UNIQUE SOLVABILITY OF THE DIRICHLET PROBLEM FOR THE EQUATION $\Delta_p u = 0$ IN THE EXTERIOR OF A PARABOLOID

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ABSTRACT. The Dirichlet problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = 0 \text{ in } \Omega, \ u|_{\partial\Omega} = f,$$

is considered in the exterior of an $n$-dimensional paraboloid, $p \in (1, n)$. The space of the traces $u|_{\Gamma}$ on the boundary of the paraboloid for functions $u$ in the class $L^1_p$ is described explicitly. This implies necessary and sufficient conditions for the existence and uniqueness of a solution to the Dirichlet problem.

§1. INTRODUCTION AND FORMULATION OF THE RESULTS

As is known (see [1, 2, 3]), the question of solvability of the Dirichlet problem for the equation $\Delta_p u = 0$ in a domain $\Omega \subset \mathbb{R}^n$ reduces to the description of the boundary traces of the functions with finite “Dirichlet integral” $\int_{\Omega} |\nabla u|^p \, dx$. By the Gagliardo theorem [4] (see also [5, 6]), the boundary traces of the elements in $W^{1,p}_1(\Omega)$ form the space $W^{1-1/p}_p(\partial\Omega)$ provided $\Omega$ is a Lipschitz domain with compact closure.

If $\Omega$ has an infinite locally Lipschitz boundary, the Gagliardo theorem gives only a local description of the traces, but, largely, this information is insufficient for a complete characterization.

In the present paper, for an unbounded $\Omega$, we introduce the space $H^{1}_p(\Omega)$ as the closure in the norm $\|\nabla(\cdot)\|_{L^p(\Omega)}$ of the set of smooth functions with bounded support in $\Omega$. We define a solution of the above Dirichlet problem in the class $H^{1}_p(\Omega)$ and give a description of the boundary trace space for $H^{1}_p$ in the exterior of a multidimensional paraboloid. Thus, we obtain necessary and sufficient conditions for the solvability of the (one-sided or two-sided) Dirichlet problem for the equation $\Delta_p u = 0$ in the exterior of a paraboloid. Earlier in [7], necessary and sufficient conditions were given for the existence and uniqueness of an energy solution of the Dirichlet problem for the Laplace equation in the exterior of a multidimensional paraboloid. In [8], necessary and sufficient conditions were found for the unique solvability of the Dirichlet problem for the equation

$$-\Delta_p u + a|u|^{p-2}u = 0, \quad (a \in L_\infty, \ a(x) \geq \text{const} > 0),$$

in the class $W^{1}_p$ inside and outside of a paraboloid.

Now we formulate the principle results of the present paper. Let $\varphi$ be a monotone increasing Lipschitz function on the half-axis $[0, \infty)$ such that

$$\varphi(0) = 0, \ \varphi'(0+0) > 0, \ \lim_{z \to \infty} \varphi'(z) = 0, \ \lim_{z \to \infty} \varphi(z) = \infty.$$
We write a point \( x \in \mathbb{R}^n \) in the form \( x = (y, z) \) with \( y \in \mathbb{R}^{n-1}, z \in \mathbb{R}^1 \). The domains \( \Omega^+ \subset \mathbb{R}^n \) are defined by

\[
\Omega^+ = \{(y, z) \in \mathbb{R}^n : z \in (0, \infty), |y| < \varphi(z)\}, \quad \Omega^- = \mathbb{R}^n \setminus \Omega^+.
\]

\( \Omega^+ \) is called an \( n \)-dimensional paraboloid, and \( \Omega^- \) is its exterior.

Let \( \Gamma \) be the common boundary of \( \Omega^+ \) and \( \Omega^- \). In each domain we consider the Dirichlet problem

\[
\begin{align*}
\text{(1.1)} & \quad - \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \text{ in } \Omega^+, \quad u|_{\Gamma} = f, \\
\text{(1.2)} & \quad - \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \text{ in } \Omega^-, \quad u|_{\Gamma} = f.
\end{align*}
\]

A function \( u \in H^1_p(\Omega^+) \) is called the solution of problem (1.1) if \( u|_{\Gamma} = f \) and

\[
\int_{\Omega^+} (|\nabla u|^{p-2} \nabla u \nabla v) \, dx = 0
\]

for all \( v \in C_0^\infty(\Omega^+) \). A solution of problem (1.2) is defined in the same way (with \( \Omega^+ \) replaced by \( \Omega^- \)). It is well known \([1, 3]\) that the above Dirichlet problem in \( \Omega^\pm \) admits a variational statement: its solution provides the minimum to the functional

\[
\Phi(u) = \int_{\Omega^\pm} |\nabla u|^p \, dx
\]

over the set of functions satisfying the boundary condition \( u|_{\Gamma} = f \). It can be shown that there exists a minimizing sequence for \( \Phi \), which is weakly convergent in \( H^1_p(\Omega^\pm) \), and its weak limit provides the minimum of \( \Phi(u) \). This implies the solvability of the Dirichlet problem if the class of functions satisfying the boundary condition is not empty. Note that the uniqueness of a solution is a consequence of the convexity of \( \Phi \). We supply the spaces \( T^\pm_p(\Gamma) \) of traces \( u|_{\Gamma} \) of the functions \( u \in H^1_p(\Omega^\pm) \) with the norms

\[
\|f\|_{T^\pm_p(\Gamma)} = \inf \{ \|u\|_{H^1_p(\Omega^\pm)} : u \in H^1_p(\Omega^\pm), u|_{\Gamma} = f \}.
\]

The space \( T_p(\Gamma) = T^+_p(\Gamma) \cap T^-_p(\Gamma) \) can be viewed as the space of the traces on \( \Gamma \) of the functions in \( H^1_p(\mathbb{R}^n) \) with the norm

\[
\|f\|_{T_p(\Gamma)} = \inf \{ \|u\|_{H^1_p(\mathbb{R}^n)} : u \in H^1_p(\mathbb{R}^n), u|_{\Gamma} = f \}.
\]

**Theorem 1.1.** For \( p \in (1, n) \), there exists a continuous linear extension operator

\[
H^1_p(\Omega^-) \rightarrow H^1_p(\mathbb{R}^n),
\]

so that the spaces \( T^-_p(\Gamma) \) and \( T^+_p(\Gamma) \) coincide with equivalence of norms.

**Remark 1.1.** In general, \( \Omega^- \) cannot be replaced by \( \Omega^+ \) in Theorem 1.1.

Consider the following example. Let \( \varphi(z) = z^{1/(n-1)} \), and let \( u(x) = z^\lambda \eta(x) \), where \( \eta \) is a smooth function in \( \mathbb{R}^n \) such that \( \eta = 0 \) in a neighborhood of the origin and \( \eta(x) = 1 \) for large \( |x| \). Then \( u \in H^1_p(\Omega^+) \) for \( 1 < p < n \) and

\[
-2(n-p)(np)^{-1} < \lambda < \min\{0, 1 - 2/p\}.
\]

At the same time, \( u \notin L_q(\Omega^+) \) for \( q = np(n - p)^{-1} \), which contradicts the extendability of \( u \) to \( \mathbb{R}^n \) with preservation of the class \( H^1_p(\mathbb{R}^n) \) because \( H^1_p(\mathbb{R}^n) \) is continuously imbedded in \( L_q(\mathbb{R}^n) \) if \( p < n \), see \([9]\).

The next statement gives a description of the space \( T^+_p(\Gamma) \) for \( p \in (1, n - 1) \).
Theorem 1.2. If \(1 < p < n - 1\), we have the following equivalence of norms:

\[
\|f\|_{T_p(\Gamma)} \sim \left( \int_\Gamma |f(x)|^p \sigma_p(z) \, ds_x + |f|_{p,\Gamma}^p \right)^{1/p},
\]

where

\[
|f|_{p,\Gamma}^p = \int_{\{x,\xi \in \Gamma : |x| < M(z, \xi)\}} |f(x) - f(\xi)|^p \frac{ds_x \, ds_\xi}{|x - \xi|^{n+p-2}},
\]

\(x = (y, z), \xi = (\eta, \zeta), ds_x, ds_\xi\) are the area elements on \(\Gamma\),

\[\sigma_p(z) = \min\{1, \varphi(z)^{1-p}\}, \quad M(z, \zeta) = \max\{\varphi(z), \varphi(\zeta), 1\}.
\]

The finiteness of the norm on the right in (1.3) is necessary and sufficient for the unique solvability of problem (1.2) and also necessary and sufficient for the unique solvability of the two-sided Dirichlet problem

\[
-\Delta_p u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Gamma, \quad u|_{\Gamma} = f.
\]

The case where \(n - 1 < p < n\) is considered in the following assertion.

Theorem 1.3. Let \(p \in (n - 1, n)\), and suppose that \(\varphi\) satisfies the additional condition \(\sup\{\varphi(2z)/\varphi(z) : z \geq 1\} < \infty\). Then

\[
\|f\|_{T_p(\Gamma)} \sim \left( \int_\Gamma |f(x)|^p \sigma_p(z) \, ds_x + |f|_{p,\Gamma}^p + \{f\}_{p,\Gamma}^p \right)^{1/p},
\]

where

\[
\{f\}_{p,\Gamma}^p = \int_{\{z, \zeta > 0 : |z - \zeta| > M(z, \zeta)\}} \frac{dz \, d\zeta}{|z - \zeta|^{2+p-n}} \int_{S_1^{(n-1)}} |f(\varphi(z)\theta, z) - f(\varphi(\zeta)\theta, \zeta)|^p \, d\theta,
\]

\[\sigma_p(z) = \min\{1, \varphi(z)^{2-n}z^{n-p-1}\}, \quad \text{and the remaining notation is the same as in the preceding theorem. The finiteness of the right-hand side in (1.5) is necessary and sufficient for the existence and uniqueness of a solution of either of problems (1.2) and (1.4)}.
\]

The case of \(p = n - 1\) requires special treatment.

