) it follows that

$$\delta(i) = \mathcal{D}(i\beta), \tag{7.1}$$

where

$$\mathcal{D}(x) = \mathrm{Fr}(x) - \mathrm{Fr}(0) \tag{7.2}$$

for any $x \in \mathbb{R}^D$. Suppose $\mathrm{Fr}(0) = 0$, i.e., $0 \in T$. Then the definition (2.2) of the fractional part $\mathrm{Fr}(x)$ and (7.2) show that

$$\mathcal{D}(x) = x \quad \text{for any} \quad x \in T. \tag{7.3}$$

If $i = 0$ or 1, then (3.16) implies

$$\delta(0) = 0, \quad \delta(1) = \mathrm{Fr}(\beta) - \mathrm{Fr}(0) = \mathrm{Fr}(\beta).$$

Thus, if $n = 1$ and $\mathrm{Fr}(0) = 0$, then the vector-valued deviation $\delta(i)$ can be expressed as

$$\delta(i) = \mathrm{Fr}(i\beta) \tag{7.4}$$

for any $i = 0, 1, 2, \ldots$, while the k-deviations have the form

$$\delta_k(i) = \mathrm{Fr}_k(i\beta) \quad \text{for} \quad k = 0, 1, \ldots, D, \tag{7.5}$$

where $\mathrm{Fr}_k(x) = l_k^* \cdot \mathrm{Fr}(x)$. Thus, (7.4) and (7.5) provide the following geometrical description of the deviations:

1) the vector-valued deviation $\delta(i)$ is the position of the T-fractional part $\mathrm{Fr}(i\beta)$ of $i\beta$ on the toric development (1.4);

2) the k-deviation $\delta_k(i)$ is related to the projection $\mathrm{Fr}_k(i\beta)$ of the vector-valued fractional part $\mathrm{Fr}(i\beta)$ to the vector l_k^*:

$$\delta_k(i) = |l_k^*| \, \mathrm{pr}_{l_k^*} \mathrm{Fr}(i\beta). \tag{7.6}$$

Using (7.6), we get the inequalities

$$|l_k^*| \underline{\mathrm{pr}}_{l_k^*} T \leq \delta_k(i) \leq |l_k^*| \overline{\mathrm{pr}}_{l_k^*} T, \tag{7.7}$$

whence

$$|\delta_k(i)| \leq |l_k^*| \max\left\{|\underline{\mathrm{pr}}_{l_k^*} T|, |\overline{\mathrm{pr}}_{l_k^*} T|\right\}. \tag{7.8}$$

Suppose that $\mathrm{Fr}(0) = 0$ lies in T. Then, by (7.8),

$$|\delta_k(i)| \leq |l_k^*| \, d_T \tag{7.9}$$

for $k = 0, 1, \ldots, D$, where d_T is the diameter (5.19) of $\bar{T}$.

Since the vectors $l_1, \dots, l_D$ are linearly independent over $\mathbb{R}$, there is an affine map $Al_k = e_k$ for $k = 1, 2, \dots, D$, where $e_1, \dots, e_D$ is a standard basis of the vector space $\mathbb{R}^D$. Since the map A is nonsingular, we can replace T with AT in all the above arguments. Thus, to simplify the notation, we may assume that $l_1 = e_1, \dots, l_D = e_D$, so that the lattice L' in (1.10) has the form

$$L' = \mathbb{Z}[e_1, \dots, e_D], \tag{7.10}$$

i.e., is a cubic lattice $\mathbb{Z}^D$ in $\mathbb{R}^D$.

Remark 7.1. In §9 we shall consider a specific example of toric developments T that are D-dimensional cubes I^D, where $I = [0, 1)$ is the unit semiinterval. Those $T = I^D$ can be constructed in purely geometric terms. Then, the lattice L' emerges naturally from the construction and in general does not coincide with the cubic lattice $\mathbb{Z}^D$. Moreover, when we substitute the new development AT for the toric development T, we have to sacrifice the geometric intuition to replace the vector fractional part $\mathrm{Fr}(x)$ (where $x = (x_1, \dots, x_D)$) with the usual fractional parts $\{x_k\}$ of the coordinates x_k.

