SOME REMARKS TO THE CORONA THEOREM

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Abstract. With the help of a fixed point theorem, in §1 it is shown that the so-called $L^\infty$- and $L^p$-corona problems are equivalent in the general situation. This equivalence extends to the case where $L^p$ is replaced by a more or less arbitrary Banach lattice of measurable functions on the circle. In §2, the corona theorem for $\ell^2$-valued analytic functions is exploited to give a new proof for the existence of an analytic partition of unity subordinate to a weight with logarithm in BMO. In §3, simple observations are presented that make it possible to pass from one sequence space to another in $L^\infty$-estimates for solutions of corona problems.

§1. CORONA PROBLEM FOR LATTICES OF MEASURABLE FUNCTIONS ON THE CIRCLE

1.1. Let $\{f_j\}$ be a sequence (finite or infinite) of bounded analytic functions on the disk $D = \{z \in \mathbb{C} : |z| < 1\}$. The classical corona problem consists in finding bounded functions $\{g_j\}$ analytic in the disk and such that

$$\sum_j f_j g_j = 1.$$  

Clearly, such functions $g_j$ may fail to exist (and if they exist, surely they are mostly not unique), so in reality metric conditions for the solvability of equation (1) are discussed, and with them, also estimates for possible solutions. For example, if we look for a solution satisfying $\sup_{z \in D} \left( \sum |g_j(z)|^2 \right)^{1/2} \leq C$, then the Cauchy inequality immediately implies a necessary solvability condition:

$$\sum_j |f_j(z)|^2 \geq \delta^2, \quad z \in D,$$

where $\delta = C^{-1}$.

Not dwelling on the well-known history of the problem (see [1, Appendix 3]; we only recall the outstanding role of L. Carleson in this range of questions), we mention Wolff’s method invented around 1980 and then improved by Gamelin, M. Rosenblum, and Tolokonnikov. At that time, this method made it possible to prove that, under conditions (2), there exist analytic functions $\{g_j\}$ satisfying (1) and such that

$$\sup_{z \in D} \left( \sum_j |g_j(z)|^2 \right)^{1/2} \leq \frac{1}{\delta} + \frac{1}{\delta^2} \left[ 7 \sqrt{\log \frac{1}{\delta} - 20 \log \delta} \right], \quad 0 < \delta < 1,$$

(this is an estimate of order $O(\delta^{-2} \log \frac{1}{\delta})$ as $\delta \to 0$); see the monograph [1] cited above. Much later, the method was further improved in [2]. In particular, the existence of a solution satisfying

$$\sup_{z \in D} \left( \sum_j |g_j(z)|^2 \right)^{1/2} \leq \frac{C_0}{\delta^2} \log \frac{1}{\delta} + \frac{1}{\delta^2},$$

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was shown there; here \( C_0 = \sqrt{1 + e^2 + \sqrt{e} + e\sqrt{2}} \approx 8.38934 \). However, this estimate is of the same order as (3) for \( \delta \to 0 \). Another feature is more important for us: in [2] the so-called \( L^2 \)-corona problem was solved first (in [2] and elsewhere it was called the \( H^2 \)-problem, but we find it appropriate to refer to the metric alone rather than to the analyticity of functions, which is assumed in any case), and then the argument was modified to yield a solution of (1) in bounded functions.

The general \( L^p \)-corona problem is formulated as follows. Let \( A \) and \( B \) be two Banach spaces, assumed finite-dimensional for the time being. (The infinite-dimensional case requires a lengthy discussion, see below.) By the data of a corona problem, we mean a bounded function analytic in the disk and taking values in the space \( \mathcal{L}(A, B) \) of operators (in symbols: \( F \in H^\infty(\mathcal{L}(A, B)) \)). Let \( 1 \leq p \leq \infty \). We say that the \( L^p \)-corona problem with data \( F \) is solvable with constant \( C \) if for every function \( x \in H^p(B) \) there exists a function \( y \in H^p(A) \) such that

\[
F(z)y(z) = x(z) \quad \text{for} \quad z \in \mathbb{D} \quad \text{and} \quad \|y\|_{H^p(A)} \leq C\|x\|_{H^p(B)}.
\]

Problem (1) under condition (2) (for the moment, we restrict ourselves to a finite number of functions \( \{f_j\} \) is of this form: put \( p = \infty \), \( A = l^2_n \), \( B = \mathbb{C} \). Then \( \mathcal{L}(A, B) = l^2_n \), so that for every \( z \) the sequence \( f_1(z), \ldots, f_n(z) \) can be viewed as a linear operator from \( A \) to \( B \). Yet in (1) we have the function identically equal to 1 on the right, whereas in the \( L^\infty \)-corona problem an arbitrary \( H^\infty \)-function may stand there. However, it is clear that if we can solve equation (1), we can solve the \( L^p \)-problem with every \( p \).

Passage in the opposite direction (from \( p \) to \( \infty \)) is not as easy in these questions. See [1, 2]: in [1] the information about the so-called “Töplitz corona theorem” should be consulted; that statement is equivalent to the \( L^2 \)-corona theorem in the classical case where \( A = B = l^2_n \) and \( F(z) = \text{diag}\{f_j(z)\} \) is a diagonal matrix. We shall show that, in fact, the \( L^p \)-corona problems for different \( p \) are all equivalent in the most general setting. We start with a very special case of what shall be proved.

**Proposition 1.** Let \( A, B \) be finite-dimensional spaces, and let \( F \in H^\infty(\mathcal{L}(A, B)) \). If the \( L^p \)-corona problem with data \( F \) is solvable with constant \( C \) for one value of \( p \), \( 1 \leq p \leq \infty \), then it is solvable for every \( p \) with the same constant.

The difficult point in the proof is passage from \( p = 1 \) to \( p = \infty \). The obstructions are overcome with the help of a fixed point theorem for multivalued mappings.

1.2. For obvious reasons, we would like to get rid of the requirement that \( A \) and \( B \) be finite-dimensional in the setting of the \( L^p \)-corona problem. However, if \( V \) is an infinite-dimensional space, there are various versions of the definition of \( H^p(V) \). For us, a precise analog of the classical definition is natural, but then the boundary values should be understood in a weak sense. It should be noted that, if \( V \) is finite-dimensional, all complications disappear and, in essence, the theory does not differ from the scalar case. It should also be noted that, in practice, infinite-dimensional results are often reduced to finite-dimensional settings by a limit passage. However, such a limit passage is not always possible.

**Definition.** For a Banach space \( V \), we denote by \( H^p(\mathbb{D}, V) \) (\( 1 \leq p \leq \infty \)) the space of \( V \)-valued functions \( F \) analytic in \( \mathbb{D} \) and such that

\[
\|F\|_{H^p(\mathbb{D}, V)} \overset{\text{def}}{=} \sup_{0 < r < 1} \left( \frac{1}{T} \int_\mathbb{T} \|F(r\zeta)\|^p_V \, dm(\zeta) \right)^{1/p} < \infty.
\]

Here and below \( m \) denotes the normalized Lebesgue measure on the unit circle \( \mathbb{T} \).
1.2.1. The case where $V$ is an ideal space of sequences. Apparently, this case suffices for the most part of applications. At the same time, it is simpler than the general case described below.

An ideal space of sequences is a Banach space $V$ of functions on the set $\mathbb{N}$ of natural numbers with the following property: if $x \in V$ and $|y(n)| \leq |x(n)|$ for all $n$, then $y \in V$ and $\|y\| \leq \|x\|$.

