ON THE DISTRIBUTION OF FRACTIONAL PARTS

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ABSTRACT. Hecke’s theorem on the distribution of fractional parts on the unit circle is generalized to the tori $\mathbb{T}^D = \mathbb{R}^D/L$ of arbitrary dimension $D$. It is proved that $|\delta_k(i)| \leq c_k n$ for $i = 0, 1, 2, \ldots$, where $\delta_k(i) = r_k(i) - \alpha_k$ is the deviation of the number $r_k(i)$ of returns in $i$ steps into $\mathbb{T}_k^D \subset \mathbb{T}^D$ for the points of an $S_\beta$-orbit from their mean value $a_k = \text{vol}(\mathbb{T}_k^D)/\text{vol}(\mathbb{T}^D)$, where $\text{vol}(\mathbb{T}_k^D)$ and $\text{vol}(\mathbb{T}^D)$ denote the volumes of the tile $T_k^D$ and of the torus $\mathbb{T}^D$. The tiles $T_k^D$ in question have the following property: for the torus $\mathbb{T}^D$ there exists a development $T^D \subset \mathbb{R}^D$ such that a shift $S_\alpha$ of the torus $\mathbb{T}^D$ is equivalent to some exchange transformation of the corresponding tiles $T_k^D$ in a partition of the development $T^D = T_0^D \sqcup T_1^D \sqcup \cdots \sqcup T_n^D$. The torus shift vectors $S_\alpha, S_\beta$ satisfy the condition $\alpha \equiv n \beta \mod L$, where $n$ is any natural number, and the constants $c_k$ in the inequalities are expressed in terms of the diameter of the development $T^D$.

INTRODUCTION

Let $\alpha$ be a real number, and let $\beta = \frac{1}{n}(\alpha + b)$, where $n = 1, 2, 3, \ldots$ and $b$ is an arbitrary integer. We define counting functions by

$$r_0(i) = \sharp\{j; \{j\beta\} < 1 - \alpha, 0 \leq j < i\},$$
$$r_1(i) = \sharp\{j; \{j\beta\} \geq 1 - \alpha, 0 \leq j < i\},$$

where $\{x\}$ denotes the fractional part of a number $x$. Consider the deviations

$$\delta_0(i) = r_0(i) - i(1 - \alpha),$$
$$\delta_1(i) = r_1(i) - i\alpha$$

of $r_0(i)$ and $r_1(i)$ from their mean values, where $1 - \alpha$ and $\alpha$ are the lengths of the intervals within which the fractional parts $\{j\beta\}$ fall. Hecke proved the following theorem concerning $\delta_0(i), \delta_1(i)$.

**Theorem 1** (Hecke [3]). Suppose $\alpha$ is irrational. Then the inequalities

$$|\delta_0(i)| \leq n, \quad |\delta_1(i)| \leq n$$

hold true for $i = 0, 1, 2, \ldots$

Our goal is to generalize the Hecke theorem to the tori $\mathbb{T}^D$ of arbitrary dimension $D$. First, observe the following analogy.

The functions $r_0(i)$ and $r_1(i)$ can be related to rotations of the circle. Let

$$S_\alpha : x \mapsto x + \alpha \mod 1$$

2010 Mathematics Subject Classification. Primary 11K60; secondary 11H06.

Key words and phrases. Hecke theorem, fractional parts distribution, mean values of deviation functions, bounded remainder sets on the torus.

Supported by RFBR (grant no. 11-01-00578-a).
be a rotation of \( C = T^1 \) by \( \alpha \). Then \( r_0(i) \) and \( r_1(i) \) can be viewed as the numbers of points

\[
S^0_\beta(0) = 0, \quad S^1_\beta(0), \ldots, \quad S^{i-1}_\beta(0)
\]

(here \( S^j_\beta \) is the \( j \)-fold iteration of the rotation \( S_\beta \)) that lie in the semiintervals \([0, 1 - \alpha) \) and \([1 - \alpha, 1) \), respectively. At the same time, \( \delta_0(i), \delta_1(i) \) can be viewed as the deviations in the problem on of the fractional parts distribution. Cutting at some point, we identify the circle with the unit semiinterval \( T^1 = [0, 1) \), and then write \( T^1 = [0, 1) \) as the union (0.4)

\[
T^1 = T^1_0 \sqcup T^1_1
\]

of two semiintervals \( T^1_0 = [0, 1 - \alpha) \), \( T^1_1 = [1 - \alpha, 1) \). A rotation of the circle (0.3) is equivalent to exchanging the semiintervals \( T^1_0 \) and \( T^1_1 \):

\[
T^1 \overset{S_\alpha}{\longrightarrow} T^1 : S(x) = x + v_k
\]

for \( x \in T^1_k \), \( k = 0, 1 \), where the shift vectors \( v_0, v_1 \) are equal to \( \alpha, \alpha - 1 \), respectively.

In the case where \( D \geq 2 \), consider a shift of the torus \( \mathbb{T}^D \approx \mathbb{R}^D / L \),

(0.5)

\[
\mathbb{T}^D \overset{S_\alpha}{\longrightarrow} \mathbb{T}^D : x \mapsto S_\alpha(x) \equiv x + \alpha \mod L,
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_D) \) is a shift vector and \( L \) is a nonsingular lattice in \( \mathbb{R}^D \). Cut the torus \( \mathbb{T}^D \) to place its development \( T^D \) in \( \mathbb{R}^D \). Now if we identify the torus \( \mathbb{T}^D \) with its development \( T^D \), then the shift (0.5) will correspond to an exchange transformation of the tiles \( T_k \) belonging to the tiling

(0.6)

\[
T^D = T^D_0 \sqcup T^D_1 \sqcup \cdots \sqcup T^D_s
\]

of this development. In nonsingular cases, the minimal number of regions in a partition of the form (0.6) is equal to \( s_{\min} = D + 1 \). For example, in the one-dimensional case we get the two semiintervals (0.4); for \( D = 2 \) we can take a planar projection of a three-dimensional cube consisting of three exchanging parallelograms.

In order to obtain a multidimensional generalization of the Hecke theorem, we shall move in the opposite direction compared to (0.2): we start with minimal tilings of the torus

(0.7)

\[
T^D = T^D_0 \sqcup T^D_1 \sqcup \cdots \sqcup T^D_D
\]

that admit an exchange transformation

(0.8)

\[
T^D \overset{S_\alpha}{\longrightarrow} T^D : x \mapsto S(x) = x + v_k
\]

for \( x \in T_k \). We need to impose the following condition on a tiling (0.7) and on shift vectors \( v_0, v_1, \ldots, v_D \): upon identifying the torus \( \mathbb{T}^D \) with its development \( T^D \), the exchange transformation (0.8) corresponds to a torus shift \( S_\alpha \) as in (0.3) for some vector \( \alpha \) uniquely determined by the transformation (0.8).

As in (0.1), we define the counting functions

\[
r_k(i) = \sharp\{ j; S^j_\beta(0) \in T^D_k; 0 \leq j < i \}
\]

for \( k = 0, 1, \ldots, D \); here \( S_\beta \) is the shift (0.3) of the torus \( \mathbb{T}^D \) by the vector

\[
\beta = \frac{1}{n}(\alpha + b_1l_1 + \cdots + b_Dl_D),
\]

where \( b_k \) are arbitrary integers and \( l_k = v_k - v_0 \) for \( k = 1, \ldots, D \). Suppose the vectors \( l_1, \ldots, l_D \) are linearly independent over \( \mathbb{R} \). For every tile \( T_k \), consider the deviation

(0.9)

\[
\delta_k(i) = r_k(i) - ia_k
\]
for \( k = 0, 1, \ldots, D \), where \( a_1, \ldots, a_D \) are the coordinates of the vector \(-\alpha\) with respect to the basis \( l_1, \ldots, l_D \), and \( a_0 = 1 - a_1 - \cdots - a_D \). In Theorem 5.1, we shall prove the following statement concerning the deviations \( \delta_k(i) \).

**Theorem 2.** Suppose that vectors \( l_1, \ldots, l_D \) are linearly independent over \( \mathbb{R} \) and that a shift vector \( \alpha \) of the torus \( T^D = \mathbb{R}^D / L \) is irrational, namely, its coordinates \( \alpha'_1, \ldots, \alpha'_D \) in some basis of the lattice \( L \) have the following property: the numbers

\[
\alpha'_1, \ldots, \alpha'_D, 1
\]

are linearly independent over \( \mathbb{Z} \). Then for any \( k = 0, 1, \ldots, D \) we have

\[
|\delta_k(i)| \leq c_k n
\]

for every \( i = 0, 1, 2, \ldots \). Here the \( c_k = c_{T,k} \) are the constants \((5.13)\) independent of \( n \) and \( i \), determined only by the size of a development of the torus \( T^D \).

Let \( A \) be the affine map that takes the vectors \( l_1, \ldots, l_D \) to the standard basis \( e_1 = (1, 0, \ldots, 0), \ldots, e_D = (0, 0, \ldots, 1) \). The image \( AT \) of a torus development \( T \) is again a development of some torus of the same dimension \( D \). Passing from \( T \) to \( AT \) preserves all the constructions above. Moreover, the constants in the inequalities \((0.11)\) are invariant under affine maps \( A \):

\[
c_{AT,k} = c_{T,k}.
\]

Hence, without loss of generality we may assume that

\[
l_1 = e_1, \ldots, l_D = e_D.
\]

Under these assumptions we prove the following theorem.

**Theorem 3** (Multidimensional Hecke theorem). Under the assumptions of Theorem 2 together with \((0.12)\), we have

\[
|\delta_k(i)| \leq d_T n \quad \text{for} \quad k = 1, \ldots, D,
\]

\[
|\delta_k(i)| \leq D d_T n \quad \text{for} \quad k = 0,
\]

where \( d_T \) is the diameter of the development \( T^D \). Moreover, if every tile \( T_k \) of the tiling \((0.7)\) is Jordan measurable, the deviations \((0.9)\) can be written in the form

\[
\delta_k(i) = r_k(i) - i \text{vol}(T_k)
\]

for \( k = 0, 1, \ldots, D \), where \( \text{vol}(T_k) \) is the volume of the tile \( T_k \).

It is informative to state the claim of this theorem analytically (rather than geometrically) in terms of the function \( \{x\} \), as in the Hecke Theorem \((0.2)\). Consider the following example in the case where \( D = 2 \). We introduce the counting function

\[
r_1(i) = \sharp\{0 \leq j < i; \{j(\beta_1 - \alpha_1\beta_2)\} + \{j\beta_2\alpha_1\} \geq 1 - \alpha_1, \{j\beta_2\} < 1 - \alpha_2\},
\]

for any vector \( \alpha = (\alpha_1, \alpha_2) \) and

\[
\beta_1 = \frac{1}{n}(\alpha_1 - b_1 - b_2\alpha_1), \quad \beta_2 = \frac{1}{n}(\alpha_2 - b_2).
\]

In the standard basis \( e_1 = (1, 0), e_2 = (0, 1) \), the counting function \((0.13)\) can be written in an equivalent form

\[
r_1(i) = \sharp\{0 \leq j < i; \ 1 - \alpha_1 \leq \{j\tilde{\beta}_1\} + \alpha_1\{j\beta_2\} < 1, \ \{j\beta_2\} < 1 - \alpha_2\},
\]

where \( \tilde{\beta}_1 = \frac{1}{n}(\alpha_1 - b_1 - \alpha_1\alpha_2) \).
Theorem 4. Suppose that the numbers $\alpha_1 - \alpha_1 \alpha_2, \alpha_2, 1$ are linearly independent over $\mathbb{Z}$. Then the deviation
\[
\delta_1(i) = r_1(i) - i(\alpha_1 - \alpha_1 \alpha_2)
\]
satisfies the inequality
\[
|\delta_1(i)| \leq (1 + \alpha_1)n
\]
for all $i = 0, 1, 2, \ldots$

The tiles $T_k$ in the tilings (0.7) are bounded remainder sets on the torus $\mathbb{T}^D$. In particular, by (0.15), the sets $T_1$ defined by the inequalities occurring in (0.14) are bounded remainder sets. In fact, they are parallelograms determined by the five independent variables $\alpha_1, \alpha_2, b_1, b_2, n$. The family $T_1$ of parallelograms contains a subfamily of parallelograms \[7\] that have fixed parameters $b_1 = 0, b_2 = 0$. All the parallelograms in $T_1$ have acute angles, and this is not a coincidence: Liardet \[4\] proved that no nontrivial rectangle on a torus can be a bounded remainder set. Bounded remainder sets with fractal boundaries in the two-dimensional case were studied previously in \[8\].