If (7.10) is true, then the dual basis (4.1) has the form

$$l_k^* = e_k^* = e_k. \tag{7.11}$$

Therefore,

$$\mathrm{pr}_{l_k^*}(x) = \mathrm{pr}_{e_k}(x)$$

and

$$\underline{\mathrm{pr}}_{l_k^*} T = \underline{\mathrm{pr}}_{e_k} T, \quad \overline{\mathrm{pr}}_{l_k^*} T = \overline{\mathrm{pr}}_{e_k} T \tag{7.12}$$

for $k = 1, \dots, D$. For $k = 0$ we have

$$l_0^* = -(e_1^* + \dots + e_D^*) = -(e_1 + \dots + e_D)$$

by (4.13). It follows that

$$\underline{\mathrm{pr}}_{l_o^*} T = -\overline{\mathrm{pr}}_e T, \quad \overline{\mathrm{pr}}_{l_0^*} T = -\underline{\mathrm{pr}}_e T, \tag{7.13}$$

where $e = e_1 + \dots + e_D = (1, \dots, 1)$. With the help of (7.11), the k-deviations $\delta_k(i)$ can be expressed as

$$\delta_k(i) = l_k^* \cdot \delta(i) = e_k \cdot \delta(i)$$

for $k = 1, \dots, D$, and

$$\delta_0(i) = l_0^* \cdot \delta(i) = -e \cdot \delta(i) \tag{7.14}$$

for $k = 0$.

Theorem 7.1. *Suppose the vectors* (2.1) *are* $l_1 = e_1, \dots, l_D = e_D$.

1. *The vector deviation* $\delta(i)$ *has the coordinates*

$$\delta(i) = (\delta_1(i), \dots, \delta_D(i))$$

with respect to the basis $e_1, \dots, e_D$, *where, as in* (7.5),

$$\delta_k(i) = \mathrm{Fr}_k(i\beta) \quad \textit{for} \quad k = 1, \dots, D,$$

while the vector T*-fractional part* $\mathrm{Fr}(x)$ *has the coordinates*

$$\mathrm{Fr}(x) = (\mathrm{Fr}_1(x), \dots, \mathrm{Fr}_D(x)) \tag{7.15}$$

in the same basis $e_1, \dots, e_D$.

2. *The* k*-deviations* $\delta_k(i)$ *satisfy the following inequalities*:

$$\underline{\mathrm{pr}}_{e_k} T \le \delta_k(i) \le \overline{\mathrm{pr}}_{e_k} T, \tag{7.16}$$

$$|\delta_k(i)| \le d_T \tag{7.17}$$

for $k = 1, \dots, D$, *and*

$$-\overline{\mathrm{pr}}_e T \le \delta_0(i) \le -\underline{\mathrm{pr}}_e T, \tag{7.18}$$

where $e = e_1 + \dots + e_D$,

$$|\delta_0(i)| \le D d_T. \tag{7.19}$$

Proof. Inequalities (7.16) for $k = 1, \dots, D$ follow from (7.7) and (7.12); inequality (7.18) follows from (7.7) and (7.13), (7.14). Inequalities (7.17) and (7.19) follow from (7.9). □

§8. Mean values of the deviations

For the vector deviation $\delta(i)$, we define its mean value by

$$\langle \delta \rangle = \lim_{N \to +\infty} \frac{1}{N} \sum_{1 \le i \le N} \delta(i) \tag{8.1}$$

if this limit exists. By (3.17), we have

$$\begin{aligned} \sum_{n < i \le N} \delta(i) &= \sum_{n < i \le N} \sum_{0 \le j \le n-1} (\mathrm{Fr}(j\beta + i\beta) - \mathrm{Fr}(j\beta)) \\ &= \sum_{0 \le j \le n-1} \sum_{n < i \le N} \mathrm{Fr}(j\beta + i\beta) - (N - n) \sum_{0 \le j \le n-1} \mathrm{Fr}(j\beta). \end{aligned} \tag{8.2}$$

Using (8.2), we can write

$$\begin{aligned} \lim_{N \to +\infty} \frac{1}{N} \sum_{1 \le i \le N} \delta(i) &= \lim_{N \to +\infty} \frac{1}{N} \sum_{n < i \le N} \delta(i) \\ &= \sum_{0 \le j \le n-1} \left(\lim_{N \to +\infty} \frac{1}{N} \sum_{0 \le i \le N} \mathrm{Fr}(j\beta + i\beta) - \sum_{0 \le j \le n-1} \mathrm{Fr}(j\beta) \right). \end{aligned} \tag{8.3}$$

Denote

$$c(j) = \lim_{N \to +\infty} \frac{1}{N} \sum_{0 \le i \le N} \mathrm{Fr}(j\beta + i\beta). \tag{8.4}$$