Thus, every function $F$ in $H^p(\mathbb{D}; V)$ can be identified with a sequence $\{f_n\}$ of scalar functions analytic in $\mathbb{D}$. Every $f_n$ belongs to the usual Hardy class $H^p(\mathbb{D})$ and, therefore, possesses boundary values $\varphi_n(\zeta) = \lim_{r \rightarrow 1} f_n(r\zeta)$ for almost every $\zeta \in \mathbb{T}$. Generally speaking, the values of the vector-valued function $\Phi = \{\varphi_n\}_{n \in \mathbb{N}}$ may fall out of $V$. However, they do not if there exists another ideal space $G$ such that $V = G^*$ with respect to the standard bilinear form $(\xi, \eta) = \sum_{n \in \mathbb{N}} \xi_n \eta_n$. As in the scalar case, the decisive argument is that $\Phi$ must coincide with (an arbitrary) $w^*$-limit point as $r \rightarrow 1$ for the functions $F_r, F_0(\zeta) = F(r\zeta), 0 \leq r < 1, \zeta \in \mathbb{T}$. If $1 < p \leq \infty$, this limit point $\Psi$ is taken in $L^p(\mathbb{T}; G)^*$, and if $p = 1$, it is taken in $C(\mathbb{T}, G)^*$ and then the theorem of the Riesz brothers is invoked. Specifically, for $p = 1$ the limit point $\Psi$ corresponds to a sequence $\{\mu_n\}$ on the circle. Clearly, the Fourier coefficients with negative indices vanish for each of these measures, so all of them are absolutely continuous with respect to Lebesgue measure.

It should be noted that by $L^r(\mathbb{T}, G)$ we mean the Banach space of strongly measurable $G$-valued functions whose $G$-norm, evaluated pointwise, is integrable in the power $r$. For $r < \infty$, the space $L^r(\mathbb{T}, G)^*$ can easily be described: it consists of $G^*$-valued functions $H = \{h_n\}_{n \in \mathbb{N}}$ on the unit circle such that all $h_n$ are measurable and the function $\zeta \mapsto ||H(\zeta)||_{G^*}, \zeta \in \mathbb{T}$, belongs to $L^r$. It is important to emphasize that, in general, elements of the conjugate space may fail to correspond to any strongly measurable $G^*$-valued function.

We shall identify $H^p(\mathbb{D}; G^*)$ with the space of boundary functions to be denoted by $H^p(\mathbb{T}, G^*)$. As usual, we do not distinguish notationally between a function and its boundary values. In general, convergence to boundary values a.e. in the norm of $G$ may fail to occur (though, componentwise, we have convergence a.e.): it suffices to consider the case of $G = l^1, G^* = l^\infty$.

The argument indicated above and involving the (scalar) theorem of Riesz brothers leads to the following statement. Some details of the proof will be given below in a more general setting (see Lemma 2).

**Lemma 1.** Suppose $G$ is an ideal space of sequences and its conjugate is also an ideal space of sequences. Then $H^1(\mathbb{T}, G^*)$ is a $w^*$-closed subspace of $C(\mathbb{T}, G)^*$. Convergence of bounded (generalized) sequences in the $w^*$-topology of this subspace is equivalent to the componentwise and pointwise convergence in the disk of the corresponding functions in $H^1(\mathbb{D}; G^*)$.

The notion of an ideal space of sequences is a particular case of the notion of a Banach lattice of measurable functions (or a Köthe space). Let $(\Omega, \Sigma, \nu)$ be a space with $\sigma$-finite measure, and let $X$ be a Banach space of measurable functions on $\Omega$ that has the following property: if $f \in X, g$ is measurable and $|g| \leq |f|$ a.e., then $g \in X$ and $\|g\| \leq \|f\|$. Then $X$ is called a lattice of measurable functions on the above measure space.

If the measure space is the circle with Lebesgue measure, we can distinguish the “analytic” subspace $X_A = X \cap N_+$ in a lattice $X$ of measurable functions; here $N_+$ is the boundary Smirnov class (see [3] and, especially, [4] for elementary properties of these
subspaces). To avoid degeneration, we impose the following condition on \( X \):

\[
\text{if } f \in X, \quad f \neq 0, \quad \text{then there exists}
\]

\[
g \in X, \quad g \geq |f|, \quad \text{with } \log g \in L^1(\mathbb{T}) \quad \text{and} \quad \|g\|_X \leq C\|f\|_X.
\]

In [3] it was shown that we always can take any number greater than 1 for the role of \( C \).

Besides (\( * \)), we usually impose the Fatou condition on \( X \): if \( f_n \in X \), \( f_n \to f \) a.e., and \( \sup_n \|f_n\| = C < \infty \), then \( f \in X \) and \( \|f\| \leq C \). This is equivalent to the relation \( X'' = X \), where \( X' \) is the set of all functionals \( \varphi \) on \( X \) that admit integral representation \( \varphi(x) = \int xy \) with some function \( y \). See, e.g., [5]. The space \( X' \) is called the order dual of \( X \). Finally, let \( X \) be a lattice of measurable functions on the circle that satisfies (\( * \)) and the Fatou condition, and let \( V \) be an ideal space of sequences, which is also subject to the Fatou condition. It is easily seen that for every sequence \( \{f_n\} \) of measurable functions on \( \mathbb{T} \), the function \( t \mapsto \|(f_1(t), f_2(t), \ldots)\|_V \) is measurable if it is finite a.e. Consequently, we can introduce the lattice \( X(V) \) of measurable functions \( F(t,n) = \{f_n(t)\}, \quad t \in \mathbb{T}, \quad n \in \mathbb{N}, \) on the space \( (\mathbb{T} \times \mathbb{N}, m \times \nu) \) (\( \nu \) is the counting measure) in which

\[
\|F\|_{X(V)} \overset{\text{def}}{=} \|\alpha\|_X < \infty, \quad \text{where } \alpha(t) = \|\{f_n(t)\}_{n \in \mathbb{N}}\|_V.
\]

We put \( X_A(V) = \{F : f_n \in X_A \text{ for all } n\} \).

It can easily be seen that if \( X = L^p \) and \( V = G^* \) for some ideal space \( G \) (by the way, then \( V \) does satisfy the Fatou condition), then we obtain the class \( H^p(\mathbb{T}; G^*) \) discussed above. However, if \( X \) is an arbitrary lattice, the metric condition that distinguishes \( X_A(G^*) \) can only be given in terms of boundary values. No description in the spirit of formula (5) may happen to be available.

Below we shall see that in statements like Proposition 1 it is natural to talk about the \( X \)-corona problem with an arbitrary lattice \( X \) rather than about the \( L^p \)-corona problem.

### 1.2.2. The space \( X_A(G^*) \) for an arbitrary Banach space \( G \)

We start with discussing the boundary values of functionals belonging to the class \( H^1(\mathbb{D}, G^*) \) (see (5)). They should be understood in the sense of the weak topology (note that we used the same framework in the preceding item). The situation was clarified as early as in [6] (see also [7]; note that the paper [8] contains a revision of the work presented in [6] and [7] from a somewhat more general viewpoint; references to other papers on the same subject and published roughly at the same time period can also be found in [8]). We describe the results in more detail.

The next statement is well known and, apparently, was reproved many times in different generality (see, e.g., the references in [6], which range from 1936 to 1959).

**Proposition 1.** Let \( G \) be a Banach space and \( K \) a compact set. Then the dual \( C(K,G)^* \) can be identified linearly and isometrically with the space \( M(K,G)^* \) of all \( G^* \)-valued countably additive regular Borel measures of bounded variation on \( K \).

