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ON $\mathcal{C}^m$-APPROXIMABILITY OF FUNCTIONS BY POLYNOMIAL SOLUTIONS OF ELLIPTIC EQUATIONS ON PLANE COMPACT SETS

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Abstract. Conditions of $\mathcal{C}^m$-approximability of functions by polynomial solutions of homogeneous elliptic equations of order n on plane compact sets are studied. For positive integers m and n such that $m \geq n-1$, new necessary and sufficient approximability conditions of a topological and metrical nature are obtained.

§1. Introduction

In this paper we are interested in the $\mathcal{C}^m$-smooth approximation (for an integer $m > 0$) of functions by polynomial solutions of homogeneous elliptic partial differential equations with constant complex coefficients on compact subsets in $\mathbb{R}^2$.

Let n be a natural number and $L(x_1, x_2)$ a homogeneous polynomial in two real variables x_1 and x_2 of degree n with constant complex coefficients, which means that

$$L(x_1, x_2) = \sum_{\substack{k,s\in\{0,\dots,n\}\\ k+s=n}} c_{k,s} x_1^k x_2^s,$$

where $c_{k,s} \in \mathbb{C}$. Assuming that L satisfies the *ellipticity condition*, i.e., $L(x_1, x_2) = 0$ if and only if $x_1 = x_2 = 0$, we define a differential operator L as follows:

$$L := L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right). \tag{1.1}$$

Thus, L is a *homogeneous elliptic differential operator on* $\mathbb{R}^2$ *of order* n *with constant complex coefficients*. In order to simplify the notation, the homogeneous polynomial $L(x_1, x_2)$ and the respective differential operator L are traditionally denoted by the same symbol; this does not cause any ambiguity.

For an arbitrary subset $E \subset \mathbb{R}^2$, we denote by $\mathcal{O}_L(E)$ the set of all complex-valued functions f, each being defined on (its own) open subset containing E and satisfying there the equation

$$Lf = 0 \tag{1.2}$$

(in the classical sense). The functions of class $\mathcal{O}_L(E)$ are said to be *L-analytic* on E.

Let $\mathcal{P}_L$ be the set of all polynomials in the variables x_1 and x_2 with complex coefficients that are solutions of equation (1.2). These polynomials are called *L-analytic polynomials*.

We shall use the following notation. Throughout, z will mean both a complex number $x_1 + ix_2$ and a point (x_1, x_2) in the plane $\mathbb{R}^2$. A pair $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, where, as usual,

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$\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, will be called a 2-index. For a 2-index α and for $z \in \mathbb{R}^2$, we put $|\alpha| := \alpha_1 + \alpha_2$, $\alpha! := \alpha_1!\alpha_2!$, and $z^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$. Define the differential operators

$$\partial_1 := \frac{\partial}{\partial x_1}, \quad \partial_2 := \frac{\partial}{\partial x_2}, \quad \partial^\alpha = \partial^{(\alpha_1,\alpha_2)} := \partial_1^{\alpha_1}\partial_2^{\alpha_2}.$$

Now we give several examples of operators L for which the respective classes of L-analytic functions and polynomials are well-known classical spaces of functions and polynomials.

Let $L = \bar\partial := \partial/\partial\bar z = \frac{1}{2}(\partial_1 + i\partial_2)$ be the standard Cauchy–Riemann operator. In this case, the L-analytic functions are usual holomorphic functions, and the L-analytic polynomials are standard polynomials in the complex variable z.

In the case where $L = \Delta$ (the Laplace operator), the L-analytic functions and polynomials are harmonic functions and polynomials, respectively.

For $L = \bar\partial^n$ and $L = \Delta^n$, $n \geq 2$, the classes of L-analytic functions and polynomials are the classes of polyanalytic and polyharmonic functions and polynomials, respectively (see, e.g., [1]).

Let X be a compact set in $\mathbb{R}^2$, let X° be its interior, and let L be an operator of the form (1.1) of order n. We consider the following problem.

Problem 1. *Suppose $n \geq 2$ and $m \in \mathbb{N}$. To find necessary and sufficient condition on a compact set X in order that each function f of class $\mathcal{C}^m$ in a neighborhood of X and satisfying $f \in \mathcal{O}_L(X^\circ)$ be approximable on X by a sequence $(p_k)_{k=1}^\infty$ of L-analytic polynomials in such a way that for all 2-indices α with $|\alpha| \leq m$ one has*

$$\partial^\alpha p_k \rightrightarrows_X \partial^\alpha f \tag{1.3}$$

as $k \to \infty$, where $\rightrightarrows_X$ means uniform convergence on X.

It should be noted that other possible settings of $\mathcal{C}^m$-approximability problems by L-analytic functions and polynomials as well as several interesting and important results on this topic and the respective bibliography can be found in [2, 3, 4, 5].

Let us study in detail the structure of polynomials by which approximation in Problem 1 is realized. For instance, in the case where $L = \bar\partial^n$, $n \geq 1$, we are dealing with polyanalytic polynomials of order n, i.e., polynomials of the form

$$\bar z^{n-1} P_{n-1}(z) + \cdots + \bar z P_1(z) + P_0(z),$$

where $P_0, \dots, P_{n-1}$ are polynomials in the complex variable.

Now, let L be an arbitrary second order operator, i.e.

$$L = (\partial_1 - \lambda_1\partial_2)(\partial_1 - \lambda_2\partial_2),$$

where the (complex) numbers λ_1, λ_2 (the roots of the corresponding characteristic equation) are not real (this is equivalent to the fact that the operator L is elliptic). If $\lambda_1 \neq \lambda_2$, then the class $\mathcal{P}_L$ consists of polynomials of the form

$$Q_1\left(\frac{\lambda_1 x_1 + x_2}{\lambda_1 - \lambda_2}\right) + Q_2\left(\frac{\lambda_2 x_1 + x_2}{\lambda_2 - \lambda_1}\right),$$

but each polynomial in the class $\mathcal{P}_L$ for $\lambda_1 = \lambda_2 = \lambda$ has the form

$$\left(x_1 - \frac{1}{\lambda}x_2\right) Q_1\left(x_1 + \frac{1}{\lambda}x_2\right) + Q_2\left(x_1 + \frac{1}{\lambda}x_2\right),$$

where Q_1 and Q_2 are usual polynomials in the complex variable (see, for instance, [6, Chapter 4, §6]). In particular, the class of all harmonic polynomials arises in the situation under consideration for $L = \Delta$. For $L = \bar\partial^2$ we are dealing with the aforesaid class of all polyanalytic polynomials of order 2 (these polynomials are traditionally called bianalytic polynomials).

If the compact set X possesses the approximation property mentioned in Problem 1, then we say that X is a compact set of $(\mathcal{C}^m, L)$-weak polynomial approximation. This terminology is related to the fact that, in general, the convergence (1.3) is weaker then that in the classical Whitney space $\mathcal{C}^m_{\rm jet}(X)$ (see, e.g., [5, §§1,2]).

It is desirable to formulate Problem 1 in terms of appropriate spaces of functions. Let Y be some subset of $\mathbb{R}^2$ and f some complex-valued function defined on Y. Put $\|f\|_Y := \sup_{z\in Y}|f(z)|$. For $Y = \mathbb{R}^2$ the index Y will be dropped.

