Алгебра и анализ
Том 24 (2012), № 3

St. Petersburg Math. J.
Vol. 24 (2013), No. 3, Pages 475–484
S 1061-0022(2013)01248-9
Article electronically published on March 21, 2013

THE STABLE CALABI–YAU DIMENSION OF THE PREPROJECTIVE ALGEBRA OF TYPE $\mathbf{L}_n$

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Abstract. It is proved that if the characteristic of the ground field is not equal to 2, then the stable Calabi–Yau dimension of the preprojective algebra of type $\mathbf{L}_n$ is equal to 5. This result contradicts certain claims by Erdmann and Skowroński related to the description of algebras whose stable Calabi–Yau dimension is 2.

Introduction

Let $\mathbf{k}$ be an algebraically closed field. We denote by $\mathbf{k}Q$ the path algebra of a quiver Q. We write paths from left to right. Let $s(\alpha)$ denote the source of an arrow α and $t(\alpha)$ its target. We denote by e_i the idempotent associated with a vertex i. An algebra of the form $A = \mathbf{k}Q/I$, where I is an admissible ideal, is called a *bound quiver* algebra.

The usual approach to the definition of preprojective algebras is the following. Let Δ be an undirected graph. Denote by $Q_{\mathcal{P}(\Delta)}$ the quiver obtained from Δ by replacing every original edge of Δ with a pair of arrows pointing in opposite directions. For an arrow a of $Q_{\mathcal{P}(\Delta)}$, we denote by $\bar{a}$ the other arrow associated with the same edge of Δ. This gives us an involution $a \mapsto \bar{a}$ on the set of arrows of $Q_{\mathcal{P}(\Delta)}$ such that $s(\bar{a}) = t(a)$, $t(\bar{a}) = s(a)$. The preprojective algebra associated with Δ is the bound quiver algebra $\mathcal{P}(\Delta) = \mathbf{k}Q_{\mathcal{P}(\Delta)}/I_\Delta$, where I_Δ is the ideal generated by the relations

$$\sum_{a\,:\,s(a)=i} a\bar{a}, \text{ where } i \text{ is a vertex } Q_\Delta.$$

The algebra $\mathcal{P}(\Delta)$ is called the *preprojective algebra of type* Δ.

However, the above definition is not quite convenient for our purposes; instead of it, we use the following modified construction introduced in [1] and [2]. Namely, instead of replacing a loop of Δ with a pair of looping arrows, we replace it with a single looping arrow on which the involution $a \mapsto \bar{a}$ acts trivially. For example, the graph

$$\mathbf{L}_n \ : \ \bullet \text{———} \bullet \text{———} \bullet \text{———} \dots \text{———} \bullet \qquad (n \geq 1 \text{ vertices})$$

corresponds to the quiver

$$Q_{\mathcal{P}(\mathbf{L}_n)} \ : \quad \varepsilon = \bar{\varepsilon} \circlearrowright 0 \underset{\bar{\alpha}_0}{\overset{\alpha_0}{\rightleftarrows}} 1 \underset{\bar{\alpha}_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \quad \dots \quad n-2 \underset{\bar{\alpha}_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1 \ ,$$

2010 *Mathematics Subject Classification.* Primary 14J35.

Key words and phrases. Preprojective algebras, self-injective algebras, Calabi–Yan dimension, stable category of modules.

Supported by RFBR (grant no. 10-01-00635); by targeted federal program "Scientific and Academic Specialists for Innovations in Russia" (grant nos. 2010-1.1-111-128-033, 14.740.11.0344); and by the St. Petersburg State University research program 6.38.74.2011 "Structure theory and geometry of algebraic groups and their applications in representation theory and algebraic K-theory".

and the ideal $I_{\mathbf{L}_n}$ is generated by the following elements:

$$\begin{aligned}&\varepsilon^2+\alpha_0\bar{\alpha}_0,\\&\bar{\alpha}_i\alpha_i+\alpha_{i+1}\bar{\alpha}_{i+1},\quad\text{where}\ \ 0\le i\le n-3,\\&\bar{\alpha}_{n-2}\alpha_{n-2}.\end{aligned}$$

Following Bondal and Kapranov [3], we say that a triangulated $\mathbf{k}$-linear hom-finite category $\mathcal{T}$ has a Serre duality if there is a triangle autoequivalence $F:\mathcal{T}\to\mathcal{T}$, called a *Serre functor*, such that for all objects T and S in $\mathcal{T}$ there exists a natural $\mathbf{k}$-linear isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(T,S)\cong D\operatorname{Hom}_{\mathcal{T}}(S,F(T))$$

(compatible with the shift functor; see [4]), where $D=\operatorname{Hom}_{\mathbf{k}}(-,\mathbf{k})$. If a Serre functor exists, it is unique up to isomorphism. Following Kontsevich [5], we say that a triangulated $\mathbf{k}$-linear hom-finite category $\mathcal{T}$ is *Calabi–Yau* if for some n the iterated shift functor $[n]$ is a Serre functor of $\mathcal{T}$. If so, then the minimal $n\ge 0$ having this property is called the Calabi–Yau dimension of $\mathcal{T}$, and is denoted by $\operatorname{CYdim}(\mathcal{T})$. If $\mathcal{T}$ is not Calabi–Yau, we set $\operatorname{CYdim}(\mathcal{T})=\infty$.

An important class of triangulated $\mathbf{k}$-linear categories of algebraic nature is formed by stable module categories $\underline{\text{mod}}$-A of finite-dimensional selfinjective $\mathbf{k}$-algebras A, where the shift is given by the inverse Ω_A^{-1} of Heller's syzygy functor Ω_A. In this case, following Erdmann and Skowroński [2], we define the stable Calabi–Yau dimension of A to be the Calabi–Yau dimension of $\underline{\text{mod}}$-A, and we write briefly $\underline{\text{CY}}\dim(A)=\operatorname{CYdim}(\underline{\text{mod}}\text{-}A)$.