From a modern viewpoint, this statement can be deduced from general duality theorems for operator spaces, without a repetition of the proof of the Riesz representation theorem for the space of scalar functions. The arguments can even be presented in a way involving duality of operator spaces only implicitly. Specifically, the fact that, for any \( \lambda \in M(K,G^*) \), the norm of the corresponding functional on \( C(K,G) \) is equal to the total variation of \( \lambda \) is proved with the help of the same easy procedure as in the scalar case. The difficult part is to find a representing measure for an arbitrary functional \( \Phi \in C(K,G)^* \). But such a functional generates a linear operator

\[
T_\Phi : C(K) \to G^*, \quad (T_\Phi(f))(x) = \Phi(f \cdot x), \quad f \in C(K), \quad x \in G.
\]
If \( f_1, \ldots, f_n \in C(K) \), \( \sum_{j=1}^{n} |f_j| \leq 1 \), and \( x_1, \ldots, x_n \) are elements of the unit ball of \( G \), then for every complex numbers \( \zeta_1, \ldots, \zeta_n \in \mathbb{T} \) we have

\[
\left| \sum_{j=1}^{n} \zeta_j T_{\Phi}(f_j)(x_j) \right| = \left| \Phi \left( \sum_{j=1}^{n} \zeta_j f_j \circ x_j \right) \right| \leq \left\| \Phi \right\| \left\| \sum_{j=1}^{n} \zeta_j f_j x_j \right\|_{C(K,G)} \leq \left\| \Phi \right\|, 
\]

whence we obtain \( \sum_{j=1}^{n} \|T_{\Phi}(f_j)\| \leq \|\Phi\| \). Consequently, \( T_{\Phi} \) is an absolutely summing operator with constant \( C \). By the Pietsch theorem (see, e.g., [13]), it admits the factorization

\[
C(K) \xrightarrow{\text{id}} L^1(\mu) \xrightarrow{\|\cdot\|} G^*,
\]

where \( \mu \) is a (regular Borel) probability measure on \( K \) and \( U \) is a bounded linear operator, \( \|U\| \leq \|\Phi\| \). The measure \( \lambda \) representing \( \Phi \) is given by \( \lambda(e) = U(\chi_e) \) for every Borel subset \( e \) of \( K \).

Now, let \( F \in H^1(\mathbb{D}; G^*) \). We put \( F_r(\zeta) = F(r\zeta), \zeta \in \mathbb{T}, 0 \leq r < 1 \), and consider a \( w^* \)-limit point \( \lambda \) for the family of measures \( F_r \) dm in \( M(\mathbb{T}; G^*) \). The Fourier coefficients of \( \lambda \) with negative indices vanish (the integrals defining the Fourier coefficients and, more generally, the integrals \( \int \varphi \, d\lambda \), where \( \varphi \) is a scalar function, are understood in the Gelfand sense: we mean the element \( x \mapsto \int \varphi \, d(x, \lambda), x \in G, \) of \( G^* \)). By the main result of [9], there exists a \( G \)-measurable \( G^* \)-valued function \( f \) such that \( \lambda(e) = \int_e f \, dm \) for all Borel sets \( e \subset \mathbb{T} \) (again, the integral is understood in a weak sense). Next, the scalar-valued function \( t \mapsto \|f(t)\|_{G^*} \) may fail to be measurable. However, the family of scalar functions

\[
\{ t \mapsto |\langle x, f(t) \rangle | : x \in G, \|x\| \leq 1 \}
\]

has supremum in the lattice \( S(\mathbb{T}) \) of all measurable functions on \( \mathbb{T} \) (this is true for arbitrary \( G \)-measurable \( G^* \)-valued functions \( f \), not only for those arising in the above construction). Denote this supremum by \( \alpha_F \). It turns out that \( \|F\|_{H^1(\mathbb{D}; G^*)} = \|\alpha_F\|_{L^1(\mathbb{T})} \). See the discussion and references in [6] about these facts.

It should be noted that if \( G \) is separable, then the function \( t \mapsto \|f(t)\|_{G^*} \) is measurable; then it coincides with \( \alpha_F \) a.e.

We identify \( H^1(\mathbb{D}, G^*) \) with the space \( H^1(\mathbb{T}, G^*) \) of all boundary functions \( f \) described above. The following generalization of Lemma 1 holds.

**Lemma 2.** Let \( G \) be a Banach space. Then \( H^1(\mathbb{T}, G^*) \) is a \( w^* \)-closed subspace of \( M(\mathbb{T}, G^*) = C(\mathbb{T}, G^*) \). Consequently, the unit ball of \( H^1(\mathbb{T}, G^*) \) is \( w^* \)-compact. Convergence of (generalized) sequences in this ball in the topology \( \sigma(M(\mathbb{T}, G^*), C(\mathbb{T}, G)) \) is equivalent to the pointwise \( w^* \)-convergence in the disk for the corresponding \( G^* \)-valued analytic functions. If \( G \) is separable, then the \( w^* \)-topology on the unit ball of \( H^1(\mathbb{T}, G^*) \) is metrizable. A sequence \( f_n \) in this ball converges (in the \( w^* \)-topology) to a function \( f \in H^1(\mathbb{T}, G^*) \) if and only if, for the corresponding analytic functions \( F_n \) and \( F \), the sequence \( \langle x, F_n(\cdot) \rangle \) converges to \( \langle x, F(\cdot) \rangle \) uniformly on compact subsets of \( \mathbb{D} \) for every \( x \in G \).

**Proof.** Maybe, only the last assertion (about the meaning of convergence in case \( G \) is separable) needs to be explained. In this case, the closed balls of \( G^* \) are metrizable (and, of course, compact) in the topology \( \sigma(G^*, G) \). Repeating the proof of the Montel theorem with small variations (clearly, we may restrict ourselves to usual rather than generalized sequences in this case), we see that the ball of \( H^1(\mathbb{D}, G^*) \) is compact in the topology of “uniform \( \sigma(G^*, G) \)-convergence on compact subsets of \( \mathbb{D} \)” (the meaning of this term is clear from the last part of the statement). This topology is not weaker than the topology of pointwise \( \sigma(G^*, G) \)-convergence; therefore it is equivalent to the latter. \( \square \)

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1 Functions coinciding a.e. are identified, so that in fact we deal with the lattice of classes of equivalent measurable functions.
Finally, let $X$ be a Banach lattice of measurable functions on the circle satisfying (\ast) and the Fatou condition, and let $G$ be a Banach space. We want to define the space $X_A(\mathbb{D}, G^*)$. This can be done quickly in terms of $H^1(\mathbb{D}, G)$. Specifically, let $X'$ be the order dual of $X$ ($X'$ satisfies (\ast) and the Fatou condition together with $X$, see [5] and [6]). By definition, a $G^*$-valued function $F$ analytic in $\mathbb{D}$ belongs to $X_A(\mathbb{D}, G^*)$ if $YF \in H^1(\mathbb{T}, G^*)$ for every $y \in X'_A$, where $Y$ denotes the analytic function in the Smirnov class whose boundary values coincide with $y$. Next,

\[(6) \quad \|F\|_{X_A(\mathbb{D}; G^*)} \overset{\text{def}}{=} \sup \{ \|YF\|_{H^1(\mathbb{D}; G^*)} : y \in X', \|y\|_{X'} \leq 1 \}.\]

For future use, we need to discuss the boundary values of functions in $X_A(\mathbb{D}; G^*)$. We shall see that they can be viewed as $G^*$-valued $G$-measurable functions if we agree not to distinguish scalar-equivalent functions. (In fact, this identification is required also in the treatment of $H^p(\mathbb{D}, G^*)$, but we did not attract the reader’s attention to that up to this point.) Two functions $f_1, f_2 : \mathbb{T} \to G^*$ are said to be $G$-equivalent if $\langle g, f_1(\cdot) \rangle = \langle g, f_2(\cdot) \rangle$ a.e. for every $g \in G$.

