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ON TIGHT SPHERICAL DESIGNS

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ABSTRACT. Let X be a tight t-design of dimension n, and let $t = 5$ or $t = 7$ (the open cases). An investigation of the lattice generated by X by using arithmetic theory of quadratic forms allows one to exclude infinitely many values of n.

§1. INTRODUCTION

Spherical designs were introduced in 1977 by Delsarte, Goethals, and Seidel in [5]; shortly afterward they were studied by Eiichi Bannai in a series of papers (see [1, 2, 3] to mention only a few of them). A spherical t-design is a finite subset X of the sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid (x, x) = 1\}$$

such that every polynomial on $\mathbb{R}^n$ of total degree at most t has the same average over X as over the entire sphere. Of course the most interesting t-designs are those of minimal cardinality. If $t = 2m$ is even, then any spherical t-design $X \subset S^{n-1}$ satisfies

$$|X| \geq \binom{n-1+m}{m} + \binom{n-2+m}{m-1},$$

and if $t = 2m + 1$ is odd, then

$$|X| \geq 2\binom{n-1+m}{m}.$$

A t-design X for which equality occurs is called a *tight* t-design.

Tight t-designs in $\mathbb{R}^n$ with $n \geq 3$ are very rare. In [1] and [2] it was shown that such tight designs only exist if $t \leq 5$ or $t = 7, 11$. The tight t-designs with $t = 1, 2, 3$ as well as $t = 11$ are classified completely, whereas their classification for $t = 4, 5, 7$ is still an open problem. It is known that the existence of a tight 4-design in dimension $n - 1$ is equivalent to the existence of a tight 5-design in dimension n, so the open cases are $t = 5$ and $t = 7$. It is also well known that tight spherical t-designs X for odd values of t are antipodal, i.e., $X = -X$ (see [5]).

There are certain numerical conditions on the dimension of such tight designs. A tight 5-design $X \subset S^{n-1}$ can only exist if either $n = 3$ and X is the set of 12 vertices of a regular icosahedron, or $n = (2m + 1)^2 - 2$ for an integer m [5, 1, 2]. Existence is only known for $m = 1, 2$, and these designs are unique. Using lattices [4] excludes the next two open cases $m = 3, 4$ as well as infinitely many of other values of m. Here we exclude infinitely many other cases, including $m = 6$.

There are similar results for tight 7-designs. Such designs only exist if $n = 3d^2 - 4$, where the only known cases are $d = 2, 3$ and the corresponding designs are unique. The paper [4] excludes the cases of $d = 4, 5$ and also gives partial results on the interesting case of $d = 6$, which still remains open. For odd values of d, we use characteristic vectors

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of the associated odd lattice of odd determinant to show that d is either $\pm 1 \pmod{16}$ or $\pm 3 \pmod{32}$ (see Theorem 3.5). We also exclude infinitely many even d in Theorem 3.3.

§2. General identities

We always deal with antipodal sets and write them as a disjoint union

$$X \dot\cup -X \subset S^{n-1}(d) = \{x \in \mathbb{R}^n \mid (x,x) = d\} \text{ with } s := |X| \in \mathbb{N}.$$

By the theory developed in [7], the set $X \dot\cup -X$ is a 7-design if and only if for all $\alpha \in \mathbb{R}^n$ we have

$$(D6)(\alpha) : \sum_{x\in X} (x,\alpha)^6 = \frac{3\cdot 5sd^3}{n(n+2)(n+4)}(\alpha,\alpha)^3.$$