For a function f of class $\mathcal{C}^m$ and for $s \in \{0, 1, \ldots, m\}$, we define

$$\nabla^s f := \big(\partial^{(s,0)}f, \partial^{(s-1,1)}f, \ldots, \partial^{(1,s-1)}f, \partial^{(0,s)}f\big),$$
$$\mathcal{D}^m f := \big(f, \nabla f, \ldots, \nabla^m f\big),$$

and set $M := (m + 1)(m + 2)/2$. For $s \in \mathbb{Z}_+$, $s \le m$, we also put

$$\|\nabla^s f\|_Y := \max_{0\le t\le s} \|\partial^{(s-t,t)}f\|_Y.$$

Let X be a compact set in $\mathbb{R}^2$, and let $\mathcal{C}(X)$ be the space of all complex-valued continuous functions on X endowed with the uniform norm $\|\cdot\|_X$. The norm $\|h\|_{X,M}$ of the element $h = \big(h_0, \ldots, h_{M-1}\big) \in \mathcal{C}(X)^M$ is defined as usual:

$$\|h\|_{X,M} := \max_{0\le\ell\le M-1} \|h_\ell\|_X.$$

Let $\mathcal{C}^m_w(X)$ denote the closure in $\mathcal{C}(X)^M$ of the subspace $\{(\mathcal{D}^m f)_{|X} \;:\; f \in \mathcal{C}^m(\mathbb{R}^2)\}$. Then $\mathcal{C}^m_w(X)$ is the space of "functions" the "weak" class $\mathcal{C}^m$ on the compact set X. This space is endowed with the norm induced from $\mathcal{C}(X)^M$.

Observe that an element $h = \{h_0, \ldots, h_{M-1}\} \in \mathcal{C}(X)^M$ belongs to $\mathcal{C}^m_w(X)$ if and only if for any $\varepsilon > 0$ there exists a function $f \in \mathcal{C}^m(\mathbb{R}^2)$ such that

$$\|(h_{M(s-1)}, \ldots, h_{M(s)}) - \nabla^s f\|_X < \varepsilon$$

for $s = 0, \ldots, m$ and $M(s) = (s + 1)(s + 2)/2$. Let $\ell \in \{1, \ldots, M - 1\}$, and let $s \in \{0, \ldots, m\}$ and $t \in \{0, \ldots, s\}$ be such that $\ell = M(s - 1) + t$. Since $h_\ell = \partial^{(s-t,t)}f$ in the classical sense on X°, then, if X° is dense in X, the functions h_ℓ for $\ell \in \{1, \ldots, M - 1\}$ are determined by the function h_0 uniquely.

We introduce the spaces $\mathcal{A}^m_L(X)$ and $\mathcal{P}^m_L(X)$ as the $\mathcal{C}(X)^M$-*closures* of the subspaces

$$\{(\mathcal{D}^m f)_{|X} \;:\; f \in \mathcal{C}^m(\mathbb{R}^2) \cap \mathcal{O}_L(X^\circ)\} \quad\text{and}\quad \{(\mathcal{D}^m p)_{|X} \;:\; p \in \mathcal{P}_L\},$$

respectively.

Thus, X is a compact set of $(\mathcal{C}^m, L)$-weak polynomial approximation if and only if $\mathcal{A}^m_L(X) = \mathcal{P}^m_L(X)$, and in Problem 1 it is required to describe all such compact sets.

In the present paper, Problem 1 is solved in the case where $m \ge n-1$. For $m = n-1$ the corresponding approximability criterion is proved in Theorem 2.1 below. It states that a compact set X is a compact set of $(\mathcal{C}^{n-1}, L)$-weak polynomial approximation if and only if the set $\mathbb{R}^2 \setminus X$ is connected. Similar results were obtained previously in [7, Theorem 1], [8, Theorem 3.2] and [9, Theorem 2.1] in the cases of $\mathcal{C}^1$-approximation by harmonic polynomials and by polynomial solutions of general second order elliptic equations, and for $\mathcal{C}^{n-1}$-approximation by polyanalytic polynomials of order n, respectively. Thus, our Theorem 2.1, Theorem 1 in [7], Theorem 3.2 in [8], and Theorem 2.1 in [9] may be viewed as natural analogs of the classical Mergelyan theorem (see [10]) about uniform approximability of holomorphic functions by polynomials in a complex variable.

In the case where $m > n - 1$, the necessary and sufficient approximability condition in Problem 1 is given by Theorem 2.4. This condition is as follows: $\overline{X^\circ} = X$ and the set $\mathbb{R}^2 \setminus X$ is connected.

For $0 < m < n - 1$ Problem 1 remains unsolved.

The following problem on uniform approximability of functions by L-analytic polynomials also remains unsolved in the general case.

Problem 2. *To find necessary and sufficient conditions on a compact set X in order that each function f continuous on X and L-analytic on X° admit uniform approximation on X by some sequence of L-analytic polynomials.*

For operators L of order $n = 1$, solution of Problem 2 follows immediately from Mergelyan's theorem: the necessary and sufficient condition for the desired approximation property is the connectedness of the set $\mathbb{R}^2 \setminus X$.

We also recall the classical Walsh–Lebesgue theorem (see, e.g., [5, 1]), which says that the necessary and sufficient approximability condition in Problem 2 for $L = \Delta$ is as follows: $\partial X = \partial \widehat{X}$, where $\widehat{X}$ is the union of X and all bounded (connected) components of the set $\mathbb{R}^2 \setminus X$. The compact sets X such that $\partial X = \partial \widehat{X}$ are called *Carathéodory compact sets*. Such compact sets have appeared naturally in various problems of approximation theory.

For general operators of order $n \geq 2$ different from the Laplace operator, practically, the only well-studied case in Problem 2 is that of $L = \bar{\partial}^n$, or, in other words, the case of uniform approximation by polyanalytic polynomials. The approximability criterion for Carathéodory compact sets was obtained in this case in [11], and several useful necessary and sufficient conditions for uniform approximability of functions by polyanalytic polynomials on non-Carathéodory compact sets were discovered in [11, 12, 13, 14]. All these results are formulated in terms of special analytic properties of the compact sets where approximation is considered and, in general, they depend substantially on the order of the operator under consideration.

Furthermore, the approximability criterion for Carathéodory compact sets and several necessary and sufficient approximability conditions for more general compact sets of special type were obtained in Problem 2 also for general operators L of order $n = 2$ (see [15, 16, 17]).

In §2 we formulate and discus the main results of the paper (Theorems 2.1 and 2.2) and their consequences. Proofs of these results are given in §3.

§2. Main results and their corollaries

Let $n \geq 2$ be a fixed integer and L an operator of the form (1.1) of order n. We have the following necessary and sufficient condition for $\mathcal{C}^{n-1}$-approximability of functions by L-analytic polynomials.

Theorem 2.1. *Let $X \subset \mathbb{R}^2$ be a compact set. The following conditions are equivalent:*

(a) *for each function $f \in \mathcal{C}^{n-1}(\mathbb{R}^2) \cap \mathcal{O}_L(X^\circ)$ there exists a sequence $(p_k)_{k=1}^\infty$ of L-analytic polynomials such that*
$$\partial^\alpha p_k \rightrightarrows_X \partial^\alpha f$$
as $k \to \infty$ for all $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \leq n-1$;

(b) *the set $\mathbb{R}^2 \setminus X$ is connected.*

Let $B(a, r)$ be an open disk centered at a point $a \in \mathbb{R}^2$ and of radius $r > 0$. If $B = B(a, r)$, then $qB = B(a, qr)$, $q > 0$.