Since

$$\sum_{0 \le i \le N} \mathrm{Fr}(j\beta + i\beta) = \sum_{0 \le i \le N} \mathrm{Fr}(j\beta) - \sum_{0 \le i < j} \mathrm{Fr}(j\beta) + \sum_{N < i \le N+j} \mathrm{Fr}(j\beta),$$

the limit in (8.4) does not depend on $j = 0, 1, 2, \dots$. Therefore,

$$c(j) = c(0) \tag{8.5}$$

for any $j = 0, 1, 2, \dots$. From (8.1), (8.3), and (8.4) it follows that the mean value $\langle \delta \rangle$ has the form

$$\langle \delta \rangle = \sum_{0 \le j \le n-1} (c(0) - \mathrm{Fr}(j\beta)). \tag{8.6}$$

Suppose the development T has finite diameter $d_T < \infty$ and its closure $\bar{T}$ is Jordan measurable. Then the indicator function

$$\chi_{\bar{T}}(x) = \begin{cases} 1 & \text{if } x \in \bar{T}, \\ 0 & \text{if } x \notin \bar{T} \end{cases}$$

of the set $\bar{T} \subset \mathbb{R}^D$ is integrable, and its integral is equal to the volume of $\bar{T}$:

$$\mathrm{vol}(\bar{T}) = \int_{\mathbb{R}^D} \chi_{\bar{T}}(x) dx.$$

Our assumptions imply that the limit in (8.4) exists and is equal to

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{0\le i\le N}\mathrm{Fr}(i\beta)=\frac{1}{\mathrm{vol}(\bar{T})}\int_{\bar{T}}x\,dx \tag{8.7}$$

(see, e.g., [9, 10]). Note that the vector-valued integral

$$c_T=\frac{1}{\mathrm{vol}(\bar{T})}\int_{\bar{T}}x\,dx \tag{8.8}$$

can be interpreted geometrically as the barycenter of $\bar{T}$. Using (8.4) and (8.8), we get

$$c(0)=c_T. \tag{8.9}$$

Theorem 8.1. *Suppose the following conditions are satisfied: the vectors $l_1,\dots,l_D$ (see (1.9)) are linearly independent over $\mathbb{R}$, α is an irrational shift of the torus $\mathbb{T}^D$, the development T as in (1.4) has finite diameter d_T, and its closure $\bar{T}$ is Jordan measurable.*

1. *The mean value $\langle\delta\rangle$ (see (8.1)) of the vector deviation $\delta(i)$ exists and can be computed as follows:*

$$\langle\delta\rangle=\sum_{0\le j\le n-1}(c_T-\mathrm{Fr}(j\beta)), \tag{8.10}$$

where c_T is the barycenter (8.8) of $\bar{T}$ and the difference $c_T-\mathrm{Fr}(x)$ in (8.10) is the centered fractional part of $x\in\mathbb{R}^D$.

2. *For any $k=0,1,\dots,D$, the mean value*

$$\langle\delta_k\rangle=\lim_{N\to+\infty}\frac{1}{N}\sum_{1\le i\le N}\delta_k(i) \tag{8.11}$$

of the k-deviations $\delta_k(i)=l_k^*\cdot\delta(i)$ exists and is equal to

$$\langle\delta_k\rangle=\sum_{0\le j\le n-1}(c_{T,k}-\mathrm{Fr}_k(j\beta)), \tag{8.12}$$

where $\mathrm{Fr}_k(j\beta)=l_k^*\cdot\mathrm{Fr}_k(j\beta)$ and

$$c_{T,k}=l_k^*\cdot c_T=|l_k^*|\,\mathrm{pr}_{l_k^*}c_T.$$

Proof. Note that (8.10) follows from (8.6) and (8.9).

To prove (8.12), observe that (8.10) implies

$$l_k^*\cdot\langle\delta\rangle=\sum_{0\le j\le n-1}(l_k^*\cdot c_T-l_k^*\cdot\mathrm{Fr}(j\beta)),$$

while the definition (8.11) yields

$$l_k^*\cdot\langle\delta\rangle=\lim_{N\to+\infty}\frac{1}{N}\sum_{1\le i\le N}l_k^*\cdot\delta(i). \tag{8.13}$$

Using (8.1) and (8.13), we get the relationship

$$l_k^*\cdot\langle\delta\rangle=\langle l_k^*\cdot\delta\rangle=\langle\delta_k\rangle \tag{8.14}$$

between the mean values of the vector deviation $\delta(i)$ and of the k-deviations $\delta_k(i)$. Combining (8.10) and (8.14), we arrive at (8.12). □

If $n=1$, the shift vector β is equal to the starting torus shift vector α. Therefore, we can rewrite (8.10) and (8.12) as follows:

$$\langle\delta\rangle=c_T-\mathrm{Fr}(0), \tag{8.15}$$

$$\langle\delta_k\rangle=c_{T,k}-\mathrm{Fr}_k(0)\ \text{ for }\ k=0,1,\dots,D. \tag{8.16}$$