As in [6], the graphs $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, and the graph $\mathbf{L}_n$ introduced above are called *generalized Dynkin graphs.* In [1, 2], Erdmann and Skowroński described the algebras of stable Calabi–Yau dimension 2. In particular, Proposition 3.4 in [2] asserts that all preprojective algebras of generalized Dynkin type not equal to $\mathbf{A}_1$, $\mathbf{A}_2$, or $\mathbf{L}_1$ have stable Calabi–Yau dimension 2.

The main result of the present paper is the following theorem, which implies that, in fact, Proposition 3.4 in [2] is valid only for the ground field of characteristic 2. Specifically, we prove the following statement.

Theorem 4.2. *Let $n\ge 2$. Then*:

- *if* $\operatorname{char}(\mathbf{k})\ne 2$, *then* $\underline{\text{CY}}\dim(\mathcal{P}(\mathbf{L}_n))=5$;
- *if* $\operatorname{char}(\mathbf{k})=2$, *then* $\underline{\text{CY}}\dim(\mathcal{P}(\mathbf{L}_n))=2$.

As a preliminary step, we show that the algebra $\mathcal{P}(\mathbf{L}_n)$ is symmetric and that the Loewy length of every nonzero projective module is equal to $2n$ (Proposition 3.1). In order to obtain this result, we introduce a basis of the algebra $\mathcal{P}(\mathbf{L}_n)$ (Proposition 5.3).

§1. The stable module category and the stable Calabi–Yau dimension

We fix an algebraically closed field $\mathbf{k}$ and assume that all algebras considered in this section are finite-dimensional $\mathbf{k}$-algebras, and that all modules are finitely generated right modules. We denote by mod-A the category of finitely generated right A-modules over an algebra A.

Let M, N be A-modules. We denote by $\mathcal{P}(M,N)$ the subset of $\operatorname{Hom}_A(M,N)$ consisting of homomorphisms of the form $M\to P\to N$, where P is a projective module. The sets $\mathcal{P}(M,N)$ form an ideal of the category mod-A. The stable module category of an algebra A is the quotient category of mod-A by this ideal. We denote the stable module category by $\underline{\text{mod}}$-A and the set of morphisms from M to N in the category $\underline{\text{mod}}$-A by $\underline{\operatorname{Hom}}_A(M,N)$. For a morphism f in the category mod-A, let $\underline{f}$ be the corresponding morphism in the category $\underline{\text{mod}}$-A.

The set $\mathcal{P}(M,N)$ can be defined in a different way. In particular, the following proposition holds true.

Proposition 1.1. *Let M, N be A-modules, and let $f \in \mathrm{Hom}_A(M,N)$. The following conditions are equivalent*:

(1) $f \in \mathcal{P}(M,N)$;
(2) *there exists a homomorphism $g : M \to P_N$ such that $f = \sigma_N g$, where $\sigma_N : P_N \twoheadrightarrow N$ is a projective cover of the module N*;
(3) *for any epimorphism $h : L \twoheadrightarrow N$, there exists $g : M \to L$ such that $f = hg$.*

Proof. The equivalence (1) $\Leftrightarrow$ (3) was proved in [8, IV, 1.5], and the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious. $\square$

Assume that A is a selfinjective algebra. Then it can be proved that the syzygy functor $\Omega : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A$ is an autoquasiequivalence (see [9]). Moreover, the structure of a triangulated category can be introduced on $\underline{\mathrm{mod}}\text{-}A$; the shift functor related to this structure is the inverse to the Heller syzygy functor $\Omega^{-1} : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A$, see [10].

The selfinjectivity of the algebra A implies that of the enveloping algebra $A^e = A^{\mathrm{op}} \otimes A$. The category of A^e-modules is isomorphic to the category of A-bimodules. Hence, we can easily define the stable bimodule category $\underline{\mathrm{bimod}}\text{-}A$ with the shift functor $\Omega_{A^e} : \underline{\mathrm{bimod}}\text{-}A \to \underline{\mathrm{bimod}}\text{-}A$. If N is a left-right projective A-bimodule, then the functor $- \otimes_A N : \mathrm{mod}\text{-}A \to \mathrm{mod}\text{-}A$ sends projective modules to projective modules. Therefore, it induces an endofunctor on the stable module category. This functor will be denoted by $- \underline{\otimes} N : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A$. It is easy to check that $\Omega^n \cong - \underline{\otimes}\, \Omega^n_{A^e}(A)$.

As was said above, the stable Calabi–Yau dimension of an algebra A is the Calabi–Yau dimension of the category $\underline{\mathrm{mod}}\text{-}A$:

$$\underline{\mathrm{CY}}\dim(A) = \mathrm{CY}\dim(\underline{\mathrm{mod}}\text{-}A).$$

In [2, Corollary 1.3], Erdmann and Skowroński proved that $\underline{\mathrm{CY}}\dim(A) = d$, where $d \geq 0$ is the smallest number such that $\Omega^{d+1} \cong \underline{\nu}^{-1}$; here $\underline{\nu}^{-1} : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}A$ is the functor induced by the inverse of Nakayama functor. It is well known that for a symmetric algebra A we have an isomorphism $\nu \cong \mathrm{Id}_{\mathrm{mod}\text{-}A}$. Consequently, the following statements is true (see [2, Corollary 1.4]).

Proposition 1.2. *Let A be a symmetric algebra. Then $\underline{\mathrm{CY}}\dim(A) = d$ if and only if $d \geq 0$ is the smallest number such that there is an isomorphism $\Omega^{d+1} \cong \mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A}$.*

Corollary 1.3. *Suppose A is a symmetric algebra, $d = \underline{\mathrm{CY}}\dim(A) < \infty$, and $\Omega^n \cong \mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A}$. Then $d+1$ divides n.*

§2. Socle and radical series of a graded bound quiver algebra

Let $\mathbf{k}Q$ be the path algebra of a quiver Q. We view it as a graded algebra, where $|e_i| = 0$ and $|a| = 1$ for $i \in Q_0$, $a \in Q_1$. If $I \triangleleft \mathbf{k}Q$ is a homogeneous ideal, the grading on $\mathbf{k}Q$ induces a grading on the quotient algebra $A = \mathbf{k}Q/I$.