Theorem 1.4. Let \(p = n - 1 > 1\), and suppose that \(\sup\{\varphi(2z)/\varphi(z) : z \geq 1\} < \infty\). Then

\[
\|f\|_{T_p(\Gamma)} \sim \left( \int_\Gamma |f(x)|^p \sigma_p(z) \, ds_x + \langle f \rangle_{p,\Gamma}^p \right)^{1/p},
\]

where

\[
\langle f \rangle_{p,\Gamma}^p = \int_{\{x,\xi \in \Gamma : 2^{-1} < |z/\zeta| < 2\}} |f(x) - f(\xi)|^p P \left( \frac{|z - \zeta|}{M(z, \zeta)} \right) \frac{ds_x \, ds_\xi}{|x - \zeta|^{n+p-2}},
\]

\[P(t) = 1 + t^{p-2}/((\log(1 + t)))^p, \quad \sigma_p(z) = \min\{1, (\varphi(z) \log(z/\varphi(z))^{1-p}\},
\]

and the remaining notation is the same as in Theorem 1.2. The finiteness of the norm on the right in (1.6) is necessary and sufficient for the unique solvability of either of problems (1.2) and (1.4).
§2. Extension operator: $H^1_p(\Omega^-) \rightarrow H^1_p(\mathbb{R}^n)$

In what follows, $B_r$ (or $B_r^{(n)}$) denotes an open ball in $\mathbb{R}^n$ of radius $r$ centered at the origin. The sphere in $\mathbb{R}^n$ with the same center and the same radius is denoted by $S_r^{(n-1)}$.

Let $G$ be a domain in $\mathbb{R}^n$. Then $C^\infty_0(G)$ is the set of infinitely differentiable functions with compact support in $G$. By $\nabla u$ we mean the gradient of $u$. The symbol $L^1_p(G)$ designates the space of functions in $L_{p,loc}(G)$ with gradient in $L_p(G)$, normed by

$$
\|u\|_{L^1_p(G)} = \left(\|u\|_{L^p(G)}^p + \|\nabla u\|_{L^p(G)}^p\right)^{1/p},
$$

where $g$ is an inner subdomain of $G$, i.e. $\bar{g}$ is a compact subset of $G$. The change of an inner subdomain $g$ induces an equivalent norm, see [10] 1.1.13.

Let $G$ be unbounded. We define $H^1_p(G)$ as the closure in the norm $\|\nabla(\cdot)\|_{L^p(G)}$ of the set of smooth functions on $G$ with bounded support in $G$. In accordance with [9], for $p < n$ the space $H^1_p(\mathbb{R}^n)$ is the intersection $L^1_p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $q = pn(n-p)^{-1}$, and the norm in $H^1_p(\mathbb{R}^n)$ is equivalent to any of the norms

$$
\|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \text{or} \quad \|u\|_{L^q(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad q = pn(n-p)^{-1}.
$$

The symbol $a \sim b$ means that the ratio of the positive quantities $a, b$ is bounded above and below by positive constants independent of $a, b$. In what follows we denote by $c$ positive constants (whose values may change within the same chain of inequalities) depending only on $n, p,$ and $\Gamma$.

**Proof of Theorem 1.1.** The relations

$$
\lim_{z \to \infty} \varphi'(z) = \lim_{z \to \infty} \varphi(z)/z = 0,
$$

imply that for some $z_0 > 1$, the following requirements (2.1) are fulfilled for all $z \geq z_0$:

(2.1) $\varphi(z) > 1, \varphi(z)/z \leq 1/2, \varphi(z) \leq 1/2, \varphi(z) \log(z/\varphi(z)) \geq 1.$

In the sequel $z_0$ is assumed to be fixed.

A sequence $\{z_k\}$ is defined by the rule

$$
z_{k+1} = z_k + \varphi(z_{k+1}), \quad k = 0, 1, \ldots.
$$

Clearly, $\{z_k\}$ is monotone increasing and $z_k \to \infty$. Furthermore,

$$
1 - \frac{z_k}{z_{k+1}} = \frac{\varphi(z_{k+1})}{z_{k+1}} \to 0,
$$

$$
1 - \frac{\varphi(z_k)}{\varphi(z_{k+1})} = \frac{1}{\varphi(z_{k+1})} \int_{z_k}^{z_{k+1}} \varphi'(t) dt \to 0,
$$

and, in particular,

$$
\lim_{k \to \infty} \frac{z_{k+1}}{z_k} = \lim_{k \to \infty} \frac{\varphi(z_{k+1})}{\varphi(z_k)} = 1.
$$

Consider the “cells”

$$
\Omega^-_k = \{(y, z) : z \in (z_k, z_{k+1}), \quad 1 < |y|/\varphi(z) < 2\}, \quad k = 1, 2, \ldots.
$$

The transformation

(2.2) $x = (y, z) \mapsto \mathcal{K}_x(x) = X = (Y, Z), \quad Y = \frac{y}{\varphi(z)}, \quad Z = \frac{z - z_{k-1}}{z_{k+1} - z_{k-1}},
$

maps $\Omega^-_k$ onto the cylinder

$$
G = \{(Y, Z) : Z \in (0, 1), \quad 1 < |Y| < 2\},
$$
and the domain
\[ \Omega_k = \{ x = (y, z) \in \mathbb{R}^n : z \in (0, 1), \ |y| < 2\varphi(z) \} \]
on onto the cylinder
\[ D = \{ (Y, Z) : Z \in (0, 1), \ |Y| < 2 \} . \]

For brevity, we set \( \varphi(z_k) = \varphi_k \). It is readily seen that
\[
(2.4) \quad dx \sim \varphi_k^n dX, \quad |\nabla x u| \sim \varphi_k |\nabla_X (u \circ x_k^{-1})|.
\]

Let \( E \) be a continuous linear extension operator: \( W^1_p(G) \to W^1_p(D) \), i.e., \( Eu|_G = u \)
for all \( u \in W^1_p(G) \). Then the operator
\[ W^1_p(\Omega^-_k) \ni u \mapsto E_k u = (E(u \circ x_k^{-1})) \circ x_k \in W^1_p(\Omega^-_k) \]
is an extension operator, and relations (2.3) show that
\[
(2.5) \quad c \| E_k u \|_{L^p(\Omega^-_k)} \leq \| u \|_{L^p(\Omega^-)} + \varphi_k \| \nabla u \|_{L^p(\Omega^-_k)},
\]
\[
(2.6) \quad c \| \nabla (E_k u) \|_{L^p(\Omega^-_k)} \leq \varphi_k^{-1} \| u \|_{L^p(\Omega^-_k)} + \| \nabla u \|_{L^p(\Omega^-_k)}.
\]

Now we turn to construction of the required bounded extension operator: \( H^1_p(\Omega^-) \to H^1_p(\mathbb{R}^n) \). Let \( \{\mu_k\}_{k \geq 1} \) be a smooth partition of unity for the interval \([z_1, \infty)\) subordinate to the covering \( \{(z_{k-1}, z_{k+1})\}_{k \geq 1} \). We may assume that \( 0 \leq \mu_k \leq 1, \mu_k \in C_0^\infty(z_{k-1}, z_{k+1}) \), and also
\[
(2.7) \quad \text{dist}(\text{supp} \mu_k, \mathbb{R}^n \setminus (z_{k-1}, z_{k+1})) \sim \varphi_k, \quad |\mu_k'| \leq c \varphi_k^{-1}, \quad k = 1, 2, \ldots ,
\]
where \( \varphi_k = \varphi(z_k) \). We add one more element to the set \( \{\mu_k\}_{k \geq 1} \) by putting \( \mu_0 \in C^\infty[0, \infty) \), \( \mu_0(z) = 0 \) for \( z > z_1 \), \( \mu_0(z) = 1 \) for \( z < z_0 \), and \( \mu_0 = 1 - \mu_1 \) in \([z_0, z_1] \). It is clear that
\[
\sum_{k=0}^{\infty} \mu_k(z) = 1, \quad z \in [0, \infty).
\]

We introduce yet another cell
\[ \Omega^-_0 = \{ x \in \Omega^-_1 : z \in (z_0, z_1) \} \cup \{ x : z \in (-1, z_1), \ |y| < 2\varphi(z_0) \} \setminus \overline{\Omega}^+ . \]

Since \( \Omega^-_0 \) is a Lipschitz domain, there exists a bounded extension operator
\[ E_0 : W^1_p(\Omega^-_0) \to W^1_p(\mathbb{R}^n), \]
see [11] Chapter VI.

Next, we consider a collection of functions \( \{\alpha_k\} \) with the following properties:
\[ \alpha_k \in C_0^\infty(z_{k-1}, z_{k+1}), \quad |\alpha_k'| \leq c \varphi_k^{-1}, \quad k \geq 1, \quad \alpha_i \mu_i = \mu_i, \ i \geq 0. \]

Let \( u \) be an arbitrary element in \( H^1_p(\Omega^-) \). For \( x \in \mathbb{R}^n \setminus \Omega^- \), we put
\[
(2.7) \quad v(x) = \sum_{k=0}^{\infty} \mu_k(z) \bar{u}_k, \quad w(x) = \sum_{k=0}^{\infty} \mu_k(z) (E_k(\alpha_k(u - \bar{u}_k))(x),
\]
where \( \bar{u}_k \) is the mean value of \( u \) in \( \Omega^-_k \), and the general term of the second sum in
(2.7) equals zero if \( z \notin (z_{k-1}, z_{k+1}) \). Now we are ready to define the required extension operator \( H^1_p(\Omega^-) \ni u \mapsto E u \in H^1_p(\mathbb{R}^n) \). It will be given as follows:
\[ (E u)(x) = \begin{cases} u(x) & \text{if } x \in \Omega^-, \\ v(x) + w(x) & \text{if } x \in \mathbb{R}^n \setminus \Omega^- . \end{cases} \]

To justify the last formula, we must check the following three assertions.