It is easily seen that, in the definition of $X_A(\mathbb{D}, G^*)$ and in formula (6), we can restrict ourselves to functions $y$ for which $Y$ is outer, i.e., $\log |Y|$ is the Poisson integral of $\log |y|$. Let $\Phi_Y$ be the boundary function for $YF \in H^1(\mathbb{D}, G^*)$. Since the function $\Phi_Y$ is $G$-measurable, the function $f_Y = y^{-1}\Phi_Y$ is also $G$-measurable. On the other hand, $\langle x, F(r\zeta) \rangle \to \langle x, f_Y(\zeta) \rangle$ a.e. for every $x \in G$. Indeed, this follows from the relations $Y(r\zeta) \to y(\zeta)$ a.e. and $\langle x, Y(r\zeta)F(r\zeta) \rangle \to \langle x, \Phi_Y(\zeta) \rangle$ a.e. Thus, the functions $f_Y$ are scalar-equivalent for different $Y$. Consequently, the function

$$\alpha_F \overset{\text{def}}{=} \sup_{x \in G, \|x\| \leq 1} |\langle x, f_Y \rangle|$$

(the supremum is taken in the lattice $S(\mathbb{T})$) is also independent of $Y$. The above discussion of the space $H^1(\mathbb{D}, G^*)$ easily shows that $\alpha_F \in L^1(\mathbb{T})$ for every function $y \in X'$ with integrable logarithm, moreover, we have an appropriate norm estimate. Consequently, $\|\alpha_F\|_X \leq \|F\|_{X_A(\mathbb{D}; G^*)}$ by condition (\ast) for $X'$ (we have used also the fact that $X'$ norms $X$ whenever $X$ satisfies the Fatou condition, see [5]). Next, it is quite easy to understand that $\log \alpha_F \in L^1(\mathbb{T})$ if $f \neq 0$, and therefore we can construct a (scalar) outer function $\Psi_F$ with modulus equal to $\alpha_F$ on the boundary.

**Lemma 3.** If $F$ is in $X_A(\mathbb{D}; G^*)$, then $F\Psi_F^{-1} \in H^\infty(\mathbb{D}; G^*)$ and $\|F\Psi_F^{-1}\|_{H^\infty(\mathbb{D}; G^*)} \leq 1$.

**Proof.** Indeed, this is a consequence of the boundary maximum of the modulus principle. Let $y \in X'_A$ be such that the corresponding function $Y$ on the disk is outer. Then

$$|\langle x, y(\zeta)f(\zeta) \rangle| \leq |y(\zeta)|\alpha_F(\zeta) \quad \text{a.e. for } |\zeta| = 1,$$

where $f$ is some $G$-measurable function representing the boundary values for $F$ and $x \in G$, $\|x\| \leq 1$. The two sides of this inequality involve the moduli of boundary values of functions in $H^1$; moreover, the function for the right-hand side is equal to $Y\Psi_F$ and is outer. \hfill \Box

If $G$ and $G^*$ are ideal spaces of sequences, then the spaces introduced in this subsection coincide with those described in Subsection 1.2.1.

1.3. Now, finally, we can state the main result of this section in full generality. Let $G_1$ and $G_2$ be Banach spaces. By the data of a corona problem, we mean an analytic function $F \in H^\infty(\mathbb{D}; L(G_1^*, G_2^*))$, whose values in the disk are $w^*$-continuous linear operators from $G_1^*$ to $G_2^*$ (in other words, each operator is adjoint to some operator from $G_2$ to $G_1$). Let $X$ be a lattice of measurable functions on the circle satisfying (\ast) and the Fatou
Proof. Statement (c) is an easy consequence of Lemma 3. Indeed, let it is not solvable for any $C$ if for every function condition. We say that the $X$-corona problem with the data $F$ is solvable with constant $C$ if for every function $\tau \in X_A(\mathbb{D}; G_2^*)$ there exists $\rho \in X_A(\mathbb{D}; G_1^*)$ such that

$$F(z)\rho(z) = \tau(z), \quad z \in \mathbb{D}, \quad \text{and} \quad \|\rho\|_{X_A(\mathbb{D}; G_1^*)} \leq C\|\tau\|_{X_A(\mathbb{D}; G_2^*)}. \quad (7)$$

**Theorem 1.**

(a) If the $X$-corona problem with data $F$ is solvable with constant $C$, then the $L^1$-corona problem with the same data is solvable with constant $C$.

(b) If the $L^1$-corona problem with data $F$ is solvable with constant $C$ and $G_1$ is separable, then the $L^\infty$-corona problem with the same data is solvable with constant $C$.

(c) If the $L^\infty$-corona problem with data $F$ is solvable with constant $C$, then the $X$-corona problem with the same data is solvable with constant $C$.

Omitting not quite important details, we may formulate the theorem as follows: if $G_1$ is separable, then, for fixed data $F$, either the $X$-corona problem is solvable for all $X$ or it is not solvable for any $X$.

**Proof.** Statement (c) is an easy consequence of Lemma 3. Indeed, let $\tau \in X_A(\mathbb{D}; G_2^*)$, then the function $\tau\Psi_\tau^{-1}$ belongs to $H^\infty(\mathbb{D}; G_2^*)$. Solving the $L^\infty$-corona problem for this function, we find $\rho \in H^\infty(\mathbb{D}; G_1^*)$ with $F(z)\rho(z) = \tau(z)\Psi_\tau^{-1}(z)$ for $z \in \mathbb{D}$. Clearly, $\rho \cdot \Psi_\tau$ solves the $X$-corona problem for the right-hand side $\tau$.

Statement (a) is only slightly more complicated. Let $\tau \in H^1(\mathbb{D}; G_2^*)$. We want to solve the corona problem (7) with this function as the right-hand side. If $\tau = 0$, then $\rho = 0$ is a solution. If $\tau \neq 0$, we consider the boundary values for $\tau$ and the corresponding function $\alpha_\tau$. (See the discussion of vector classes $H^1$ in Subsection 1.2.2.) So, $\alpha_\tau \in L^1(\mathbb{T})$ and $\|\alpha_\tau\|_{L^1} = \|\tau\|_{H^1(\mathbb{D}; G_2^*)}$. Next, clearly, $\log \alpha_\tau \in L^1(\mathbb{T})$. By the Lozanovskii factorization theorem (see [9]), we have $\alpha_\tau = \psi_0$ for some nonnegative $u \in X$, $v_0 \in X'$; moreover, we may assume that $\|u\|_{X'}\|v_0\|_{X'} \leq (1 + \varepsilon)\|\alpha_\tau\|_{L^1}$, where $\varepsilon$ is an arbitrary positive number fixed beforehand. We have already mentioned that $X'$ satisfies $(\ast)$ because $X$ does; see [3, 4]. Therefore, there exists $v \in X'$ such that the corresponding function $V$ is outer and $|v| \geq |v_0|$, $\|v\|_{X'} \leq (1 + \varepsilon)\|v_0\|_{X'}$. We put $\varphi = \tau V^{-1}$. Since $\alpha_\varphi = |\alpha_\tau\tau^{-1}| \leq u$, it is easily seen that $\varphi \in X_A(\mathbb{D}; G_2^*)$. Solving the $X$-corona problem with the right-hand side $\varphi$, we find a function $\psi \in X_A(\mathbb{D}; G_1^*)$ with $F\psi = \varphi$ and $\|\psi\|_{X_A(\mathbb{D}; G_1^*)} \leq C\|\varphi\|_{X_A(\mathbb{D}; G_2^*)}$. But then $\rho = V\psi$ solves the $L^1$-problem with the right-hand side $\tau$ and constant $C$. \qed

To prove statement (b), we need a fixed-point theorem for multivalued mappings.

**Theorem** (Fan Ky–Kakutani; see [10]). Let $B$ be a compact convex set in a Hausdorff locally convex space, and let $T : B \to 2^B$ be a mapping such that all sets $T(b)$, $b \in B$, are nonempty, convex, closed, and the graph $\{(b, d) : d \in T(b), \ b \in B\}$ of $T$ is closed in $B \times B$. Then there exists $b \in B$ such that $b \in T(b)$.