Let $X \subset \mathbb{R}^2$ be a compact set. For $z \in \mathbb{R}^2$ and $r > 0$, we define $d(z, r, X)$ to be the least upper bound for the diameters of all connected components of the set $B(z, r) \setminus X$, and put
$$\theta(X) := \inf\left\{ \frac{d(z, r, X)}{r} \;:\; z \in \partial X,\; r > 0 \right\}.$$
The proof of Theorem 2.1 is based on the following statement.

Theorem 2.2. *Let X be a compact set in $\mathbb{R}^2$ with $\theta(X) > 0$. Then for each function $f \in \mathcal{C}^{n-1}(\mathbb{R}^2) \cup \mathcal{O}_L(X^\circ)$ there exists a sequence of functions $(f_k)_{k=1}^\infty$, $f_k \in \mathcal{O}_L(X)$, such that*

$$\partial^\alpha f_k \rightrightarrows_X \partial^\alpha f \tag{2.1}$$

as $k \to \infty$ for all $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \le n-1$.

Note that for the space $\mathcal{C}^{n-1}_{\mathrm{jet}}(X)$ the validity of analogs of Theorems 2.1 and 2.2 remains in question.

An immediate consequence of Theorem 2.2 is the following statement, the formulation of which does not involve the quantity $\theta(X)$, and which seems useful for the investigation of certain specific examples.

Corollary 2.3. *If the lower bound of the diameters of all connected components of the set $\mathbb{R}^2 \setminus X$ is positive, then for each function f belonging to the class $\mathcal{C}^{n-1}$ in a neighborhood of X and satisfying the condition $f \in \mathcal{O}_L(X^\circ)$, there exists a sequence of functions $(f_k)_{k=1}^\infty \subset \mathcal{O}_L(X)$ such that $\partial^\alpha f_k \rightrightarrows_X \partial^\alpha f$ as $k \to \infty$ for all $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \le n-1$.*

The next result follows immediately from [2, Theorem 1] and from the proof of Theorem 2.1.

Theorem 2.4. *Let X be a compact set in $\mathbb{R}^2$, and let $m \ge n$ be an integer. The following conditions are equivalent*:

(a) *for each function $f \in \mathcal{C}^m(\mathbb{R}^2) \cap \mathcal{O}_L(X^\circ)$ there exists a sequence $(p_k)_{k=1}^\infty$ of L-analytic polynomials such that $\partial^\alpha p_k \rightrightarrows_X \partial^\alpha f$ as $k \to \infty$ for all $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \le m$;*
(b) *$\overline{X^\circ} = X$ and the set $\mathbb{R}^2 \setminus X$ is connected.*

Note that all claims of Theorems 2.1, 2.2, and 2.4, as well as Corollary 2.3 are new, at least for operators L of order $n \ge 3$, except for the case of $L = \bar{\partial}^n$. As has been mentioned above, for operators of order $n = 2$ the corresponding results were proved in [8], while the case of $L = \bar{\partial}^n$, $n \ge 2$, was treated in [9].

Since the concepts of a classical and distributional solutions of equation (1.2) have the same meaning in the class of continuous functions (see [18, Theorem 18.1]), the uniform limit of a sequence of solutions of (1.2) in some domain is a solution again (in the same domain), and therefore, the condition $f \in \mathcal{O}_L(X^\circ)$ is a natural necessary condition for approximability of functions by solutions of equation (1.2) in all approximation problems considered in Theorems 2.1, 2.2 and 2.4, and in Corollary 2.3.

At the end of this section we discuss the following problem.

Problem 3. *Let $N = n(n+1)/2$, where n is the order of the operator L. Find necessary and sufficient conditions on a compact set X in order that each element $(h_0, \dots, h_{N-1}) \in \mathcal{C}(X)^N$ admit approximation on X by some sequence $(p_k)_{k=1}^\infty$ of L-analytic polynomials in such a way that*

$$\mathcal{D}^{n-1} p_k \rightrightarrows_X \big(h_0, h_1, \dots, h_{N-1}\big), \quad k \to \infty.$$

In the case of operators L of order $n = 2$, the solution of this problem was obtained in [8, Theorem 3.4] in terms of geometrical and topological properties of X. The result reads as follows.

The following conditions on a compact set $X \subset \mathbb{R}^2$ are equivalent:

(a) *for each element $(h_0, h_1, h_2) \in (\mathcal{C}(X))^3$ there exists a sequence $(p_k)_{k=1}^\infty$ of L-analytic polynomials such that*

$$p_k \rightrightarrows_X h_0, \quad \nabla p_k \rightrightarrows_X (h_1, h_2), \quad k \to \infty;$$

(b) *the set* $\mathbb{R}^2 \setminus X$ *is connected, and moreover,* X *contains no rectifiable Jordan arc.*

In order to prove this, it suffices to note that the space $\{(h, \nabla h)|_X : h \in \mathcal{C}^1(\mathbb{R}^2)\}$ is dense in $\mathcal{C}(X)^3$ if and only if X contains no rectifiable Jordan arc (see [19, Theorem in Subsection 3.3]) and apply Theorem 2.1 for $n = 2$.

In the case of operators L of order $n > 2$, Problem 3 remains unsolved. Taking Theorem 2.1 into account we should have been able to resolve this problem on the basis of a certain description of the compact sets X in $\mathbb{R}^2$ for which the space $\{(\mathcal{D}^{n-1}h)_{|X} : h \in \mathcal{C}^{n-1}(\mathbb{R}^2)\}$ is dense in $\mathcal{C}(X)^N$. However, we do not know of any description of this sort.

§3. Proofs

3.1. Preliminaries. For the reader's convenience and unifying the notation, in this subsection we recall several properties of the operator L and of solutions of equation (1.2) that we shall use in what follows. Also, here we introduce the Vitushkin localization operator corresponding to L and study its properties.

We shall denote by A, A_0, A_1, $\dots$ positive constants whose values in formulas may vary, and by q, q_0, q_1, $\dots$ constants the values of which are fixed throughout the text.

We recall some notation. Let Y be a subset in $\mathbb{R}^2$ and f a complex-valued function defined on Y. Then $\omega_Y(f,\delta) := \sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in Y,\ |z_1 - z_2| \le \delta\}$ is the modulus of continuity of f on Y. If $Y = \mathbb{R}^2$, then the subscript Y will be dropped. For a function f of class $\mathcal{C}^m$ and $s \in \{0, \dots, m\}$, we define

$$\omega_Y(\nabla^s f, \delta) := \max_{0 \le t \le s} \omega_Y(\partial^{(s-t,t)} f, \delta).$$

As usual, $\operatorname{Supp} T$ denotes the support of a distribution T, $\langle T | \varphi \rangle$ is the value of a distribution T at a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, and $T_1 * T_2$ stands for the convolution of distributions T_1 and T_2.