Applying the Laplace operator to $(D6)(\alpha)$, we get

$$(D4)(\alpha) : \sum_{x\in X} (x,\alpha)^4 = \frac{3sd^2}{n(n+2)}(\alpha,\alpha)^2,$$

$$(D2)(\alpha) : \sum_{x\in X} (x,\alpha)^2 = \frac{sd}{n}(\alpha,\alpha).$$

Substituting $\alpha = \sum_{i=1}^6 \xi_i\alpha_i$ in $(D6)$, $(D4)$, and $(D2)$, we find that, for all $\alpha, \beta \in \mathbb{R}^n$,

$$(D11) \qquad \sum_{x\in X} (x,\alpha)(x,\beta) = \frac{sd}{n}(\alpha,\beta),$$

$$(D13) \qquad \sum_{x\in X} (x,\alpha)(x,\beta)^3 = \frac{3sd^2}{n(n+2)}(\alpha,\beta)(\beta,\beta),$$

$$(D22) \qquad \sum_{x\in X} (x,\alpha)^2(x,\beta)^2 = \frac{sd^2}{n(n+2)}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)),$$

$$(D15) \qquad \sum_{x\in X} (x,\alpha)(x,\beta)^5 = \frac{3\cdot 5sd^3}{n(n+2)(n+4)}(\beta,\beta)^2(\alpha,\beta),$$

$$(D24) \qquad \sum_{x\in X} (x,\alpha)^2(x,\beta)^4 = \frac{3sd^3}{n(n+2)(n+4)}((\beta,\beta)^2(\alpha,\alpha) + 4(\alpha,\beta)^2(\beta,\beta)),$$

$$(D33) \qquad \sum_{x\in X} (x,\alpha)^3(x,\beta)^3 = \frac{3sd^3}{n(n+2)(n+4)}(2(\alpha,\beta)^3 + 3(\alpha,\alpha)(\beta,\beta)(\alpha,\beta)).$$

Similarly, $X \dot\cup -X$ is a spherical 5-design if an only if $(D4)$ and $(D2)$ are valid for any $\alpha \in \mathbb{R}^n$. Then we obtain $(D11)$, $(D13)$, and $(D22)$.

We shall consider the lattice $\Lambda := \langle X \rangle$ and $\alpha \in \Lambda^*$. Then (α, x) is integral for all $x \in X$. This yields certain integrality conditions for the norms and inner products of elements in Λ^*.

Lemma 2.1. *If $X \dot\cup -X \subset S^{n-1}(d)$ is a spherical 5-design, then*

$$\frac{sd}{12n}(\alpha,\alpha)\Big(\frac{d}{n+2}(\alpha,\alpha) - 1\Big) \in \mathbb{Z}$$

and

$$\frac{sd}{6n}(\alpha,\beta)\Big(\frac{d}{n+2}(\alpha,\alpha) - 1\Big) \in \mathbb{Z}$$

for all $\alpha, \beta \in \Lambda^*$.

Proof. Let $x \in X$ and put $k := (x, \alpha)$. Then $k^4 - k^2$ is a multiple of 12, whence $\frac{1}{12}\sum_{x\in X}(x,\alpha)^4 - (x,\alpha)^2 \in \mathbb{Z}$, which yields the first divisibility condition. Similarly, $k^3 - k$ is a multiple of 6, whence

$$\frac{1}{6}\sum_{x\in X}(x,\beta)\big((x,\alpha)^3 - (x,\alpha)\big) = \frac{1}{6}(D13 - D11) \in \mathbb{Z}. \qquad \square$$

Similarly,

$$(\beta,x)(\alpha,x)\big((\alpha,x)^2-1\big)\big((\alpha,x)^2-4\big) = (\beta,x)(\alpha,x)^5 - 5(\beta,x)(\alpha,x)^3 + 4(\beta,x)(\alpha,x)$$

is divisible by 5 consecutive integers, so that this quantity is a multiple of 120 for any $\alpha, \beta \in \Lambda^*$ and $x \in X$.

Moreover, $(\alpha, x)((\alpha, x)^2 - 1)$ is divisible by 3 consecutive integers and, therefore, is a multiple of 6; hence,

$$(\beta,x)\big((\beta,x)^2-1\big)(\alpha,x)\big((\alpha,x)^2-1\big) = (\beta,x)(\alpha,x)\big((\beta,x)^2(\alpha,x)^2 - (\beta,x)^2 - (\alpha,x)^2 + 1\big)$$

is divisible by 36. Summing over all $x \in X$, we see that the right-hand side of $D15 - 5D13 + 4D11$ is a multiple of 120 and that $D33 - D13 - D31 + D11$ is divisible by 36.