Our method is geometrical. It resembles that of French mathematicians (Rauzy \[6\], Ferenczi \[2\], etc.).

This technique allows us 1) to construct bounded remainder sets on the torus $\mathbb{T}^D$ for any dimension $D$; 2) to find explicit estimates for the deviations $\delta_k(i)$ on these sets, namely, approximate bounds as in (0.15) (see Theorem 5.1) and sharp bounds for deviations (see Theorems 6.1 and 7.1); 3) to compute the mean values $\langle \delta_k \rangle$ of the deviations $\delta_k(i)$ (see Theorem 8.1).

§1. Exchange transformations for toric developments

Let
\[
\mathbb{T}^D \simeq \mathbb{R}^D/L
\]
be a $D$-dimensional torus, where $L$ is a nonsingular lattice in the real coordinate space $\mathbb{R}^D$. In other words, the lattice $L$ has dimension $D$ over $\mathbb{R}$. Suppose we are given a shift
\[
\mathbb{T}^D \xrightarrow{S_{\alpha}} \mathbb{T}^D : x \mapsto S_\alpha(x) \equiv x + \alpha \text{ mod } L
\]
of the torus. A subset $T$ of $\mathbb{R}^D$ is called a development of the torus $\mathbb{T}^D$ if the restriction of the canonical projection
\[
\mathbb{R}^D \xrightarrow{\text{mod } L} \mathbb{T}^D : x \mapsto x \text{ mod } L
\]
onto $T \subset \mathbb{R}^D$ gives a bijection
\[
T \xrightarrow{\text{mod } L} \mathbb{T}^D.
\]

Suppose $T$ is a toric development satisfying the following condition: there exists a partition
\[
T = T_0 \sqcup T_1 \sqcup \cdots \sqcup T_D
\]
of $T$ such that after identifying the torus $\mathbb{T}^D$ with its development $T$ via (1.3), the map
\[
T \xrightarrow{S_\alpha} T
\]
induced by the shift (1.2) is equivalent to a $(D + 1)$-exchange transformation of the development $T$:
\[
S_\alpha(x) = x + v(x),
\]
where a shift vector $v(x)$ depends on $x \in T$ and satisfies
MULTIDIMENSIONAL HECKE THEOREM

(1.7) \[ v(x) = v_k \text{ if } x \in T_k, \]

where \( v_0, v_1, \ldots, v_D \) are some fixed vectors.

**Examples of toric developments.**

1. The one-dimensional torus \( \mathbb{T}^1 \) is the same as the unit circle \( C \); a torus shift \( S_\alpha \) is a rotation of \( C \) by the angle \( \alpha \). If we identify the circle \( C \) with the unit interval \( I = [0, 1) \), the rotation \( S_\alpha \) becomes equivalent to the exchange of the two semiintervals \( [0, 1 - \alpha), [1 - \alpha, 1) \) inside \( I \).

2. For \( D = 2 \), we can choose a development \( T \) of the torus \( \mathbb{T}^2 \) to be, for instance, a hexagon divided into three parallelograms, and their exchange corresponds to some rotation of the torus \( \mathbb{T}^2 \).

3. For the general \( D \), the construction of exchange transformations for a development \( T \) of the torus \( \mathbb{T}^D \) will be discussed in §9.

Note that (1.2), (1.6), and (1.7) imply

(1.8) \[ v_k \equiv \alpha \mod L \text{ for } k = 0, 1, \ldots, D, \]

and the vectors

(1.9) \[ l_k = v_k - v_0, \quad k = 1, \ldots, D, \]

lie in the period lattice \( L \) of the torus (1.1). These vectors generate a sublattice

(1.10) \[ L' = \mathbb{Z}[l_1, \ldots, l_D] \]

of the lattice \( L \), where \( l_1, \ldots, l_D \) are generators of \( L' \) over \( \mathbb{Z} \).

A shift \( S_\alpha \) of the torus (1.2) is said to be *irrational* if the numbers

(1.11) \[ \alpha'_1, \ldots, \alpha'_D, 1 \]

are linearly independent over \( \mathbb{Z} \), where \( \alpha'_1, \ldots, \alpha'_D \) are the coordinates of the shift vector \( \alpha \) with respect to some basis of the period lattice \( L \).

**Proposition 1.1.** Suppose that a shift \( S_\alpha \) (1.2) of the torus \( \mathbb{T}^D \) (1.1) and its development \( T \) satisfy the following conditions:

1. the shift \( S_\alpha \) is irrational;
2. the closure \( \overline{T} \) of the net \( T \) has dimension \( \dim \overline{T} = D \), and therefore, \( \overline{T} \) is not contained in any subspace \( \mathbb{R}^{D'} \subset \mathbb{R}^D \) of dimension \( D' < D \);
3. the development \( T \) has finite diameter \( \text{d_}T = \sup_{x, x' \in T} |x - x'|. \)

Then

\[ \text{rank } L' = \text{rank } L = D, \]

where \( \text{rank } M \) denotes the rank of a lattice \( M \subset \mathbb{R}^D \), which is equal to the dimension of the lattice \( M \) over \( \mathbb{R} \).

**Proof.** Suppose the contrary. Assume that for the lattice \( L' \) we have

(1.12) \[ L' \subset \mathbb{R}^{D'}, \text{ where } D' < D. \]

**Case 1.** Suppose \( \alpha \in \mathbb{R}^{D'} \). Then (1.12) together with the definition (1.6) implies the inclusion \( \overline{\text{Orb}_{S_\alpha}(0)} \subset \mathbb{R}^{D'} \). Therefore, we have \( \overline{\text{Orb}(0)}_{S_\alpha} \subset \mathbb{R}^{D'} \). Now assumption 1), the theorem about the ergodicity of a torus shift [10, p. 66], and (1.5) imply

\[ \overline{\text{Orb}(0)} = \overline{T}. \]

It follows that the closure \( \overline{T} \) of the development is contained in the subspace \( \mathbb{R}^{D'} \), which contradicts 2).
Case 2. Now suppose \( \alpha \notin \mathbb{R}^{D'} \). Then
\begin{equation}
S_{\alpha}^i(0) \subset i\alpha + \mathbb{R}^{D'}
\end{equation}
for every \( i = 0, 1, 2, \ldots \). Since \( \alpha \notin \mathbb{R}^{D'} \), the inclusion (1.13) shows that the point \( S_{\alpha}^i(0) \) tends to infinity as \( i \to +\infty \). By (1.6), this contradicts the fact that
\begin{equation}
S_{\alpha}^i(0) \subset T
\end{equation}
and the assumption \( d_T < \infty \) about the diameter of the development \( T \). This completes the proof. \( \square \)

§2. VECTOR T-FRACTIONAL PART

From now on we assume that \( L' \) is a spanning lattice, namely, that
\begin{equation}
\text{the family } l_1, \ldots, l_D \text{ has rank } D \text{ over } \mathbb{R}.
\end{equation}

For any \( x \in \mathbb{R}^D \), we denote by \( Fr(x) \) its vector \( T \)-fractional part
\begin{equation}
Fr(x) = x',
\end{equation}
where \( x' \equiv x \mod L \) and \( x' \in T \). This notion is well defined because there is a partition
\begin{equation}
\mathbb{R}^D = \bigsqcup_{l \in L} T[l]
\end{equation}
of \( \mathbb{R}^D \) into the sets
\begin{equation}
T[l] = T + l = \{ x + l; x \in T \}
\end{equation}
obtained by shifts of \( T \) by the vectors \( l \).

We often write \( Fr_T(x) \) instead of \( Fr(x) \) to emphasize its dependence on the development \( T \). The point (or the vector) \( Fr_T(x) \) is called the \( T \)-fractional part of \( x \in \mathbb{R}^D \).

Proposition 2.1. Let
\begin{equation}
\Delta Fr(x) = Fr(x + \alpha) - Fr(x)
\end{equation}
be a vector-valued difference function with step \( \alpha \), where \( \alpha \) is a shift vector of the torus \( \mathbb{T}^D \). Then
\begin{equation}
\Delta Fr(x) = v(x)
\end{equation}
for any \( x \in \mathbb{R}^D \), where
\begin{equation}
v(x) = \alpha + l(x),
\end{equation}
and \( l(x) = l_k \) for \( x \in T_k, k = 1, \ldots, D \), while \( l(x) = l_0 = 0 \) for \( x \in T_0 \). Here the \( l_k \) are the vectors as in (1.9).

Proof. For any \( x \) in the development \( T \), we have
\begin{equation}
S_{\alpha}(x) = x + v(x)
\end{equation}
and \( v(x) = v_k \) for \( x \in T_k, k = 0, 1, \ldots, D \). By (1.8), we have \( v(x) = \alpha + l(x) \), where \( l(x) = l_k \) for \( x \in T_k \) and \( k = 0, 1, \ldots, D \). Therefore, (2.6) implies
\begin{equation}
S_{\alpha}(x) = x + \alpha + l(x).
\end{equation}
Moreover, for any \( x \) in \( T \) its image \( x + \alpha + l(x) \) also lies in the toric development \( T \). It follows that
\begin{equation}
Fr(x + \alpha) = x + \alpha + l(x) = x + v(x)
\end{equation}
for any \( x \in T \). In order to prove (2.4), note that
\begin{equation}
x + \alpha \equiv x + \alpha + l(x) \mod L,
\end{equation}
where
where \( l(x) \in L \), and that \( 1.6 \) implies
\[
(2.9) \quad x + \alpha + l(x) \in T.
\]
From \( 2.7 \) it follows that
\[
\Delta \text{Fr}(x) = \text{Fr}(x + \alpha) - \text{Fr}(x)
= x + \alpha + l(x) - x = \alpha + l(x) = v(x)
\]
for any \( x \in T \).

Now consider the general case: \( x \in \mathbb{R}^D \). Since we have the partition \( 2.3 \), every \( x \) can be written in the form \( x = x' + l \) for some \( x' \in T, l \in L \). Hence,
\[
(2.10) \quad \text{Fr}(x) = x'.
\]
Now \( 2.8 \) and \( 2.9 \) imply
\[
(2.11) \quad x + \alpha \equiv x' + \alpha + l \equiv x' + \alpha + l(x) + l \mod L,
\]
where \( x' + \alpha + l(x) \) lies in the development \( T \). By \( 2.11 \),
\[
(2.12) \quad \text{Fr}(x + \alpha) = x' + \alpha + l(x)
\]
for any \( x \in \mathbb{R}^D \). Using \( 2.10 \) and \( 2.12 \), we get
\[
\Delta \text{Fr}(x) = \text{Fr}(x + \alpha) - \text{Fr}(x)
= x' + \alpha + l(x) - x' = \alpha + l(x) = v(x);
\]
therefore, once again we obtain \( 2.4 \):
\[
\Delta \text{Fr}(x) = v(x),
\]
where \( v(x) = \alpha + l(x), \) for any \( x \in T \).