1. If the support of \( u \) is bounded in \( \Omega^- \), then the support of \( E u \) is bounded in \( \mathbb{R}^n \).
2. For the same \( u \), the function \( E u \) has a locally integrable gradient in \( \mathbb{R}^n \).
3. For the same $u$, the following estimate is true:

\[(2.8) \quad \|\nabla (\mathcal{E}u)\|_{L^p(\mathbb{R}^n)} \leq c \|\nabla u\|_{L^p(\Omega^-)}.\]

1. If $\text{supp } u$ is bounded in $\Omega^-$, then each sum in (2.7) has a bounded number of nonzero terms. Hence, $\text{supp } \mathcal{E}u$ is bounded in $\Omega^+$, and consequently, in $\mathbb{R}^n$, because $\text{supp } \mathcal{E}u \subset \text{supp } u \cup (\text{supp } \mathcal{E}u \cap \Omega^+)$. 

2. Each term of the two sums in (2.7) has a locally integrable gradient in $\mathbb{R}^n$, and, furthermore, (2.7) implies that for $x \in \bigcup_{k=0}^{\infty} \Omega_k^-$ we have

\[v(x) + w(x) = \sum_{k=0}^{\infty} (\mu_k \bar{u}_k + \mu_k \alpha_k (u - \bar{u}_k)) = \sum_{k \geq 0} \mu_k u = u.\]

Thus, $\nabla (\mathcal{E}u) \in L^p_{\text{loc}}(\mathbb{R}^n)$.

3. Inequality (2.8) follows from the inequalities

\[(2.9) \quad \|\nabla v\|_{L^p(\Omega^+)} \leq c \|\nabla u\|_{L^p(\Omega^-)},\]

\[(2.10) \quad \|\nabla w\|_{L^p(\Omega^+)} \leq c \|\nabla u\|_{L^p(\Omega^-)}\]

verified below.

Turning to (2.9), observe that if $x = (y, z) \in \Omega^+$, then either $z \in (0, z_0]$ or $z \in (z_k, z_{k+1})$ for some $k \geq 0$. In the first case $v(x) = \bar{u}_0$, whence $\nabla v = 0$. In the second case $x \in \text{supp } \mu_i$ only if $i = k, k + 1$, and for these $x$ we have

\[v(x) = \mu_k (z) \bar{u}_k + \mu_{k+1} (z) \bar{u}_{k+1} = \bar{u}_k + \mu_{k+1} (\bar{u}_{k+1} - \bar{u}_k).\]

Let $\delta_k = (z_k, z_{k+1})$. Then

\[\|\nabla v\|_{L^p(\{x \in \Omega^+ : z \in \delta_k\})} \leq c \varphi_k^{-p} \|\bar{u}_{k+1} - \bar{u}_k\|^p.\]

The right-hand side of the last inequality is not greater than

\[c \varphi_k^{-p} \|\bar{u}_{k+1} - \bar{u}_k\|_{L^p(\{x \in \delta_k : 1 < |y|/\varphi(z) < 2\})}.\]

Hence,

\[\|\nabla v\|_{L^p(\{x \in \Omega^+ : z \in \delta_k\})} \leq c \varphi_k^{-p} \sum_{i=k}^{k+1} \|u - \bar{u}_i\|_{L^p(\Omega_k^-)}^p.\]

Using the Poincaré inequality in $\Omega_k^-$, we arrive at the estimate

\[\|\nabla v\|_{L^p(\{x \in \Omega^+ : z \in \delta_k\})} \leq c \|\nabla u\|_{L^p(\Omega_k^- \cup \Omega_{k+1}^-)}^p.\]

Summing over $k$ gives (2.9).

Now we turn to (2.10). From (2.7) it follows that

\[\|\nabla w\|_{L^p(\Omega^+)} \leq c \sum_{k \geq 0} \varphi_k^{-p} \|E_k(\alpha_k (u - \bar{u}_k))\|_{L^p(\Omega^+)}^p + c \sum_{k \geq 0} ||\nabla E_k(\alpha_k (u - \bar{u}_k))\|_{L^p(\Omega^+)}^p.\]

By combining (2.4), (2.5), and the Poincaré inequality in $\Omega_k^-$, we dominate the right-hand side of the last inequality by the quantity

\[c \sum_{k \geq 0} \|\nabla u\|_{L^p(\Omega_k^-)}^p,\]

and (2.10) follows.

We have shown that the linear map

\[u \mapsto \mathcal{E}u \in H^1_p(\mathbb{R}^n),\]

defined on the set of smooth (on $\Omega^-$) functions $u$ with bounded support in $\Omega^-$, is continuous, and that $\mathcal{E}u|_{\Omega^-} = u$ for every $u$. Since the functions $u$ described above form
§3. Weighted estimates for traces of functions on $\Gamma$

In this section we establish that the trace space $T_p(\Gamma)$ is continuously imbedded in some weighted space $L_{p,\sigma}(\Gamma)$ with an explicitly defined weight function $\sigma$.

Let $r < R < \infty$. By $A_{r,R}$ we mean the spherical shell $A_{r,R} = B_R \setminus B_r$ in $\mathbb{R}^n$.

**Lemma 3.1.** Let $u \in W^1_p(B_R)$, and let $r \in (0, R/2)$. Then

$$
(3.1) \quad \|u\|_{L_p(S_{r}^{n-1})}^p \leq c \Lambda_{p,n}(r, R) \left( \|\nabla u\|_{L_p(B_R)}^p + R^{-p} \|u\|_{L_p(A_{R/2,R})}^p \right),
$$

where

$$
(3.2) \quad \Lambda_{p,n}(r, R) = \begin{cases} 
    r^{p-1} & \text{if } 1 < p < n, \\
    (r \log(R/r))^{p-1} & \text{if } p = n, \\
    r^{n-1}R^{p-n} & \text{if } p > n.
\end{cases}
$$

**Proof.** First, we establish the inequality

$$
(3.3) \quad \|u\|_{L_p(S_{r}^{n-1})}^p \leq c \Lambda_{p,n}(r, R) \|\nabla u\|_{L_p(B_R)}^p
$$

when $u_{|S_r} = 0$. Indeed, applying the Newton–Leibnitz formula and the Hölder inequality, we find

$$
\|u\|_{L_p(S_{r}^{n-1})}^p \leq r^{n-1} \int_{S_{r}^{n-1}} d\theta \left( \int_r^R |u(q, \theta)|^p \frac{d\theta}{q} \right)^{p-1} \leq r^{n-1} \int_{S_{r}^{n-1}} d\theta \left( \int_r^R |u(q, \theta)|^p \frac{d\theta}{q} \right)^{p-1}.
$$

Since the right-hand side of the last inequality does not exceed that of (3.3), estimate (3.3) follows. □

Let

$$
\psi \in C^\infty([0, \infty)), \quad \psi(t) = 1 \text{ for } t \in [0, 1/2], \quad \psi(t) = 0 \text{ for } t \geq 1.
$$

We put $\psi_R(x) = \psi(|x|/R)$, $x \in \mathbb{R}^n$, and insert $\psi_R u$ in (3.3) in place of $u$. Then we obtain (3.1) for all $u \in W^1_p(B_R)$.

**Lemma 3.2.** Let

$$
(3.4) \quad \sigma_p(z) = \begin{cases} 
    \min\{1, \varphi(z)^{1-p}\} & \text{if } p \in (1, n-1), \\
    \min\{1, (\varphi(z) \log(z/\varphi(z))^{1-p}\} & \text{if } p = n-1, \\
    \min\{1, \varphi(z)^{2-n}z^{n-1-p}\} & \text{if } p \in (n-1, n).
\end{cases}
$$

If $p \in (1, n)$, then for all $u \in H^1_p(\mathbb{R}^n)$ we have

$$
\int_{\Gamma'} |u(x)|^p \sigma_p(z) \, dx \leq c \|\nabla u\|_{L_p(\mathbb{R}^n)}^p.
$$

**Proof.** Suppose $z_0$ satisfies (2.1), and let $\Gamma' = \{x \in \Gamma : z < z_0\}$. Clearly, $\Gamma' \subset B_{2z_0}$ and $\sigma_p(z) \leq c$ for $z < z_0$. An application of Gagliardo’s theorem gives

$$
\int_{\Gamma'} |u(x)|^p \sigma_p(z) \, dx \leq c \|u\|_{W^1_p(B_{2z_0})}^p.
$$
Since the norm on the right is not greater than $c \| \nabla u \|_{L^p(R^n)}^p$, the proof of the lemma reduces to the proof of the estimate

$$
\int_{\Gamma \setminus \Gamma'} |u(x)|^p \sigma_p(z) \, ds_x \leq c \| \nabla u \|_{L^p(R^n)}^p.
$$

Let $y = (r, \theta)$ be the spherical coordinates in $R^{n-1}$. Then

$$
ds_x \sim \varphi(z)^{n-2} \, dz \, d\theta = dz \, dS_{\varphi(z)}^{(n-2)}, \quad z > z_0,
$$

whence

$$
\int_{\Gamma \setminus \Gamma'} |u(x)|^p \sigma_p(z) \, ds_x \sim \int_{z_0}^{\infty} \sigma_p(z) \varphi(z)^{n-2} \, dz \int_{S_{\varphi(z)}^{(n-2)}} |u(\varphi(z), \theta, z)|^p \, d\theta.
$$