Let M be an A-module. We denote by $\mathrm{rad}^k(M)$ the kth term of the radical series $M = \mathrm{rad}^0(M) \supseteq \mathrm{rad}^1(M) \supseteq \ldots$, and by $\mathrm{soc}^k(M)$ the kth term of the socle series $0 = \mathrm{soc}^0(M) \subseteq \mathrm{soc}^1(M) \subseteq \ldots$. For convenience, we set $\mathrm{soc}^k(M) = 0$ and $\mathrm{rad}^k(M) = M$ for $k < 0$.

The following fact appears to be well known, but since we were unable to find a proper reference, it will be presented with a proof.

Proposition 2.1. *Let I be a homogeneous admissible ideal of the path algebra $\mathbf{k}Q$, let $A = \mathbf{k}Q/I$, and let $P_i = e_iA$. Then*

$$\operatorname{rad}^k(P_i) = \bigoplus_{l\geq k} e_iA_l.$$

Moreover, if A is self-injective and the Loewy length of the module P_i is equal to m, then

$$\operatorname{soc}^k(P_i) = \operatorname{rad}^{m-k}(P_i).$$

Proof. We denote by $J := (Q_1)$ the ideal of $\mathbf{k}Q$ generated by arrows. It is easily seen that $J^k = \bigoplus_{l\geq k} \mathbf{k}Q_l$, where $\mathbf{k}Q_l$ is the lth homogeneous component of the path algebra. The image of J in the algebra A coincides with its radical. Therefore, multiplying the relation $\operatorname{rad}^k(A) = \bigoplus_{l\geq k} A_l$ by e_i we obtain the first statement of the proposition.

Since the Loewy length of the module P_i is equal to m, we have $\operatorname{rad}^m(P_i) = 0$ and $\operatorname{rad}^{m-1}(P_i) \neq 0$. It is easy to check that $\operatorname{soc}^k(P_i) = \{x \in P_i \mid \forall a \in A_k\ xa = 0\}$. It follows that $\operatorname{rad}^{m-k}(P_i) \leq \operatorname{soc}^k(P_i)$. Since the algebra A is selfinjective, $\operatorname{soc}(P_i)$ is a simple module. Moreover, $\operatorname{rad}^{m-1}(P_i) \leq \operatorname{soc}(P_i)$, whence $\operatorname{soc}(P_i) = \operatorname{rad}^{m-1}(P_i)$.

Now we use induction on k to show that $\operatorname{soc}^k(P_i) = \operatorname{rad}^{m-k}(P_i)$. Assuming that $\operatorname{soc}^k(P_i) = \operatorname{rad}^{m-k}(P_i)$, we prove that $\operatorname{soc}^{k+1}(P_i) = \operatorname{rad}^{m-k-1}(P_i)$. We already know that $\operatorname{rad}^{m-k-1}(P_i) \leq \operatorname{soc}^{k+1}(P_i)$. We assume that $k < m-1$, otherwise the required relation is obvious. Consider an element $x \in P_i \setminus \operatorname{rad}^{m-k-1}(P_i)$. In order to complete the proof, it suffices to show that $x \notin \operatorname{soc}^{k+1}(P_i)$. We can present x as $x = x_l + x_{>l}$, where $x_l \in e_iA_l \setminus \{0\}$ and $x_{>l} \in \operatorname{rad}^{l+1}(P_i)$, for some $l < m-k-1$. Since $x_l \notin \operatorname{soc}(P_i)$, there exists an element $a \in A_1$ such that $x_la \neq 0$. Consequently, $x_la \in A_{l+1} \setminus \{0\}$. Since $l+1 < m-k$, it follows that $x_la \notin \operatorname{rad}^{m-k}(P_i) = \operatorname{soc}^k(P_i)$. Consequently, there exists an element $b \in A_k$ such that $x_lab \neq 0$. Since the elements x_lab and $x_{>l}ab$ lie in different direct summands, $x_lab \neq 0$ gives us $xab = x_lab + x_{>l}ab \neq 0$. Since $ab \in A_{k+1}$, it follows that $xA_{k+1} \neq 0$. Finally, $x \notin \operatorname{soc}^{k+1}(P_i)$. □

§3. The preprojective algebra of type $\mathbf{L}_n$

Recall that $\mathbf{L}_n$ denotes the following graph.

$$\mathbf{L}_n\ :\ \bullet \text{———} \bullet \text{———} \bullet \text{———} \dots \text{———} \bullet \qquad (n \geq 1 \text{ vertices})$$

In this section we present several results on the preprojective algebra of type $\mathbf{L}_n$. Recall that it is the bound quiver algebra $\mathcal{P}(\mathbf{L}_n) = \mathbf{k}Q_{\mathcal{P}(\mathbf{L}_n)}/I_{\mathbf{L}_n}$ of the following quiver with relations:

$$\varepsilon=\bar{\varepsilon}\circlearrowright 0 \underset{\bar{\alpha}_0}{\overset{\alpha_0}{\rightleftarrows}} 1 \underset{\bar{\alpha}_1}{\overset{\alpha_1}{\rightleftarrows}} 2\ \dots\ n-2 \underset{\bar{\alpha}_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1\ , \qquad \begin{array}{l} \bar{\alpha}_i\alpha_i + \alpha_{i+1}\bar{\alpha}_{i+1}\ (0 \leq i \leq n-3), \\ \varepsilon^2 + \alpha_0\bar{\alpha}_0,\ \ \bar{\alpha}_{n-2}\alpha_{n-2}. \end{array}$$

The proof of the following statement is rather bulky, and we give it separately at the end of the paper.

Proposition 3.1. *$\mathcal{P}(\mathbf{L}_n)$ is a finite-dimensional symmetric algebra, and the Loewy length of any nonzero projective $\mathcal{P}(\mathbf{L}_n)$-module is equal to $2n$.*

It is easy to show that the ideal $I_{\mathbf{L}_n}$ is a homogeneous ideal of $\mathbf{k}Q_{\mathcal{P}(\mathbf{L}_n)}$; consequently, the algebra $\mathcal{P}(\mathbf{L}_n)$ has a natural grading.