This concludes the proof. \( \square \)

§3. Total vector deviation

We define a vector-valued function \( \delta(i) \) as follows:
\[
(3.1) \quad \delta(i) = \sum_{0 \leq j < i} \Delta \text{Fr}(j\beta)
\]
for \( i = 0, 1, 2, \ldots \), where
\[
(3.2) \quad \beta = \alpha_n(b) = \frac{1}{n}(\alpha + b \circ l),
\]
n is any natural number,
\[
(3.3) \quad b \circ l = b_1l_1 + \cdots + b_Dl_D,
\]
and \( b = (b_1, \ldots, b_d) \in \mathbb{Z}^D \) is a vector with integral coordinates. Formula \( 3.3 \) shows that \( b \circ l \) lies in the lattice \( L' \).

Using \( 2.4 \), we can write \( 3.1 \) in the form
\[
(3.4) \quad \delta(i) = \sum_{0 \leq j < i} (\alpha + l(j\beta)) = i\alpha + \sum_{0 \leq j < i} l(j\beta) = i\alpha + \sum_{0 \leq k \leq D} \sum_{0 \leq j < i \subseteq T_k} l(j\beta).
\]
Since \( \text{Fr}(j\beta) \in T_k \) implies \( l(j\beta) = l_k \), and \( l_0 = 0 \) by the definition \( 2.5 \), we can use \( 3.4 \) for \( \delta(i) \) to obtain
\[
\delta(i) = i\alpha + \sum_{0 \leq k \leq D} l_k \sum_{0 \leq j < i \subseteq T_k} 1,
\]
or equivalently,
\[ \delta(i) = i\alpha + r_1(i)\lambda_1 + \cdots + r_D(i)\lambda_D, \]
where
\[ r_k(i) = \sharp \{ \text{Fr}(j\beta) \in T_k; 0 \leq j < i \}. \]

In terms of shifts of the torus, we can rewrite the counting function \((3.6)\) in the form
\[ r_k(i) = \sharp \{ j; S_{\beta}(0) \in T_k; 0 \leq j < i \}, \]
where \(S_{\beta}\) is the shift
\[ T^D \xrightarrow{S_{\beta}} T^D : x \mapsto x + \beta \mod L \]
of the torus by the vector \(\beta\) defined by \((3.2)\).

Here is another way to obtain the sum \((3.1)\). Using \((3.1)\), we can write \(\delta(i)\) as the difference of two sums:
\[ \delta(i) = \sum_{0 \leq j < i} \text{Fr}(j\beta + \alpha) - \sum_{0 \leq j < i} \text{Fr}(j\beta). \]

Relation \((3.2)\) implies that
\[ \alpha \equiv n\beta \mod L. \]
Combining \((3.10)\) and the definition \((2.2)\) of the vector \(T\)-fractional part \(\text{Fr}(x)\), we get
\[ \text{Fr}(j\beta + \alpha) = \text{Fr}(j\beta + n\beta). \]

Now \((3.11)\) implies
\[ \sum_{0 \leq j < i} \text{Fr}(j\beta + \alpha) = \sum_{0 \leq j < i} \text{Fr}(j\beta + n\beta) = \sum_{n \leq j < i + n} \text{Fr}(j\beta). \]

Using \((3.9)\) and \((3.12)\), we get another expression for \(\delta(i)\):
\[ \delta(i) = \sum_{n \leq j < i + n} \text{Fr}(j\beta) - \sum_{0 \leq j < i} \text{Fr}(j\beta). \]

Suppose \(i > n\). Then, by \((3.13)\),
\[ \delta(i) = \sum_{n \leq j < i + n} \text{Fr}(j\beta) - \sum_{0 \leq j < n} \text{Fr}(j\beta) - \sum_{0 \leq j \leq n - 1} (\text{Fr}(j\beta + i\beta) - \text{Fr}(j\beta)). \]

**Proposition 3.1.** Suppose a shift vector \(\beta\) of \(S_{\beta}\) as in \((3.8)\) is given by \((3.2)\), and \(\delta(i)\) is the sum \((3.1)\). Then
\[ \delta(i) = i\alpha + r_1(i)\lambda_1 + \cdots + r_D(i)\lambda_D, \]
where \(r_k(i)\) is the number of returns of the initial point \(x = 0\) into the tile \(T_k\) of the development \(T\) under the action of \(S_{\beta}\).

Apart from \((3.15)\), we can write \(\delta(i)\) in one of the following two forms:
\[ \delta(i) = \sum_{0 \leq j < i} (\text{Fr}(j\beta + n\beta) - \text{Fr}(j\beta)) \]
for \(0 \leq i \leq n\) and
\[ \delta(i) = \sum_{0 \leq j \leq n - 1} (\text{Fr}(j\beta + i\beta) - \text{Fr}(j\beta)) \]
for \(i > n\). Here \(\text{Fr}(x) = \text{Fr}_T(x)\) is given by \((2.2)\) and is equal to the vector \(T\)-fractional part of \(x \in \mathbb{R}^D\).
Proof. Identity (3.16) follows immediately from (3.1) and (3.10), while (3.17) follows from (3.14). \qed

Remark 3.1. In \S 5 we shall show that under some assumptions we have the following bounds on the absolute value $|\delta(i)|$:

$$|\delta(i)| = o(i) \quad \text{as} \quad i \to +\infty.$$ 

Therefore, using (3.15), we can view $\delta(i)$ as the total vector deviation from $i\alpha$ of the distribution of points in the orbit

$$(3.18) \quad \text{Orb}_{S,i}(0) = \{S^i_\beta(0) \equiv i\beta \mod L; i = 0, 1, 2, \ldots \}$$

over all tiles $T_1, \ldots, T_D$ of the development $T$.

\section{Colored $k$-deviations}

Suppose $L'$ is a spanning lattice (2.1). Then its basis $l_1, \ldots, l_D$ has a dual basis $l^*_1, \ldots, l^*_D$ defined by

$$(4.1) \quad l_k^* \cdot l_m = \delta_{k,m},$$

where $x = (x_1, \ldots, x_D)$ and $y = (y_1, y_D)$ are in $\mathbb{R}^D$, and $x \cdot y$ denotes the scalar product $x \cdot y = x_1y_1 + \cdots + x_Dy_D$.

Using (3.15) and the dual basis (4.1), we get

$$(4.2) \quad l_k^* \cdot \delta(i) = r_k(i) - il_k^* \cdot \alpha \quad \text{for} \quad k = 1, \ldots, D.$$ 

Note that (4.2) can be written in the form

$$(4.3) \quad \delta_k(i) = r_k(i) - ia_k, \quad k = 1, \ldots, D.$$ 

Here the $\delta_k(i)$ are given by

$$(4.4) \quad \delta_k(i) = l_k^* \cdot \delta(i).$$

Condition (2.1) on the vectors $l_1, \ldots, l_D$ together with (4.1) implies that the coefficients $a_k$ are uniquely determined by the equation

$$(4.5) \quad -\alpha = a_1l_1 + \cdots + a_Dl_D.$$ 

The number $\delta_k(i)$ is called the deviation of the distribution of points in the orbit $\text{Orb}_{S,i}(0)$ for the tile $T_k \subset T$. Short version: $k$-deviation, or colored deviation.

From (4.1) and (4.4) it follows that the vector-valued function $\delta(i)$ is related to the colored deviations $\delta_k(i)$ by the identity

$$(4.6) \quad \delta(i) = \delta_1(i)l_1 + \cdots + \delta_D(i)l_D.$$ 

Therefore, it is natural to call $\delta(i)$ the total vector deviation of the distribution of points in the orbit $\text{Orb}_{S,i}(0)$ with respect to the tiles $T_1, \ldots, T_D$ of the development $T$.

It remains to compute the deviation $\delta_0(i)$ with respect to the tile $T_0$. Note that the counting functions $r_0(i), r_1(i), \ldots, r_D(i)$ are not independent, because the definitions (3.6) and (3.7) show that

$$(4.7) \quad r_0(i) + r_1(i) + \cdots + r_D(i) = i$$

for $i = 0, 1, 2, \ldots$. Let $a_0$ be a number such that

$$(4.8) \quad a_0 + a_1 + \cdots + a_D = 1.$$ 

Using (4.7) and (4.8), we get

$$(4.9) \quad [r_0(i) - ia_0] + [r_1(i) - ia_1] + \cdots + [r_D(i) - ia_D] = 0.$$
As in (4.13) for the deviations $\delta_1(i), \ldots, \delta_D(i)$, we can define the zeroth colored deviation as follows:

\begin{equation}
\delta_0(i) = r_0(i) - ia_0.
\end{equation}

By (4.13) and (4.9), the colored deviations satisfy $\delta_0(i) + \delta_1(i) + \cdots + \delta_D(i) = 0$ for all $i = 0, 1, 2, \ldots$. Therefore,

\begin{equation}
\delta_0(i) = -\delta_1(i) - \cdots - \delta_D(i).
\end{equation}

On the other hand, relations (4.4) and (4.11) show that

\begin{equation}
\delta_0(i) = -l_1^* \cdot \delta(i) - \cdots - l_D^* \cdot \delta(i) = -(l_1^* + \cdots + l_D^*) \cdot \delta(i).
\end{equation}

Now, put

\begin{equation}
l_0^* = -(l_1^* + \cdots + l_D^*).
\end{equation}

Using (4.12), we get

\begin{equation}
\delta_0(i) = l_0^* \cdot \delta(i).
\end{equation}

Comparing this with (4.4), we see that symmetry is restored for the colored deviations with respect to all tiles $T_0, T_1, \ldots, T_D$ in the development $T$. This symmetry has been broken because we distinguished the vector $v_0$ in order to define the basis (1.9) of the lattice $L'$, see (1.10).

**Proposition 4.1.** Suppose $L'$ is a complete lattice (1.10) and

\begin{equation}
\delta_k(i) = r_k(i) - ia_k \quad \text{for} \quad k = 0, 1, \ldots, D,
\end{equation}

where the numbers $a_1, \ldots, a_D$ are the coordinates of $-\alpha$ with respect to the basis $l_1, \ldots, l_D$, and $a_0 = 1 - a_1 - \cdots - a_D$. Then for the $k$-deviations (4.15) we have

\begin{equation}
\delta_0(i) + \delta_1(i) + \cdots + \delta_D(i) = 0
\end{equation}

for all $i = 0, 1, 2, \ldots$;

\begin{equation}
\delta_k(i) = l_k^* \cdot \delta(i) \quad \text{for} \quad k = 0, 1, \ldots, D,
\end{equation}

where $l_1^*, \ldots, l_D^*$ is the basis dual to the basis $l_1, \ldots, l_D$ of $L'$ (see (4.1)), the vector $l_0^*$ is given by (4.13), and the vector deviation $\delta(i)$ is given by (4.1). Moreover, the deviations $\delta(i)$ and $\delta_k(i)$ are related to each other by the formula

\begin{equation}
\delta(i) = \delta_1(i) l_1 + \cdots + \delta_D(i) l_D
\end{equation}

for all $i = 0, 1, 2, \ldots$.

**Proof.** Using (4.3), (4.10), and (4.4), we get (4.16) and (4.17) for the $k$-deviations $\delta_k(i)$; (4.6) implies formula (4.18).