It is well known (see, e.g., Theorem 7.1.20 in [20]) that the operator L (or equation (1.2)) has a fundamental solution Φ of the form

$$\Phi(z) = E(z) - P(z) \log |z|, \tag{3.1}$$

where $E(\cdot)$ is a real-analytic function on $\mathbb{R}^2 \setminus \{0\}$ homogeneous of order $n-2$, and $P(\cdot)$ is a homogeneous polynomial (in the variables x_1 and x_2) of degree $n-2$ (if $n < 2$, then $P \equiv 0$). Moreover, for $z \ne 0$ and for all $\alpha \in \mathbb{Z}_+^2$, the following obvious estimate is true:

$$\bigl|\partial^\alpha \Phi(z)\bigr| \le A \alpha! |z|^{n-2-|\alpha|} (\log |z| + 1). \tag{3.2}$$

Together with the fundamental solution Φ, we need the fundamental solution Φ_δ that is defined for $\delta > 0$ as follows:

$$\Phi_\delta(z) = E(z) - P(z) \log \frac{|z|}{\delta}.$$

Then $\Phi(z) - \Phi_\delta(z) = P(z) \log(1/\delta)$ is a homogeneous polynomial of degree $n-2$. Moreover, for $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| > n-2$ we have

$$\partial^\alpha \Phi_\delta(z) = \partial^\alpha \Phi(z),$$

and if $\alpha \in \mathbb{Z}_+^2$, $|\alpha| \le n-2$ and $|z| \le \delta$, then

$$|\partial^\alpha \Phi_\delta(z)| \le A \delta^{n-2-|\alpha|}. \tag{3.3}$$

We also recall that the solutions of equation (1.2) admit Laurent type expansions. Let Φ be any fundamental solutions for the operator L. Taking a distribution T supported in the disk $B = B(a, r)$, we define the function $f = \Phi * T$. Then there exists $q_0 = q_0(L) \geq 1$ such that

$$f(z) = \sum_{|\alpha|\geq 0}^{\infty} c_\alpha \partial^\alpha \Phi(z-a), \tag{3.4}$$

where the series converges in the space $\mathcal{C}^\infty(\mathbb{R}^2 \setminus q_0 B)$ (see [2] and [21, §2.2]). The coefficients $c_\alpha = c_\alpha(f, a)$, $\alpha \in \mathbb{Z}_+^2$, are calculated by the formula

$$c_\alpha = c_\alpha(f,a) = \frac{(-1)^{|\alpha|}}{\alpha!} \big\langle T(w) | (w-a)^\alpha \big\rangle. \tag{3.5}$$

Suppose that the fundamental solution Φ for the operator L is chosen so that

$$\max_{|z|=1} \big\{ |E(z)|, |P(z)| \big\} \leq A,$$

and the fundamental solution Φ_δ, $\delta > 0$, is constructed by this Φ.

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. We define the Vitushkin localization operator

$$V_\varphi : \mathcal{C}_0^\infty(\mathbb{R}^2)' \to \mathcal{C}_0^\infty(\mathbb{R}^2)'$$

for L (see [2, 22]) by the formula

$$V_\varphi f = \Phi * (\varphi L f).$$

Observe that $V_\varphi f = V_{\varphi,\delta} f + P_1$, where $V_{\varphi,\delta} f = \Phi_\delta * (\varphi L f)$ is the localization operator constructed by the fundamental solution Φ_δ, and $P_1 = \log(1/\delta) P * (\varphi L f)$ is the polynomial of degree at most $n-2$. Since $LP_1 = 0$, it follows that P_1 is an L-analytic polynomial.

Now we establish several properties of the operator $V_{\varphi,\delta}$. The following lemma is an immediate analog of [8, Proposition 2.4] and [9, Lemma 2.3].

Lemma 3.1. *Let $f \in \mathcal{C}_0^{n-1}(\mathbb{R}^2)$, and let $R > 2$ be such that* $\operatorname{Supp}(f) \subset B(0, R)$. *Then for any $a \in B(0, R)$, any $\delta > 0$, and $\varphi \in \mathcal{C}_0^\infty(B(a, \delta))$, the function $F := V_{\varphi,\delta} f$ possesses the following properties.*

1. *$F \in \mathcal{C}_{\mathrm{loc}}^{n-1}(\mathbb{R}^2)$, and $LF = 0$ on the set $\mathbb{R}^2 \setminus \big(\operatorname{Supp}(Lf) \cap \operatorname{Supp}(\varphi)\big)$.*
2. *For any $\lambda \geq 1$ and all 2-indices α with $|\alpha| \leq n-1$, we have*

$$\|\partial^\alpha F\|_{B(a,\lambda\delta)} \leq A k(\lambda) \delta^n \omega(\nabla^{n-1} f, \delta) \|\nabla^{|\alpha|+1} \varphi\|, \tag{3.6}$$

where $k(\lambda) = (\lambda+1)^{n-2} \ln(\lambda+1)$.

3. *For any 2-index α we have the estimate*

$$|c_\alpha(F, a)| \leq A \delta^{|\alpha|+2} \omega(\nabla^{n-1} f, \delta) \|\nabla \varphi\| / \alpha!. \tag{3.7}$$

Proof. We present the arguments for the case where $f \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. The general case follows by regularization. Assertion (1) can be proved in a standard way, see, for instance, the proof of Proposition 2.4 in [8].

Observe that

$$\|\nabla^\ell \varphi\| \leq A \delta^{m-\ell} \|\nabla^m \varphi\| \tag{3.8}$$

for $\ell = 0, 1, \ldots, m$, $m \in \mathbb{N}$.

By the definition of F, we have (all derivatives in angular brackets $\langle \cdot | \cdot \rangle$ are taken with respect to the variable w):

$$F(z) = \big\langle \Phi_\delta(w-z) | \varphi(w) L f(w) \big\rangle = \big\langle \Phi_\delta(w-z) | \varphi(w) L f_a(w) \big\rangle,$$

where

$$f_a(w) = f(w) - \sum_{|\alpha|=0}^{n-1} \frac{\partial^\alpha f(a)}{\alpha!} (w-a)^\alpha.$$

Until the end of the proof of this lemma, we shall assume without loss of generality that $L = \partial_1 L_1$, where L_1 is some differential operator of order $n-1$.

Let $\alpha \in \mathbb{Z}_+^2$ be such that $|\alpha| \le n-1$. We have

$$\begin{aligned}|\partial^\alpha F(z)| &= |\langle \partial^\alpha \varPhi_\delta(w-z)|\varphi(w)Lf_a(w)\rangle| \\ &= |\langle \partial_1(\varphi(w)\partial^\alpha \varPhi_\delta(w-z))|L_1 f_a(w)\rangle| \\ &= |\langle \partial_1\varphi(w)\partial^\alpha\varPhi_\delta + \varphi(w)\partial^{(\alpha_1+1,\alpha_2)}\varPhi_\delta(w-z)|L_1 f_a(w)\rangle| \\ &\le |\langle \partial_1\varphi(w)\partial^\alpha\varPhi_\delta(w-z)|L_1 f_a(w)\rangle| + |\langle \varphi(w)\partial^{(\alpha_1+1,\alpha_2)}\varPhi_\delta(w-z)|L_1 f_a(w)\rangle|.\end{aligned} \tag{3.9}$$

If $|\alpha| \le n-2$, then all terms in the last expression for $\partial^\alpha F$ are regular and can be estimated straightforwardly by using (3.2), (3.3), (3.8) if we take into account the form of $\varPhi_\delta$ and the fact that the functions E and P are homogeneous of degree $n-2$. In particular, if $z \in B(a, \lambda\delta)$, then

$$|\langle \partial_1\varphi(w)\partial^\alpha\varPhi_\delta(w-z)|L_1 f_a(w)\rangle| \le \delta^{n-|\alpha|}\|\nabla\varphi\|\omega(\delta)\max_{z\in B(0,\lambda)}\int_{B(0,1)}|\partial^\alpha\varPhi_\delta(w-z)|\,d\mathfrak{m}_2(w),$$

where $\mathfrak{m}_2$ denotes the standard two-dimensional Lebesgue measure. Thus, estimate (3.6) is true if $|\alpha| \le n-2$.