Next we apply Lemma 3.1 to the function $u(\cdot, z)$ for $R = z, r = \varphi(z)$. This results in the inequality

$$
c \int_{\Gamma \setminus \Gamma'} |u(x)|^p \sigma_p(z) \, ds_x
\leq \int_{z_0}^{\infty} \sigma_p(z) \Lambda_{p,n-1}(\varphi(z), z) \, dz \left( \int_{|y| < z} |\nabla_y u(y, z)|^p \, dy + z^{-p} \int_{\frac{1}{2} < |y| < z} |u(y, z)|^p \, dy \right),
$$

where $\Lambda$ is as defined in (3.2). Since $\sigma_p(z) \Lambda_{p,n-1}(\varphi(z), z) \leq 1$, the right-hand side of the last inequality is not greater than

$$
c \| \nabla u \|_{L^p(R^n)}^p + c \| u/|x| \|_{L^p(R^n)}^p.
$$

The Hardy inequality shows that the last sum does not exceed the expression on the right in (3.5), which concludes the proof of the lemma. \[\square\]

§4. Description of the Space $T_p(\Gamma)$ for $p < n - 1$

A description of $T_p(\Gamma)$ is given in Theorem 1.2 whose proof can be found at the end of the present section; now we state two lemmas to facilitate this proof. The first lemma is about some consequences of the Gagliardo theorem and the Sobolev theorem on equivalent norms in $L^1_p$. Some statements of Lemma 4.1 might have been known, but for the author it was difficult to cite an appropriate reference.

**Lemma 4.1.** Let $G$ be a domain in $R^n$ ($n \geq 2$) with compact closure and Lipschitz boundary $S = \partial G$.

(i) The following seminorms are equivalent ($p \in (1, \infty)$):

$$
[f]_S = \left( \iint_{S \times S} |f(x) - f(\xi)|^p \, ds_x \, ds_\xi \right)^{1/p},
$$

$$
(f)_S = \inf \{ \| \nabla u \|_{L_p(G)} : u \in W^1_p(G), \ u|_S = f \},
$$

$$
\{f\}_S = \inf \{ \| \nabla u \|_{L_p(R^n)} : u \in L^1_p(R^n), \ u|_S = f \}.
$$

The constants in the relation

$$
[f]_S \sim (f)_S \sim \{f\}_S
$$

depend only on $S, p,$ and $n$.

(ii) Let $1 < p < n$. If $S'$ is a measurable subset of $S$ with positive area and the mean value $f'$ of the function $f$ in $S'$ equals zero, then

$$
[f]_S \sim \inf \{ \| \nabla u \|_{L_p(R^n)} : u \in H^1_p(R^n), u|_S = f \}
$$

with constants depending only on $S, S', p,$ and $n$. 

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Proof. First we observe that if $[f]_S < \infty$, then
\[ \int_S |f(x) - f(\xi)|^p \, ds_\xi < \infty \]
for almost all $x \in S$. It follows that $f \in L^p(S)$, so that the finiteness of the seminorm (4.1) is equivalent to the finiteness of the norm
\[ \|f\|_{W^{1-1/p}_p(S)} = \|f\|_{L^p(S)} + [f]_S. \]

Let $u \in W^{1}_p(G)$, $u|_S = f$, and let $\bar{u}$ be the mean value of $u$ in $G$. Combining the Gagliardo theorem with the Poincaré inequality in $G$, we obtain
\[ [f]_S = [f - \bar{u}]_S \leq c \|u - \bar{u}\|_{W^1_p(G)} \leq c \|\nabla u\|_{L^p(G)}. \]

Thus, we have established that $[f]_S \leq c (f)_S$. The inequality $(f)_S \leq \{f\}_S$ is obvious. To conclude the proof of (4.2), we must check that $\{f\}_S \leq c [f]_S$. Combining the Gagliardo theorem and the Stein extension theorem, see [11, Chapter VI], we see that there exists a continuous linear extension operator $E : W^{1-1/p}_p(S) \to W^{1}_p(\mathbb{R}^n)$. Let $\bar{f}$ be the mean value of $f$ in $S$. We put
\[ u = \bar{f} + E(f - \bar{f}). \]

Then $u \in L^1_p(\mathbb{R}^n)$, $u|_S = f$, and
\[ \|\nabla u\|_{L^p(\mathbb{R}^n)} \leq c [f - \bar{f}]_S + c \|f - \bar{f}\|_{L^p(S)} \leq c [f]_S. \]

Hence, (4.2) follows.

(ii). If $f' = 0$, then
\[ \|f\|_{L^p(S)} = \|f - f'\|_{L^p(S)} \leq c [f]_S. \]

Let $E$ be the same extension operator as above. We have
\[ \|\nabla (Ef)\|_{L^p(\mathbb{R}^n)} \leq c [f]_S. \]

Since $Ef \in H^1_p(\mathbb{R}^n)$, it follows that the right-hand side of (4.3) does not exceed $c [f]_S$. The reverse inequality is valid because (4.2) implies that $[f]_S \leq c \{f\}_S$. \hfill \square

Remark 4.1. Statement (ii) in Lemma 4.1 is not true unless $f' = 0$. For example, if $S = S_1^{(n-1)}$ and $f = 1$, the left-hand side of (4.3) is zero, and the infimum on the right is attained for a solution of the two-sided Dirichlet problem
\[ \Delta_p u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus S, \quad u|_S = 1, \]
i.e., for
\[ u(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ \frac{|x|^p}{|x|^{n-1}} & \text{if } |x| > 1, \end{cases} \]
and consequently, is positive.

Now, we study some norms (and seminorms) for functions on $\Gamma$ whose finiteness characterizes the trace space $T^p_\Gamma$ in the case where $p < n - 1$.

Let $\{z_k\}_{k \geq 0}$ and $\{\mu_k\}_{k \geq 1}$ be the same numerical sequence and the partition of unity that were introduced in the proof of Theorem 1.1. Let
\[ \Gamma_k = \{ x \in \Gamma : z \in (z_{k-1}, z_{k+1}) \}, \quad k = 1, 2, \ldots. \]
We define the following norms and seminorms for $f \in L_{p, \text{loc}}(\Gamma)$:

$$
|f|_{p, \Gamma} = \left( \int \int_{x, \xi \in \Gamma : |\xi - z| < M(z, \zeta)} |f(x) - f(\xi)|^p \frac{ds_x \, ds_\xi}{|x - \xi|^{n+p-2}} \right)^{1/p},
$$

$$
\|f\|_{p, \Gamma} = \left( \int_{\Gamma} |f(x)|^p \sigma_p(z) \, ds_x \right)^{1/p},
$$

where $x = (y, z)$, $\xi = (\eta, \zeta)$, $M(z, \zeta) = \max \{\varphi(z), \varphi(\zeta), 1\}$, and $\sigma_p(z)$ is as in (3.4). We shall also use the seminorms $|.|_{p, \Gamma_k}$ defined by the right-hand side of (4.1) with $S$ replaced by $\Gamma_k$.

**Lemma 4.2.** Suppose $p \in (1, n - 1)$, $f \in L_{p, \text{loc}}(\Gamma)$, $f(y, z) = 0$ for $z < z_1$. Then

$$
(4.4) \quad \|f\|_{p, \Gamma} + |f|_{p, \Gamma} \sim \|f\|_{p, \Gamma} + \left( \sum_{k \geq 1} |\mu_k f|_{p, \Gamma_k}^p \right)^{1/p}.
$$

**Proof.** Suppose that the left-hand side of (4.4) is finite and fix $x, \xi \in \Gamma$. Clearly,

$$
|f(x) - f(\xi)| = \sum_{k \geq 1} (\mu_k(z) f(x) - \mu_k(\zeta) f(\xi)).
$$

Observe that the number of nonzero terms in this sum is uniformly bounded with respect to $x, \xi$; so it follows that

$$
|f(x) - f(\xi)|^p \leq c \sum_{k \geq 1} |\mu_k(z) f(x) - \mu_k(\zeta) f(\xi)|^p,
$$

whence

$$
(4.5) \quad |f|_{p, \Gamma}^p \leq c \sum_{k \geq 1} \int_{|\xi - z| < M(z, \zeta)} |\mu_k(z) f(x) - \mu_k(\zeta) f(\xi)|^p \frac{ds_x \, ds_\xi}{|x - \xi|^{n+p-2}}.
$$

The integrand in the double integral is nonzero only if either $z, \zeta \in (z_{k-1}, z_{k+1})$ or one of the points $z, \zeta$ (let it be $z$) lies in supp $\mu_k$, and the other ($\zeta$) does not belong to $(z_{k-1}, z_{k+1})$. Using (2.6), we can dominate the double integral in (4.5) by the quantity

$$
c [\mu_k f]_{p, \Gamma_k}^p + c \int_{\Gamma_k} |f(x)|^p \, ds_x \int_{c \varphi(z_k) < |\xi - z| < M(z, \zeta)} |x - \xi|^{2-n-p} \, ds_\xi
$$

with constants independent of $k$. Finally, the general term of the sum in (4.5) does not exceed

$$
c [\mu_k f]_{p, \Gamma_k}^p + c \int_{\Gamma_k} |f(x)|^p \varphi(z_k)^{p-1} \, ds_x.
$$

Thus, the left-hand side (4.4) is not greater than a constant times the right-hand side.