Lemma 2.2. *If $X \,\dot\cup\, -X \subset S^{n-1}(d)$ is a spherical 7-design then, for all $\alpha, \beta \in \Lambda^*$,*

$$\frac{1}{120}(\alpha,\beta)\left(\frac{3\cdot 5sd^2}{n(n+2)}(\alpha,\alpha)\left(\frac{d}{n+4}(\alpha,\alpha)-1\right) + 4\frac{sd}{n}\right) \in \mathbb{Z}$$

and

$$\frac{1}{36}(\alpha,\beta)\left(\frac{3sd^2}{n(n+2)}\left(\frac{d}{n+4}(2(\alpha,\beta)^2 + 3(\alpha,\alpha)(\beta,\beta)) - (\alpha,\alpha) - (\beta,\beta)\right) + \frac{sd}{n}\right) \in \mathbb{Z}.$$

§3. Tight spherical 7-designs

Let $X\dot\cup{-X} \subset S^{n-1}(d)$ be a tight spherical 7-design. Then $n = 3d^2-4$, $(x, y) \in \{0, \pm 1\}$ for all $x \neq y \in X$, and $s := |X| = n(n + 1)(n + 2)/6$.

Let $\Lambda = \langle X\rangle$ be the lattice generated by the set X, and put $\Gamma := \Lambda^*$. Then Λ is an integral lattice, and Λ is even if d is even. Substituting these values in the formulas of Lemma 2.2, we obtain the following statement.

Lemma 3.1. *For all $\alpha, \beta \in \Gamma$, we have*

$$\big((d^3-d)/240\big)(\alpha,\beta)\big(12d^2 - 8 - 15d(\alpha,\alpha) + 5(\alpha,\alpha)^2\big) \in \mathbb{Z}$$

and

$$\big((d^3-d)/72\big)(\alpha,\beta)\big(3(\alpha,\alpha)(\beta,\beta) - 3d\big((\alpha,\alpha)+(\beta,\beta)\big) + 2(\alpha,\beta)^2 + (3d^2-2)\big) \in \mathbb{Z}.$$

For a prime p, let v_p denote the p-adic valuation on $\mathbb{Q}$.

Corollary 3.2 (Improvement of [4, Lemma 4.2])**.**

(i) *Let $p \geq 5$ be a prime. If $v_p(d^3 - d) \leq 2$, then $v_p\big((\alpha,\alpha)\big) \geq 0$ for all $\alpha \in \Gamma$.*
(ii) *If $v_3(d^3 - d) \leq 4$, then $v_3\big((\alpha,\alpha)\big) \geq 0$ for all $\alpha \in \Gamma$.*
(iii) *If $v_2(d^3 - d) \leq 6$, then $v_2\big((\alpha,\alpha)\big) \geq 0$ for all $\alpha \in \Gamma$.*
(iv) *If d is even but not divisible by 8, then $v_2\big((\alpha,\alpha)\big) \geq 1$ for all $\alpha \in \Gamma$.*
(v) *If d is even but not divisible by 32, then $v_2\big((\alpha,\beta)\big) \geq 0$ for all $\alpha, \beta \in \Gamma$.*
(vi) *If d is odd and $v_2(d^2 - 1) \leq 4$, then $v_2\big((\alpha,\beta)\big) \geq 0$ for all $\alpha, \beta \in \Gamma$.*

Proof. Parts (i), (iii), and (iv) are the same as in [4, Lemma 4.2] and follow from the first congruence in Lemma 3.1.

For (ii), we use the second congruence in the special case where $\alpha = \beta$. Using the assumption, we obtain $v_3\big((d^3-d)/72\big) \le 4-2 \le 2$. If $v_3\big((\alpha,\alpha)\big) \le -1$, then

$$v_3\big(5(\alpha,\alpha)^3 - 6d(\alpha,\alpha)^2 + (3d^2-2)(\alpha,\alpha)\big) = v_3\big((\alpha,\alpha)^3\big) \le -3,$$

contradicting the fact that the product is integral.