Now we consider the case where $|\alpha| = n-1$. In this case the estimate for $\partial^\alpha F$ is more difficult. First, we note that

$$\varphi(w)Lf_a(w) = L(\varphi(w)f_a(w)) - \mathcal{L}(\varphi(w), f_a(w)),$$

where $\mathcal{L}$ is some differential expression involving the derivatives of φ of order at most n and derivatives of f_a of order at most $n-1$. Since $\varPhi_\delta$ is the fundamental solution for L, we have

$$\begin{aligned}\langle \partial^\alpha\varPhi_\delta(z-w)|L(\varphi(w)f_a(w))\rangle &= (-1)^{n+|\alpha|}\langle L\varPhi_\delta(z-w)|\partial^\alpha(\varphi(w)f_a(w))\rangle \\ &= (-1)^{n+|\alpha|}\partial^\alpha(\varphi(z)f_a(z)),\end{aligned}$$

which yields

$$|\langle \partial^\alpha\varPhi_\delta(z-w)|L(\varphi(w)f_a(w))\rangle| \le |\partial^\alpha(\varphi(z)f_a(z))|.$$

It remains to observe that

$$\|\partial^\alpha(\varphi f_a)\|_{B(a,\delta)} \le A\delta^n\omega(\nabla^{n-1}f,\delta)\|\nabla^{|\alpha|+1}\varphi\|,$$

because if $\tau \in \mathbb{Z}_+^2$ satisfies $|\tau| \le |\alpha|$, then

$$\|\partial^\tau f_a\|_{B(a,\delta)} \le A\delta^{n-1-|\tau|}\omega(\nabla^{n-1}f,\delta). \tag{3.10}$$

Since each term in $\mathcal{L}(\varphi, f_a)$ has the form $k\partial^\sigma\varphi(w)\partial^\tau f_a(w)$, where $\sigma,\tau \in \mathbb{Z}_+^2$, $|\sigma| \le n$, $|\tau| \le n-1$, and k is a constant coefficient, relations (3.8) and (3.10) show that the expression $|\langle\partial^\alpha\varPhi_\delta(z-w)|\mathcal{L}(\varphi(w), f_a(w))\rangle|$ for $z \in B(a,\lambda\delta)$ can be estimated in the same way as the second term in (3.9):

$$|\langle\partial^\alpha\varPhi_\delta(z-w)|\mathcal{L}(\varphi(w), f_a(w))\rangle| \le Ak(\lambda)\delta^n\omega(\nabla^{n-1}f,\delta)\|\nabla^n\varphi\|.$$

Therefore, estimate (3.6) is established for all 2-indices α such that $|\alpha| \le n-1$. It remains to verify (3.7). Let $\alpha \in \mathbb{Z}_+^2$ be an arbitrary 2-index. Then

$$\begin{aligned}\alpha!|c_\alpha(F,a)| &= |\langle\varphi(w)Lf(w)|(w-a)^\alpha\rangle| \\ &= |\langle L_1 f_a(w)|\partial_1(\varphi(w)(w-a)^\alpha)\rangle| \le A\delta^{|\alpha|+2}\omega(\nabla^{n-1}f,\delta)\|\nabla\varphi\|,\end{aligned}$$

where we have used the assumption $L = \partial_1 L_1$ once again. So, the lemma is proved completely. □

3.2. Now we proceed to the proof of our main results. We start with the proof of Theorem 2.2, and then turn to Theorem 2.1.

Proof of Theorem 2.2. Put $\theta := \theta(X)$; then $\theta > 0$. In the present proof, all constants $A, A_0, A_1, \dots$ and $q, q_0, q_1, \dots$ (see the beginning of Subsection 3.1) may depend not only on L, but also on θ.

Without loss of generality we assume that the function f is compactly supported. Let $f \in \mathcal{C}_0^{n-1}(\mathbb{R}^2) \cap \mathcal{O}_L(X^\circ)$. We set $\omega(\delta) := \omega(\nabla^{n-1} f, \delta)$, $\delta > 0$. Take a number $R > 2$ such that $\operatorname{Supp} f \cup X \subset B(0, R/2)$.

For an arbitrary $\delta \in (0,1)$, we consider the standard δ-partition of unity, that is, for each 2-index $j \in \mathbb{Z}^2$, we set $a_j := j\delta = (j_1\delta, j_2\delta)$ and choose functions $\varphi_j \in \mathcal{C}_0^\infty(B(a_j,\delta))$ such that $0 \le \varphi_j \le 1$, $\|\nabla^k \varphi_j\| \le A\delta^{-k}$ for $k = 0, \dots, n$ and $\sum_{j \in \mathbb{Z}^2_+} \varphi_j \equiv 1$.

For each $j \in \mathbb{Z}^2$ we consider the function $f_j := \Phi_\delta * (\varphi_j L f)$. In accordance with Lemma 3.1, the functions f_j possess the following properties. First, $f_j \in \mathcal{C}^{n-1}_{\mathrm{loc}}(\mathbb{R}^2)$, and each f_j is L-analytic on X° (where the function f is such) and outside the disk $B(a_j, \delta)$. Next, the f_j satisfy the estimates

$$\|\partial^\alpha f_j\|_{B(a_j, q\delta)} \le A\delta^{n-1-|\alpha|}\omega(\delta), \quad \alpha \in \mathbb{Z}^2_+, \ |\alpha| \le n-1, \tag{3.11}$$

where $q \ge 1$ a number to be specified later, and

$$|c_\alpha(f_j, a_j)| \le A\delta^{|\alpha|+1}\omega(\delta)/\alpha!, \quad \alpha \in \mathbb{Z}^2_+. \tag{3.12}$$

Moreover (see, e.g., Lemma 1 in [23]),

$$f(z) = \sum_{j \in \mathbb{Z}^2_+} f_j(z),$$

and this sum is finite because $f_j \equiv 0$ if $\operatorname{Supp}(\varphi_j) \cap \operatorname{Supp}(Lf) = \varnothing$ (in particular, if $B(0,R) \cap B(a_j,\delta) = \varnothing$). Thus, we have represented f as a sum of functions f_j with singularities localized in the disks $B(a_j,\delta)$, and, in order to approximate f in the desired way, it suffices to approximate each f_j with satisfactory accuracy. Note that if $B(a_j,\delta) \subset X^\circ$, then $f_j \equiv 0$, while if $B(a_j,\delta) \cap X = \varnothing$, then f_j is L-analytic in a neighborhood of X. So we must approximate only the functions f_j with indices j belonging to the set $\mathcal{J} = \{j \in \mathbb{Z}^2 : B(a_j,\delta) \cap \partial X \ne \varnothing\}$.

We proceed to constructing the required approximants. By the definition of $\theta = \theta(X)$, we can find a smooth Jordan arc $\gamma_j \subset B(a_j, 2\theta^{-1}\delta)$ such that $\operatorname{diam}\gamma_j = \delta$, $\gamma_j \cap X = \varnothing$, and the distance between its endpoints is δ. In what follows we shall assume that $\delta \in (0, \theta/4)$ is sufficiently small, so that $B(a_j, 2\theta^{-1}\delta) \subset B(0,R)$.