If the point $\mathrm{Fr}(0) = 0$ lies in T, then (8.15) and (8.16) simplify to

$$\langle \delta \rangle = c_T, \tag{8.17}$$

$$\langle \delta_k \rangle = c_{T,k} \ \text{ for } \ k = 0, 1, \dots, D. \tag{8.18}$$

If, moreover, the barycenter c_T of T is inside T, then the translation $x \mapsto x - c_T$ maps T into the domain $T' = T - c_T$ with the barycenter

$$c_{T'} = 0$$

lying inside T'. Therefore, in the case where $c_T \in T$, there is no loss of generality in assuming that T has the barycenter

$$c_T = 0. \tag{8.19}$$

Since (8.19) implies that $\mathrm{Fr}(0) = 0$, we can rewrite (8.17) and (8.18) as follows:

$$\langle \delta \rangle = 0, \ \ \langle \delta_k \rangle = 0 \ \text{ for } \ k = 0, 1, \dots, D. \tag{8.20}$$

§9. Exchanging unit cube

9.1. General construction of an exchanging cube. We start with dimension $D = 1$. Let T^1 be the unit semiinterval $I^1 = [0, 1)$. Consider the partition

$$T^1 = T^1_0 \sqcup T^1_1 \tag{9.1}$$

of T^1 into two semiintervals

$$T^1_0 = [0, 1 - \alpha), \quad T^1_1 = [1 - \alpha, 1). \tag{9.2}$$

We define the exchange on T^1 by

$$T^1 \xrightarrow{S^1} T^1 \ : \ S^1(x) = x + v^1_k \ \text{ for } \ x \in T^1_k, \tag{9.3}$$

where $v_0 = \alpha$, $v_1 = \alpha - 1$.

We consider the semiinterval T^1 together with the partition (9.1) as a colored unit semiinterval $T^1 = I^1_{\mathrm{col}}$.

Now suppose $D \geq 2$. We construct an exchange cube T^D by induction on D. Suppose we have a $(D-1)$-dimensional unit cube $T^{D-1} = I^{D-1}_{\mathrm{col}}$ with a partition

$$T^{D-1} = T^{D-1}_0 \sqcup \dots \sqcup T^{D-1}_{D-1}.$$

Moreover, suppose there is a one-dimensional partition $T^1 = T^1_0 \sqcup T^1_1$ with $\alpha = \alpha_D$ in (9.2).

We define the D-dimensional cube T^D and its partition by putting

$$T^D = T^{D-1} \times T^1 = T^D_0 \sqcup T^D_1 \sqcup \dots \sqcup T^D_D i \tag{9.4}$$

where

$$rT^D_k = T^{D-1}_k \times T^1_0 \ \text{ for } \ k = 0, \dots, D-1$$

and

$$T^D_D = T^{D-1} \times T^1_1$$

are Cartesian products of the corresponding sets. To define exchange vectors for the cube T^D as in (9.4), we may assume that the spaces $\mathbb{R}^{D-1}$ and $\mathbb{R}^1$ are embedded in the space $\mathbb{R}^D = \mathbb{R}^{D-1} \times \mathbb{R}^1$ as follows:

$$\begin{gathered} \mathbb{R}^{D-1} \ni x^{D-1} \hookrightarrow (x^{D-1}, 0) \in \mathbb{R}^D, \\ \mathbb{R}^1 \ni x^1 \hookrightarrow (0, x^1) \in \mathbb{R}^D. \end{gathered} \tag{9.5}$$

The exchange on T^D will be defined by

$$T^D \xrightarrow{S^D} T^D \ : \ S^D(x) = x + v^D_k \ \text{ for } \ x \in T^D_k,$$

where

$$v_k^D = v_k^{D-1} + v_0^1 \quad \text{for} \quad k = 0, \dots, D-1 \tag{9.6}$$

and

$$v_D^D = v_1^1. \tag{9.7}$$

Again, we see that the partition (9.4) gives rise to a colored D-dimensional cube $T^D = I^D_{\mathrm{col}}$.