To check (v), we use (iii) to see that $v_2((\alpha,\alpha)) \ge 0$ for all $\alpha \in \Gamma$. Then the second congruence yields $v_2\big(\frac{d}{4}(\alpha,\beta)^3\big) \ge 0$. Since $v_2(d) < 5$, we obtain $v_2\big((\alpha,\beta)\big) \ge 0$.

The last assertion (vi) is obtained by the same argument. □

Using this observation, we can extend [4, Theorem 4.3], which only treats the case where $v_2(d) = 2$.

Theorem 3.3. *Assume that $v_p(d^3-d) \le 2$ for all primes $p \ge 5$ and that $v_3(d^3-d) \le 4$. If $v_2(d)$ is equal to $2, 3,$ or 4, then no tight spherical 7-design exists in dimension $n = 3d^2-4$.*

Proof. Since Γ is integral by Corollary 3.2, Λ is an even unimodular lattice of dimension $n \equiv 4 \pmod 8$, which gives a contradiction. □

A similar argument allows us to deduce the following lemma from Corollary 3.2.

Lemma 3.4. *If d is odd and $v_2(d^2-1) \le 4$, then Λ is an odd lattice of odd determinant. If, moreover, $v_p(d^3-d) \le 2$ for all primes $p \ge 5$ and $v_3(d^3-d) \le 4$, then $\Lambda = \Lambda^*$ is an odd unimodular lattice.*

In particular if d is odd and $d \not\equiv \pm 1 \pmod{16}$, then Λ is an odd lattice of odd determinant. Over the 2-adic numbers there is an orthogonal basis,

$$\Lambda \otimes \mathbb{Z}_2 \cong \langle b_1,\dots,b_n\rangle_{\mathbb{Z}_2} \text{ with } (b_i,b_j) = 0, (b_k,b_k) = 1, (b_n,b_n) = 1+\delta \in \{1,3,5,7\}$$

for $1 \le i \ne j \le n$, $k = 1,\dots,n-1$. Such a lattice contains *characteristic vectors.* These are elements $\alpha \in \Lambda \otimes \mathbb{Z}_2$ such that

$$(\alpha,\lambda) \equiv (\lambda,\lambda) \pmod 2 \text{ for all } \lambda \in \Lambda \otimes \mathbb{Z}_2.$$

In terms of the basis as above, the characteristic vectors in Λ are of the form

$$\alpha = \sum_{i=1}^{n} a_i b_i \quad \text{with} \quad a_i \in 1+2\mathbb{Z}_2, \quad \text{and} \quad (\alpha,\alpha) \equiv n+\delta \pmod 8.$$

Theorem 3.5. *Let $X \dot\cup -X$ be a tight 7-design of dimension $3d^2-4$ with odd d. Assume that $d \not\equiv \pm 1 \pmod{16}$. Then either $d \equiv 3 \pmod{32}$ and $\det(\Lambda) \in (\mathbb{Z}_2^*)^2$, or $d \equiv -3 \pmod{32}$ and $\det(\Lambda) \in 3(\mathbb{Z}_2^*)^2$. If, moreover, $v_p(d^3-d) \le 2$ for all primes $p \ge 5$ and $v_3(d^3-d) \le 4$, then $d \not\equiv -3 \pmod{16}$.*

Proof. Let $\Lambda = \langle X\rangle_{\mathbb{Z}_2}$, and let $\alpha \in \Lambda$ be a characteristic vector of Λ of norm $(\alpha,\alpha) = n+\delta-8a$ for some $a \in \mathbb{Z}_2$ and $\delta \in \{0,2,4,6\}$. Then $(\alpha,\lambda) \equiv (\lambda,\lambda) \pmod 2$ for all $\lambda \in \Lambda$; in particular, (α,x) is odd for all $x \in X$. For $k > 0$, let