**Remark 4.1.** In the next section we shall prove the following asymptotic bound for the $k$-deviations $\delta_k(i)$ (see Theorem 5.1):

\begin{equation}
\delta_k(i) = o(i) \quad \text{as} \quad i \to +\infty,
\end{equation}

where $k = 0, 1, \ldots, D$. From (4.15) and (4.19) it follows that

\begin{equation}
\lim_{i \to +\infty} \frac{r_k(i)}{i} = a_k.
\end{equation}

Combining this with the definition of the counting functions (3.3), we get

\begin{equation}
a_k \geq 0 \quad \text{for all} \quad k = 0, 1, \ldots, D.
\end{equation}

Therefore, $a_k$ is the mean value of the number of the points in $\text{Orb}_{S, \rho}(0)$ that lie in the tile $T_k$. 

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§5. Approximate bounds for $k$-deviations

For $x \in \mathbb{R}^D$, define a vector-valued function $\mathcal{D}$ by

$$
\mathcal{D}(x) = \sum_{0 \leq j \leq n-1} \left( \text{Fr}(j\beta + x) - \text{Fr}(j\beta) \right).
$$

The definition (2.2) of the $T$-fractional part $\text{Fr}(x)$ implies that $\mathcal{D}(x + l) = \mathcal{D}(x)$ for any vector $l$ in the lattice $L$, which means that $\mathcal{D}(x)$ is a periodic function on the vector space $\mathbb{R}^D$ with the spanning period lattice $L$. We can rewrite equation (3.17) in terms of the function $\mathcal{D}(x)$:

$$
\delta(i) = \mathcal{D} \circ \text{Fr}(i\beta) = \mathcal{D}(\text{Fr}(i\beta)),
$$

i.e., $\delta(i)$ is the composition of $\text{Fr}(x)$ and $\mathcal{D}(x)$.

Remark 5.1. Suppose $S_\alpha$ is an irrational shift (1.2) of the torus $\mathbb{T}^D$. The definition (1.11) shows that the shift $S_\beta$, where $\beta$ is as in (3.2), is also irrational. It is known [9] that in this case the points of the orbit (3.18) are dense and equidistributed on the development of the torus $\mathbb{T}$. Hence, in order to determine the behavior of the vector-valued deviation $\delta(i)$ for $i = 0, 1, 2, \ldots$, it suffices to write $\delta(i)$ as the composition (5.1) and estimate the value of $\mathcal{D}(x)$, where $x$ runs over the development of $\mathbb{T}$.

Let $i > n$. We introduce the difference function

$$
\Delta(x, y) = \text{Fr}(x + y) - \text{Fr}(y);
$$

by (2.2), it has the periodicity property

$$
\Delta(x + l, y + l') = \Delta(x, y)
$$

with respect to the periods $l, l'$ in the lattice $L$. Using (5.2), we can rewrite the function $\mathcal{D}(x)$ in the form

$$
\mathcal{D}(x) = \sum_{0 \leq j \leq n-1} \Delta(x, j\beta).
$$

By (5.2), for any $x, y \in \mathbb{R}^D$ we have

$$
\Delta(x, y) \in T_{\Delta},
$$

where

$$
T_{\Delta} = T - T = \{t - t'; t, t' \in T\}
$$

is the difference subset of the development $T$. The definition (5.5) implies that the difference set $T_{\Delta}$ is symmetric,

$$
T_{\Delta} \rightarrow T_{\Delta} : t \mapsto -t,
$$

with respect to the origin of $\mathbb{R}^D$. Using (5.3) and (5.4), we see that

$$
\mathcal{D}(x) \in nT_{\Delta}
$$

for any $x$ in $\mathbb{R}^D$, where multiplication by $n$ on the right-hand side of (5.6) means the homothety $t \mapsto nt$ with coefficient $n = 1, 2, 3, \ldots$. By (5.4), for any $x, y \in \mathbb{R}^D$ we have

$$
l_k^* \cdot \Delta(x, y) \in (l_k^* \cdot T)_{\Delta}, \quad k = 0, 1, \ldots, D,
$$

because $l_k^* \cdot (T)_{\Delta} = (l_k^* \cdot T)_{\Delta}$. We put

$$
\mathcal{D}_k(x) = l_k^* \cdot \mathcal{D}(x);
$$

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then, using (5.6) and (5.8), for any \(x \in \mathbb{R}^D\) we get
\[
(5.9) \quad D_k(x) \in n(l_k^* \cdot T)_\Delta, \quad k = 0, 1, \ldots, D.
\]
For any subset \(X \subset \mathbb{R}^D\), we define the boundary values
\[
(5.10) \quad \text{pr}_l X = \inf_{x \in X} \text{pr}_l x, \quad \text{pr}_l X = \sup_{x \in X} \text{pr}_l x
\]
of the orthogonal projections
\[
\text{pr}_l x = \frac{l \cdot x}{|l|}
\]
of the vectors \(x \in X\) to the vector \(l\), where \(|l| = \sqrt{l \cdot l}\) is the length of \(l\).

Relations (5.9) and (5.10) show that
\[
(5.11) \quad |D_k(x)| \leq n |l_k^*| (\text{pr}_l^* T - \text{pr}_l T)
\]
for any \(x \in \mathbb{R}^D\) and \(k = 0, 1, \ldots, D\).

By Proposition 3.1, the vector-valued deviation \(\delta(i)\) can be expressed as a composition:
\[
\delta(i) = D \circ \text{Fr}(i\beta) \quad \text{for} \quad i > n.
\]
Therefore, using (5.11), we obtain the inequalities
\[
(5.12) \quad |\delta_k(i)| \leq c_k n \quad \text{for} \quad i > n,
\]
where
\[
(5.13) \quad c_k = c_{T,k} = |l_k^*| (\text{pr}_l^* T - \text{pr}_l T)
\]
are the constants independent of \(n, i\); they depend solely on the size of the development \(T\) (see (1.4)) of the torus \(T^D\).

It remains to consider the case of \(0 \leq i \leq n\), where, by (3.16), the vector-valued deviation \(\delta(i)\) has the form
\[
\delta(i) = \sum_{0 \leq j < i} (\text{Fr}(j\beta + n\beta) - \text{Fr}(j\beta))
\]
or, in terms of the difference function (5.2),
\[
(5.14) \quad \delta(i) = \sum_{0 \leq j < i} \Delta(n\beta, j\beta).
\]
In this case instead of (5.12) we can use (5.7) and (5.14) to obtain the inequalities
\[
(5.15) \quad |\delta_k(i)| \leq c_k i \quad \text{for} \quad 0 \leq i \leq n.
\]

**Theorem 5.1.** Suppose the vectors \(l_1, \ldots, l_D\) are linearly independent over \(\mathbb{R}\). Then:

1) for any \(k = 0, 1, \ldots, D\), we have
\[
(5.16) \quad |\delta_k(i)| \leq c_k n
\]
for all \(i = 0, 1, 2, \ldots, \), where the \(c_k = c_{T,k}\) are the constants (5.13) independent of \(n\) and \(i\) and determined by the size of the toric development \(T\), see (1.4);

2) if, moreover, \(\alpha\) is an irrational shift (1.11), and every tile \(T_k\) in the tiling (1.4) of the development of \(T\) is Jordan measurable, then the \(k\)-deviations \(\delta_k(i)\) defined by (4.3) and (4.14) have the form
\[
(5.17) \quad \delta_k(i) = r_k(i) - ia_k, \quad \text{where} \quad a_k = \frac{\text{vol} T_k}{\text{vol} T},
\]
for \(k = 0, 1, \ldots, D\). Here \(\text{vol} T_k\) and \(\text{vol} T\) denote the volumes of the tiles \(T_k\) and of the whole development of \(T\).
Proof. Inequality (5.16) follows from (5.12) and (5.15), while (5.17) follows from (4.3), (4.14), and (4.20). □

Now we estimate the absolute value of the total vector-valued deviation $\delta(i)$. Using (5.1) and (5.6), we get
\[
\delta(i) \in nT_{\Delta}.
\]
Therefore,
\[
(5.18) \quad |\delta(i)| \leq d_T n
\]
for all $i = 0, 1, 2, \ldots$, where
\[
(5.19) \quad d_T = \sup_{t, t' \in T} |t - t'|
\]
is the diameter of the development of $T$.

As above, using (5.6) and (5.8), see that $\delta_k(i) \in n^* \cdot T_{\Delta}$. Therefore,
\[
(5.20) \quad |\delta_k(i)| \leq |n^*| \cdot d_T n
\]
for $k = 0, 1, \ldots, D$.

Assume once again that the vectors $l_1, \ldots, l_D$ are linearly independent over $\mathbb{R}$. Let $A$ be a nonsingular affine map on the vector space $\mathbb{R}^D$. Consider the action of this map on the torus
\[
A : \mathbb{T}_L^D \simeq \mathbb{R}^D / L \rightarrow \mathbb{T}_{AL}^D \simeq \mathbb{R}^D / AL.
\]
We obtain the following substitutions for the basic elements of the construction in §1:
\[
L \rightarrow AL, \quad \alpha \rightarrow A\alpha; \quad T \rightarrow AT, \quad L' \rightarrow AL', \quad v_k \rightarrow Av_k, \quad v_k \rightarrow Av_k.
\]
Consider the action of the affine map $A$ on the vector-valued deviation $\delta(i) = \delta_T(i)$. The definition (3.1) shows that
\[
(5.21) \quad \delta_{AT}(i) = A\delta_T(i)
\]
for $i = 0, 1, 2, \ldots$. Formula (5.21) means that $A$ acts on the vector-valued deviation $\delta_T(i)$ in the same way as on the vectors $x \mapsto Ax$ of the vector space $\mathbb{R}^D$. The choice of the starting basis $l_1, \ldots, l_D$ is arbitrary: we can use it to simplify the construction of exchanging toric developments (1.4). Therefore, we conclude that the vector deviation $\delta(i) = \delta_T(i)$ has no natural scale.

Now, consider the action of the affine map $A$ on the $k$-deviations $\delta_k(i)$. Suppose we have started with some other basis $Al_1, \ldots, Al_D$. By (4.18),
\[
(5.22) \quad \delta_{AT}(i) = \delta_{AT,1}(i) Al_1 + \cdots + \delta_{AT,D}(i) Al_D.
\]
Now apply (4.18) again for the basis $l_1, \ldots, l_D$. Considering the action of $A$ on (4.18), we get
\[
(5.23) \quad A\delta_T(i) = \delta_{T,1}(i) Al_1 + \cdots + \delta_{T,D}(i) Al_D.
\]
The left-hand sides of (5.22) and (5.23) are equal by (5.21). Therefore, the right-hand sides of (5.22) and (5.23) are also equal. Together with the linear independence of $Al_1, \ldots, Al_D$, this implies
\[
\delta_{AT,k}(i) = \delta_{T,k}(i)
\]
for all $k = 0, 1, \ldots, D$. As we can see, unlike the vector-valued deviation $\delta_T(i)$ (see (5.21)), the colored $k$-deviations $\delta_{T,k}(i)$ are invariant under the action of the affine map $A$.\]
A. Therefore, they do not depend on the choice of the basis $l_1, \ldots, l_D$. The constants in (5.13) are also invariant,
\begin{equation}
\tag{5.24}
c_{AT,k} = c_{T,k},
\end{equation}
and so are the coordinates (4.13),
\begin{equation}
\tag{5.25}
a_{AT,k} = a_{T,k}
\end{equation}
for any $k = 0, 1, \ldots, D$.