In our setting, the unit ball of $H^1(\mathbb{T}, G_1^*)$ with $w^*$-topology (see Lemma 2) will be taken for $B$.

Let $\tau \in H^\infty(\mathbb{D}; G_2^*)$. Next, let $e$ be an arbitrary finite subset of the unit sphere of $G_1$. It suffices to prove the following lemma.

**Lemma 4.** For every $\delta > 0$ there is a function $\rho \in C(\delta) \cdot B$ such that $F(z)\rho(z) = \tau(z)$ and $|\langle x, \rho(z) \rangle| \leq (1 + \delta)C$ for all $z \in \mathbb{D}$ and $x \in e$.

Indeed, suppose that the lemma is proved. We denote the set of functions $\rho$ described in it by $A_\delta$ ($\delta > 0$ is fixed for the moment). The set $A_\delta$ is $w^*$-closed in the ball $C(\delta) \cdot B$ and, consequently, $w^*$-compact. By Lemma 4, it is nonempty. If $e_1 \supset e_2$, then $A_{e_1} \subset A_{e_2}$,
so the sets $A_e$ form a nesting family. Every point $\rho$ of the (nonempty, by compactness) intersection of the $A_e$ has the property

$$|(x, \rho(z))| \leq (1 + \delta)C, \quad x \in G_1, \quad \|x\| = 1;$$

consequently, $\|\rho\|_{H^\infty(\mathbb{D}, G_1^1)} \leq C(1 + \delta)$. Surely, $\rho$ solves equation (7). To get rid of $\delta$, we again pass to the limit, by using the compactness of the balls of $H^\infty(\mathbb{D}, G_1^1)$. Thus, we have obtained the claim of Theorem 1.

To prove the lemma, we denote by $P_r$ the operator of convolution with the Poisson kernel $\theta \mapsto \frac{1-r^2}{1-2r\cos\theta + r^2}$, $0 \leq r < 1$; it is defined on the functions (scalar or vector-valued) for which this operation makes sense. For example, if $f \in H^1(\mathbb{T}; G^*$) and $F$ is the function in $H^1(\mathbb{D}; G^*)$ for which $f$ serves as the boundary function, then the integral in the formula for convolution is a Gelfand integral (i.e., is understood in a weak sense) and we have $(P_r f)(\zeta) = F(r \zeta)$, $\zeta \in \mathbb{T}$. In particular, $P_r$ has norm 1 as an operator from $H^1(\mathbb{T}, G^*)$ to the space of strongly measurable functions $L^1(\mathbb{T}; G^*)$.

So, we fix $\delta > 0$ and $0 \leq r < 1$. For $g \in B$, we put

$$u(\zeta) = \max_{x \in e} |\langle x, (P_r g)(\zeta)\rangle|, \quad \zeta \in \mathbb{T},$$

$$f^{(r)}_g = \frac{1}{C(1 + \delta)} \exp[\log(\delta + u) + i\mathcal{H}(\log(\delta + u))],$$

where $\mathcal{H}$ is the harmonic conjugation operator. (The dependence of $u$ on $r$ has not been reflected in the notation.) We have

$$(8) \quad |f^{(r)}_g| \geq \frac{\delta}{C(1 + \delta)} \quad \text{and} \quad \|f^{(r)}_g\|_{H^1(\mathbb{T})} \leq \frac{1}{C(1 + \delta)}(\delta + 1) = \frac{1}{C},$$

the first inequality holds true in the entire disk $\mathbb{D}$. We define a mapping $T : B \to 2^B$ by

$$T(g) = \{h \in B : Fh = f^{(r)}_g\}.$$

Surely, the formula $Fh = f^{(r)}_g\tau$ should be interpreted as pointwise equality inside the disk. Since the $L^1$-corona problem is solvable with constant $C$ by assumption, the set $T(g)$ is not empty. Clearly, it is also convex and $w^*$-closed. We show that the graph of $T$ is closed. By Lemma 2, we can restrict ourselves to usual convergent sequences: let $g_n \in B$, $h_n \in T(g_n)$ ($n = 1, 2, \ldots$) and $h_n \overset{w^*}{\to} h$, $g_n \overset{w^*}{\to} g$. We must prove that $h \in T(g)$.

The sets $T(g_n)$ are defined in terms of the functions $f^{(r)}_{g_n}$, which, in their turn, are constructed in terms of the functions $u_n$,

$$u_n(\zeta) = \max_{x \in e} |\langle x, (P_r g)(\zeta)\rangle|, \quad \zeta \in \mathbb{T}.$$

Since $e$ is finite, Lemma 2 shows that $u_n(\zeta) \to u(\zeta)$ uniformly in $\zeta \in \mathbb{T}$. But then the functions $\mathcal{H}(\log(\delta + u_n))$ converge to $\mathcal{H}(\log(\delta + u))$ in $L^2(\mathbb{T})$, whence we see that $f^{(r)}_{g_n} \to f^{(r)}_g$ uniformly on compact subsets of $\mathbb{D}$. Then $h_n$ also converge to $h$ uniformly on compact subsets of the disk, so we can pass to the limit in the formula $Fh_n = f^{(r)}_{g_n}\tau$, which proves that the graph is closed.

By the fixed point theorem, there exists $g_r \in B$ such that

$$(9) \quad Fg_r = f^{(r)}_g\tau.$$

We recall that $r \in [0, 1)$ was arbitrary, i.e., we have constructed $g_r$ for every such $r$. For some sequence $r_n \to 1$ the functions $g_{r_n}$ $w^*$-converge to a function $g \in B$. We form the functions $v_n = \max_{x \in e} |\langle x, P_{r_n} g_{r_n}\rangle|$ and $a_n = \log(\delta + v_n)$. The sequence $a_n$ is uniformly bounded in $L^2$; consequently, passing to a subsequence, we may assume that $a_n \to a \in L^2$ weakly. Then $\mathcal{H}a_n \to \mathcal{H}a$ weakly in $L^2$. Thus, the analytic functions
Some Remarks to the Corona Theorem

Let \( a_n + iHa_n \) converge to \( a + iHa \) uniformly on compact subsets of the disk \( \mathbb{D} \). Clearly, \( a \geq \log \delta \).

So, the outer functions \( f_{gr_n}^{(r_n)} \) converge uniformly on compact subsets of \( \mathbb{D} \) to the outer function \( \Phi = \frac{1}{C(1+\delta)} \exp(a + iHa) \), and from (9) and (8) we deduce that

\[
Fg = \Phi \tau \quad \text{and} \quad |\Phi(z)| \geq \frac{\delta}{C(1+\delta)}, \quad z \in \mathbb{D}.
\]

Passing to a subsequence once again, we may assume that the functions \( h_{r_n} = P_{r_n}g_{r_n} \) converge in the \( w^* \)-topology to a function \( h \in B \). For every \( x \in G_1 \) and every \( r, 0 \leq r < 1 \), we have

\[
\langle x, (P, h)(\cdot) \rangle = \lim_{n \to \infty} P_r \langle x, h_{r_n}(\cdot) \rangle = \lim_{n \to \infty} P_r \langle x, g_{r_n}(\cdot) \rangle + \lim_{n \to \infty} (P_{r_n} - id)P_r \langle x, g_{r_n}(\cdot) \rangle.
\]

By Lemma 2, the functions \( P_r \langle x, g_{r_n}(\cdot) \rangle \) converge uniformly to \( \langle x, P_r g \rangle \); for this reason, the second summand in the last expression is zero, and \( \langle x, P_r g \rangle = \langle x, P_r h \rangle \) for all \( x \in G \) and \( r < 1 \). Thus, \( g \) and \( h \) are \( G_1 \)-equivalent and (both) are the boundary functions of one and the same \( G_1^* \)-valued analytic function \( V \) in the unit ball of \( H^1(\mathbb{D}; G_1^*) \). Clearly, \( FV = \Phi \tau \) (this is the first relation in (10)).