For each $j \in \mathcal{J}$ we construct a compact set K_j such that $\gamma_j \subset K_j^\circ$, $K_j \subset B(a_j, 2\theta^{-1}\delta)$ and $K_j \cap X = \varnothing$, and a function g_j such that $Lg_j = 0$ outside of K_j, and

$$\|\partial^\alpha g_j\|_{B(a_j,q\delta)\setminus K_j} \le A\delta^{n-1-|\alpha|}\omega(\delta), \quad \alpha \in \mathbb{Z}^2_+, \ |\alpha| \le n-1, \tag{3.13}$$

$$|c_\alpha(g_j, a_j)| \le A\delta^{|\alpha|+1}\omega(\delta)/\alpha!, \quad \alpha \in \mathbb{Z}^2_+, \tag{3.14}$$

where the $c_\alpha(g_j, a_j)$ are coefficients of the expansion (3.4) of the functions g_j with respect to the fundamental solution Φ_δ. Furthermore, the function g_j should be constructed in such a way that

$$c_\alpha(f_j, a_j) = c_\alpha(g_j, a_j), \quad \alpha \in \mathbb{Z}^2_+, \ |\alpha| \le n. \tag{3.15}$$

In order to define the desired functions g_j, we use a special modification of Lemma 3.1 in [8]. Let Γ be a smooth Jordan curve of diameter δ in the disk $B(0, 2\delta)$, $\delta > 0$. Then there exists a function g_Γ such that $\Delta g_\Gamma = 0$ outside Γ (i.e., g_Γ is harmonic outside Γ), and

$$\|g_\Gamma\|_{B(0,2R)\setminus\Gamma} \le A(|\ln\delta| + \ln R),$$
$$\|\partial^\alpha g_\Gamma\|_{\mathbb{R}^2\setminus\Gamma} \le A\delta^{-|\alpha|} \quad \text{for} \quad |\alpha| \le n+2.$$

Moreover, for $|z| > 2q_1\delta$, $q_1 \ge 1$, we have

$$g_\Gamma(z) = \frac{1}{2\pi}\ln|z| + \sum_{k=1}^{\infty}\frac{d_{1,k}}{z^k} + \sum_{k=1}^{\infty}\frac{d_{2,k}}{\overline{z}^k},$$

and the coefficients $d_{1,k}$ and $d_{2,k}$, $k \in \mathbb{N}$, satisfy

$$|d_{1,k}| \le A\delta^k, \quad |d_{2,k}| \le A\delta^k.$$

In order to construct this function g_Γ, we need to repeat the construction used in the proof of Lemma 3.1 in [8] for $L = \Delta$ with the following modification: instead of the functions h_s, $s = 1, 2$, employed in that proof, we consider the functions

$$h_s(z) = C_s\big(z^{n+2}(z-b_s)^{n+2}\sqrt{z(z-b_s)} - P_s(z)\big), \quad s = 1, 2.$$

Here we keep the entire notation from the proof of Lemma 3.1 in [8], the branch of the root is assumed to be holomorphic outside Γ_s and equivalent to z as $z \to \infty$, and the constants C_s and the polynomials P_s, $s = 1, 2$, are chosen so that $\lim_{z\to\infty} zh_s(z) = 1$.

Let a_j^* be the initial point of the curve γ_j, and let $\gamma_j^* := \gamma_j - a_j^* = \{\zeta - a_j^* : \zeta \in \gamma_j\}$. Since $\theta > 0$, there exist $q_2 \ge 1$ and $r_j > 0$ such that $\gamma_j + \overline{B(0, r_j)} \subset B(a_j^*, q_2\delta)\setminus X$. We take a function $\rho \in \mathcal{C}_0^\infty(B(0,1))$ such that $\int \rho(z)\,d\mathfrak{m}_2(z) = 1$, and put $\rho_j(z) := \rho\big(zr_j^{-1}\big)r_j^{-2}$.

Now we introduce the functions $\widetilde{g}_j := \Delta g_{\gamma_j^*} * \rho_j$ and $\psi_j := k_j\Phi_\delta * \widetilde{g}_j$, where k_j is a constant to be specified later.

Since $L\psi_j = k_j\widetilde{g}_j$ and $K_j := \operatorname{Supp}\widetilde{g}_j \subset \gamma_j + B(0, r_j)$, the function ψ_j is L-analytic in a neighborhood of the compact set X. Moreover,

$$\|\partial^\alpha\psi_j\|_{B(a_j, q\delta)\setminus K_j} \le A\delta^{n-2-|\alpha|}$$

for $|\alpha| \le n$, and if $|z - a_j^*| > 2q_3\delta$, $q_3 \ge 1$, then

$$\psi_j(z) = \Phi_\delta(z - a_j^*) + \sum_{|\alpha|\ge 1}\mu_{\alpha,j}\partial^\alpha\Phi_\delta(z - a_j^*), \tag{3.16}$$

where the coefficients μ_α satisfy

$$|\mu_{\alpha,j}| \le A\delta^{|\alpha|}/\alpha!. \tag{3.17}$$

The constant k_j in the definition of the function ψ_j is chosen so that the coefficient of $\Phi_\delta(z - a_j^*)$ in the expansion (3.16) is equal to 1. Put also $\mu_{(0,0)} = 1$. Finally, we put $q := \max\{q_0, q_2, q_3\}$.

We seek the functions g_j, $j \in \mathfrak{J}$, in the following form:

$$g_j(z) := \sum_{|\alpha|\le n}\beta_{\alpha,j}\partial^\alpha\psi_j(z). \tag{3.18}$$

The coefficients $\beta_{\alpha,j}$ for $0 \le |\alpha| \le n$ are uniquely determined by conditions (3.15), which can be expressed in the form

$$c_\alpha(f_j, a_j^*) = \sum_{\tau_1=0}^{\alpha_1}\sum_{\tau_2=0}^{\alpha_2}\beta_{\tau,j}\mu_{\alpha-\tau,j}, \tag{3.19}$$

where τ and $\alpha - \tau$ are the 2-indices (τ_1, τ_2) and $(\alpha_1 - \tau_1, \alpha_2 - \tau_2)$, respectively. Then (3.19) is a nondegenerate system of linear equations.

We claim that for $0 \le |\alpha| \le n$ the coefficients $\beta_{\alpha,j}$ admit the estimate

$$|\beta_{j,\alpha}| \le A\delta^{|\alpha|+1}\omega(\delta). \tag{3.20}$$

Indeed, for all $\alpha \in \mathbb{Z}_+^2$ relations (3.5) and (3.12) imply that

$$\begin{aligned}
\alpha!|c_\alpha(f_j, a_j^*)| &= \big|\langle \varphi_j(w) Lf(w) | ((w-a_j) + (a_j - a_j^*))^\alpha\rangle\big| \\
&\le \sum_{\tau_1=0}^{\alpha_1}\sum_{\tau_2=0}^{\alpha_2} C_{\alpha_1}^{\tau_1} C_{\alpha_2}^{\tau_2} \tau! |c_\tau(f_j, a_j)|\, |a_j - a_j^*|^{|\alpha|-|\tau|} \\
&\le \sum_{\tau_1=0}^{\alpha_1}\sum_{\tau_2=0}^{\alpha_2} C_{\alpha_1}^{\tau_1} C_{\alpha_2}^{\tau_2} A_1 \delta^{|\tau|+1}\omega(\delta) |a_j - a_j^*|^{|\alpha|-|\tau|} \qquad (3.21)\\
&\le A_1\delta\omega(\delta) \sum_{\tau_1=0}^{\alpha_1} C_{\alpha_1}^{\tau_1}\delta^{\tau_1}\Big(\frac{2\delta}{\theta}\Big)^{\alpha_1-\tau_1} \sum_{\tau_2=0}^{\alpha_2} C_{\alpha_2}^{\tau_2}\delta^{\tau_2}\Big(\frac{2\delta}{\theta}\Big)^{\alpha_2-\tau_2} \\
&= A_1\delta\omega(\delta)\Big(1+\frac{2}{\theta}\Big)^{\alpha_1+\alpha_2}\delta^{\alpha_1+\alpha_2} \le A\delta^{|\alpha|+1}\omega(\delta).
\end{aligned}$$

Now, (3.20) follows from (3.19) with the help of (3.17) and (3.21).