Now we turn to the reverse inequality

$$
(4.6) \quad \sum_{k \geq 1} |\mu_k f|_{p, \Gamma_k}^p \leq c \|f\|_{p, \Gamma}^p + c |f|_{p, \Gamma}^p.
$$

Note that

$$
|\mu_k(z) f(x) - \mu_k(\zeta) f(\xi)|^p \leq c |f(x) - f(\xi)|^p + c |f(x)|^p |\mu_k(z) - \mu_k(\zeta)|^p
$$

and

$$
|\mu_k(z) - \mu_k(\zeta)|^p \leq c \varphi_k^{-p} |\zeta - z|^p.
$$

Putting for brevity $\varphi_k = \varphi(z_k)$ and

$$
g(x, \xi) = |f(x) - f(\xi)|^p |x - \xi|^{2-n-p},
$$

we obtain

$$
(4.7) \quad \sum_{k \geq 1} |\mu_k f|_{p, \Gamma_k}^p \leq c \left( \int_{\Gamma} |f(x)|^p g(x, \xi) \, ds_x \right)^{1/p} + c \|f\|_{p, \Gamma}^p.
$$

Since $|f|_{p, \Gamma} \sim \|f\|_{p, \Gamma}$, we obtain

$$
\|f\|_{p, \Gamma} + |f|_{p, \Gamma} \sim \|f\|_{p, \Gamma} + \left( \sum_{k \geq 1} |\mu_k f|_{p, \Gamma_k}^p \right)^{1/p}.
$$

This proves the lemma.
we obtain
\[
c \sum_{k \geq 1} \left| \mu_k f \right|^p_{p, \Gamma_k} \leq \frac{1}{\varphi_k} \int_{\Gamma_k} |f(x)|^p \, ds_x \int_{\Gamma_k} |x - \xi|^{2-n} \, ds_{\xi} + \sum_{k \geq 1} \int_{\Gamma_k} ds_x \int_{\xi \in \Gamma : |z - \xi| < M(z, \zeta)} g(x, \xi) \, ds_{\xi} + \sum_{k \geq 1} \int_{x, \xi \in \Gamma : |z - \xi| > \varphi_k} g(x, \xi) \, ds_x \, ds_{\xi}.
\]
(4.7)

Since
\[
\int_{\Gamma_k} |x - \xi|^{2-n} \, ds_{\xi} \leq c \varphi_k
\]
for \( x \in \Gamma_k \), the first sum on the right in (4.7) is not greater than
\[
\sum_{k \geq 1} \int_{\Gamma_k} |f(x)|^p \frac{ds_x}{\varphi(z)^{p-1}} \leq c \|f\|^p_{p, \Gamma}.
\]
The second sum does not exceed \( c \|f\|^p_{p, \Gamma} \) because every point \( x \in \Gamma \) belongs to at most two sets in the collection \( \{\Gamma_k\}_{k \geq 1} \). The third sum on the right in (4.7) is dominated by the quantity
\[
c \sum_{k \geq 1} \varphi_k^{2-n-p} \int_{\Gamma_k \times \Gamma_k} (|f(x)|^p + |f(\xi)|^p) \, ds_x \, ds_{\xi},
\]
which is not greater than \( c \|f\|^p_{p, \Gamma} \) in view of (4.8). This concludes the proof of (4.6) and the lemma.

\( \square \)

**Proof of Theorem 1.2.** Let \( u \in H^1_p(\Omega^-) \) and \( f = u|_\Gamma \). By Theorem 1.1, we may assume that \( u \in H^1_p(\mathbb{R}^n) \). Since \( \Gamma \) is a locally Lipschitz surface, the trace on \( \Gamma \) of a function \( u \in H^1_p(\mathbb{R}^n) \) is locally characterized by the Gagliardo theorem. With the help of this theorem and a finite partition of unity, the description of the space \( T_p(\Gamma) \) reduces to the case where \( u = 0 \) in a neighborhood of the origin. Without loss of generality we may assume that \( u = 0 \) in the cylinder \( \{x : |y| < z_1, |z| < z_1\} \).

By Lemma 3.2, we have
\[
\|f\|_{p, \Gamma} \leq c \|\nabla u\|_{L_p(\mathbb{R}^n)}
\]
(4.9)
(here and below we use the notation from the preceding lemma). Let \( \psi \in C^\infty[0, \infty) \) be such that \( \psi(t) = 1 \) for \( t \in [0, 2] \) and \( \psi(t) = 0 \) for \( t \geq 3 \). A sequence \( \{u_k\}_{k \geq 1} \) is defined by
\[
\mathbb{R}^n \ni x = (y, z) \mapsto u_k(x) = \mu_k(z) \psi_k(y) u(x),
\]
where \( \psi_k(y) = \psi(|y|/\varphi_k) \), \( \varphi_k = \varphi(z_k) \), and \( \{\mu_k\}_{k \geq 1} \) is a partition of unity as in the proof of Theorem 1.1. It is readily seen that \( u_k \) is supported in the cylinder
\[
D_k = \{x : z \in (z_{k-1}, z_{k+1}), |y| < 3 \varphi_k\}
\]
and \( u_k|_{\Gamma} = \mu_k f \).

The transformation \( x \mapsto \kappa_k x = X \) defined by (2.2) maps
\[
\Gamma_k = \{x \in \Gamma : z \in (z_{k-1}, z_{k+1})\}
\]
to the lateral surface of the cylinder
\[
Q = \{(Y, Z) : |Y| < 1, Z \in (0, 1)\}.
\]
Let $v_k = u_k \circ \kappa_k^{-1}$. Then $v_k$ is defined in a neighborhood of $\bar{Q}$, $v_k = 0$ in a neighborhood of the bases of $Q$, and the trace of $v_k$ on the lateral surface of $Q$ is $g_k = (\mu_k f) \circ \kappa_k^{-1}$. We extend $g_k$ to be zero on the bases of $Q$. By Lemma 4.1,

$$[g_k]_{p, \partial Q} \leq c \|\nabla v_k\|_{L_p(Q)}.$$ 

Returning to $x = \kappa_k^{-1}X$, we obtain

$$\int_{\Gamma} |f(x)|^p \sigma_p(z) \, ds_x + |f|^p_{p, \Gamma} \leq c \|\nabla u\|^p_{L_p(\mathbb{R}^n)}.$$ 

Next, we check the estimate

$$\sum_{k \geq 1} \|\nabla u_k\|^p_{L_p(\mathbb{R}^n)} \leq c \|\nabla u\|^p_{L_p(\mathbb{R}^n)}.$$ 

Indeed, since $u_k = 0$ outside $D_k$, we have

$$\sum_{k \geq 1} \|\nabla u_k\|^p_{L_p(\mathbb{R}^n)} \leq c \sum_{k \geq 1} \|\nabla u\|^p_{L_p(D_k)} + c \sum_{k \geq 1} \|u/\varphi_k\|^p_{L_p(D_k)} \leq c \|\nabla u\|^p_{L_p(\mathbb{R}^n)} + c \|u/|x|\|^p_{L_p(\mathbb{R}^n)},$$

and the right-hand side of the last inequality does not exceed the right-hand side of (4.11) by the Hardy inequality. So, (4.11) is true. By combining (4.9)–(4.11) and Lemma 4.2, we arrive at

$$\int_{\Gamma} |f(x)|^p \sigma_p(z) \, ds_x + |f|^p_{p, \Gamma} \leq c \|\nabla u\|^p_{L_p(\mathbb{R}^n)}.$$ 

To complete the proof of the theorem, it remains to verify the reverse inequality. Let $f$ be a function on $\Gamma$ ($f = 0$ for $z < z_1$) such that the expression on the left in (4.12) is finite. We need to construct a function $u \in H^1_p(\mathbb{R}^n)$, $u|_{\Gamma} = f$, that satisfies the inequality reverse to (4.12). With the help of the transformation (2.2), we define $g_k = (\mu_k f) \circ \kappa_k^{-1}$ on the lateral surface $S$ of the cylinder $Q$. The finiteness of the left-hand side in (4.12) implies that $[g_k]_{p,S} < \infty$ for $k \geq 1$. We extend $g_k(Y,Z)$ by zero to the bases $Z = 0$, $Z = 1$ of $Q$. Then $g_k$ is defined on $\partial Q$ and

$$[g_k]_{p,\partial Q} \leq c [g_k]_{p,S}.$$ 

By Lemma 4.1, there is a function $v_k \in H^1_p(\mathbb{R}^n)$ such that $v_k|_{S} = g_k$ and

$$\|\nabla v_k\|_{L_p(\mathbb{R}^n)} \leq c [g_k]_{p,S}.$$ 

Since Lipschitz continuous functions with bounded supports are multipliers in $H^1_p(\mathbb{R}^n)$, we may assume that $v_k$ is compactly supported in $\mathbb{R}^n$. Let $u_k = v_k \circ \kappa_k$. Then $u_k \in H^1_p(\mathbb{R}^n)$, the support of $u_k$ is bounded, $u_k|_{\Gamma_k} = \mu_k f$, and

$$\|\nabla u_k\|_{L_p(\mathbb{R}^n)} \leq c [\mu_k f]_{p,\Gamma_k}.$$ 

Let $\{\alpha_k\}_{k \geq 1}$ be the collection of functions described in the proof of Theorem 1.1. We define

$$u(x) = \sum_{k \geq 1} \alpha_k(z) \psi(|y|/\varphi_k) u_k(x), \quad \varphi_k = \varphi(z_k).$$

Clearly, $u(x)$ is defined in $\mathbb{R}^n$, the support of the general term of the sum lies in the cylinder $D_k = B_{3\varphi_k}^{(n-1)} \times (z_{k-1}, z_{k+1})$, and

$$u|_{\Gamma} = \sum_{k \geq 1} \alpha_k \mu_k f = \sum_{k \geq 1} \mu_k f = f.$$
We claim that $u$ is as required, i.e., $u$ satisfies the inequality reverse to (4.12). So, it remains to check the estimate
\[ \| \nabla u \|_{L^p(R^n)} \leq c \left( \sum_{k \geq 1} \| \nabla u_k \|_{L^p(R^n)}^p \right)^{1/p}. \]

The result will then follow by combining this estimate with (4.13) and Lemma 4.2. We have
\[ \| \nabla u \|_{L^p(R^n)}^p \leq c \sum_{k \geq 1} \| \nabla u_k \|_{L^p(D_k)}^p + c \sum_{k \geq 1} \frac{\| u_k \|_{L^p(D_k)}^p}{\varphi_k}. \]

Since $\varphi_k \sim |x|$ in $D_k$, the last sum over $k$ does not exceed
\[ c \sum_{k \geq 1} \frac{\| u_k \|_{L^p(R^n)}}{|x|} \leq c \sum_{k \geq 1} \| \nabla u_k \|_{L^p(R^n)}^p. \]

Here, we have used the Hardy inequality at the last step. \hfill \Box

§5. Equivalence of Some Norms for Functions on $\Gamma$

In this section we prove two lemmas on the equivalence of norms that will be used in the proof of Theorems 1.3 and 1.4.