**Proof of Theorem 3** (see the Introduction). Suppose the vectors $l_1, \ldots, l_D$ are linearly independent over $\mathbb{R}$. There is no loss of generality in assuming that
\begin{equation}
\tag{5.26}
l_1 = e_1, \ldots, l_D = e_D.
\end{equation}
In the standard basis, inequalities (5.20) can be written as
\begin{equation}
\tag{5.27}
|\delta_k(i)| \leq d_T n
\end{equation}
for $k = 1, \ldots, D$. For $k = 0$ they can be written as
\begin{equation}
\tag{5.28}
|\delta_0(i)| \leq Dd_T n,
\end{equation}
because from (5.26) and (4.13) applied to $l_0^*$ it follows that
\begin{equation*}
l_0^* = -(e_1 + \cdots + e_D) = (-1, \ldots, -1).
\end{equation*}

Suppose also that the following conditions are satisfied:
1) the basic period lattice $L$ of the torus $\mathbb{T}^D = \mathbb{R}^D/L$ has the form
\begin{equation}
\tag{5.29}
L = L',
\end{equation}
where $L' = \mathbb{Z}[e_1, \ldots, e_D]$ is a cubic lattice in $\mathbb{R}^D$;
2) $\alpha$ is an irrational shift of $\mathbb{T}^D$;
3) the toric development $T$ and all its tiles $T_k$ are Jordan measurable.

Then conditions 2) and 3) imply (see [9]) that for any $k = 0, 1, \ldots, D$ there is a frequency
\begin{equation}
\tag{5.30}
\nu_k = \lim_{i \to +\infty} \frac{r_k(i)}{i}
\end{equation}
of the number of returns of the orbit $\text{Orb}_{S_\beta}(0)$ into the tile $T_k$. Moreover, we have
\begin{equation}
\tag{5.31}
\nu_k = \text{vol } T_k,
\end{equation}
because 1) together with (5.29) implies that the volume of the toric development equals $\text{vol } T = 1$. At the same time, by (4.20) and (5.30) we have
\begin{equation}
\nu_k = a_k;
\end{equation}
therefore, (5.31) yields
\begin{equation}
\tag{5.32}
a_k = \text{vol } T_k.
\end{equation}
Moreover, using (4.5), we obtain
\begin{equation}
\tag{5.33}
\text{vol } T_k = -e_k \cdot \alpha = -\alpha_k
\end{equation}
for $k = 1, \ldots, D$, where the shift vector $\alpha = (\alpha_1, \ldots, \alpha_D)$ is expressed in the standard basis $e_1, \ldots, e_D$.

Proposition 4.1 and (5.32) show that
\begin{equation}
\tag{5.34}
\delta_k(i) = r_k(i) - i \text{ vol } T_k.
\end{equation}
Combining inequalities (5.27), (5.28), and (5.34), we obtain Theorem 3. \(\square\)
§6. Precise bounds for $k$-deviations

Let $\Delta(x, y)$ denote the difference function (5.2). It can be written as

$$\Delta(x, y) = x - l(x, y)$$

for any $x, y \in T$, where $l(x, y) = l$ is a unique vector in the lattice $L$ such that

$$x \in T[l] - y \sim x + y \in T[l],$$

with $T[l] = T + l$. Indeed, there is a unique vector $l$ in $L$ such that $x + y - l \in T[0] = T$. Therefore,

$$\text{Fr}(x + y) = x + y - l = x + \text{Fr}(y) - l$$

for any $y \in T$. Thus, we have (6.1).

The case where $i > n$. By (5.3) and (6.1), we have

$$D(x) = \sum_{0 \leq j \leq n-1} \Delta(x, j\beta) = nx - \sum_{0 \leq j \leq n-1} l(x, j\beta),$$

where for any $x \in T$ we denote by $l(x, j\beta) = l$ a unique vector in $L$ such that $x + \text{Fr}(j\beta) \in T[l]$. Looking at (6.2), we introduce the function

$$A(x) = \sum_{0 \leq j \leq n-1} l(x, j\beta).$$

Remark 6.1. Our use of the vector-valued fractional part $\text{Fr}(j\beta)$ instead of $j\beta$ itself corresponds to localizing the situation: we are interested only in the tiles $T[l]$ adjacent to the tile $T = T[0]$.

From now on we assume that the toric development $T$ and all its tiles $T_k$ are unions of finitely many arbitrary polygons.

We define a set of vectors:

$$L(j\beta) = \{ l \in L; T \cap (T_l - \text{Fr}(j\beta)) \neq \emptyset \}.$$

Note that there can be several vectors $l$ satisfying the condition in (6.3).

Consider the union of the following boundaries of polygons:

$$\partial_n = \bigcup_{0 \leq j \leq n-1} \bigcup_{l \in L(j\beta)} \partial(T_l - \text{Fr}(j\beta)),$$

where for any set $X \subset \mathbb{R}^D$ by $\partial X$ we denote the boundary of $X$:

$$\partial X = \bar{X} \setminus X^{\text{int}}.$$

Consider a partition

$$\text{Til}_n(T) = \bigcup_p C_p$$

of $T$ into a finite number of cells $C_p$, where every $C_p$ is a closed polygon (i.e., $C_p = \bar{C}_p$), hence a simply connected set. The partition $\text{Til}_n(T)$ is uniquely determined by the conditions

$$\partial(\text{Til}_n(T)) = \bar{T} \cap \partial_n,$$

and any two cells $C_p, C_{p'}$ in (6.4) have no interior points in common.

By (6.3) and (5.8),

$$D_k(x) = nl_k^* \cdot x - A_k(x).$$
where
\begin{equation}
A_k(x) = \sum_{0 \leq j \leq n-1} l_k^* \cdot l(x, j\beta) \quad \text{for} \quad k = 0, 1, \ldots, D.
\end{equation}

Using the definitions of \(l(x, j\beta)\) and \(\text{Til}_n(T)\), we get the following property of \(A_k(x)\).

**Lemma 6.1.** Suppose \(A_k(x)\) is the function defined by (6.6). Then:
1) \(A_k(x)\) is constant on every set \(C_p^\text{int}\) in (6.4);
2) all jumps of \(A_k(x)\) are contained in the boundaries \(\partial C_p\).

The first term \(nl_k^* \cdot x\) on the right-hand side of (6.5) is a nonzero linear form, which, when viewed as a function on \(C_p\), has extrema points only at the vertices \(v \in V(C_p)\) of the polygon \(C_p\).

Denote by \(\bar{V}(\text{Til}_n(T))\) the set of all vertices of all the polygons \(C_p\) in the partition \(\text{Til}_n(T)\). For any vertex \(v\) in \(\bar{V}(\text{Til}_n(T))\), put
\begin{equation}
A_{k, C_p} = A_k(x_{C_p}^\text{int}),
\end{equation}
where \(x_{C_p}^\text{int}\) is any interior point of \(C_p\). By (6.6), we have
\begin{equation}
A_{k, C_p} = \sum_{0 \leq j \leq n-1} l_k^* \cdot l(x_{C_p}^\text{int}, j\beta).
\end{equation}

For every vertex \(v\) in \(\bar{V}(\text{Til}_n(T))\), consider the following two extremal quantities:
\begin{equation*}
\min_k(v) = \min_{C_p \in \text{Til}_n(T)} A_{k, C_p}, \quad \max_k(v) = \max_{C_p \in \text{Til}_n(T)} A_{k, C_p}.
\end{equation*}
Let
\begin{equation}
m_k' = \inf_{x \in \bar{V}} D_k(x), \quad M_k' = \sup_{x \in \bar{V}} D_k(x).
\end{equation}

Lemma 6.1 and relations (6.5) and (6.7) show that
\begin{align}
m_k' &= \min_{v \in \bar{V}(\text{Til}_n(T))} (nl_k^* \cdot v - \max_k(v)), \\
M_k' &= \max_{v \in \bar{V}(\text{Til}_n(T))} (nl_k^* \cdot v - \min_k(v))
\end{align}
for any \(i > n\).

**The case where** \(0 \leq i \leq n\). Let
\begin{equation}
D_{k, i} = \sum_{0 \leq j < i} l_k^* \cdot \Delta(n\beta, j\beta),
\end{equation}
and let
\begin{equation}
m_k'' = \min_{0 \leq i \leq n} D_{k, i}, \quad M_k'' = \max_{0 \leq i \leq n} D_{k, i}.
\end{equation}

Proposition 3.1 and relations (6.3), (6.9), and (6.10) imply the following result.

**Theorem 6.1.** Suppose the vectors \(l_1, \ldots, l_D\) are linearly independent over \(\mathbb{R}\). Then, for any \(k = 0, 1, \ldots, D\), the deviations \(\delta_k(i)\) satisfy the inequalities
\begin{equation}
m_k \leq \delta_k(i) \leq M_k
\end{equation}
for any \(i = 0, 1, 2, \ldots\), where
\begin{equation}
m_k = \min\{m_k', m_k''\}, \quad M_k = \max\{M_k', M_k''\}.
\end{equation}
Moreover, the bounds in (6.11) are sharp, i.e., for any \(\varepsilon > 0\) there exist \(i_1\) and \(i_2\) such that
\begin{equation}
m_k + \varepsilon \geq \delta_k(i_1), \quad \delta_k(i_2) \geq M_k - \varepsilon.
\end{equation}
Remark 6.2. The computational complexity of the sharp bounds $m_k, M_k$ in (6.11) depends on the number $c_{n,D}(T)$ of the cells $C_p$ in the partitions $T_{il}(T)$, see (6.4). Choosing the $D$-dimensional unit cube $I^D$ (see §9) as a toric development $T$, we have the following estimates for $c_{n,D}(T)$:

$$c_{n,D}(T) = O(n^D) \text{ as } n, D \to +\infty.$$ 

Remark 6.3. The sharp bounds (6.11) are useful, e.g., in solving the problem of embedding lattices into quasiperiodic partitions [11], and in constructing quasiperiodic sequences close to periodic ones, or balanced words (see, e.g., [1, 5]).

§7. The case where $n = 1$

For $n = 1$, the deviations $\delta(i)$ have the simplest behavior, which is completely determined by the geometry of the toric development $T$ as in (1.4). When $n > 1$, the deviation $\delta(i)$ is seriously affected by the metric properties of the shift vector $\alpha$, see (1.2).

So, let $n = 1$. Then, for $i \geq 2$, from (3.17) it follows that

$$\delta(i) = D(i \beta),$$

where

$$D(x) = \text{Fr}(x) - \text{Fr}(0)$$

for any $x \in \mathbb{R}^D$. Suppose $\text{Fr}(0) = 0$, i.e., $0 \in T$. Then the definition (2.2) of the fractional part $\text{Fr}(x)$ and (7.2) show that

$$D(x) = x \text{ for any } x \in T.$$ 

If $i = 0$ or $1$, then (3.16) implies

$$\delta(0) = 0, \quad \delta(1) = \text{Fr}(\beta) - \text{Fr}(0) = \text{Fr}(\beta).$$

Thus, if $n = 1$ and $\text{Fr}(0) = 0$, then the vector-valued deviation $\delta(i)$ can be expressed as

$$\delta(i) = \text{Fr}(i \beta)$$

for any $i = 0, 1, 2, \ldots$, while the $k$-deviations have the form

$$\delta_k(i) = \text{Fr}_k(i \beta) \text{ for } k = 0, 1, \ldots, D,$$

where $\text{Fr}_k(x) = l_k^* \cdot \text{Fr}(x)$. Thus, (7.4) and (7.5) provide the following geometrical description of the deviations:

1) the vector-valued deviation $\delta(i)$ is the position of the $T$-fractional part $\text{Fr}(i \beta)$ of $i \beta$ on the toric development (1.4);

2) the $k$-deviation $\delta_k(i)$ is related to the projection $\text{Fr}_k(i \beta)$ of the vector-valued fractional part $\text{Fr}(i \beta)$ to the vector $l_k^*$:

$$\delta_k(i) = |l_k^*| \text{pr}_k \text{Fr}(i \beta).$$

Using (7.6), we get the inequalities

$$|l_k^*| \text{pr}_k T \leq \delta_k(i) \leq |l_k^*| \text{pr}_k T,$$

whence

$$|\delta_k(i)| \leq |l_k^*| \text{max} \{ |\text{pr}_k T|, |\text{pr}_k T| \}.$$ 

Suppose that $\text{Fr}(0) = 0$ lies in $T$. Then, by (7.8),

$$|\delta_k(i)| \leq |l_k^*| d_T$$

for $k = 0, 1, \ldots, D$, where $d_T$ is the diameter (5.19) of $T$. 