$$n_k := |\{x \in X \mid (x,\alpha) = \pm k\}|.$$

Then (D2), (D4), (D6) yield

$$(D0) \qquad \sum n_k = |X| = (1/2)(3d^2-4)(3d^2-2)(d^2-1),$$

$$(D2) \qquad \sum k^2 n_k = (1/2)(3d^2-2)(d^2-1)d(n+\delta-8a),$$

$$(D4) \qquad \sum k^4 n_k = (3/2)(d^2-1)d^2(n+\delta-8a)^2,$$

$$(D6) \qquad \sum k^6 n_k = (5/2)(d^2-1)d(n+\delta-8a)^3.$$

Now $n_k \neq 0$ only for odd k. If k is odd, then (k^2-1) is a multiple of 8 and $(k^2-1)(k^2-9)$ is a multiple of $8 \cdot 16$. Now $(k^2-1)(k^2-9)(k^2-25) = k^6 - 35k^4 + 259k^2 - 225$ is a multiple of $2^{10}3^2 5$; in particular,

$$\text{(a)} \qquad 2^{-7}\big((D4) - 10(D2) + 9(D0)\big) \in \mathbb{Z}$$

and

$$\text{(b)} \qquad 2^{-10}\big((D6) - 35(D4) + 259(D2) - 225(D0)\big) \in \mathbb{Z}.$$

We substitute $d = 16b + r$ for $r = \pm 3, \pm 5, \pm 7$ in these congruences to obtain polynomials in a the coefficients of which are polynomials in b. The contradictions we obtain in the respective cases are listed below the table:

$r =$	3	5	7	-7	-5	-3
$\delta = 0$	$(c0)$	$(a2)$	$(b1)$	$(a1)$	$(c2)$	$(a2)$
$\delta = 2$	$(a2)$	$(c2)$	$(a1)$	$(b1)$	$(a2)$	$(c0)$
$\delta = 4$	$(c1)$	$(a2)$	$(b2)$	$(a1)$	$(c1)$	$(a2)$
$\delta = 6$	$(a2)$	$(c1)$	$(a1)$	$(b2)$	$(a2)$	$(c1)$

(a) In congruence (a), the coefficients of a and a^2 are in $\mathbb{Z}[b]$ but the constant coefficient is
 (a1) $p(b) + \frac{b}{2} + \frac{x}{4}$ with $p(b) \in \mathbb{Z}[b]$ and x odd;
 (a2) $p(b) + \frac{b}{2} + \frac{x}{8}$ with $p(b) \in \mathbb{Z}[b]$ and x odd.

(b) In congruence (b), the coefficients of a, a^2, and a^3 are in $\mathbb{Z}[b]$, but the constant coefficient is
 (b1) $p(b) + \frac{1}{2}$ with $p(b) \in \mathbb{Z}[b]$;
 (b2) $p(b) + \frac{b}{2} + \frac{x}{4}$ with $p(b) \in \mathbb{Z}[b]$ and x odd.

(c) In congruence (b), the coefficient of a^3 is in $\mathbb{Z}[b]$, those of a and a^2 are in $\frac{1}{2} + \mathbb{Z}[b]$, but the constant coefficient is
 (c0) $p(b) + \frac{b}{2}$ with $p(b) \in \mathbb{Z}[b]$. Here we can only deduce that b is even;
 (c1) $p(b) + \frac{x}{8}$ with $p(b) \in \mathbb{Z}[b]$ and x odd;
 (c2) $p(b) + \frac{b}{2} + \frac{x}{4}$ with $p(b) \in \mathbb{Z}[b]$ and x odd.

Hence, only the cases where $r = 3$, $\delta = 0$ and $r = -3$, $\delta = 2$ are possible, and then b is even. □

To summarize, we list a few small values that are excluded by Theorems 3.5 and 3.3.