Next, since the functions \( f_{gr_n}^{(r_n)} \) are outer, in \( \mathbb{D} \) we have the estimate

\[
|\langle x, h_{r_n}(\cdot) \rangle| \leq C(1 + \delta)|f_{gr_n}^{(r_n)}(\cdot)|, \quad x \in e.
\]

Passing to the limit over \( n \), we obtain \( |\langle x, V(z) \rangle| \leq C(1 + \delta)|\Phi(z)|, z \in \mathbb{D} \). Therefore, the function \( \rho = V\Phi^{-1} \) satisfies the inequality \( |\langle x, \rho(\cdot) \rangle| \leq C(1 + \delta), x \in e \) (we recall that \( \Phi \) is also outer). Now (10) implies that

\[
F\rho = \tau \quad \text{and} \quad \|\rho\|_{H^1(\mathbb{D}; G_1^*)} \leq \frac{C(1 + \delta)}{\delta}.
\]

This proves Lemma 4 with \( C(\delta) = \frac{C(1+\delta)}{\delta} \).

\[\] §2. Analytic partition of unity

In this section, the action will develop on the measure space \( (\mathbb{T} \times \Omega, m \times \mu) \), where \( \mu \) is a \( \sigma \)-finite measure on the set \( \Omega \). The next notion plays an important role in certain questions related to interpolation of analytic Hardy classes.

**Definition.** Let \( w \) be a nonnegative measurable function on \( \mathbb{T} \times \Omega \). A sequence \( \{\varphi_j\}_{j \in \mathbb{Z}}, \varphi_j \in H^\infty(\mathbb{T} \times \Omega) \), is called an analytic partition of unity with constants \( C > 0, 0 < \varepsilon \leq 1 \), and \( \lambda > 1 \) subordinate to \( w \) if

\[
\sum_{j \in \mathbb{Z}}|\varphi_j|^\varepsilon \leq C, \quad |\varphi_j|^\varepsilon w \leq C\lambda^j \quad \text{a.e.,} \quad j \in \mathbb{Z},
\]

\[
\sum_{j \in \mathbb{Z}}|\varphi_j|^\varepsilon \lambda^j \leq Cw \quad \text{a.e.},
\]

\[
\sum_{j \in \mathbb{Z}} \varphi_j = 1 \quad \text{a.e.}
\]

By definition, the space \( H^\infty(\mathbb{T} \times \Omega) \) consists of functions \( F \) on \( \mathbb{T} \times \Omega \) bounded a.e. and lying in \( H^\infty(\mathbb{T}) \) with respect to the first variable. By the way, this space coincides with the set of boundary functions for the class \( H^\infty(\mathbb{D}; L^\infty(\Omega)) \) introduced in §1.

The meaning of conditions (11)–(14) is that the \( \varphi_j \) behave like the characteristic functions of the sets \( \{\lambda^j, -1 \leq w < \lambda^j\} \), but they are analytic in the first variable. There is a well-known relationship between the condition \( \log w(\cdot, \omega) \in \text{BMO}(\mathbb{T}) \) and the existence of an analytic partition of unity.
Theorem 2 (see [11]). Let $w$ be a nonnegative measurable function on $\mathbb{T} \times \Omega$. The following conditions are equivalent:

1) $\operatorname{ess sup}_{\omega \in \Omega} \| \log w(\cdot, \omega) \|_{\text{BMO}} = d < \infty$;

2) there exists an analytic partition of unity with some (equivalently, with arbitrary fixed) $\varepsilon > 0$ and $\lambda > 1$ subordinate to $w$.

Moreover, $d$ and the constant $C$ in the definition of an analytic partition of unity can be estimated in terms of each other and the parameters $\lambda$ and $\varepsilon$.

Formally, in [11] this result was proved for weights on the circle $\mathbb{T}$, but the arguments apply to the measure space $\mathbb{T} \times \Omega$. The implication 2) $\Rightarrow$ 1) had been known before the paper [11] was written. The converse implication was proved in a somewhat tricky way, by using a specific “interpolatory stability” of the couple $(H^\infty(w_0), H^\infty(w_1))$ under the condition $\log \| w \|_1 \in \text{BMO}$. See the references in [11] and the paper [12].

In [13], there is a proof of such interpolation stability based on the classical corona theorem whose data is a couple $(f_1, f_2)$ of scalar functions (that is, the equation to be solved is $f_1 g_1 + f_2 g_2 = 1$). We are going to show that the implication 1) $\Rightarrow$ 2) can be deduced directly from the corona theorem with $l^2$-valued data. More precisely, we shall use the following statement.

Proposition 2. Let $f_j \in H^\infty(\mathbb{T} \times \Omega)$, and let $\delta^2 \leq \sum_{j \in \mathbb{Z}} |f_j(z)|^2 \leq 1$, $z \in \mathbb{D}$. Then there exist functions $g_j \in H^\infty(\mathbb{T} \times \Omega)$ such that $\sum_{j \in \mathbb{Z}} g_j h_j = 1$ and $\sum_{j \in \mathbb{Z}} |g_j(z, \omega)|^2 \leq C(\delta)^2$ a.e.

In fact, $C(\delta) = O(\delta^{-2} \log \delta^{-1})$ as in (3), (3'). For the proof, it suffices, e.g., to examine the arguments of [1] Appendix 3 and to see that the second variable does not undermine them.

Proof of the implication 1) $\Rightarrow$ 2) in Theorem 2. Suppose $w$ satisfies the condition

$$\operatorname{ess sup}_{\omega \in \Omega} \| \log w(\cdot, \omega) \|_{\text{BMO}} = d < \infty.$$  

Using the so-called Garcia norm on BMO (see, for example, [14], Chapter 6, Theorem 1.2]), we conclude that there exists a constant $c$ (depending only on $d$) such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \log w(\theta, \omega) - \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log w(\tau, \omega)) P_z(\tau) d\tau \right| P_z(\theta) d\theta \leq c,$$

for all $z \in \mathbb{D}$ and almost every $\omega \in \Omega$, where $P_z$ is the Poisson kernel for the point $z \in \mathbb{D}$. Consider the outer functions $f_j = \exp(\alpha_j + i \mathcal{H} \alpha_j)$, $j \in \mathbb{Z}$, where $\mathcal{H}$ is the harmonic conjugation operator and $\alpha_j = \log[(\lambda^{\frac{1}{2}} w^{-\frac{1}{2}}) \wedge (\lambda^{\frac{1}{2}} w^{\frac{1}{2}})]$. The function under the logarithm sign is $\beta_j \wedge \beta_j^{-1}$, where $\beta_j = \lambda^{-\frac{1}{2}} w^{\frac{1}{2}}$, thus $\alpha_j \leq 0$ and $-\alpha_j = |\log \beta_j|$.