Relations (3.18) and (3.20) show that (3.13) and (3.14) are valid with a_j replaced by a_j^*. Estimate (3.14) for the coefficients $c_\alpha(g_j, a_j)$ is derived from the corresponding estimate for the coefficients $c_\alpha(g_j, a_j^*)$ like estimate (3.21) is derived from (3.12).

Thus, we have constructed functions g_j possessing all the desired properties. To finish the proof of the theorem, it suffices to check that for all 2-indices α with $|\alpha| \le n-1$ we have

$$\Big\|\sum_{j\in\mathcal{J}}(\partial^\alpha f_j - \partial^\alpha g_j)\Big\|_X \le A\ln R\,\delta^{n-1-|\alpha|}\omega(\delta), \tag{3.22}$$

because the right-hand side of (3.22) tends to zero as $\delta \to 0$.

Identity (3.15) and estimates (3.12) and (3.14) imply that, for all z such that $|z-a_j| > q\delta$ and for all $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \le n-1$,

$$\big|\partial^\alpha f_j(z) - \partial^\alpha g_j(z)\big| \le \frac{A\ln R\,\delta^{n+2}\omega(\delta)}{|z-a_j|^{3+|\alpha|}}. \tag{3.23}$$

For $|z-a_j| < q\delta$, $z \notin K_j$, from (3.11) and (3.13) it follows that

$$\big|\partial^\alpha f_j(z) - \partial^\alpha g_j(z)\big| \le A\delta^{n-1-|\alpha|}\omega(\delta), \quad \alpha\in\mathbb{Z}_+^2,\ |\alpha|\le n-1. \tag{3.24}$$

Now, consider an arbitrary point $z \in X$ and an arbitrary 2-index α such that $|\alpha| \le n-1$. In order to prove (3.22), we use the well-known method of "layerwise" summation (see Lemma 1 in [22, §4] or the proof of Proposition 2.2 in [9]). Denote by $[t]$ the integral part of a number t. Using estimates (3.23) and (3.24), we arrive at the following inequalities:

$$\begin{aligned}
&\Big|\sum_{j\in\mathcal{J}}\big(\partial^\alpha f_j(z) - \partial^\alpha g_j(z)\big)\Big| \\
&\quad\le \sum_{\substack{j\in\mathcal{J},\\ |z-a_j|<q\delta}} \big|\partial^\alpha f_j(z) - \partial^\alpha g_j(z)\big| + \sum_{m=[q]}^{\infty}\ \sum_{\substack{j\in\mathcal{J},\\ m\delta\le|z-a_j|<(m+1)\delta}} \big|\partial^\alpha f_j(z) - \partial^\alpha g_j(z)\big| \\
&\quad\le A_1\delta^{n-1-|\alpha|}\omega(\delta) + A_2\ln R\,\delta^{n+2}\omega(\delta)\sum_{m=[q]}^{\infty}\frac{A_0 m}{(m\delta)^{3+|\alpha|}} \\
&\quad\le \delta^{n-1-|\alpha|}\omega(\delta)\ln R\Big(A_1 + A_3\sum_{m=[q]}^{\infty}\frac{1}{m^{2+|\alpha|}}\Big) \le A\delta^{n-1-|\alpha|}\omega(\delta)\ln R,
\end{aligned}$$

where we have used the fact that the number N_m of indices j such that $m\delta \le |z-a_j| < (m+1)\delta$ admits the estimate $N_m \le A_0 m$.

So, inequality (3.22) is proved. Now, for the role of an approximant for the function f we can take some function of class $\mathcal{C}^{n-1}(\mathbb{R}^2)$ coinciding with the function $\sum_{j\in\mathcal{J}} g_j + \sum_{j\notin\mathcal{J}} f_j$ in a neighborhood of the compact set X. □

Proof of Theorem 2.1. First, we prove the implication $(a) \Rightarrow (b)$. The case of the general operator L can be reduced by a real linear (nondegenerate) change of variables in $\mathbb{R}^2$ to the case where $L = \bar\partial L_1$, where L_1 is some homogeneous elliptic operator of order $n-1$. Lemma 3.1 in [24] shows that if $\varPhi$ is the fundamental solution of the form (3.1) for the operator L, then $L_1\varPhi$ is a homogeneous functions of order -1 real analytic in $\mathbb{R}^2\setminus\{0\}$. Thus, the operator L has a fundamental solution $\varPhi$ such that $L_1\varPhi = 1/(\pi z)$ (we recall that $1/(\pi z)$ is the fundamental solution for the Cauchy–Riemann operator $\bar\partial$).

Arguing by contradiction, we assume that the set $\mathbb{R}^2\setminus X$ is disconnected and G is some bounded connected component of this set. Without loss of generality we also assume that $0\in G$; then $d := \operatorname{dist}(0,\partial G) > 0$. We have $X \subset B(0,R)$, where $R = \operatorname{diam}(X)$. Consider a function f that is equal to $\varPhi$ on the set $\{\frac{d}{2} < |z| < R\}$ and is continued arbitrarily up to a function of class $\mathcal{C}_0^\infty(\mathbb{R}^2)$. Assume that there exists a sequence $(p_j)_{j=1}^\infty$ of L-analytic polynomials with $\partial^\alpha p_j \rightrightarrows_X \partial^\alpha f$ for all $\alpha\in\mathbb{Z}_+^2$ such that $0\le|\alpha|\le n-1$. Then the polynomials $\widetilde{p}_j := L_1 p_j$ are usual polynomials in a complex variable and the sequence $(\widetilde{p}_j)_{j=1}^\infty$ converges uniformly on X to the function $L_1\varPhi = 1/(\pi z)$. Thus, on ∂G the function $1/z$ can be uniformly approximated by polynomials in the complex variable, which is clearly impossible.