Corollary 3.2. *Let $A = \mathcal{P}(\mathbf{L}_n)$, and let A_l be the lth homogeneous component of A. Then*

$$\operatorname{rad}^k(P_i) = \operatorname{soc}^{2n-k}(P_i) = \bigoplus_{l\geq k} e_iA_l.$$

Proof. This follows from Propositions 2.1 and 3.1. □

Let $\varphi : A \to A$ be an automorphism of the algebra A. Consider a functor res_φ : mod-$A \to$ mod-A. By definition, it modifies the structure of every right A-module by the formula $m \underset{\varphi}{*} a = m\varphi(a)$. At the same time, it leaves unchanged both the underlying vector space structure and the set-theoretical structure of every morphism. We denote $M_\varphi = \mathrm{res}_\varphi(M)$. A similar functor $\mathrm{res}_{\mathrm{Id},\varphi}$: bimod-$A \to$ bimod-A can be defined on the category of bimodules which only "twists" the right module structure of a bimodule by the formula $\mathrm{res}_{\mathrm{Id},\varphi}(M) = M_\varphi$. It is easy to check the isomorphism $\mathrm{res}_\varphi \cong - \otimes_A A_\varphi$.

Proposition 3.3. *Let $A = \mathcal{P}(\mathbf{L}_n)$. Then $\Omega^3_{A^e}(A) \cong A_\tau$, where $\tau : A \to A$ is the automorphism given by the formulas $\tau(e_i) = e_i$, $\tau(a) = -a$, for all vertices i and arrows a of the quiver $Q_{\mathcal{P}(\mathbf{L}_n)}$.*

This statement was mentioned without proof in [7, Proposition 2.1]. It fixes an inaccuracy in the statement of [1, Proposition 2.3]. Nevertheless, the proof of the latter fact presented in that paper is correct.

§4. The main theorem

Proposition 4.1. *If $n \geq 2$ and* $\mathrm{char}(\mathbf{k}) \neq 2$, *then* $\underline{\mathrm{CY}}\dim(\mathcal{P}(\mathbf{L}_n)) \neq 2$.

Proof. We set $A = \mathcal{P}(\mathbf{L}_n)$. Consider the quotient module $M = P_0/\operatorname{rad}^2(P_0)$. Since $\operatorname{soc}^2(M) = M$, for any homomorphism $g : M \to P_0$ we have the inclusion $\mathrm{Im}(g) \leq \operatorname{soc}^2(P_0)$. Therefore, for $n \geq 2$, Corollary 3.2 shows that $\operatorname{soc}^2(P_0) = \operatorname{rad}^{2n-2}(P_0) \leq \operatorname{rad}^2(P_0)$. Hence $\mathrm{Im}(g) \leq \operatorname{rad}^2(P_0)$ for any homomorphism $g : M \to P_0$, and it follows that any homomorphism of the form $M \to P_0 \twoheadrightarrow M$, where $P_0 \twoheadrightarrow M$ denotes the canonical projection, is zero. Together with Proposition 1.1, this yields $\mathcal{P}(M, M) = 0$, whence $\underline{\mathrm{Hom}}_A(M, M) \cong \mathrm{Hom}_A(M, M)$.

The automorphism τ sends P_0 to P_0. Consider its restriction $\tau_{P_0} : P_0 \to (P_0)_\tau$. It is easily seen that τ_{P_0} is an isomorphism of modules sending $\operatorname{rad}^2(P_0)$ to $\operatorname{rad}^2(P_0)_\tau$. Therefore, it induces an isomorphism $M \cong M_\tau$.

Now assume that $\underline{\mathrm{CY}}\dim(A) = 2$, contrary to the claim. Then Propositions 1.2 and 3.1 give an isomorphism of functors $\Omega^3 \cong \mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A}$. We denote by $\underline{\mathrm{res}}_\tau$: $\underline{\mathrm{mod}}$-$A \to \underline{\mathrm{mod}}$-$A$ the functor induced by res_τ on the stable module category. Since $\Omega^3_{A^e}(A) \cong A_\tau$, it follows that

$$\underline{\mathrm{res}}_\tau \cong -\underline{\otimes} A_\tau \cong -\underline{\otimes}\Omega^3_{A^e}(A) \cong \Omega^3_A \cong \mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A}\,.$$

Therefore, $\mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A} \cong \underline{\mathrm{res}}_\tau$.

We denote this isomorphism by $f : \mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A} \to \underline{\mathrm{res}}_\tau$.

The component $f_M : M \to M_\tau$ of f at M is an isomorphism in the category $\underline{\mathrm{mod}}$-A. We choose its representative $f'_M : M \to M_\tau$ in the category mod-A. Since $\mathcal{P}(M_\tau, M_\tau) \cong \mathcal{P}(M, M) = 0$, it follows that $f'_M : M \to M_\tau$ is an isomorphism in the category mod-A.

The submodule $\operatorname{rad}^2(P_0)$ is invariant under the endomorphism of multiplication by the loop ε, which we denote by $\varepsilon\,\cdot : P_0 \to P_0$. Consequently, this endomorphism induces an endomorphism on M, still denoted by $\varepsilon\cdot : M \to M$. Since $f : \mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A} \to \underline{\mathrm{res}}_\tau$ is a natural transformation, the following diagram is commutative in the category $\underline{\mathrm{mod}}$-A:

$$\begin{tikzcd} M \arrow[r, "\underline{\varepsilon\,\cdot}"] \arrow[d, "f_M"', "\cong"] & M \arrow[d, "f_M"', "\cong"] \\ M_\tau \arrow[r, "\underline{\varepsilon\,\cdot}"] & M_\tau \end{tikzcd}$$

Since $\mathcal{P}(M, M_\tau) \cong \mathcal{P}(M, M) = 0$, the following diagram is commutative in the category mod-A:

$$\begin{tikzcd} M \arrow[r, "\varepsilon\,\cdot"] \arrow[d, "f'_M"', "\cong"] & M \arrow[d, "f'_M"', "\cong"] \\ M_\tau \arrow[r, "\varepsilon\,\cdot"] & M_\tau \end{tikzcd}$$