9.2. One-dimensional case. Let us apply the constructions of §5 to the exchanging semiinterval T^1 as in (9.1). By (1.9), we have $l_1 = v_1 - v_0 = -1$. Thus, the lattice (1.10) has the form $L' = \mathbb{Z}[l_1] = \mathbb{Z}$, whence

$$S^1_\alpha \,:\, x \to x + \alpha \bmod 1$$

is a rotation of the unit circle $\mathbb{T}^1 \simeq \mathbb{R}^1/L$, where $L = L' = \mathbb{Z}$. Therefore,

$$\mathrm{Fr}^1(x) = \{x\}$$

is the fractional part of x. Using (4.1) and (4.13), we get

$$l_1^* = -1, \quad l_0^* = -l_1^* = 1, \tag{9.8}$$

so that the counting functions $r_0^1(i)$ and $r_1^1(i)$ look like this:

$$r_0^1(i) = \sharp\{j; \{j\beta\} < 1 - \alpha, 0 \le j < i\},$$
$$r_1^1(i) = \sharp\{j; \{j\beta\} \ge 1 - \alpha, 0 \le j < i\}.$$

Consider the k-deviations

$$\delta_k^1(i) = r_k^1(i) - i a_k \quad \text{for} \quad k = 0, 1, \tag{9.9}$$

where, in accordance with (4.5) and (9.8),

$$a_1 = -l_1^* \cdot \alpha = \alpha, \quad a_0 = 1 - a_1 = 1 - \alpha.$$

Therefore, we have the following equations for the k-deviations (9.9):

$$\delta_0^1(i) = r_0^1(i) - i(1-\alpha), \quad \delta_1^1(i) = r_1^1(i) - i\alpha. \tag{9.10}$$

In order to estimate the values of the k-deviations $\delta_0^1(i)$, $\delta_1^1(i)$, we can use Theorem 5.1. The following crude bounds are implied by (5.16):

$$|\delta_k^1(i)| \le c_k^1 n \quad \text{for} \quad k = 0, 1,$$

where, by (5.13),

$$c_k^1 = |l_k^*|(\overline{\mathrm{pr}}_{l_k^*} T^1 - \underline{\mathrm{pr}}_{l_k^*} T^1) = 1.$$

Thus, we obtain Hecke's estimates [3] for the deviations (9.10):

$$|\delta_0^1(i)| \le n, \quad |\delta_1^1(i)| \le n. \tag{9.11}$$

9.3. Two-dimensional case: n is arbitrary. Now $T^2 = I^2_{\mathrm{col}}$ is the unit square with the partition

$$T^2 = T^2_0 \sqcup T^2_1 \sqcup T^2_2$$

into the tiles

$$\begin{aligned} T^2_0 &= \{x = (x_1, x_2); 0 \le x_1 < 1 - \alpha_1, 0 \le x_2 < 1 - \alpha_2\}, \\ T^2_1 &= \{x = (x_1, x_2); 1 - \alpha_1 \le x_1 < 1, 0 \le x_2 < 1 - \alpha_2\}, \\ T^2_2 &= \{x = (x_1, x_2); 0 \le x_1 < 1, 1 - \alpha_2 \le x_2 < 1\}. \end{aligned} \tag{9.12}$$

The exchange vectors (9.6) and (9.7) are

$$v_0 = (\alpha_1, \alpha_2), \quad v_1 = (\alpha_1 - 1, \alpha_2), \quad v_2 = (0, \alpha_2 - 1);$$

therefore,

$$l_1 = v_1 - v_0 = (-1, 0), \quad l_2 = v_2 - v_0 = (-\alpha_1, -1) \tag{9.13}$$

are basis vectors of the lattice L'. It follows that in this case the exchange of the unit square T^2 is equivalent to the shift

$$S^2_\alpha \,:\, x \mapsto x + \alpha \bmod L \tag{9.14}$$

of the two-dimensional torus $\mathbb{T}^2 \simeq \mathbb{R}^2/L$ by the vector $\alpha = (\alpha_1, \alpha_2)$, where

$$L = \mathbb{Z}[l_1, l_2] = L'.$$

Reduction in a Cartesian basis. For any $x \in \mathbb{R}^2$, the fractional part $\mathrm{Fr}^2(x)$ is defined by

$$\mathrm{Fr}^2(x) = x', \quad \text{where} \quad x' \equiv x \bmod L, \ x' \in T^2.$$

We rewrite the fractional part $\mathrm{Fr}^2(x)$ in terms of the fractional parts $\{x_1\}$, $\{x_2\}$. Using (9.13), we can write the period lattice L as

$$L = \mathbb{Z}[m_1, m_2], \quad \text{where} \quad m_1 = -l_1 = (1, 0), \ m_2 = -l_2 = (\alpha_1, 1). \tag{9.15}$$

Now consider the basis $\{m_1, m_2\}$ and the new coordinates $\widetilde{x}_1$ and $\widetilde{x}_2$:

$$x = (x_1, x_2) = \widetilde{x}_1 m_1 + \widetilde{x}_2 m_2. \tag{9.16}$$

It follows that for the coordinates $\widetilde{x}_1$ and $\widetilde{x}_2$ we have

$$\widetilde{x}_1 = x_1 - \alpha_1 x_2, \quad \widetilde{x}_2 = x_2. \tag{9.17}$$