In what follows we assume that the function $\varphi$ describing our paraboloid satisfies the additional condition
\[ \sup \{ \varphi(2z)/\varphi(z) : z \geq 1 \} < \infty. \]

We need a new partition of unity, different from that constructed in Theorem 1.1. Suppose $z_0$ satisfies (2.1). We define a sequence $\{ t_k \}_{k \geq 0}$ by
\[ t_0 = z_0, \quad t_{k+1} = 2t_k, \quad k = 0, 1, \ldots. \]

Let $\{ \lambda_k \}_{k \geq 1}$ be a smooth partition of unity for $[t_1, \infty)$ subordinate to the covering $\{ (t_{k-1}, t_{k+1}) \}_{k \geq 1}$. We can construct $\{ \lambda_k \}$ so as to ensure the conditions
\[ \text{dist} \left( \text{supp} \lambda_k, R^1 \setminus (t_{k-1}, t_{k+1}) \right) \sim t_k, \quad |\lambda_k'| \leq c t_k^{-1}, \quad k = 1, 2, \ldots. \]

Let $p \in (n-1, n)$. We introduce the following norms (and seminorms) for $f \in L^p_{\text{loc}}(\Gamma)$. The norm $\| f \|_{p, \Gamma}$ and the seminorm $|f|_{p, \Gamma}$ are defined in the same way as in the preceding section (before Lemma 4.2). However, it should be noted that the weight function $\sigma_p$ occurring in the weighted norm
\[ \| f \|_{p, \Gamma} = \left( \int_\Gamma |f(x)|^p \sigma_p(z) \, dx \right)^{1/p} \]

is defined by (3.4) and depends on $p$.

Yet another seminorm $\{ f \}_{p, \Gamma}$ is defined by
\[ \{ f \}_{p, \Gamma}^p = \int_{z, \zeta > 0} \int_{z+\zeta > M(z, \zeta)} |f(\varphi(z)\theta, \zeta) - f(\varphi(z)\theta, \zeta)|^p \, d\theta, \]

where, as usual, $M(z, \zeta) = \max\{1, \varphi(z), \varphi(\zeta)\}$.

**Lemma 5.1.** Suppose $n > 2$ and $p \in (n-1, n)$, and let (5.1) be fulfilled. If $f \in L^p_{\text{loc}}(\Gamma)$ and $f(y, z) = 0$ for $z < t_1$, then
\[ \| f \|_{p, \Gamma} + |f|_{p, \Gamma} + \{ f \}_{p, \Gamma} \sim \| f \|_{p, \Gamma} + \sum_{k \geq 1} |\lambda_k f|_{p, \Gamma} + \sum_{k \geq 1} \{ \lambda_k f \}_{p, \Gamma}, \]

where $\{ \lambda_k \}$ is the partition of unity described above.
Proof. The same argument as in Lemma 4.2 leads to the estimate
\[ |f(x) - f(\xi)|^p \leq c \sum_{k \geq 1} |\lambda_k(z) f(x) - \lambda_k(\xi) f(\xi)|^p \]
for any \( x, \xi \in \Gamma \). Hence,
\[ |f|_{p, \Gamma}^p + \{f\}_{p, \Gamma}^p \leq c \sum_{k \geq 1} |\lambda_k f|_{p, \Gamma}^p + c \sum_{k \geq 1} \{\lambda_k f\}_{p, \Gamma}^p. \]
Thus, for the proof of the lemma it suffices to establish the inequality
\[ \sum_{k \geq 1} |\lambda_k f|_{p, \Gamma}^p + \sum_{k \geq 1} \{\lambda_k f\}_{p, \Gamma}^p \leq c (\|f\|_{p, \Gamma}^p + |f|_{p, \Gamma}^p + \{f\}_{p, \Gamma}^p). \]

Let \( x = (y, z), \xi = (\eta, \zeta) \in \Gamma \), and let \( \zeta > z \). It is clear that
\[ \sum_{k \geq 1} |\lambda_k(z) f(x) - \lambda_k(\zeta) f(\xi)|^p \leq c |f(x) - f(\xi)|^p + c |f(\xi)|^p \sum_{k \geq 1} |\lambda_k(z) - \lambda_k(\zeta)|^p. \]
To bound the last sum, first we suppose that \( z < \zeta \leq 2z \). Then the number of nonzero terms in this sum is bounded uniformly with respect to \( z, \zeta \). By using (5.2), we obtain
\[ \sum_{k \geq 1} |\lambda_k(z) - \lambda_k(\zeta)|^p \leq c |z - \zeta|^p / \zeta^p. \]
If \( \zeta > 2z \), then the right-hand side of (5.5) is equivalent to a constant, so that (5.5) is true in this case as well. Inequalities (5.4) and (5.5) imply
\[ \sum_{k \geq 1} |\lambda_k f|_{p, \Gamma}^p \leq c \|f\|_{p, \Gamma}^p + c \int_{\Gamma} |f(\xi)|^p \frac{d\xi}{\zeta^p} \int_{0 < \zeta - 2z < \varphi(\xi)} |x - \xi|^{2-n} ds_x. \]
Since the last integral over \( x \) is not greater than \( c \varphi(\zeta) \) and
\[ \varphi(\zeta) \zeta^{-p} \leq c \sigma_p(\zeta), \]
the double integral above does not exceed \( c \|f\|_{p, \Gamma}^p \). So, we have established the estimate
\[ \sum_{k \geq 1} |\lambda_k f|_{p, \Gamma}^p \leq c (\|f\|_{p, \Gamma}^p + |f|_{p, \Gamma}^p). \]
To conclude the proof of (5.3) (and of the lemma), we need to check the inequality
\[ \sum_{k \geq 1} \{\lambda_k f\}_{p, \Gamma}^p \leq c (\|f\|_{p, \Gamma}^p + \{f\}_{p, \Gamma}^p). \]
Put \( x = (\varphi(\zeta) \theta, z), \xi = (\varphi(\zeta) \theta, \zeta) \) in (5.4). Using inequality (5.5) and the relation
\[ ds_x \sim \varphi(\zeta)^{n-2} d\zeta d\theta, \]
we obtain
\[ \sum_{k \geq 1} \{\lambda_k f\}_{p, \Gamma}^p \leq c \{f\}_{p, \Gamma}^p + c \int_{\Gamma} |f(\xi)|^p \frac{d\xi}{\varphi(\zeta)^{n-2} \zeta^p} \int_0^{\zeta - \varphi(\zeta)} (\zeta - z)^{n-2} dz. \]
The double integral on the right is not greater than
\[ c \int_{\Gamma} |f(\xi)|^p \varphi(\zeta)^{2-n} \zeta^{n-1-p} ds_x = c \int_{\Gamma} |f(\xi)|^p \sigma_p(\zeta) ds_x. \]
Combining this with the preceding inequality, we get (5.6), concluding the proof of the lemma. \( \square \)
Next, we study some equivalent norms in the case where \( p = n - 1 \). The norm \( \|f\|_{p,\Gamma} = \|f(\sigma_p)^{1/p}\|_{L_p(\Gamma)} \) is introduced in the same way as above with 
\[ \sigma_p(z) = \min\{1, (\varphi(z) \log(\varphi(z)))^{1-p}\}. \]
Furthermore, we consider the seminorms
\[
(f)_{p,\Gamma} = \left( \iint_{\Gamma \times \Gamma} |f(x) - f(\xi)|^p \, P\left( \frac{|x - \zeta|}{M(z, \zeta)} \right) \frac{d\xi}{|x - \xi|^{n+p-2}} \right)^{1/p},
\]
\[
\langle f \rangle_{\Gamma} = \left( \iint_{x,\xi \in \Gamma: 2^{-1} < z/\xi < 2} |f(x) - f(\xi)|^p \, P\left( \frac{|x - \zeta|}{M(z, \zeta)} \right) \frac{d\xi}{|x - \xi|^{n+p-2}} \right)^{1/p},
\]
where \( x = (y, z), \xi = (\eta, \zeta), M(z, \zeta) = \max\{\varphi(z), \varphi(\zeta), 1\} \), \( d\xi, d\zeta \) are the area elements on \( \Gamma \), and
\[ P(t) = 1 + t^{2p-2}(\log(1 + t))^{-p}. \]