We set $\widehat{e}_0 := e_0 + \mathrm{rad}^2(P_0) \in M$, $\widehat{\varepsilon} := \varepsilon + \mathrm{rad}^2(P_0) \in M$. Let $f'_M(\widehat{e}_0) = \lambda\widehat{e}_0 + r$, where $\lambda \in \mathbf{k}$ and $r \in \mathrm{rad}(M)$. It is easy to check that $(\varepsilon\,\cdot)(r) = 0$ and $r\varepsilon = 0$. Therefore,

$$f'_M(\widehat{\varepsilon}) = \big(f'_M \circ (\varepsilon\,\cdot)\big)(\widehat{e}_0) = \big((\varepsilon\,\cdot) \circ f'_M\big)(\widehat{e}_0) = (\varepsilon\,\cdot)(f'_M(\widehat{e}_0)) = \lambda\widehat{\varepsilon}.$$

On the other hand,

$$f'_M(\widehat{\varepsilon}) = f'_M(\widehat{e}_0) \underset{\tau}{*} \varepsilon = -f'_M(\widehat{e}_0)\varepsilon = -\lambda\widehat{\varepsilon}.$$

Since $\mathrm{char}(\mathbf{k}) \neq 2$, the relation $\lambda\widehat{\varepsilon} = f'_M(\widehat{\varepsilon}) = -\lambda\widehat{\varepsilon}$ shows that $\lambda = 0$, which contradicts the assumption that f'_M is an isomorphism. □

Theorem 4.2. *Let $n \geq 2$. Then*:

- *if* $\mathrm{char}(\mathbf{k}) \neq 2$, *then* $\underline{\mathrm{CY}}\dim(\mathcal{P}(\mathbf{L}_n)) = 5$;
- *if* $\mathrm{char}(\mathbf{k}) = 2$, *then* $\underline{\mathrm{CY}}\dim(\mathcal{P}(\mathbf{L}_n)) = 2$.

Proof. In [2] it was shown that $\underline{\mathrm{CY}}\dim(A) = 0$ if and only if A is a Nakayama algebra of Loewy length at most two; and $\underline{\mathrm{CY}}\dim(A) = 1$ if and only if $A \cong \mathrm{Mat}_m\big(\mathbf{k}[x]/(x^l)\big)$ where $l \geq 3$. Therefore, $\underline{\mathrm{CY}}\dim(\mathcal{P}(\mathbf{L}_n)) \geq 2$.

We set $A := \mathcal{P}(\mathbf{L}_n)$. If $\mathrm{char}(\mathbf{k}) = 2$, then $\tau = \mathrm{Id}_A$. Hence, Proposition 3.3 shows that $\Omega^3_{A^e}(A) \cong A$, so that $\Omega^3 \cong \mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A}$. Then $\underline{\mathrm{CY}}\dim(\mathcal{P}(\mathbf{L}_n)) = 2$.

If $\mathrm{char}(\mathbf{k}) \neq 2$, then the inequality $\underline{\mathrm{CY}}\dim(\mathcal{P}(\mathbf{L}_n)) \geq 2$ and Proposition 4.1 imply that $\underline{\mathrm{CY}}\dim(A) \geq 3$. By Proposition 3.3, we have $\Omega^6_{A^e}(A) \cong A_{\tau^2} = A$, and it follows that $\Omega^6 \cong \mathrm{Id}_{\underline{\mathrm{mod}}\text{-}A}$. Therefore, using Proposition 1.3, we see that $\underline{\mathrm{CY}}\dim(A) + 1$ divides 6. Since $\underline{\mathrm{CY}}\dim(A) \geq 3$, we obtain $\underline{\mathrm{CY}}\dim(A) = 5$. □

§5. Proof of Proposition 3.1

In the present section we need the notion of a *simple graph.* Informally speaking, it is an undirected graph that has no loops and at most one edge between any pair of vertices. However, for technical reasons it will be more convenient for us to employ the following formal definition. Under this approach, a simple graph is not a graph, and we are going to clearly distinguish between these two concepts. For example, $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{L}_n$ are graphs, but not simple graphs.

Let X be a set. We denote by $\mathrm{F}(X)$ the free $\mathbf{k}$-vector space generated by X, by $\bigwedge^2(X)$ the set of two-element subsets of the set X, and by $|X|$ the cardinality of the set X.

A *simple graph* is a pair $\Gamma = (\Gamma_0, \varrho)$, where Γ_0 is a set and $\varrho \subseteq \bigwedge^2(\Gamma_0)$. We call an element of Γ_0 a vertex of Γ, and an element of ϱ an edge of Γ. A simple graph Γ is *bipartite* if Γ_0 can be presented as a disjoint union $\Gamma_0 = U_0 \sqcup U_1$ such that $a, b \in U_i \Rightarrow \{a, b\} \notin \varrho$. In the latter case we call U_i a part of Γ.

Let $\pi_0(\Gamma)$ denote the set of connected components of a simple graph Γ, and let $[\gamma]$ be the component containing $\gamma \in \Gamma_0$.

Lemma 5.1. *Let $\Gamma = (\Gamma_0, \varrho)$ be a bipartite simple graph with parts U_0, U_1, and let V_ϱ be a subspace of $\mathrm{F}(\Gamma_0)$ generated by the set $\{\gamma + \gamma' \mid \{\gamma, \gamma'\} \in \varrho\}$. Then the following sequence is exact*:

$$0 \longrightarrow V_\varrho \longrightarrow \mathrm{F}(\Gamma_0) \xrightarrow{\theta} \mathrm{F}(\pi_0(\Gamma)) \longrightarrow 0,$$

where the first map is induced by inclusion, and the second map is given by the formula $\theta(\gamma) = (-1)^i[\gamma]$, *for* $\gamma \in U_i$.