Corollary 3.6. *There is no tight* 7*-design of dimension* $n = 3d^2 - 4$ *for*

$$d \in \{4, 5, 7, 8, 9, 11, 12, 13, 16, 19, 20, 21, \dots\}.$$

§4. Tight spherical 5-designs

Assume that $d = 2m + 1$ and that $X \dot\cup -X$ is a tight spherical 5-design in dimension $n = d^2 - 2$. Then $|X| = n(n+1)/2$, and scaling so that $(x, x) = d$ for all $x \in X$, we have $(x, y) = \pm 1$ for $x \neq y \in X$ and $\Lambda := \langle X \rangle$ is an odd integral lattice. With these values, formula $(D4)$ looks like this:

$$(D4) \qquad \sum_{x \in X} (x, \alpha)^4 = 6m(m+1)(\alpha, \alpha)^2.$$

Lemma 4.1 (see [4, Lemma 3.6])**.** *Assume that* $m(m+1)$ *is not divisible by the square of a prime* $p \geq 5$*. Then* $(\alpha, \alpha) \in \mathbb{Z}[1/6]$ *for all* $\alpha \in \Lambda^*$*.*

Substituting the special values in the formula of Lemma 2.1, we immediately obtain the following.

Lemma 4.2 (see [4, Lemma 3.3]). *For all $\alpha \in \Lambda^*$,*

$$\frac{1}{6}m(m+1)(\alpha,\alpha)\big(3(\alpha,\alpha)-(2m+1)\big) \in \mathbb{Z}.$$

Corollary 4.3. *If $m(m+1)$ is not a multiple of 8, then $(\alpha,\alpha) \in \mathbb{Z}_2$ is 2-integral for all $\alpha \in \Lambda^*$.*

Now we treat the Sylow 3-subgroup $D_3 := \mathrm{Syl}_3(\Lambda^*/\Lambda)$.

Lemma 4.4. *Assume that $m(m+1)$ is not a multiple of 9. Then $|D_3| \in \{1,3\}$.*

Proof. Assume that $D_3 \neq 1$. Since D_3 is a regular quadratic 3-group, it contains an anisotropic element $\alpha + \Lambda \in \Lambda^*/\Lambda$ with $(\alpha,\alpha) = \frac{p}{q}$ and $3 \mid q$. By $(D4)$, the denominator q is not divisible by 9; in particular, the exponent of D_3 is 3 and $(\alpha,\alpha) = \frac{p}{3}$ with a 3-adic unit $p \equiv \pm 1 \pmod 3$. Now Lemma 4.2 gives

$$\frac{1}{18}m(m+1)p\big(p-(2m+1)\big) \in \mathbb{Z}.$$

Since $m(m+1)$ is not a multiple of 9, this implies that $p \equiv (2m+1) \pmod 3$. If $|D_3| > 3$, then the regular quadratic $\mathbb{F}_3$-space D_3 is universal, representing also elements $\frac{p}{3}$ with $p \not\equiv (2m+1) \pmod 3$. This is a contradiction. So, $|D_3| = 1$ or $|D_3| = 3$. □

Let Λ_+ be the even sublattice of $\Lambda = \langle X \rangle$. Then $\Lambda = \Lambda_+ \dot\cup \Lambda_-$ with $\Lambda_- = x + \Lambda_+$ for any $x \in X$. Since (x,y) is odd for all $x \in X$, we have

$$\Lambda_+ = \left\{ \sum_{x \in X} c_x x \;\middle|\; c_x \in \mathbb{Z}, \sum_{x\in X} c_x \text{ even} \right\},$$

and $(\alpha, x) \in 2\mathbb{Z}$ for any $\alpha \in \Lambda_+$ and $x \in X$. Therefore, $\Lambda_+ \subset 2\Lambda^*$ and the lattice $\Gamma := \frac{1}{\sqrt{2}}\Lambda_+$ is an integral lattice of dimension n.

The next lemma is an improvement of [4, Lemma 3.6].