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Since the vectors \( l_1, \ldots, l_D \) are linearly independent over \( \mathbb{R} \), there is an affine map 
\( \lambda_k = e_k \) for \( k = 1, 2, \ldots, D \), where \( e_1, \ldots, e_D \) is a standard basis of the vector space \( \mathbb{R}^D \).
Since the map \( \lambda \) is nonsingular, we can replace \( T \) with \( AT \) in all the above arguments. 
Thus, to simplify the notation, we may assume that \( l_1 = e_1, \ldots, l_D = e_D \), so that the lattice \( L' \) in (7.10) has the form
\[
L' = \mathbb{Z}[e_1, \ldots, e_D],
\]
i.e., is a cubic lattice \( \mathbb{Z}^D \) in \( \mathbb{R}^D \).

Remark 7.1. In §9 we shall consider a specific example of toric developments \( T \) that are 
\( D \)-dimensional cubes \( I^D \), where \( I = [0, 1] \) is the unit semiinterval. Those \( T = I^D \) can be constructed in purely geometric terms. Then, the lattice \( L' \) emerges naturally from the construction and in general does not coincide with the cubic lattice \( \mathbb{Z}^D \). Moreover, when we substitute the new development \( AT \) for the toric development \( T \), we have to sacrifice the geometric intuition to replace the vector fractional part \( \text{Fr}(x) \) (where \( x = (x_1, \ldots, x_D) \)) with the usual fractional parts \( \{x_k\} \) of the coordinates \( x_k \).

If (7.10) is true, then the dual basis (7.11) has the form
\[
l_k^* = e_k^* = e_k.
\]
Therefore,
\[
\text{pr}_{l_k^*}(x) = \text{pr}_{e_k}(x)
\]
and
\[
\text{pr}_{l_k^*}T = \text{pr}_{e_k}T, \quad \text{pr}_{l_0^*}T = \text{pr}_{e_0}T
\]
for \( k = 1, \ldots, D \). For \( k = 0 \) we have
\[
l_0^* = -(e_1^* + \cdots + e_D^*) = -(e_1 + \cdots + e_D)
\]
by (7.13). It follows that
\[
\text{pr}_{l_0^*}T = -\text{pr}_{e}T, \quad \text{pr}_{l_0^*}T = -\text{pr}_{e}T,
\]
where \( e = e_1 + \cdots + e_D = (1, \ldots, 1) \). With the help of (7.11), the \( k \)-deviations \( \delta_k(i) \) can be expressed as
\[
\delta_k(i) = l_k^* \cdot \delta(i) = e_k \cdot \delta(i)
\]
for \( k = 1, \ldots, D \), and
\[
\delta_0(i) = l_0^* \cdot \delta(i) = -e \cdot \delta(i)
\]
for \( k = 0 \).

Theorem 7.1. Suppose the vectors (2.1) are \( l_1 = e_1, \ldots, l_D = e_D \).

1. The vector deviation \( \delta(i) \) has the coordinates
\[
\delta(i) = (\delta_1(i), \ldots, \delta_D(i))
\]
with respect to the basis \( e_1, \ldots, e_D \), where, as in (7.5),
\[
\delta_k(i) = \text{Fr}_k(i\beta) \quad \text{for} \quad k = 1, \ldots, D,
\]
while the vector \( T \)-fractional part \( \text{Fr}(x) \) has the coordinates
\[
\text{Fr}(x) = (\text{Fr}_1(x), \ldots, \text{Fr}_D(x))
\]
in the same basis \( e_1, \ldots, e_D \).

2. The \( k \)-deviations \( \delta_k(i) \) satisfy the following inequalities:
\[
\text{pr}_{e_k}T \leq \delta_k(i) \leq \text{pr}_{e_k}T,
\]
(7.16)
\[
|\delta_k(i)| \leq d_T
\]
(7.17)
for \( k = 1, \ldots, D \), and
\[
-\text{pr}_e T \leq \delta_0(i) \leq -\text{pr}_e T,
\]
where \( e = e_1 + \cdots + e_D \),
\[
|\delta_0(i)| \leq Dd_T.
\]

**Proof.** Inequalities (7.16) for \( k = 1, \ldots, D \) follow from (7.7) and (7.12); inequality (7.18) follows from (7.7) and (7.13), (7.14). Inequalities (7.17) and (7.19) follow from (7.9). □

### §8. Mean values of the deviations

For the vector deviation \( \delta(i) \), we define its mean value by
\[
\langle \delta \rangle = \lim_{N \to +\infty} \frac{1}{N} \sum_{1 \leq i \leq N} \delta(i)
\]
if this limit exists. By (3.17), we have
\[
\sum_{n<i \leq N} \delta(i) = \sum_{n<i \leq N} \sum_{0 \leq j \leq n-1} (\text{Fr}(j\beta + i\beta) - \text{Fr}(j\beta))
\]
\[
= \sum_{0 \leq j \leq n-1} \sum_{n<i \leq N} \text{Fr}(j\beta + i\beta) - (N-n) \sum_{0 \leq j \leq n-1} \text{Fr}(j\beta).
\]
Using (8.2), we can write
\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{1 \leq i \leq N} \delta(i) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n<i \leq N} \left( \sum_{0 \leq j \leq n-1} \text{Fr}(j\beta + i\beta) - \sum_{0 \leq j \leq n-1} \text{Fr}(j\beta) \right).
\]
Denote
\[
c(j) = \lim_{N \to +\infty} \frac{1}{N} \sum_{0 \leq i \leq N} \text{Fr}(j\beta + i\beta).
\]
Since
\[
\sum_{0 \leq i \leq N} \text{Fr}(j\beta + i\beta) = \sum_{0 \leq i \leq N} \text{Fr}(j\beta) - \sum_{0 \leq i < j} \text{Fr}(j\beta) + \sum_{N<i \leq N+j} \text{Fr}(j\beta),
\]
the limit in (8.4) does not depend on \( j = 0, 1, 2, \ldots \). Therefore,
\[
c(j) = c(0)
\]
for any \( j = 0, 1, 2, \ldots \). From (8.1), (8.3), and (8.4) it follows that the mean value \( \langle \delta \rangle \) has the form
\[
\langle \delta \rangle = \sum_{0 \leq j \leq n-1} (c(0) - \text{Fr}(j\beta)).
\]
Suppose the development \( T \) has finite diameter \( d_T < \infty \) and its closure \( \bar{T} \) is Jordan measurable. Then the indicator function
\[
\chi_{\bar{T}}(x) = \begin{cases} 
1 & \text{if } x \in \bar{T}, \\
0 & \text{if } x \notin \bar{T}
\end{cases}
\]
of the set \( \bar{T} \subset \mathbb{R}^D \) is integrable, and its integral is equal to the volume of \( \bar{T} \):
\[
\text{vol}(\bar{T}) = \int_{\mathbb{R}^D} \chi_{\bar{T}}(x) dx.
\]
Our assumptions imply that the limit in (8.4) exists and is equal to

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{0 \leq i \leq N} \text{Fr}(i\beta) = \frac{1}{\text{vol}(T)} \int_T x\,dx
\]

(see, e.g., [9, 10]). Note that the vector-valued integral

\[
c_T = \frac{1}{\text{vol}(T)} \int_T x\,dx
\]

can be interpreted geometrically as the barycenter of $\bar{T}$. Using (8.4) and (8.8), we get

\[
c(0) = c_T.
\]

**Theorem 8.1.** Suppose the following conditions are satisfied: the vectors $l_1, \ldots, l_D$ (see (1.9)) are linearly independent over $\mathbb{R}$, $\alpha$ is an irrational shift of the torus $T^D$, the development $T$ as in (1.4) has finite diameter $d_T$, and its closure $\bar{T}$ is Jordan measurable.

1. The mean value $\langle \delta \rangle$ (see (8.1)) of the vector deviation $\delta(i)$ exists and can be computed as follows:

\[
\langle \delta \rangle = \sum_{0 \leq j \leq n-1} (c_T - \text{Fr}(j\beta)),
\]

where $c_T$ is the barycenter (8.8) of $\bar{T}$ and the difference $c_T - \text{Fr}(x)$ in (8.10) is the centered fractional part of $x \in \mathbb{R}^D$.

2. For any $k = 0, 1, \ldots, D$, the mean value

\[
\langle \delta_k \rangle = \lim_{N \to +\infty} \frac{1}{N} \sum_{1 \leq i \leq N} \delta_k(i)
\]

of the $k$-deviations $\delta_k(i) = l_k^* \cdot \delta(i)$ exists and is equal to

\[
\langle \delta_k \rangle = \sum_{0 \leq j \leq n-1} (c_{T,k} - \text{Fr}_k(j\beta)),
\]

where $\text{Fr}_k(j\beta) = l_k^* \cdot \text{Fr}_k(j\beta)$ and $c_{T,k} = l_k^* \cdot c_T = |l_k^*| \text{pr}_{l_k^*} c_T$.

**Proof.** Note that (8.10) follows from (8.6) and (8.9).

To prove (8.12), observe that (8.10) implies

\[
l_k^* \cdot \langle \delta \rangle = \sum_{0 \leq j \leq n-1} (l_k^* \cdot c_T - l_k^* \cdot \text{Fr}(j\beta)),
\]

while the definition (8.11) yields

\[
l_k^* \cdot \langle \delta \rangle = \sum_{0 \leq j \leq n-1} l_k^* \cdot \delta(i).
\]

Using (8.11) and (8.13), we get the relationship

\[
l_k^* \cdot \langle \delta \rangle = l_k^* \cdot \langle \delta_k \rangle = \langle \delta_k \rangle
\]

between the mean values of the vector deviation $\delta(i)$ and of the $k$-deviations $\delta_k(i)$. Combining (8.10) and (8.14), we arrive at (8.12). $\square$

If $n = 1$, the shift vector $\beta$ is equal to the starting torus shift vector $\alpha$. Therefore, we can rewrite (8.10) and (8.12) as follows:

\[
\langle \delta \rangle = c_T - \text{Fr}(0),
\]

\[
\langle \delta_k \rangle = c_{T,k} - \text{Fr}_k(0) \quad \text{for} \quad k = 0, 1, \ldots, D.
\]
If the point \( \text{Fr}(0) = 0 \) lies in \( T \), then (8.15) and (8.16) simplify to
\[
\langle \delta \rangle = c_T, \\
(8.17) \quad \langle \delta_k \rangle = c_{T,k} \text{ for } k = 0, 1, \ldots, D.
\]

If, moreover, the barycenter \( c_T \) of \( T \) is inside \( T \), then the translation \( x \mapsto x - c_T \) maps \( T \) into the domain \( T' = T - c_T \) with the barycenter
\[
c_{T'} = 0
\]
lying inside \( T' \). Therefore, in the case where \( c_T \in T \), there is no loss of generality in assuming that \( T \) has the barycenter
\[
(8.19) \quad c_T = 0.
\]

Since (8.19) implies that \( \text{Fr}(0) = 0 \), we can rewrite (8.17) and (8.18) as follows:
\[
(8.20) \quad \langle \delta \rangle = 0, \quad \langle \delta_k \rangle = 0 \text{ for } k = 0, 1, \ldots, D.
\]

§9. Exchanging unit cube

9.1. General construction of an exchanging cube. We start with dimension \( D = 1 \).
Let \( T^1 \) be the unit semiinterval \( I^1 = [0, 1) \). Consider the partition
\[
(9.1) \quad T^1 = T^1_0 \sqcup T^1_1
\]
of \( T^1 \) into two semiintervals
\[
(9.2) \quad T^1_0 = [0, 1 - \alpha), \quad T^1_1 = [1 - \alpha, 1).
\]
We define the exchange on \( T^1 \) by
\[
(9.3) \quad T^1 \xrightarrow{S^1} T^1 : S^1(x) = x + v_1^1 \text{ for } x \in T^1_k,
\]
where \( v_0 = \alpha, v_1 = \alpha - 1 \).