Proof. Choose an element $\omega_C \in C$ in any component $C \in \pi_0(\Gamma)$. Let $\gamma \in \Gamma_0$. We denote by $d(\gamma)$ the length of a shortest path from $\omega_{[\gamma]}$ to γ. Let $x = \sum \lambda_\gamma \gamma \in \mathrm{F}(\Gamma_0)$, and put

$$h_0(x) = \sup\{d(\gamma) \mid \lambda_\gamma \neq 0\}, \qquad h_1(x) = \big|\{\gamma \mid \lambda_\gamma \neq 0 \wedge d(\gamma) = h_0(x)\}\big|.$$

We set the supremum of the empty set equal to zero. Therefore, we get a map

$$h : \mathrm{F}(\Gamma_0) \to \mathbb{N}_0 \times \mathbb{N}_0, \quad h(x) = (h_0(x), h_1(x)).$$

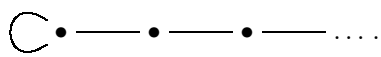
Here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and the set $\mathbb{N}_0 \times \mathbb{N}_0$ is viewed as a well-ordered set relative to the lexicographic ordering. It is easy to check that $h(x) = (0, 0)$ if and only if $x = 0$.

Obviously, θ is an epimorphism and $V_\varrho \subseteq \mathrm{Ker}(\theta)$. Therefore, it suffices to prove that $\mathrm{Ker}(\theta) \subseteq V_\varrho$. We consider $x \in \mathrm{Ker}(\theta)$ and choose $y \in V_\varrho$ such that $h(x - y)$ is the smallest possible element in the set $\mathbb{N}_0 \times \mathbb{N}_0$. We shall prove that $y = x$. Suppose the contrary; then $0 \neq x - y = \sum \lambda_\gamma \gamma$.

Case 1: $h_0(x - y) = 0$. Then $\lambda_\gamma = 0$ for all $\gamma \neq \omega_{[\gamma]}$. It follows that $x - y = \sum \lambda_{\omega_C} \omega_C$, and also $\lambda_{\omega_C} \neq 0$ for some C, because $x - y \neq 0$. Consequently, $\theta(x - y) = \sum \pm\lambda_{\omega_C} C \neq 0$, which contradicts the assumption $x - y \in \mathrm{Ker}(\theta)$.

Case 2: $h_0(x - y) > 0$. Then we consider γ such that $d(\gamma) = h_0(x - y)$ and $\lambda_\gamma \neq 0$. Since $d(\gamma) > 0$, there exists a vertex γ' connected by an edge with the vertex γ, but lying closer to the vertex $\omega_{[\gamma]}$, i.e., $d(\gamma') < d(\gamma)$. Consider $y' = y + \lambda_\gamma(\gamma + \gamma') \in V_\varrho$. It is easily seen that $h(x - y') < h(x - y)$, contradicting the assumption. □

Now we consider an auxiliary infinite graph (which is not a simple graph):

$\mathbf{L}_\infty$: 

This graph is associated with the infinite-dimensional (not unital) preprojective algebra $\mathcal{P}(\mathbf{L}_\infty) = \mathbf{k}Q_{\mathcal{P}(\mathbf{L}_\infty)}/I_{\mathbf{L}_\infty}$, where

$$Q_{\mathcal{P}(\mathbf{L}_\infty)} : \qquad \varepsilon = \bar{\varepsilon} \circlearrowright 0 \underset{\bar{\alpha}_0}{\overset{\alpha_0}{\rightleftarrows}} 1 \underset{\bar{\alpha}_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\bar{\alpha}_2}{\overset{\alpha_2}{\rightleftarrows}} \ \dots$$

$$I_{\mathbf{L}_\infty} : \qquad \begin{array}{c} \varepsilon^2 + \alpha_0\bar{\alpha}_0, \\ \bar{\alpha}_i\alpha_i + \alpha_{i+1}\bar{\alpha}_{i+1} \ \text{ for } \ i \geq 0. \end{array}$$

Denoting by $\Gamma_0(\mathbf{L}_\infty)$ the set of paths of the quiver $Q_{\mathcal{P}(\mathbf{L}_\infty)}$, we endow this set with a structure of a bipartite simple graph. The set of edges $\varrho_{\mathbf{L}_\infty}$ of this graph consists of the following two-element sets:

$$\{u\varepsilon^2 v, u\alpha_0\bar{\alpha}_0 v\}, \ \{u\bar{\alpha}_i\alpha_i v, u\alpha_{i+1}\bar{\alpha}_{i+1} v\},$$

where $i \geq 0$ and u, v are some paths. Let U_0 be a set of all paths such that the number of their arrows of the form α_{2k} $(k \geq 0)$ is even, and let $U_1 = \Gamma(\mathbf{L}_\infty) \setminus U_0$. It is easily seen that $\Gamma_{\mathbf{L}_\infty} = (\Gamma_0(\mathbf{L}_\infty), \varrho_{\mathbf{L}_\infty})$ is a bipartite simple graph with parts U_0 and U_1; moreover, $V_{\varrho_{\mathbf{L}_\infty}} = I_{\mathbf{L}_\infty}$, whence $\mathcal{P}(\mathbf{L}_\infty) \cong \mathrm{F}(\pi_0(\Gamma_{\mathbf{L}_\infty}))$.

We set

$$\begin{aligned} l_{k,m} &:= \bar{\alpha}_{k-1}\bar{\alpha}_{k-2}\dots\bar{\alpha}_m && \text{if } \ k > m \ \text{ and } \ l_{k,k} := e_k, \\ r_{m,k} &:= \alpha_m\alpha_{m+1}\dots\alpha_{k-1} && \text{if } \ k > m \ \text{ and } \ r_{k,k} := e_k, \\ l_k &:= l_{k,0}, \qquad r_k := r_{0,k}. \end{aligned}$$

Lemma 5.2.