Lemma 4.5. *Assume that $m(m+1)$ is not divisible by the square of an odd prime and that m is odd and $(m+1)$ is not a multiple of 8. Then for any $x \in X$ we have*

$$\Gamma^*/\Gamma = \left\langle \frac{1}{\sqrt{2}}x + \Gamma \right\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. For odd primes p, the Sylow p-subgroup of Γ^*/Γ is isomorphic to that of Λ^*/Λ and, hence, to $\{0\}$ for $p \geq 5$, and either $\{0\}$ or $\mathbb{Z}/3\mathbb{Z}$ for $p = 3$. Clearly $\alpha := \frac{1}{\sqrt{2}}x \in \Gamma^*$ has order 2 modulo Γ. Moreover,

$$\Gamma^* = \sqrt{2}\Lambda_+^* = \langle \alpha, \sqrt{2}\Lambda^* \rangle$$

is an overlattice of $\sqrt{2}\Lambda^*$ of index 2. By Corollary 4.3, $(\beta,\beta) \in 2\mathbb{Z}_2$ for all elements $\beta \in \sqrt{2}\Lambda^*$, and since $x \in \Lambda$, we get $(\beta,\alpha) \in \mathbb{Z}$ for all $\beta \in \sqrt{2}\Lambda^*$. Since the Sylow 2-subgroup D_2 of Γ^*/Γ is a regular quadratic 2-group, and $D_2 \cap \sqrt{2}\Lambda^*/\Gamma$ is in the radical of this group, we see that $D_2 = \langle \alpha + \Gamma \rangle \cong \mathbb{Z}/2\mathbb{Z}$. To exclude the case where $D_3 = \mathbb{Z}/3\mathbb{Z}$, we use the fact that Γ is an even lattice and, hence, the Gauss sum

$$G(\Gamma) := \frac{1}{\sqrt{2\cdot 3^t}} \sum_{d \in \Gamma^*/\Gamma} \exp(2\pi i q(d))$$

for the quadratic group $(\Gamma^*/\Gamma, q)$ with $q(z+\Gamma) := \frac{1}{2}(z,z) + \mathbb{Z}$ is equal to

$$G(\Gamma) = \exp\left(\frac{2\pi i}{8}\right)^n = \exp\left(\frac{2\pi i}{8}\right)^{-1}$$

by the Milgram–Braun formula. Clearly, $G(\Gamma)$ is the product of the Gauss sums of its Sylow subgroups, $G(\Gamma) = G_2 G_3$ with

$$G_2 = \frac{1}{\sqrt{2}}\Big(1 + \exp\Big(2\pi i \frac{2m+1}{4}\Big)\Big) = \frac{1-i}{\sqrt{2}} = \exp\Big(\frac{2\pi i}{8}\Big)^{-1} = G(\Gamma),$$

because m is odd. This implies that $G_3 = 1$. Then [6, Corollary 5.8.3] shows that D_3 cannot be anisotropic, and, hence, $D_3 = \{0\}$ by Lemma 4.4. ☐

Theorem 4.6 (see also [4, Theorem 3.10] for one case)**.** *Assume that $m(m+1)$ is not divisible by the square of an odd prime, and that m is even but not divisible by* 8. *Then* $\Gamma^*/\Gamma \cong \mathbb{Z}/6\mathbb{Z}$ *and* $m \equiv -1 \pmod 3$.

Proof. With the same proof as above, we obtain $G(\Gamma) = \exp(\frac{2\pi i}{8})^{-1}$ and $G_2 = \exp(\frac{2\pi i}{8})$, whence $G_3 = -i$. Then [6, Corollary 5.8.3] shows that $D_3 = \langle \beta + \Gamma \rangle$ with $3(\beta, \beta) \equiv 1 \pmod 3$. Let $\lambda := \sqrt{2}\beta \in \Lambda^*$. Then $(\lambda, \lambda) = \frac{p}{3}$ with $p \equiv 2 \pmod 3$. Then the integrality condition in Lemma 4.2 implies that

$$m(m+1)(2m-1) \in 9\mathbb{Z}_3$$

is a multiple of 9. It follows that $m \not\equiv 1 \pmod 3$ as it was already observed in [4], and it also follows that $m \not\equiv 0 \pmod 3$. ☐

Corollary 4.7. $m \neq 3, 4, 6, 10, 12, 22, 28, 30, 34, 42, 46, \dots$.

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