If we replace x in (9.16) by

$$\widetilde{x}' = \{\widetilde{x}_1\} m_1 + \{\widetilde{x}_2\} m_2,$$

we see that $\widetilde{x}' \equiv x \bmod L$, while (9.16) and (9.17) show that

$$\widetilde{x}' = \{x_1 - \alpha_1 x_2\} m_1 + \{x_2\} m_2 = \big(\{x_1 - \alpha_1 x_2\} + \{x_2\}\alpha_1, \{x_2\}\big).$$

In order to embed $\widetilde{x}'$ into the unit square T^2, which is a fundamental domain with respect to the lattice L, we replace $\widetilde{x}'$ additionally by

$$x' = \big(\{\{x_1 - \alpha_1 x_2\} + \{x_2\}\alpha_1\}, \{x_2\}\big).$$

After this, the first coordinate of $\widetilde{x}'$ could possibly be reduced by 1, i.e.,

$$x' = \widetilde{x}' - (1, 0) = \widetilde{x}' - m_1.$$

Consequently, in any case we have

$$x' \equiv \widetilde{x}' \bmod L.$$

Since $x' \in T^2$ by construction, we see that, for any $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\mathrm{Fr}^2(x) = \big(\{\{x_1 - \alpha_1 x_2\} + \{x_2\}\alpha_1\}, \{x_2\}\big), \tag{9.18}$$

which is the reduction formula for the fractional part $\mathrm{Fr}^2(x)$ in the standard basis $e_1 = (1,0)$, $e_2 = (0,1)$.

Relations (3.6) and (3.7) imply that the counting function $r_k^2(i)$, where $k = 0,1,2$, is equal to the number of j such that $0 \le j < i$ and, moreover,

for $r_0^2(i)$,

$$\big\{\{j(\beta_1 - \alpha_1\beta_2)\} + \{j\beta_2\}\alpha_1\big\} < 1 - \alpha_1, \quad \{j\beta_2\} < 1 - \alpha_2; \tag{9.19}$$

for $r_1^2(i)$,

$$\big\{\{j(\beta_1 - \alpha_1\beta_2)\} + \{j\beta_2\}\alpha_1\big\} \ge 1 - \alpha_1, \{j\beta_2\} < 1 - \alpha_2;$$

for $r_2^2(i)$,

$$\{j\beta_2\} \ge 1 - \alpha_2.$$

Reduction in a basis of the period lattice L. Suppose $E = \{e_1, e_2\}$ is the standard basis and L is the lattice (9.15) with a basis $\{m_1, m_2\}$. The coordinates of x_i with respect to the basis E are related to the coordinates of $\widetilde{x}_i$ in the basis $\{m_1, m_2\}$ by the formulas

$$x = (x_1, x_2)_E = (\widetilde{x}_1, \widetilde{x}_2)_L, \ \text{ where } \widetilde{x}_1 = x_1 - \alpha_1 x_2, \ \widetilde{x}_2 = x_2,$$
$$\alpha = (\alpha_1, \alpha_2)_E = (\widetilde{\alpha}_1, \widetilde{\alpha}_2)_L, \ \text{ where } \widetilde{\alpha}_1 = \alpha_1 - \alpha_1\alpha_2, \ \widetilde{\alpha}_2 = \alpha_2.$$

Here we write the indices at the coordinates $(x_1, x_2)_E$ and $(\widetilde{x}_1, \widetilde{x}_2)_L$ in order to indicate the corresponding basis. Since

$$e_1 = (1,0)_L, \quad e_2 = (-\alpha_1, 1)_L,$$

the toric development T^2 transforms into a new development $\widetilde{T}^2$:

$$\widetilde{T}^2 = \{x \in \mathbb{R}^2; 0 \le \widetilde{x}_1 + \alpha_1\widetilde{x}_2 < 1, 0 \le \widetilde{x}_2 < 1\}, \tag{9.20}$$

while, by (9.12), the tiles T_0^2, T_1^2, T_2^2 transform to the tiles

$$\begin{aligned} \widetilde{T}_0^2 &= \{x \in \mathbb{R}^2; 0 \le \widetilde{x}_1 + \alpha_1\widetilde{x}_2 < 1 - \alpha_1, 0 \le \widetilde{x}_2 < 1 - \alpha_2\}, \\ \widetilde{T}_1^2 &= \{x \in \mathbb{R}^2; 1 - \alpha_1 \le \widetilde{x}_1 + \alpha_1\widetilde{x}_2 < 1, 0 \le \widetilde{x}_2 < 1 - \alpha_2\}, \\ \widetilde{T}_2^2 &= \{x \in \mathbb{R}^2; 1 - \alpha_2 \le \widetilde{x}_2 < 1\}, \end{aligned} \tag{9.21}$$

respectively.