(1) *Two paths of the quiver $Q_{\mathcal{P}(\mathbf{L}_\infty)}$ lie in one and the same connected component of the simple graph $\Gamma_{\mathbf{L}_\infty}$ if and only if their sources, targets, lengths, and the parities of the numbers of occurrences of ε coincide.*

(2) *In every component $\Gamma_{\mathbf{L}_\infty}$ there is only one path of the form $l_{i,j}r_{j,k}$ or $l_i\varepsilon^t r_k$. In particular, the images of these paths form a basis in the algebra $\mathcal{P}(\mathbf{L}_\infty)$.*

Proof. It is easy to show that if two paths of the quiver $Q_{\mathcal{P}(\mathbf{L}_\infty)}$ are connected by an edge in the simple graph $\Gamma_{\mathbf{L}_\infty}$, then their sources, targets, lengths and parities of numbers of occurrences of ε coincide. Consequently, the whole connected component of a path consists of paths with the same sources, targets, lengths, and the parities of the numbers of occurrences of ε. It suffices to prove that if the sources, targets, lengths, and parities of the numbers of occurrences of ε of two paths coincide, then they lie in the same connected component. In order to prove this, we show that in every connected component of $\Gamma_{\mathbf{L}_\infty}$ there is a path of the form $l_{i,j}r_{j,k}$ or $l_i\varepsilon^t r_k$. This will complete the proof, because a path of the form $l_{i,j}r_{j,k}$ or $l_i\varepsilon^t r_k$ is determined by its source, target, length, and the parity of the number of occurrences of ε.

Consider any path w. From the definition it is clear that we can always replace a subpath of the form $\alpha_i\bar{\alpha}_i$ by the subpath $\bar{\alpha}_{i-1}\alpha_{i-1}$ or ε^2, remaining in the same connected component. First, we replace the subpaths of this form with the largest possible i, then we repeat this operation, until we get a path without subpaths of the form $\alpha_i\bar{\alpha}_i$. It is easy to check that any path without such subpaths is a path of the form $l_{i,j}r_{j,k}$ or $l_i\varepsilon^t r_k$. Therefore, in any connected component there is a path of the form $l_{i,j}r_{j,k}$ or $l_i\varepsilon^t r_k$. This completes the proof. □

Proposition 5.3.

(1) *The union of the following three sets of paths forms a basis of $\mathcal{P}(\mathbf{L}_n)$. This basis will be denoted by $\mathcal{B}_n$.*
   - $l_{ij}r_{jk}$, *where* $i+k-j \le n-1$, $i,k \ge j \ge 0$;
   - $l_i\varepsilon^{2t}r_k$, *where* $i+k+t \le n-1$ *and* $i,k \ge 0$, $t \ge 1$;
   - $l_i\varepsilon^{2t+1}r_k$, *where* $\max(i,k)+t \le n-1$, $i,k,t \ge 0$.

(2) *All remaining paths of the form $l_{i,j}r_{j,k}$ and $l_i\varepsilon^t r_k$ vanish in $\mathcal{P}(\mathbf{L}_n)$.*

(3) *Two paths with the same sources, targets, lengths, and the parities of the numbers of occurrences of ε are equal up to a sign in $\mathcal{P}(\mathbf{L}_n)$.*

Proof. Consider the ideal $I_n \lhd \mathbf{k}Q_{\mathcal{P}(\mathbf{L}_\infty)}$ generated by the idempotents e_i with $i \ge n$. It is easily seen that $\mathcal{P}(\mathbf{L}_n) \cong \mathbf{k}Q_{\mathcal{P}(\mathbf{L}_\infty)}/(I_n + I_{\mathbf{L}_\infty})$. Consider the epimorphism $\theta : \mathbf{k}Q_{\mathcal{P}(\mathbf{L}_\infty)} \twoheadrightarrow \mathrm{F}(\pi_0(\Gamma_{\mathbf{L}_\infty}))$ occurring in Lemma 5.1. It is easy to check that $\theta(I_n) = \theta(I_n + I_{\mathbf{L}_\infty})$ is the vector space generated by the connected components of paths passing through vertices $i \ge n$. Since θ is an epimorphism, it follows that

$$\mathcal{P}(\mathbf{L}_n) \cong \mathbf{k}Q_{\mathcal{P}(\mathbf{L}_\infty)}/(I_n + I_{\mathbf{L}_\infty}) \cong \mathrm{F}(\pi_0(\Gamma_{\mathbf{L}_\infty}))/\theta(I_n).$$

Therefore, Lemma 5.2 implies that two paths with the same source, target, length, and the parity of the numbers of occurrences of ε are equal up to a sign in $\mathcal{P}(\mathbf{L}_n)$. Furthermore, using this isomorphism we see that the set of paths of the form $l_{i,j}r_{j,k}$ and $l_i\varepsilon^t r_j$ such that their connected components contain no path passing through a vertex $i \ge n$ is a basis of $\mathcal{P}(\mathbf{L}_n)$. The remaining paths of the form $l_{i,j}r_{j,k}$ and $l_i\varepsilon^t r_j$ are equal to zero in $\mathcal{P}(\mathbf{L}_n)$. It suffices to check that the paths mentioned in the first statement of the proposition are all paths of the form $l_{i,j}r_{j,k}$ and $l_i\varepsilon^t r_j$ such that their connected components contain no path passing through a vertex $i \ge n$. This technical statement can be checked with the help of Lemma 5.2. We leave this verification to the reader. □

We set

$$a_i := \sum_{l=1}^{i} l = \frac{i(i+1)}{2}.$$

Lemma 5.4. *The following relations hold true in the algebra* $\mathcal{P}(\mathbf{L}_n)$:

- $r_i l_{ik} = (-1)^{a_i - a_k} \varepsilon^{2(i-k)} r_k$;
- $r_{ki} l_i = (-1)^{a_i - a_k} l_k \varepsilon^{2(i-k)}$.

This lemma is proved by double induction on $i-k$ and on i, with the use of the relations $\alpha_i \bar{\alpha}_i = -\bar{\alpha}_{i-1}\alpha_{i-1}$ and $\alpha_0 \bar{\alpha}_0 = -\varepsilon^2$. We leave this proof to the reader.