We consider the semiinterval \( T^1 \) together with the partition (9.1) as a colored unit semiinterval \( T^1 = I^1_{\text{col}} \).

Now suppose \( D \geq 2 \). We construct an exchange cube \( T^D \) by induction on \( D \). Suppose we have a \((D - 1)\)-dimensional unit cube \( T^{D-1} = I^{D-1}_{\text{col}} \) with a partition
\[
T^{D-1} = T^{D-1}_0 \sqcup \cdots \sqcup T^{D-1}_{D-1}.
\]
Moreover, suppose there is a one-dimensional partition \( T^1 = T^1_0 \sqcup T^1_1 \) with \( \alpha = \alpha_D \) in (9.2).

We define the \( D \)-dimensional cube \( T^D \) and its partition by putting
\[
(9.4) \quad T^D = T^{D-1} \times T^1 = T^D_0 \sqcup T^D_1 \sqcup \cdots \sqcup T^D_{D-1}
\]
where
\[
(9.5) \quad rT^D_k = T^{D-1}_k \times T^1_0 \quad \text{for } k = 0, \ldots, D - 1,
\]
and
\[
T^D_D = T^{D-1} \times T^1_1
\]
are Cartesian products of the corresponding sets. To define exchange vectors for the cube \( T^D \) as in (9.4), we may assume that the spaces \( \mathbb{R}^{D-1} \) and \( \mathbb{R}^1 \) are embedded in the space \( \mathbb{R}^D = \mathbb{R}^{D-1} \times \mathbb{R}^1 \) as follows:
\[
\mathbb{R}^{D-1} \ni x^{D-1} \leftrightarrow (x^{D-1}, 0) \in \mathbb{R}^D, \quad \mathbb{R}^1 \ni x^1 \leftrightarrow (0, x^1) \in \mathbb{R}^D.
\]
The exchange on \( T^D \) will be defined by
\[
T^D \xrightarrow{S^D} T^D : S^D(x) = x + v^D_k \quad \text{for } x \in T^D_k,
\]
where
\begin{equation}
(9.6) \quad v_k^D = v_k^{D-1} + v_0^1 \quad \text{for} \quad k = 0, \ldots, D - 1
\end{equation}
and
\begin{equation}
(9.7) \quad v_D^D = v_1^1.
\end{equation}

Again, we see that the partition (9.4) gives rise to a colored \(D\)-dimensional cube \(T^D = I^D_{\text{col}}\).

9.2. One-dimensional case. Let us apply the constructions of §5 to the exchanging semiinterval \(T^1\) as in (9.1). By (1.9), we have \(l_1 = v_1 - v_0 = -1\). Thus, the lattice (1.10) has the form 
\(L' = \mathbb{Z} \{l_1\} = \mathbb{Z}\), whence
\[S_1^1 : x \mapsto x + \alpha \mod 1\]
is a rotation of the unit circle \(T^1 \simeq \mathbb{R}^1 / L\), where \(L = L' = \mathbb{Z}\). Therefore,
\[\text{Fr}^1(x) = \{x\}\]
is the fractional part of \(x\). Using (4.1) and (4.13), we get
\begin{equation}
(9.8) \quad l_0^* = -1, \quad l_0^* = -l_1^* = 1,
\end{equation}
so that the counting functions \(r_0^1(i)\) and \(r_1^1(i)\) look like this:
\[r_0^1(i) = \# \{j; \{j \beta\} < 1 - \alpha, 0 \leq j < i\},\]
\[r_1^1(i) = \# \{j; \{j \beta\} \geq 1 - \alpha, 0 \leq j < i\}.
\]

Consider the \(k\)-deviations
\begin{equation}
(9.9) \quad \delta_k^1(i) = r_k^1(i) - ia_k \quad \text{for} \quad k = 0, 1,
\end{equation}
where, in accordance with (4.5) and (9.8),
\[a_1 = -l_1^* \cdot \alpha = \alpha, \quad a_0 = 1 - a_1 = 1 - \alpha.
\]

Therefore, we have the following equations for the \(k\)-deviations (9.9):
\begin{equation}
(9.10) \quad \delta_0^1(i) = r_0^1(i) - i(1 - \alpha), \quad \delta_1^1(i) = r_1^1(i) - i\alpha.
\end{equation}

In order to estimate the values of the \(k\)-deviations \(\delta_0^1(i), \delta_1^1(i)\), we can use Theorem 5.1. The following crude bounds are implied by (5.16):
\[|\delta_k^1(i)| \leq c_k^1 n \quad \text{for} \quad k = 0, 1,
\]
where, by (5.13),
\[c_k^1 = |l_k^*| (\text{pr}_{l_k^*} T_1^1 - \text{pr}_{l_k^*} T_1^1) = 1.
\]
Thus, we obtain Hecke’s estimates [3] for the deviations (9.10):
\begin{equation}
(9.11) \quad |\delta_0^1(i)| \leq n, \quad |\delta_1^1(i)| \leq n.
\end{equation}
9.3. Two-dimensional case: \( n \) is arbitrary. Now \( T^2 = I_{\text{col}}^2 \) is the unit square with the partition

\[
T^2 = T^2_0 \sqcup T^2_1 \sqcup T^2_2
\]

into the tiles

\[
T^2_0 = \{ x = (x_1, x_2); 0 \leq x_1 < 1 - \alpha_1, 0 \leq x_2 < 1 - \alpha_2 \}, \tag{9.12}
\]

\[
T^2_1 = \{ x = (x_1, x_2); 1 - \alpha_1 \leq x_1 < 1, 0 \leq x_2 < 1 - \alpha_2 \}, \tag{9.13}
\]

\[
T^2_2 = \{ x = (x_1, x_2); 0 \leq x_1 < 1, 1 - \alpha_2 \leq x_2 < 1 \}. \tag{9.14}
\]

The exchange vectors (9.6) and (9.7) are

\[
v_0 = (\alpha_1, \alpha_2), \quad v_1 = (\alpha_1 - 1, \alpha_2), \quad v_2 = (0, \alpha_2 - 1);
\]

therefore,

\[
l_1 = v_1 - v_0 = (-1, 0), \quad l_2 = v_2 - v_0 = (-\alpha_1, -1)
\]

are basis vectors of the lattice \( L' \). It follows that in this case the exchange of the unit square \( T^2 \) is equivalent to the shift

\[
S^2_\alpha : x \mapsto x + \alpha \mod L
\]

of the two-dimensional torus \( T^2 \simeq \mathbb{R}^2/L \) by the vector \( \alpha = (\alpha_1, \alpha_2) \), where

\[
L = \mathbb{Z}[l_1, l_2] = L'.
\]

**Reduction in a Cartesian basis.** For any \( x \in \mathbb{R}^2 \), the fractional part \( \text{Fr}^2(x) \) is defined by

\[
\text{Fr}^2(x) = x', \quad \text{where} \quad x' \equiv x \mod L, \quad x' \in T^2.
\]

We rewrite the fractional part \( \text{Fr}^2(x) \) in terms of the fractional parts \( \{ x_1 \}, \{ x_2 \} \). Using (9.13), we can write the period lattice \( L \) as

\[
L = \mathbb{Z}[m_1, m_2], \quad \text{where} \quad m_1 = -l_1 = (1, 0), \quad m_2 = -l_2 = (\alpha_1, 1).
\]

Now consider the basis \( \{ m_1, m_2 \} \) and the new coordinates \( \tilde{x}_1 \) and \( \tilde{x}_2 \):

\[
x = (x_1, x_2) = \tilde{x}_1 m_1 + \tilde{x}_2 m_2. \tag{9.15}
\]

It follows that for the coordinates \( \tilde{x}_1 \) and \( \tilde{x}_2 \) we have

\[
\tilde{x}_1 = x_1 - \alpha_1 x_2, \quad \tilde{x}_2 = x_2. \tag{9.16}
\]

If we replace \( x \) in (9.16) by

\[
\tilde{x}' = \{ \tilde{x}_1 \} m_1 + \{ \tilde{x}_2 \} m_2,
\]

we see that \( \tilde{x}' \equiv x \mod L \), while (9.15) and (9.16) show that

\[
\tilde{x}' = \{ x_1 - \alpha_1 x_2 \} m_1 + \{ x_2 \} m_2 = \{ \{ x_1 - \alpha_1 x_2 \} + \{ x_2 \} \alpha_1, \{ x_2 \} \}.
\]

In order to embed \( \tilde{x}' \) into the unit square \( T^2 \), which is a fundamental domain with respect to the lattice \( L \), we replace \( \tilde{x}' \) additionally by

\[
x' = \{ \{ x_1 - \alpha_1 x_2 \} + \{ x_2 \} \alpha_1, \{ x_2 \} \}.
\]

After this, the first coordinate of \( \tilde{x}' \) could possibly be reduced by 1, i.e.,

\[
x' = \tilde{x}' - (1, 0) = \tilde{x}' - m_1.
\]

Consequently, in any case we have

\[
x' \equiv \tilde{x}' \mod L.
\]

Since \( x' \in T^2 \) by construction, we see that, for any \( x = (x_1, x_2) \in \mathbb{R}^2 \),

\[
\text{Fr}^2(x) = \{ \{ x_1 - \alpha_1 x_2 \} + \{ x_2 \} \alpha_1, \{ x_2 \} \}. \tag{9.17}
\]
which is the reduction formula for the fractional part \( Fr^2(x) \) in the standard basis \( e_1 = (1,0), e_2 = (0,1) \).