Remark 9.1. The reductions (9.20)–(9.21) in the basis of the period lattice L (see (9.15)) is aimed at stating the bounds for the tiles in the two-dimensional case in terms of the usual (one-dimensional) fractional parts $\{\widetilde{x}_1\}$, $\{\widetilde{x}_2\}$.

Using formula (3.2) for the shift vector β of $\mathbb{T}^2 = \mathbb{R}^2/L$, we can compute its coordinates

$$\beta = \frac{1}{n}(\alpha + b_1 l_1 + b_2 l_2) \tag{9.22}$$

with respect to the same basis (9.15) of the lattice L. We have

$$\beta = (\widetilde{\beta}_1, \widetilde{\beta}_2)_L = \widetilde{\beta}_1 m_1 + \widetilde{\beta}_2 m_2.$$

Therefore,

$$\widetilde{\beta}_1 = \frac{1}{n}(\widetilde{\alpha}_1 - b_1), \quad \widetilde{\beta}_2 = \frac{1}{n}(\widetilde{\alpha}_2 - b_2).$$

In terms of the basis of the period lattice L, the functions $r_k^2(i)$, $k = 0,1,2$, are equal to the number of j's such that $0 \le j < i$ and, in accordance with (9.21),

for $r_0^2(i)$,

$$\begin{cases} \{j\widetilde{\beta}_1\} + \alpha_1\{j\widetilde{\beta}_2\} < 1 - \alpha_1 \ \text{ or } \ \{j\widetilde{\beta}_1\} + \alpha_1\{j\widetilde{\beta}_2\} \ge 1, \\ \{j\widetilde{\beta}_2\} < 1 - \alpha_2; \end{cases} \tag{9.23}$$

for $r_1^2(i)$,

$$\begin{cases} 1-\alpha_1 \le \{j\widetilde{\beta}_1\} + \alpha_1\{j\widetilde{\beta}_2\} < 1, \\ \{j\widetilde{\beta}_2\} < 1-\alpha_2; \end{cases}$$

for $r_2^2(i)$,

$$\{j\widetilde{\beta}_2\} \ge 1-\alpha_2.$$

Now, using (4.3) and (4.14), we can write the k-deviations as follows:

$$\delta_k^2(i) = r_k^2(i) - ia_k \quad \text{for} \quad k=0,1,2,$$

where the counting functions $r_k^2(i)$ are as in (9.19) or, equivalently, in (9.23): $a_k = -l_k^* \cdot \alpha$ for $\alpha = (\alpha_1, \alpha_2)$. From (9.13) it follows that $l_1 = (-1, 0)$, $l_2 = (-\alpha_1, -1)$, whence

$$l_1^* = (-1, \alpha_1), \quad l_2^* = (0, -1), \quad l_0^* = -l_1^* - l_2^* = (1, 1-\alpha_1). \tag{9.24}$$

Therefore,

$$a_1 = \alpha_1 - \alpha_1\alpha_2 = \widetilde{\alpha}_1, \quad a_2 = \alpha_2 = \widetilde{\alpha}_2,$$

and (4.8) allows us to get the following identity for the coefficient a_0:

$$a_0 = 1 - \widetilde{\alpha}_1 - \widetilde{\alpha}_2 = (1-\alpha_1)(1-\alpha_2).$$

In order to obtain approximate bounds for the k-deviations $\delta_k^2(i)$, we apply Theorem 5.1. We have

$$|\delta_k^2(i)| \le c_k^2 n \quad \text{for } k=0,1,2, \tag{9.25}$$

where

$$c_k^2 = |l_k^*|(\overline{\mathrm{pr}}_{l_k^*} T^2 - \underline{\mathrm{pr}}_{l_k^*} T^2). \tag{9.26}$$

Since in our case the toric development T^2 is the unit square $I^2 = [0,1) \times [0,1)$, we can use (9.24) to obtain the following explicit expressions for the constants (9.26):

$$c_0 = 2-\alpha_1, \quad c_1 = 1+\alpha_1, \quad c_2 = 1.$$

Combining this with (9.25) shows that in the two-dimensional case we have the following approximate estimates for the k-deviations:

$$|\delta_0^2(i)| \le (2-\alpha_1)n, \quad |\delta_1^2(i)| \le (1+\alpha_1)n, \quad |\delta_2^2(i)| \le n \tag{9.27}$$

for all $i = 0, 1, 2, \ldots$.