Proof of Proposition 3.1. We set $A := \mathcal{P}(\mathbf{L}_n)$. Let $P_i = e_i A$ be a projective indecomposable A-module associated with $i \in Q_0$. Lemma 5.3 implies that any path of length at least $2n-1$ vanishes. On the other hand, for any $i \in \{0, \dots, n-1\}$, in A there exists a nonzero path $l_i \varepsilon^{2n-2i-1} r_i \in P_i$ of length $2n-1$. Consequently, Loewy length of every projective indecomposable module is equal to $2n$. We set $s_i := l_i \varepsilon^{2n-2i-1} r_i$; then s_i is a longest path with the source and target i. Consequently, $s_i \in \operatorname{soc}_A(e_i A) \cap \operatorname{soc}_{A^{\mathrm{op}}}(A e_i)$.

We prove that $\operatorname{soc}_A(P_i)$ is a one-dimensional module generated by s_i. It is easy to check that for any path $w \in \mathcal{B}_n \setminus \{s_i \mid i = 0..n-1\}$ there exists an arrow α such that $w\alpha \neq 0$. Consider an element $a \in P_i \setminus \langle s_i \rangle$. We express it as a linear combination of the basis vectors of $\mathcal{B}_n$ and choose any shortest path w in this linear combination. Let i be the target of w, and let $\lambda \neq 0$ be its coefficient in the linear combination. It is easy to check that there is at most one path in $\mathcal{B}_n$ starting at i and ending at j with a given length. Consequently, $ae_j = \lambda w + r$, where $r \in \operatorname{rad}^{|w|+1}(A)$. We consider an arrow α such that $w\alpha \neq 0$. Then $a\alpha = \lambda w\alpha + r\alpha$. Therefore, $\lambda w\alpha \in A_{|w|+1}$ and $r\alpha \in \operatorname{rad}^{|w|+2}(A)$. Consequently, Proposition 2.1 shows that $\lambda w\alpha \neq 0$ implies $a\alpha = \lambda w\alpha + r\alpha \neq 0$. It follows that $a \notin \operatorname{soc}(P_i)$, whence $\operatorname{soc}(P_i) = \langle s_i \rangle$. By analogy, we prove that $\operatorname{soc}_{A^{\mathrm{op}}}(A e_i) = \langle s_i \rangle$. Hence, the algebra A is self-injective by Theorem 13.4.2 in [11].

Now we prove that A is a symmetric algebra. For this, we construct a symmetric Frobenius form, i.e., a linear map $\eta : A \to \mathbf{k}$ such that the map $A \otimes A \to \mathbf{k}$ given by $a \otimes b \mapsto \eta(ab)$ is a symmetric nondegenerate bilinear form. We define $\eta : A \to \mathbf{k}$ on the basis $\mathcal{B}_n$ by the formula

$$\eta(b) = \begin{cases} (-1)^{a_i} & \text{if } \ b = s_i, \\ 0 & \text{if } \ b \neq s_i \text{ for any } \ i. \end{cases}$$

It is well known and easy to check that, for self-injective bound quiver algebras, if $\eta|_{\operatorname{soc}(P_i)} \neq 0$ then η is a Frobenius form.

It remains to prove that η is a symmetric bilinear form. We need to check that $\eta(ab) = \eta(ba)$ for every $a, b \in A$. It suffices to check this on the basis. Consider $b_1, b_2 \in \mathcal{B}_n$. We are only interested in the case where $s(b_1) = t(b_2)$ and $t(b_1) = s(b_2)$, $|b_1| + |b_2| = 2n-1$, and the parities of the number of occurrences of ε in b_1 and b_2 are different, because otherwise $b_1 b_2$ and $b_2 b_1$ are equal up to a sign to paths of the form $l_{ij} r_{jk}, l_i \varepsilon^t r_k$ except s_i, or zero, and therefore $\eta(b_1 b_2) = 0 = \eta(b_2 b_1)$. Suppose $i = s(b_1) = t(b_2)$, $j = t(b_1) = s(b_2)$, $|b_1| + |b_2| = 2n-1$, and the parities of the number of occurrences of ε in b_1 and b_2 are different. Since the situation is symmetric, we may assume that the number of occurrences of ε in b_1 is odd, and the number of those in b_2 is even. Then only two cases are possible.

(1) $b_1 = l_i\varepsilon^{2t+1}r_j$, $b_2 = l_{jk}r_{ki}$, where $|b_1| + |b_2| = 2(i+j+t-k)+1 = 2n-1$. Using Lemma 5.4, we get

$$b_1b_2 = l_i\varepsilon^{2t+1}r_jl_{jk}r_{ki} = (-1)^{a_j-a_k}l_i\varepsilon^{2(t+j-k)+1}r_i = (-1)^{a_j-a_k}s_i,$$
$$b_2b_1 = l_{jk}r_{ki}l_i\varepsilon^{2t+1}r_j = (-1)^{a_i-a_k}l_j\varepsilon^{2(t+i-k)+1}r_i = (-1)^{a_i-a_k}s_j.$$

It follows that $\eta(b_1b_2) = (-1)^{a_j-a_k+a_i} = \eta(b_2b_1)$.

(2) $b_1 = l_i\varepsilon^{2t+1}r_j, b_2 = l_j\varepsilon^{2t}r_i$, where $|b_1| + |b_2| = 2(2t+k+i)+1 = 2n-1$. Using Lemma 5.4, we get

$$b_1b_2 = l_i\varepsilon^{2t+1}r_jl_j\varepsilon^{2t}r_i = (-1)^{a_j}l_i\varepsilon^{2(2t+j)+1}r_i = (-1)^{a_j}s_i,$$
$$b_2b_1 = l_j\varepsilon^{2t}r_il_i\varepsilon^{2t+1}r_j = (-1)^{a_i}l_j\varepsilon^{2(2t+i)+1}r_j = (-1)^{a_i}s_j.$$

Consequently, $\eta(b_1b_2) = (-1)^{a_j+a_i} = \eta(b_2b_1)$. □

Acknowledgments

I wish to express my sincere gratitude to A. I. Generalov and M. A. Antipov for their interest in this work and helpful remarks.

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Received 14/OCT/2011

Translated by THE AUTHOR