Relations (3.6) and (3.7) imply that the counting function \( r_k^2(i) \), where \( k = 0,1,2 \), is equal to the number of \( j \) such that \( 0 \leq j < i \) and, moreover, for \( r_0^2(i) \),

\[
(9.19) \quad \{j(\beta_1 - \alpha_1 \beta_2)\} + \{j\beta_2\} \alpha_1 < 1 - \alpha_1, \quad \{j\beta_2\} < 1 - \alpha_2;
\]

for \( r_1^2(i) \),

\[
(9.20) \quad \{j(\beta_1 - \alpha_1 \beta_2)\} + \{j\beta_2\} \alpha_1 \geq 1 - \alpha_1, \quad \{j\beta_2\} < 1 - \alpha_2;
\]

for \( r_2^2(i) \),

\[
(9.21) \quad \{j\beta_2\} \geq 1 - \alpha_2.
\]

**Reduction in a basis of the period lattice \( L \).** Suppose \( E = \{e_1, e_2\} \) is the standard basis and \( L \) is the lattice (9.15) with a basis \( \{m_1, m_2\} \). The coordinates of \( x \), with respect to the basis \( E \) are related to the coordinates of \( \bar{x} \) in the basis \( \{m_1, m_2\} \) by the formulas

\[
x = (x_1, x_2)_E = (\bar{x}_1, \bar{x}_2)_L, \quad \text{where} \quad \bar{x}_1 = x_1 - \alpha_1 x_2, \quad \bar{x}_2 = x_2,
\]

\[
\alpha = (\alpha_1, \alpha_2)_E = (\bar{\alpha}_1, \bar{\alpha}_2)_L, \quad \text{where} \quad \bar{\alpha}_1 = \alpha_1 - \alpha_1 \alpha_2, \quad \bar{\alpha}_2 = \alpha_2.
\]

Here we write the indices at the coordinates \( (x_1, x_2)_E \) and \( (\bar{x}_1, \bar{x}_2)_L \) in order to indicate the corresponding basis. Since

\[
e_1 = (1,0)_L, \quad e_2 = (-\alpha_1, 1)_L,
\]

the toric development \( T^2 \) transforms into a new development \( \tilde{T}^2 \): 

\[
(9.20) \quad \tilde{T}^2 = \{x \in \mathbb{R}^2; 0 \leq \bar{x}_1 + \alpha_1 \bar{x}_2 < 1, 0 \leq \bar{x}_2 < 1\},
\]

while, by (9.12), the tiles \( T_0^2, T_1^2, T_2^2 \) transform to the tiles

\[
(9.21) \quad \tilde{T}_0^2 = \{x \in \mathbb{R}^2; 0 \leq \bar{x}_1 + \alpha_1 \bar{x}_2 < 1 - \alpha_1, 0 \leq \bar{x}_2 < 1 - \alpha_2\},
\]

\[
\tilde{T}_1^2 = \{x \in \mathbb{R}^2; 1 - \alpha_1 \leq \bar{x}_1 + \alpha_1 \bar{x}_2 < 1, 0 \leq \bar{x}_2 < 1 - \alpha_2\},
\]

\[
\tilde{T}_2^2 = \{x \in \mathbb{R}^2; 1 - \alpha_2 \leq \bar{x}_2 < 1\},
\]

respectively.

**Remark 9.1.** The reductions (9.20)–(9.21) in the basis of the period lattice \( L \) (see (9.15)) is aimed at stating the bounds for the tiles in the two-dimensional case in terms of the usual (one-dimensional) fractional parts \( \{\bar{x}_1\}, \{\bar{x}_2\} \).

Using formula (3.2) for the shift vector \( \beta \) of \( T^2 = \mathbb{R}^2 / L \), we can compute its coordinates

\[
(9.22) \quad \beta = \frac{1}{n}(\alpha + b_1 l_1 + b_2 l_2)
\]

with respect to the same basis (9.15) of the lattice \( L \). We have

\[
\beta = (\tilde{\beta}_1, \tilde{\beta}_2)_L = \tilde{\beta}_1 m_1 + \tilde{\beta}_2 m_2.
\]

Therefore,

\[
\tilde{\beta}_1 = \frac{1}{n}(\bar{\alpha}_1 - b_1), \quad \tilde{\beta}_2 = \frac{1}{n}(\bar{\alpha}_2 - b_2).
\]

In terms of the basis of the period lattice \( L \), the functions \( r_k^2(i) \), \( k = 0,1,2 \), are equal to the number of \( j \)'s such that \( 0 \leq j < i \) and, in accordance with (9.21), for \( r_0^2(i) \),

\[
(9.23) \quad \{j\tilde{\beta}_1\} + \alpha_1 \{j\tilde{\beta}_2\} < 1 - \alpha_1 \quad \text{or} \quad \{j\tilde{\beta}_1\} + \alpha_1 \{j\tilde{\beta}_2\} \geq 1,
\]

\[
\{j\tilde{\beta}_2\} < 1 - \alpha_2;
\]
for $r_1^2(i)$,
\[
\begin{cases}
1 - \alpha_1 \leq \{j\tilde{\beta}_1\} + \alpha_1\{j\tilde{\beta}_2\} < 1, \\
\{j\tilde{\beta}_2\} < 1 - \alpha_2;
\end{cases}
\]
for $r_2^2(i)$,
\[
\{j\tilde{\beta}_2\} \geq 1 - \alpha_2.
\]

Now, using (4.3) and (4.14), we can write the $k$-deviations as follows:
\[
\delta_k^2(i) = r_k^2(i) - ia_k \quad \text{for} \quad k = 0, 1, 2,
\]
where the counting functions $r_k^2(i)$ are as in (9.19) or, equivalently, in (9.23): $a_k = -l^*_k \cdot \alpha$ for $\alpha = (\alpha_1, \alpha_2)$. From (9.13) it follows that $l_1 = (-1, 0)$, $l_2 = (-\alpha_1, -1)$, whence
\[
l_1^* = (-1, \alpha_1), \quad l_2^* = (0, -1), \quad l_0^* = -l_1^* - l_2^* = (1, 1 - \alpha_1).
\]

Therefore,
\[
a_1 = \alpha_1 - \alpha_1 \alpha_2 = \tilde{\alpha}_1, \quad a_2 = \alpha_2 = \tilde{\alpha}_2,
\]
and (4.8) allows us to get the following identity for the coefficient $a_0$:
\[
a_0 = 1 - \tilde{\alpha}_1 - \tilde{\alpha}_2 = (1 - \alpha_1)(1 - \alpha_2).
\]

In order to obtain approximate bounds for the $k$-deviations $\delta_k^2(i)$, we apply Theorem 5.1. We have
\[
|\delta_k^2(i)| \leq c_k^2 n \quad \text{for} \quad k = 0, 1, 2,
\]
where
\[
c_k^2 = |l_k^*| (\overline{pr}_{l_k^*} T^2 - \overline{pr}_{l_k^*} T^2).
\]
Since in our case the toric development $T^2$ is the unit square $I^2 = [0, 1) \times [0, 1)$, we can use (9.24) to obtain the following explicit expressions for the constants (9.26):
\[
c_0 = 2 - \alpha_1, \quad c_1 = 1 + \alpha_1, \quad c_2 = 1.
\]
Combining this with (9.25) shows that in the two-dimensional case we have the following approximate estimates for the $k$-deviations:
\[
|\delta_0^2(i)| \leq (2 - \alpha_1)n, \quad |\delta_1^2(i)| \leq (1 + \alpha_1)n, \quad |\delta_2^2(i)| \leq n
\]
for all $i = 0, 1, 2, \ldots$.

Remark 9.2.
1. The last estimate in (9.27) does not depend on the choice of the shift vector $\alpha = (\alpha_1, \alpha_2)$; again, it is basically the same as Hecke’s estimate [3].
2. The first two estimates in (9.27) do not depend on the choice of the coordinate $\alpha_2$ of the shift vector $\alpha = (\alpha_1, \alpha_2)$ of $T^2 = \mathbb{R}^2 / L$.
3. The second estimate in (9.27) corresponds to the deviation $\delta_2^2(i)$ for the two-dimensional tile $\tilde{T}_2^2$ that is a skew-angle parallelogram, see the definition (9.21). Some particular cases of these parallelograms were previously found by Szüsz [7].
9.4. The two-dimensional case: \( n = 1 \). We use the development of \( \tilde{T}^2 \) and the results of \( \S 6 \) to obtain sharp bounds for the \( k \)-deviations \( \delta_k^2(i) \) in the case where for the shift vector \( \beta \) as in (9.22) we have \( n = 1 \).

The passage to the coordinates with respect to the basis \( \{m, m_2\} \) of the period lattice \( L = \mathbb{Z}[m_1, m_2] \) is equivalent to the passage to the square lattice \( \mathbb{Z}^2 = \mathbb{Z}[e_1, e_2] \). Under the map \( m_1 \mapsto e_1 \), \( m_2 \mapsto e_2 \), the vectors \( l_k, k = 0, 1, 2 \), are taken to the vectors

\[
\begin{align*}
    l_1 & \mapsto -e_1, \\
    l_2 & \mapsto -e_2, \\
    l_0 & = -l_1 - l_2 \mapsto e_1 + e_2,
\end{align*}
\]

respectively, and by (4.4) and (4.13), the dual vectors \( l_k^* \), \( k = 0, 1, 2 \), are taken to

\[
\begin{align*}
    l_1^* & \mapsto -e_1, \\
    l_2^* & \mapsto -e_2, \\
    l_0^* & \mapsto e_1 + e_2.
\end{align*}
\]

From (9.20) it follows that the development of the torus \( \tilde{T}^2 \) is a parallelogram. Hence, in order to find extrema of the projections \( \text{pr}_{l_k^*} x, x \in \tilde{T}^2 \), it suffices to consider only the vertices \( x = v \) of \( V(\tilde{T}^2) \). Looking through the vertices in \( V(\tilde{T}^2) \), we get the projections

\[
\begin{align*}
    \{\text{pr}_{v_1 + v_2}; v \in V(\tilde{T}^0)\} & = \{0, 1, 2 - \alpha_1, 1 - \alpha_1\}, \\
    \{\text{pr}_{-v_1}; v \in V(\tilde{T}_1^2)\} & = \{0, -1, \alpha_1 - 1, \alpha_1\}, \\
    \{\text{pr}_{-v_2}; v \in V(\tilde{T}_2^2)\} & = \{0, -1\}.
\end{align*}
\]

Thus, by (9.28) and Proposition 6.1, for \( n = 1 \) the \( k \)-deviations \( \delta_k^2(i) \) satisfy

\[
0 \leq \delta_0^2(i) \leq 2 - \alpha_1, \quad -1 \leq \delta_1^2(i) \leq \alpha_1, \quad -1 \leq \delta_2^2(i) \leq 0.
\]

Remark 9.3. Comparing (9.29) and (9.27), we see that the bounds in (9.29) for \( \delta_2^2(i) \), \( \delta_1^2(i) \) depend only on the first coordinate of the shift vector \( \alpha = (\alpha_1, \alpha_2) \), while the bounds for \( \delta_0^2(i) \) do not depend on \( \alpha \).

Mean values of deviations. Theorem 8.1 and relation (8.17) show that the total vector deviation \( \delta^2(i) \) has the mean value

\[
\langle \delta^2 \rangle = c_T = \left( \frac{1}{2}, \frac{1}{2} \right).
\]

It is easily seen that the mean value (9.30) is in fact the barycenter of the unit square \( T^2 = I^2 \). In the same way, we can use (8.18) to compute the mean values of the \( k \)-deviations \( \delta_k^2(i) \):

\[
\langle \delta_k^2 \rangle = l_k^* \cdot c_T \quad \text{for} \quad k = 0, 1, 2.
\]

Using (9.30), we find that the mean values in (9.31) can explicitly be written as

\[
\begin{align*}
    \langle \delta_0^2 \rangle & = 1 - \frac{\alpha_1}{2}, \quad \langle \delta_1^2 \rangle = -\frac{1}{2} + \frac{\alpha_1}{2}, \quad \langle \delta_2^2 \rangle = -\frac{1}{2}.
\end{align*}
\]

The right-hand sides in (9.32) coincide with the midpoints of the intervals for the sharp bounds in (9.29).

References


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Received 20/DEC/ 2010

Translated by A. LUZGAREV