**Lemma 5.2.** Suppose \( p = n - 1 > 1 \), \( f \in L_{p,\text{loc}}(\Gamma) \), and \( f(y, z) = 0 \) for \( z < 2z_0 \), where \( z_0 \) satisfies (2.1). Also, let \( \{\lambda_k\} \) be the partition of unity described at the beginning of the present section. Then
\[
\|f\|_{p,\Gamma} + (f)_{p,\Gamma} \sim \|f\|_{p,\Gamma} + \langle f \rangle_{p,\Gamma} \sim \|f\|_{p,\Gamma} + \left( \sum_{k \geq 1} \langle \lambda_k f \rangle_{p,\Gamma}^p \right)^{1/p}.
\]

**Proof.** Let \( r = |x - \xi| \). We have
\[
\int_{\{\xi \in \Gamma: \zeta > 2z\}} \frac{d\xi}{r^{n+p-2}} \leq c \int_{2z}^{\infty} \varphi(z)^{p-1} \frac{d\zeta}{\zeta^{p-1}} \leq c \sigma_p(z), \ z > z_0;
\]
\[
\int_{\{x \in \Gamma: z|\zeta| < 2\}} \frac{d\xi}{r^{n+p-2}} \leq c \int_0^{\infty} \varphi(z)^{p-1} \frac{dz}{\zeta^{p-1}} \leq c \sigma_p(z), \ z > 2z_0;
\]
\[
\int_{\{\xi \in \Gamma: \zeta > 2z\}} rM(z, \zeta)^{2p-2} \log(1 + r/M(z, \zeta)) \, d\xi \leq c \int_{2z}^{\infty} \frac{(\zeta/\varphi(z))^{p-1}}{\log(\zeta/\varphi(z))}\zeta^{p-1} \leq c \sigma_p(z), \ z > 2z_0;
\]
\[
\int_{\{x \in \Gamma: z|\zeta| < 2\}} r\log(1 + r/M(z, \zeta)) \, d\xi \leq c \sigma_p(z), \ z > 2z_0.
\]
Thus,
\[
\iint_{\{x,\xi \in \Gamma: \zeta > 2z\}} \left( |f(x)|^p + |f(\xi)|^p \right) P\left( \frac{|\zeta - z|}{M(z, \zeta)} \right) \frac{d\xi}{r^{n+p-2}} \leq c \|f\|_{p,\Gamma}^p,
\]
and the first relation in (5.7) is established.

Now we check the second relation in (5.7). Let
\[ E = \{(x, \xi) : x, \xi \in \Gamma : 2^{-1} < z/\xi < 2\}, \]
where \( x = (y, z), \xi = (\eta, \zeta) \). Arguing in the same way as in Lemma 5.1, we obtain
\[
|f(x) - f(\xi)|^p \leq c \sum_{k \geq 1} |\lambda_k(z)f(x) - \lambda_k(\zeta)f(\xi)|^p,
\]
whence
\[
\langle f \rangle_{p,\Gamma}^p \leq c \sum_{k \geq 1} \langle \lambda_k f \rangle_{p,\Gamma}^p.
\]
So, the proof of the second relation in (5.7) reduces to the proof of the estimate
\begin{equation}
(5.8) \quad c \sum_{k \geq 1} \int_{E} |(\lambda_k f)(x) - (\lambda_k f)(\xi)|^p P \left( \frac{|z - \zeta|}{M(z, \zeta)} \right) \frac{ds_x ds_\xi}{|x - \zeta|^{n+p-2}} \leq \langle f \rangle_{p,\Gamma}^p + \|f\|_{p,\Gamma}^p.
\end{equation}

By combining (5.4) and (5.5), we see that if \((x, \xi) \in E\), then
\begin{equation}
(6.1) \quad \frac{c}{\|f\|_{p,\Gamma}^p} \left( |f(x) - f(\xi)|^p + \left( \frac{|f(\xi)|}{\zeta} \right)^p |\zeta - z|^p \right) \leq \langle f \rangle_{p,\Gamma}^p + \|f\|_{p,\Gamma}^p.
\end{equation}

Furthermore,
\begin{equation}
P \left( \frac{|z - \zeta|}{M} \right) \sim \begin{cases} 
1 & \text{if } |\zeta - z| < M, \\
\left( \frac{\zeta - z}{M} \right)^{2p-2} \left( \log \left( 1 + \frac{\zeta - z}{M} \right) \right)^{-p} & \text{if } |\zeta - z| > M,
\end{cases}
\end{equation}
where \(M = M(z, \zeta)\). Since \(M(z, \zeta) \approx \varphi(\zeta)\) for \((x, \xi) \in E\), the left-hand side in (5.8) is dominated by the expression \(c \langle (f \rangle_{p,\Gamma}^p + I_1 + I_2\rangle\), where
\begin{align*}
I_1 &= \int_{\Gamma} |f(\xi)|^p \frac{ds_\xi}{\zeta^p} \int_{\{x \in \Gamma : |\zeta - z| < M(z, \zeta)\}} |\zeta - z|^p \frac{ds_x}{|x - \zeta|^{n+p-2}}, \\
I_2 &= \int_{\Gamma} |f(\xi)|^p \frac{ds_\xi}{\zeta^p} \int_{\ell(\zeta)} \frac{|(\zeta - z)/\varphi(\zeta)|^{p-1} dz}{|\log(1 + |\zeta - z|/\varphi(\zeta))|^p},
\end{align*}
and \(\ell(\zeta) = \{ z : \varphi(\zeta) < |\zeta - z| < \zeta \}\).

Let \(J(\xi)\) be the inner integral in \(I_1\). We have
\begin{equation}
J(\xi) \leq \int_{\{x \in \Gamma : |\zeta - z| < c \varphi(\zeta)\}} |x - \zeta|^{2-n} ds_x \leq c \varphi(\zeta).
\end{equation}
Hence, \(I_1 \leq c \|f\|_{p,\Gamma}^p\).

In order to bound \(I_2\), we represent the integral over \(\ell(\zeta)\) in the form
\begin{equation}
2\varphi(\zeta) \int_1^{\zeta/\varphi(\zeta)} g(t) dt, \quad g(t) = t^{p-1} (\log(1 + t))^{-p}.
\end{equation}
Since
\begin{equation}
\int_1^a g(t) dt \leq c ag(a), \quad a > 2,
\end{equation}
it follows that
\begin{equation}
I_2 \leq c \int_{\Gamma} |f(\xi)|^p g(\zeta/\varphi(\zeta)) \frac{ds_\xi}{\zeta^{p-1}} \leq c \|f\|_{p,\Gamma}^p.
\end{equation}
Thus, inequality (5.8) is established, and this concludes the proof of the lemma. \(\square\)

§6. THE SPACE \(T_p(\Gamma)\) FOR \(p \in [n - 1, n)\)

**Proof of Theorem 1.3.** Let \(u \in H_p^1(\mathbb{R}^n)\), and let \(u|_{\Gamma} = f\). As in Theorem 1.2, we may assume that \(u = 0\) in a neighborhood of the origin. Without loss of generality, this neighborhood has the form \(B_{2z_0}^{(n-1)} \times (-2z_0, 2z_0)\), where \(z_0\) satisfies (2.1).