We view the vector-valued function as the data of a corona problem. Shortly, we shall see that this data satisfies a much stronger condition than it is required in Proposition 2. Fix $z \in \mathbb{D}$ and $\omega \in \Omega$, and denote by $j$ the integral part of the number

$$\frac{1}{\log \lambda^{\frac{1}{2}}} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log w^{\frac{1}{2}}(\tau, \omega)) P_z(\tau) d\tau.$$

We estimate the quantity $f_j(z, \omega)$ from below:

$$- \log |f_j(z, \omega)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-(\alpha_j(\theta, \omega)) P_z(\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log \beta_j(\theta, \omega)| P_z(\theta) d\theta$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \log w^{\frac{1}{2}}(\theta, \omega) - \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log w^{\frac{1}{2}}(\tau, \omega)) P_z(\tau) d\tau \right| P_z(\theta) d\theta$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log w^{\frac{1}{2}}(\tau, \omega)) P_z(\tau) d\tau - j \log \lambda^{1/\varepsilon} \right) P_z(\theta) d\theta.$$
By the definition of \( j \), the expression under the modulus sign in the last integral is dominated by \( \log \lambda^{\frac{1}{2}} \); taking (15) into account, we obtain
\[
|f_j(z, \omega)| \geq (\lambda \epsilon)^{-\frac{1}{2}} \text{ def } \eta.
\]
Therefore, for the functions \( h_j = f_j^2 \) we obtain
\[
\left( \sum_{k \in \mathbb{Z}} |h_k(z, \omega)|^2 \right)^{1/2} \geq \max_{k \in \mathbb{Z}} |f_k(z, \omega)|^2 \geq \eta^2, \quad z \in \mathbb{D}, \quad \omega \in \Omega.
\]

On the other hand, for every \( q, 0 < q < \infty \), we have
\[
\sum_{j \in \mathbb{Z}} |f_j|^q = \sum_{j \in \mathbb{Z}} \left( [\lambda^{-j/\epsilon} w^{1/\epsilon}] \land [\lambda^{j/\epsilon} w^{-1/\epsilon}] \right)^q \leq 2 \sum_{j \geq 0} \lambda^{-\frac{q}{2} j} = \frac{2}{1 - \lambda^{-\frac{q}{2}}}
\]
on \mathbb{T}. Putting \( q = 4 \), we see that the \( l^2 \)-valued analytic function \( (2^{-1}(1 - \lambda^{-\frac{q}{2}}))^{1/2} h \) satisfies the assumptions of Proposition 2. So, there exists a function \( g = \{g_j\}_{j \in \mathbb{Z}} \in H^\infty(l^2) \) such that
\[
\sum_j f_j^2 g_j = 1 \quad \text{and} \quad \|g\|_{H^\infty(l^2)} \leq C = C(c, \lambda, \epsilon).
\]
The required analytic partition of unity is provided by the functions \( \varphi_j = f_j^2 g_j, j \in \mathbb{Z} \). Moreover, we shall not need the estimate in (18) in full generality. Condition (14) has already been ensured. Next,
\[
\left\| \sum_{j \in \mathbb{Z}} |\varphi_j|^\epsilon \right\|_{L^\infty} \leq \|g\|_{L^\infty(l^\epsilon)} \|f\|_{L^\infty(l^2)} \leq C_1(c, \lambda, \epsilon)
\]
by (17) with \( q = 2 \epsilon \) and (18), and condition (11) follows. Finally,
\[
|\varphi_j|^\epsilon w \leq C_2 |f_j|^\epsilon w \leq C_2 \lambda^j w^{-1/2} = C_2 \lambda^j
\]
and (since \( |f_j| = \exp \alpha_j \leq \lambda^{-\frac{1}{2}} w^{\frac{1}{2}} \) by definition)
\[
\sum_{j \in \mathbb{Z}} \lambda^j |\varphi_j|^\epsilon \leq C_3 \sum_j (\lambda^{\frac{1}{2}} |f_j|)^\epsilon \cdot |f_j|^\epsilon \leq C_3 \sum_j w |f_j|^\epsilon \leq C_4 w,
\]
by (17). Thus, (12) and (13) are established. \( \square \)

§3. ESTIMATES WITH VARIOUS NORMS ON SPACES OF SEQUENCES

To a certain extent, the argument in §2 motivates the following question: how to estimate a solution \( g = \{g_j\} \) of the problem \( \sum g_j f_j = 1 \) if the pointwise conditions on the data \( \{f_j\} \) are formulated in terms of certain norms on sequences different from the \( l^2 \)-norm? In particular, what are reasonable restrictions on \( f = \{f_j\} \) under which we can ensure the estimate \( \sum_j |g_j(z)|^a \leq C, \quad z \in \mathbb{D} \), with small \( a \) for some solution \( \{g_j\} \)?

Estimates in various norms have been treated before. Beside the results indicated in (3) and (3’), we mention the difficult theorem by Uchiyama [15]: if \( \delta \leq \sup_z |f_j(z)| \leq 1, \quad z \in \mathbb{D} \), then there exist analytic functions \( g_j \) such that
\[
\sum_j |g_j(z)| = O(\delta^{-c}), \quad \delta \to 0,
\]
for some \( c > 0 \). In his proof, Uchiyama refined Carleson’s original method. Apparently, Wolff’s techniques are not applicable here.
In [1, Appendix 3] yet another (probably, unpublished) estimate by Tolokonnikov was cited: if \( p > 2 \) is an even integer and the data satisfies \( \delta \leq \left( \sum_j |f_j(z)|^{|p|} \right)^{1/p} \leq 1 \), then there is a solution \( \{g_j\} \) with

\[
\left( \sum_j |g_j(z)|^{p'} \right)^{1/p'} = O(\delta^{-p} \log \delta^{-1}), \quad \delta \to 0.
\]

Apparently, the condition that \( p \) is even arose here because the author of this estimate applied a version of Wolff’s techniques directly, and all such arguments require explicit calculation of the \( p \)th powers of certain sums. Among other things, below we shall see that \( (20) \) can be deduced from \( (3) \) and \( (3') \) by application of the Hölder inequality and without any additional restrictions on \( p \in \{2, \infty\} \).

Thus, we consider a corona problem data (that is, a sequence \( \{f_j\} \) of analytic functions in the disk) from the space \( H^{\infty}(l^p) \), \( 0 < p \leq \infty \). However, we shall look for a solution in the space \( H^{\infty}(l^{p'}) \), where, largely speaking, \( r \) is not linked with \( p \). A necessary condition for solvability can easily be deduced: if \( \sum f_j g_j = 1 \) and \( \|g\|_{H^{\infty}(l^p)} = C \) (\( g = \{g_j\}_{j=1}^\infty \) ), then

\[
1 \leq \|f(z)\|_{l^p} \cdot \|g\|_{H^{\infty}(l^{p'})}, \quad z \in \mathbb{D},
\]

where \( s \) is the exponent conjugate to \( r \vee 1 \). So, \( \inf_{z \in \mathbb{D}} \|f(z)\|_{l^p} \geq C^{-1} \). In this connection, it is natural to give the following definition (in it, \( \delta , p , q , r > 0 , \delta < 1 , \) and \( p , q , r \) are permitted to take the value \( +\infty \)). Consider various data \( f \) for which \( \|f(z)\|_{l^p} \leq 1 \) and \( \|f(z)\|_{l^r} \geq \delta \) for \( z \in \mathbb{D} \); we put

\[
C(\delta; p , q , r) = \sup \inf \left\{ \|g\|_{H^{\infty}(l^p)} : \sum f_j g_j = 1 \right\},
\]

where the supremum is taken over all functions \( f \) satisfying the conditions indicated above.

In these terms, formulas \( (3) \) and \( (3') \) give estimates for \( C(\delta; 2, 2, 2) \). We also mention that \( C(\delta; 2, 2, 2) \geq \max(\delta^{-1}, \delta^{-2}) \), see [1, Appendix 3]. Formula \( (19) \) is an upper estimate for \( C(\delta , \infty , \infty , 1) \) and formula \( (20) \) is an upper estimate for \( C(\delta, p, p, p') \), \( p = 2k , k \in \mathbb{N} \). Quite shortly, we shall prove that

\[
(21) \quad C(\delta; p, p, p') \leq C(\delta^{p/2}; 2, 2, 2), \quad 0 < \delta < 1, \quad p \geq 2,
\]

where \( p \) is not necessarily an even integer. For the time being, we make a series of simple observations.