We turn to the proof of the implication $(b) \Rightarrow (a)$. Suppose that the set $\mathbb{R}^2\setminus X$ is connected and take a function $f \in \mathcal{C}^{n-1}(\mathbb{R}^2)\cap\mathcal{O}_L(X^\circ)$. By Theorem 2.2, we can find a sequence $(f_k)_{k=1}^\infty \subset \mathcal{O}_L(X)$ such that (2.1) is true. It remains to approximate each function f_k by L-analytic polynomials. Assume that the function f_k is L-analytic in some neighborhood U_k of the compact set X, and let $U_k' \subset U_k$ be a neighborhood of X such that each point in $\mathbb{R}^2\setminus U_k$ can be joined with the point ∞ by some curve lying outside $\bar U_k'$. By using Runge's method (in the case of approximation by solutions of general elliptic equations, this method can be found, e.g., in [25, §3.10]) it can be proved that the function f_k can be uniformly approximated together with all partial derivatives up to the order $n-1$ by L-analytic polynomials on $\bar U_k'$, and therefore, on X. □

References

[1] M. B. Balk, *Polyanalytic functions*, Math. Res., vol. 63, Akademie–Verlag, Berlin, 1991. MR1184141 (93k:30076)

[2] J. Verdera, *$\mathcal{C}^m$-approximation by solutions of elliptic equations, and Calderon–Zygmund operators*, Duke Math. J. **55** (1987), no. 1, 157–187. MR0883668 (88e:35011)

[3] ———, *Removability, capacity and approximation*, Complex Potential Theory (Montreal, PQ, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 439, Kluwer Acad. Publ., Dordrecht, 1994, pp. 419–473. MR1332967 (96b:30086)

[4] P. V. Paramonov, *Harmonic approximations in the C^1-norm*, Mat. Sb. **181** (1990), no. 10, 1341–1365; English transl., Math. USSR-Sb. **71** (1992), no. 1, 183–207. MR1085885 (92j:31006)

[5] ———, *$\mathcal{C}^m$-approximations by harmonic polynomials on compact sets in $\mathbb{R}^n$*, Mat. Sb. **184** (1993), no. 2, 105–128; English transl., Russian Acad. Sci. Sb. Math. **78** (1994), no. 1, 231–251. MR1214947 (94d:42024)

[6] A. G. Bitsadze, *Boundary value problems second order for elliptic equations*, Nauka, Moscow, 1966; English transl., North-Holland Ser. in Appl. Math. Mech., vol. 5, North-Holland Publ. Co., Amsterdam, 1968. MR0208152 (34:7962); MR0226183 (37:1773)

[7] P. V. Paramonov, *Approximations by harmonic polynomials in $\mathcal{C}^1$-norm on compact sets in $\mathbb{R}^2$*, Izv. Ross. Akad. Nauk Ser. Mat. **57** (1993), no. 2, 113–124; English transl., Russian Acad. Sci. Izv. Math. **42** (1994), no. 2, 321–331. MR1230969 (94h:31001)

[8] P. V. Paramonov and K. Yu. Fedorovskiĭ, *On uniform and $\mathcal{C}^1$-approximability of functions on compact sets in $\mathbb{R}^2$ by solutions of second-order elliptic equations*, Mat. Sb. **190** (1999), no. 2, 123–144; English transl., Sb. Math. **190** (1999), no. 1–2, 285–307. MR1701003 (2000k:41024)

[9] K. Yu. Fedorovskiy, *$\mathcal{C}^m$-approximation by polyanalytic polynomials on compact subsets of the complex plane*, Complex Anal. Oper. Theory **5** (2011), no. 3, 671–681. MR2836315 (2012j:30108)

[10] S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, Uspekhi Mat. Nauk **7** (1952), no. 2, 31–122; English transl., Amer. Math. Soc. Transl., No. 101, Amer. Math. Soc., Providence, RI, 1954, 99 pp. MR0051921 (14:547e)

[11] J. J. Carmona, P. V. Paramonov, and K. Yu. Fedorovskiĭ, *Uniform approximation by polyanalytic polynomials and the Dirichlet problem for bianalytic functions*, Mat. Sb. **193** (2002), no. 10, 75–98; English transl., Sb. Math. **193** (2002), no. 9–10, 1469–1492. MR1937036 (2004f:30027)

[12] A. Boivin, P. M. Gauthier, and P.V. Paramonov, *On uniform approximation by n-analytic functions on closed sets in* $\mathbb{C}$, Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 3, 15–28; English transl., Izv. Math. **68** (2004), no. 3, 447–459. MR2069192 (2005m:30044)

[13] J. J. Carmona and K. Yu. Fedorovskiy, *Conformal maps and uniform approximation by polyanalytic functions*, Selected Topics in Complex Analysis, Oper. Theory: Adv. Appl., vol. 158, Birkhäuser Verlag, Basel, 2005, pp. 109–130. MR2147592 (2006b:30067)

[14] ———, *On the dependence of conditions for the uniform approximability of functions by polyanalytic polynomials on the order of polyanalyticity*, Mat. Zametki **83** (2008), no. 1, 32–38; English transl., Math. Notes **83** (2008), no. 1–2, 31–36. MR2399995 (2009c:30012)

[15] A. B. Zaĭtsev, *On the uniform approximability of functions by polynomial solutions of second-order elliptic equations on compact sets in* $\mathbb{R}^2$, Mat. Zametki **74** (2003), no. 1, 41–51; English transl., Math. Notes **74** (2003), no. 1–2, 38–48. MR2010675 (2005a:41015)

[16] ———, *On the uniform approximability of functions by polynomial solutions of second-order elliptic equations on planar compact sets*, Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 6, 85–98; English transl., Izv. Math. **68** (2004), no. 6, 1143–1156. MR2108523 (2005h:41050)

[17] ———, *Uniform approximation of second-order elliptic equations by polynomial solutions and the corresponding Dirichlet problem*, Tr. Mat. Inst. Steklova **253** (2006), 67–80; English transl., Proc. Steklov Inst. Math. **2006**, no. 2 (253), 57–70. MR2338688 (2008f:35009)

[18] J. F. Treves, *Lectures on linear partial differential equations with constant coefficients*, Notas de Matemátiea, No. 27, Inst. Mat. Pura Apl., Rio de Janeiro, 1961. MR0155078 (27:5020)

[19] S. K. Smirnov and V. P. Khavin, *Approximation and extension problems for some classes of vector fields*, Algebra i Analiz **10** (1998), no. 3, 133–162; English transl., St. Petersburg Math. J. **10** (1999), no. 3, 507–528. MR1628034 (99f:41046)

[20] L. Hörmander, *The analysis of linear partial differential operators.* I. *Distribution theory and Fourier analysis*, Grundlehren Math. Wiss., Bd. 256, Springer-Verlag, Berlin, 1983. MR0717035 (85g:35002a)

[21] N. N. Tarkhanov, *Laurent series for solutions of elliptic systems*, Nauka, Novosibirsk, 1991. (Russian) MR1226897 (94e:35013)

[22] A. G. Vitushkin, *Analytic capacity of sets in problems of approximation theory*, Uspekhi Mat. Nauk **22** (1967), no. 6, 141–199; English transl., Russian Math. Surveys **22** (1967), 139–200. MR0229838 (37:5404)

[23] P. V. Paramonov and J. Verdera, *Approximation by solutions of elliptic equations on closed subsets of euclidean space*, Math. Scand. **74** (1994), 249–259. MR1298365 (95i:41040)

[24] M. Ya. Mazalov, *A criterion for uniform approximability on arbitrary compact sets for solutions of elliptic equations*, Mat. Sb. **199** (2008), no. 1, 15–46; English transl., Sb. Math. **199** (2008), no. 1–2, 13–44. MR2410145 (2009c:35005)

[25] R. Narasimkhan, *Analysis on real and complex manifolds*, 2nd ed., Adv. Stud. Pure Math., vol. 1, North-Holland Publ. Co., Amsterdam–London, 1973. MR0346855 (49:11576)

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