First, we prove the estimate
\begin{equation}
(6.1) \quad \|f\|_{p,\Gamma}^p + |f|_{p,\Gamma}^p + \{f\}_{p,\Gamma}^p \leq \|\nabla u\|_{H_p^1(\mathbb{R}^n)}^p.
\end{equation}
The first term on the left in (6.1) is not greater than the right-hand side by Lemma 3.2. To show that the remaining terms on the left in (6.1) do not exceed the right-hand side of (6.1), we use the numerical sequence \(\{t_k\}_{k \geq 0}\) and the partition of unity \(\{\lambda_k\}_{k \geq 1}\) described at the beginning of the preceding section. Let \(\psi\) be a cut-off function satisfying
\begin{equation}
(6.2) \quad \psi \in C^\infty([0, \infty), \quad \psi|_{(0,2)} = 1, \quad \psi|_{(3, \infty)} = 0.
\end{equation}
A sequence \( \{u_k\} \) is defined by
\[
u_k(x) = \lambda_k(z)\psi_k(y)u(x), \quad \psi_k(y) = \psi(|y|/t_k), \quad k = 1, 2, \ldots.
\]
Clearly, \( u_k \) is defined in \( \mathbb{R}^n \), and its support lies in the cylinder
\[
Q_k = \{ x : z \in (t_{k-1}, t_{k+1}), \ |y| < 3t_k \}.
\]
Furthermore, \( u_k|_\Gamma = \lambda_k f, \ k = 1, 2, \ldots \) Next, we introduce the transformation
\[
x = (y, z) \mapsto \nu_k(x) = X = (Y, Z), \quad Y = \frac{y}{\varphi(z)}, \quad Z = \frac{z}{\varphi(t_k)},
\]
which maps the set \( \{x \in \Omega^- : z > 0\} \) to the exterior of the cylinder
\[
Q = \{(Y, Z) : |Y| < 1, \ Z \in \mathbb{R}^1\},
\]
and the surface \( \Gamma \setminus \{O\} \) to the lateral surface \( S \) of \( Q \). Put
\[
\nu_k|_S = g_k \quad \text{and} \quad \nu_k(Y, Z) = 0 \quad \text{if} \quad Z \notin (t_{k-1}/\varphi(t_k), t_{k+1}/\varphi(t_k)) \}. \]
By Theorem 3.6 in [12], we have
\[
\inf \{\|\nabla v\|_{L^p(\mathbb{R}^n)} : v \in L^1_p(\mathbb{R}^n), \ v|_S = g \} \sim |g|_{p,S} + \{g\}_{p,S},
\]
where
\[
|g|_{p,S} = \int \int_{X, X' \in S : |Z-Z'| < 1} |g(X) - g(X')|^p \frac{ds_X ds_{X'}}{|X-X'|^{n+p-2}},
\]
\[
\{g\}_{p,S} = \int \int_{Z, Z' \in \mathbb{R}^1 : |Z-Z'| > 1} \frac{dZ dZ'}{|Z-Z'|^{p+2-n}} \int_{S^{(n-2)}} |g(\theta, Z) - g(\theta, Z')|^p \, d\theta.
\]
In particular, (6.3) implies that
\[
|g|_{p,S} + \{g\}_{p,S} \leq c \|\nabla \nu_k\|_{L^p(\mathbb{R}^n)}.
\]
Putting \( \varphi_k = \varphi(t_k) \), we observe that the change of variables \( x \mapsto \nu_k x = X \) leads to the relations
\[
|g|_{p,S} \sim |\lambda_k f|_{p,\Gamma}, \quad \{g\}_{p,S} \sim \{\lambda_k f\}_{p,\Gamma}, \quad \|\nabla v_k\|_{L^p(\mathbb{R}^n)} \sim \|\nabla u_k\|_{L^p(\mathbb{R}^n)}.
\]
With the help of (6.5), inequality (6.4) in the variable \( x = \nu_k^{-1} X \) takes the form
\[
|\lambda_k f|_{p,\Gamma} + \{\lambda_k f\}_{p,\Gamma} \leq c \|u_k\|_{L^p(\mathbb{R}^n)}.
\]
To conclude the proof of (6.1), we need yet another estimate
\[
\sum_{k \geq 1} \|\nabla u_k\|_{L^p(\mathbb{R}^n)} \leq c \|\nabla u\|_{L^p(\mathbb{R}^n)}.
\]
Since \( \text{supp } u_k \subset Q_k \) and \( |x| \sim t_k \) for \( x \in Q_k \), it follows that
\[
\sum_{k \geq 1} \|\nabla u_k\|_{L^p(\mathbb{R}^n)} \leq c \sum_{k \geq 1} \left( \|\nabla u\|_{L^p(Q_k)} + \int_{Q_k} \frac{|u(x)|^p}{t_k} \, dx \right)
\]
\[
\leq c \|\nabla u\|_{L^p(\mathbb{R}^n)} + c \int_{\mathbb{R}^n} (|u(x)|/|x|)^p \, dx,
\]
and (6.7) follows by the Hardy inequality. Combining (6.6), (6.7), and Lemma 5.1, we arrive at (6.1).
To prove the reverse inequality, we assume that the left-hand side of (6.1) is finite. Our aim is to construct \( u \in H^1_p(\mathbb{R}^n) \) such that \( u|_\Gamma = f \) and
\[
\|\nabla u\|_{L^p_p(\mathbb{R}^n)} \leq c \|f\|^p_{p,\Gamma} + \|f\|^p_{p,\Gamma} + \{f\}^p_{p,\Gamma}.
\]
Let \( g_k = (\lambda_k f) \circ \nu^{-1}_k \). By (6.5), we have \( \|g_k\|_{p,S} + \{g_k\}_{p,S} < \infty \). Then (6.3) implies that there is a function \( v_k \in L^1_p(\mathbb{R}^n) \) satisfying \( v_k|_S = g_k \) and
\[
\|\nabla v_k\|_{L^p_p(\mathbb{R}^n)} \leq c (\|g_k\|_{p,S} + \{g_k\}_{p,S}).
\]
Furthermore, \( g_k(Y,Z) = 0 \) for \( Z \notin (\varphi_k^{-1}t_{k-1},\varphi_k^{-1}t_{k+1}) \), whence \( v_k \in H^1_p(\mathbb{R}^n) \). Since all Lipschitz continuous functions with bounded supports are multipliers in \( H^1_p(\mathbb{R}^n) \), we may assume that \( v_k \) has a bounded support in the strip \( Z \in (\varphi_k^{-1}t_{k-1},\varphi_k^{-1}t_{k+1}) \).

Let \( \{\beta_k\}_{k \geq 1} \) be a collection of functions such that
\[
\beta_k \in C_0^\infty(t_{k-1},t_{k+1}), \quad \beta_k \lambda_k = \lambda_k, \quad |\beta'_k| \leq c t_k^{-1}, \quad k = 1,2,\ldots.
\]
We define a function \( u \) on \( \mathbb{R}^n \) by
\[
u(x) = \sum_{k \geq 1} \beta_k(x) \psi_k(y) u_k(x),
\]
where \( \psi_k(y) = \psi(|y|/t_k) \), and \( \psi \) is given by (6.2). Clearly,
\[
u|_\Gamma = \sum_{k \geq 1} \psi_k \lambda_k \beta_k f = \sum_{k \geq 1} \lambda_k f = f.
\]
To conclude the proof of the theorem, it remains to check (6.8). Since the support of the general term of the sum (6.11) lies in \( Q_k \), we have
\[
\|\nabla u\|^p_{L^p_p(\mathbb{R}^n)} \leq c \sum_{k \geq 1} \|\nabla u_k\|^p_{L^p_p(Q_k)} + c \sum_{k \geq 1} t_k^{-p} \int_{Q_k} |u_k|^p \, dx.
\]
Since \( |x| \sim t_k \) for \( x \in Q_k \), the last sum is not greater than
\[
\sum_{k \geq 1} \|(|u_k|/|x|)|^p_{L^p_p(\mathbb{R}^n)}
\]
which does not exceed
\[
\sum_{k \geq 1} \|\nabla u_k\|^p_{L^p_p(\mathbb{R}^n)}
\]
by the Hardy inequality. An application of (6.10) and Lemma 5.1 gives (6.8). \( \square \)

**Proof of Theorem 1.4.** Since the argument below is similar to the proof of Theorem 1.3, we shall not give it in detail.

Suppose \( u \in H^1_p(\mathbb{R}^n) \), \( u|_\Gamma = f \), \( u = 0 \) in the same cylindrical neighborhood of the origin as in Theorem 1.3. By Lemma 3.2, the proof of the estimate
\[
\|f\|_{p,\Gamma} + \{f\}_{p,\Gamma} \leq c \|\nabla u\|_{L^p_p(\mathbb{R}^n)}
\]
reduces to the proof of (6.12) with the first term omitted. Next, we construct a sequence \( u_k = \lambda_k \psi_k u \) as in Theorem 1.3; then \( u_k \) is defined in \( \mathbb{R}^n \), \( \supp u_k \subset Q_k \), \( Q_k = B_{3t_k}(t_{k-1}) \times (t_{k-1},t_{k+1}) \). The transformation \( x \mapsto \nu_k x = X \) maps the surface \( \Gamma \setminus \{O\} \) to the lateral surface \( S = \partial Q \) of the cylinder \( Q = B_1(n-1) \times \mathbb{R}^1 \).
The further argument is closely related to the theorem on traces \( u|_S \) of functions \( u \in L^1_p(\mathbb{R}^n) \) for \( p = n - 1 \). This theorem (see [12, 3.7]) claims the equivalence of the seminorms
\[
\inf \{ \| \nabla v \|_{L^p(\mathbb{R}^n)} : v \in L^1_p(\mathbb{R}^n), \ v|_S = g \}
\]
and
\[
\langle g \rangle_{p,S} = \left( \int_{S \times S} |g(X) - g(X')|^p P(|Z - Z'|) \frac{ds_X ds_{X'}}{|X - X'|^{n+p-2}} \right)^{1/p},
\]
where \( P(t) = 1 + t^{2p-2} \log(1 + t) \). By applying the above theorem on traces of the functions of class \( L^1_p(\mathbb{R}^n) \) on a cylindrical surface, we find that there is a function \( v_k \in L^1_p(\mathbb{R}^n) \) such that \( v_k|_S = g_k \) and
\[
\| \nabla v_k \|_{L^p(\mathbb{R}^n)} \leq c \langle g_k \rangle_{p,S}.
\]
Returning to the variables \( x = \nu_k^{-1}X, \xi = \nu_k^{-1}X' \), we rewrite the preceding inequality in the form
\[
\langle \lambda_k f \rangle_{p,\Gamma} \leq c \| \nabla u_k \|_{L^p(\mathbb{R}^n)}^p.
\]
Combining this inequality, estimate (6.7) (which is verified in the same way as in Theorem 1.3), and Lemma 5.2, we obtain (6.12).

To prove the reverse inequality, we assume that the left-hand side of (6.12) is finite. Then \( \langle g_k \rangle_{p,S} < \infty \), where \( g_k = (\lambda_k f) \circ \nu_k^{-1} \). By applying the above theorem on traces of the functions of class \( L^1_p(\mathbb{R}^n) \) on a cylindrical surface, we find that there is a function \( v_k \in L^1_p(\mathbb{R}^n) \) such that \( v_k|_S = g_k \) and
\[
\| \nabla v_k \|_{L^p(\mathbb{R}^n)} \leq c \langle g_k \rangle_{p,S}.
\]
Returning to \( x = \nu_k^{-1}X \), we rewrite this inequality in the form
\[
\| \nabla u_k \|_{L^p(\mathbb{R}^n)} \leq c \langle \lambda_k f \rangle_{p,\Gamma},
\]
where \( u_k = v_k \circ \nu_k \). Let \( \{ \beta_k \} \) be the collection of functions described in Theorem 1.3, and let \( u \) be defined by (6.11). Then \( u|_\Gamma = f \) and (as was shown in Theorem 1.3) the following inequality is true:
\[
\| \nabla u \|_{L^p(\mathbb{R}^n)}^p \leq c \sum_{k \geq 1} \| \nabla u_k \|_{L^p(\mathbb{R}^n)}^p.
\]
Combing this inequality with (6.13) and Lemma 5.2, we arrive at the inequality reverse to (6.12). The proof of the theorem is complete. \( \square \)

References


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Received 13/SEP/2011

Translated by THE AUTHOR