It is easily seen that the function \( C(\delta; p, q, r) \) is (nonstrictly) monotone increasing in the argument \( q \) and (nonstrictly) monotone decreasing in \( p , r , \) and \( \delta \). Considering the data \( f = \{f_j\} \), where \( f_1 = 1 \) and \( f_j = 0 \) for \( j > 1 \), we see that \( C(\delta; p, q, r) \geq 1 \) for all values of the parameters.

Next, \( C(\delta; p, q, r) = \infty \) for \( r \geq 1 , \ p \leq q \) and \( r' > p \). Indeed, for \( N \in \mathbb{N} \), define the data \( f \) by \( f_k(z) \equiv N^{-1/p} \) for \( 1 \leq k \leq N \) and \( f_k(z) = 0 \) for \( k > N \). Then \( \|f(\cdot )\|_{l^r} = 1 \) and \( \|f(\cdot )\|_{l^p} = N^{\frac{1}{q} - \frac{1}{p'}} \). If \( C(\delta; p, q, r) < \infty \), then for every \( \varepsilon > 0 \) there exists \( g \in H^{\infty}(l^r) \) such that \( \|g\|_{H^{\infty}(l^{p'})} \leq (1 + \varepsilon) C(\delta; p, q, r) \) and \( \sum f_j g_j = 1 \). But then

\[
1 \leq \|g(\cdot )\|_{l^r} \cdot \|f(\cdot )\|_{l^p} \leq (1 + \varepsilon) C(\delta; p, q, r) N^{\frac{1}{q} - \frac{1}{p'}},
\]

which is impossible because \( r' > p \). By monotonicity, we see that if \( 1 \leq p \leq q \), then \( C(\delta; p, q, r) \) is infinite for every \( r < p' \).

In particular, if \( q \geq p \geq 1 \), no \( l^r \)-estimate with very small \( r \) is possible. Below, among other things, we shall see that the situation for \( q < p \) is different. In essence, a key role is played by a trick used in §2 in the proof of the existence of an analytic partition of unity.
\textbf{Theorem 3.} Suppose that $0 < \delta < 1$, $t > 1$, $0 < p$, $q < \infty$. Then for $s = \frac{rq}{r(t-1)+q}$ (consequently, also for all greater values of $s$) we have

\begin{equation}
C(\delta; p, q, s) \leq C\left(\delta^t; \frac{p}{t}, \frac{q}{t}, r\right).
\end{equation}

\textit{Proof.} Let the data $f = \{f_j\}$ of the corona problem satisfy $\|f(z)\|_{t^p} \geq \delta$, $\|f(z)\|_{t^q} \leq 1$ for $z \in \mathbb{D}$, and let $f_j = \theta_j h_j$ be the inner-outer factorization of $f_j$ (the outer function is $h_j$). Suppose that the quantity $C(\delta^t, \frac{p}{t}, \frac{q}{t}, r)$ is finite, and consider the corona problem with the data $\{\theta_j h_j^t\}$. Since

\begin{equation}
\sum_j (|\theta_j(z)| |h_j(z)|^t)^\frac{q}{t} = \sum_j (|\theta_j(z)|)^\frac{q}{t} |h_j(z)|^t \geq \sum_j (|\theta_j(z)|^p |h_j(z)|^p)^\frac{q}{p} \geq \delta^p
\end{equation}

for $z \in \mathbb{D}$, and $\sum_j (|\theta_j(z)| |h_j(z)|^t)^{q/t} = \sum_j |h_j(z)|^q \leq 1$ for $z \in \mathbb{T}$, we see that for every $\varepsilon > 0$ there exist analytic functions $a_j$ in the disk with

\begin{equation}
1 = \sum_j a_j \theta_j h_j^t = \sum_j f_j \cdot (a_j h_j^{t-1}), \quad \|a\|_{H^\infty(t^q)} \leq (1 + \varepsilon) C\left(\delta^t; \frac{p}{t}, \frac{q}{t}, r\right).
\end{equation}

Thus, the functions $\{a_j h_j^{t-1}\}$ form a solution of the corona problem with the data $f$; we estimate this solution by the Hölder inequality with exponents $u$ and $u'$:

\begin{equation}
\sum_j (|a_j(\cdot)| |h_j(\cdot)|^t)^s \leq \left(\sum_j |a_j(\cdot)|^s u\right)^{1/u} \left(\sum_j |h_j(\cdot)|^{s(t-1)u'}^t\right)^{1/u'}.
\end{equation}

We require that $su = r$, $s(t-1)u' = q$. Simple calculations show that then $s = \frac{rq}{r(t-1)+q}$ and the right-hand side of the last inequality is dominated by

\begin{equation}
\left(\sum_j |a_j(\cdot)|^r\right)^\frac{q}{t} \leq \left(1 + \varepsilon\right) C\left(\delta^t; \frac{p}{t}, \frac{q}{t}, r\right)^s.
\end{equation}

Putting $r = \frac{p}{t} = \frac{q}{t} = 2$ in the theorem, we arrive at the announced estimate (21).

Next, it is clear that we can make $s$ as small as we wish if $q$ is taken small. For example, let $p$ and $t$ satisfy $p/t = 2$. Take $q \leq p$ and $r = 2$, then it turns out that $C(\delta; p, q, s) \leq C(\delta^2, 2, 2, 2) \leq C(\delta^2, 2, 2, 2) < \infty$. If, say $p = 4$ and $t = 2$, we obtain $s = \frac{2q}{2q+q}$. For small $q$, an effect similar to that in \S 2 arises, but under less restrictive assumptions on the data than in (16) and (17).

The theorem can be supplemented by yet another quite simple estimate: if $0 \leq \alpha < 1$ and $\alpha q < p < \infty$, then

\begin{equation}
C(\delta; p, q, r) \leq C\left(\frac{\delta - \frac{p}{p - \alpha q}}{1 - \alpha}, q, r\right).
\end{equation}

For $\alpha = 1$, this inequality is still true in the form

\begin{equation}
C(\delta; p, q, r) \leq C\left(\frac{\delta}{p - q}; \infty, q, r\right).
\end{equation}

Indeed, for $\alpha = 0$ formula (23) is a tautology. Consider certain data $f = \{f_j\}$ of the corona problem with $\delta^p \leq \sum |f_j(\cdot)|^p$ and $\sum |f_j(\cdot)|^q \leq 1$. For $\alpha = 1$, formula (24) follows from the inequality

\begin{equation}
\delta^p \leq \sum_j |f_j(\cdot)|^p \leq \|f(\cdot)\|_{t^p}^{-q} \sum_j |f_j(\cdot)|^q \leq \|f(\cdot)\|_{t^q}^{p-q},
\end{equation}

and for $\alpha < 1$ formula (23) follows from the Hölder inequality

\begin{equation}
\delta^p \leq \sum_j |f_j(\cdot)|^p = \sum_j |f_j(\cdot)|^{\alpha q} |f_j(\cdot)|^{p-\alpha q} \leq \left(\sum_j |f_j(\cdot)|^q\right)^\alpha \left(\sum_j |f_j(\cdot)|^{\frac{p}{1-\alpha}q}\right)^{1-\alpha},
\end{equation}
whence
\[
\left( \sum_j |f_j(\cdot)| \left( \frac{p-q}{1-\alpha} \right)^{\frac{1-\alpha}{p-\alpha}} \right)^{\frac{1}{p-\alpha}} \geq \delta^{\frac{p}{p-q}}.
\]

It should be noted that it is not known whether \( C(\delta; p, q, r) \), \( 0 < \delta < 1 \), is finite for \( p = q < 2 \) and \( r \geq p' \) and also for \( p = q = \infty \) and \( r < 1 \).

REFERENCES


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