Remark 9.2.

1. The last estimate in (9.27) does not depend on the choice of the shift vector $\alpha = (\alpha_1, \alpha_2)$; again, it is basically the same as Hecke's estimate [3].

2. The first two estimates in (9.27) do not depend on the choice of the coordinate α_2 of the shift vector $\alpha = (\alpha_1, \alpha_2)$ of $\mathbb{T}^2 = \mathbb{R}^2/L$.

3. The second estimate in (9.27) corresponds to the deviation $\delta_1^2(i)$ for the two-dimensional tile $\widetilde{T}_1^2$ that is a skew-angle parallelogram, see the definition (9.21). Some particular cases of these parallelograms were previously found by Szüsz [7].

9.4. The two-dimensional case: $n = 1$**.** We use the development of $\widetilde{T}^2$ and the results of §6 to obtain sharp bounds for the k-deviations $\delta_k^2(i)$ in the case where for the shift vector β as in (9.22) we have $n = 1$.

The passage to the coordinates with respect to the basis $\{m_, m_2\}$ of the period lattice $L = \mathbb{Z}[m_1, m_2]$ is equivalent to the passage to the square lattice $\mathbb{Z}^2 = \mathbb{Z}[e_1, e_2]$. Under the map $m_1 \mapsto e_1$, $m_2 \mapsto e_2$, the vectors l_k, $k = 0, 1, 2$, are taken to the vectors

$$l_1 \to -e_1, \quad l_2 \to -e_2, \quad l_0 = -l_1 - l_2 \to e_1 + e_2,$$

respectively, and by (4.1) and (4.13), the dual vectors l_k^*, $k = 0, 1, 2$, are taken to

$$l_1^* \to -e_1, \quad l_2^* \to -e_2, \quad l_0^* =\to e_1 + e_2.$$

From (9.20) it follows that the development of the torus $\widetilde{T}^2$ is a parallelogram. Hence, in order to find extrema of the projections $\mathrm{pr}_{l_k^*} x$, $x \in \widetilde{T}^2$, it suffices to consider only the vertices $x = v$ of $V(\widetilde{T}^2)$. Looking through the vertices in $V(\widetilde{T}^2)$, we get the projections

$$\begin{aligned} \{\mathrm{pr}_{e_1+e_2} v; v \in V(\widetilde{T}_0^2)\} &= \{0, 1, 2 - \alpha_1, 1 - \alpha_1\}, \\ \{\mathrm{pr}_{-e_1} v; v \in V(\widetilde{T}_1^2)\} &= \{0, -1, \alpha_1 - 1, \alpha_1\}, \\ \{\mathrm{pr}_{-e_2} v; v \in V(\widetilde{T}_2^2)\} &= \{0, -1\}. \end{aligned} \tag{9.28}$$

Thus, by (9.28) and Proposition 6.1, for $n = 1$ the k-deviations $\delta_k^2(i)$ satisfy

$$0 \le \delta_0^2(i) \le 2 - \alpha_1, \quad -1 \le \delta_1^2(i) \le \alpha_1, \quad -1 \le \delta_2^2(i) \le 0. \tag{9.29}$$

Remark 9.3. Comparing (9.29) and (9.27), we see that the bounds in (9.29) for $\delta_0^2(i)$, $\delta_1^2(i)$ depend only on the first coordinate of the shift vector $\alpha = (\alpha_1, \alpha_2)$, while the bounds for $\delta_2^2(i)$ do not depend on α.

Mean values of deviations. Theorem 8.1 and relation (8.17) show that the total vector deviation $\delta^2(i)$ has the vector mean value

$$\langle \delta^2 \rangle = c_T = \Big(\frac{1}{2}, \frac{1}{2}\Big). \tag{9.30}$$

It is easily seen that the mean value (9.30) is in fact the barycenter of the unit square $T^2 = I^2$. In the same way, we can use (8.18) to compute the mean values of the k-deviations $\delta_k^2(i)$:

$$\langle \delta_k^2 \rangle = l_k^* \cdot c_T \quad \text{for} \quad k = 0, 1, 2. \tag{9.31}$$

Using (9.30), we find that the mean values in (9.31) can explicitly be written as

$$\langle \delta_0^2 \rangle = 1 - \frac{\alpha_1}{2}, \quad \langle \delta_1^2 \rangle = -\frac{1}{2} + \frac{\alpha_1}{2}, \quad \langle \delta_2^2 \rangle = -\frac{1}{2}. \tag{9.32}$$

The right-hand sides in (9.32) coincide with the midpoints of the intervals for the sharp bounds in (9.29).

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