CAUCHY-TYPE INTEGRALS AND SINGULAR MEASURES

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Abstract. In an earlier paper by the author it was shown that, in the case of rank-two commutators the problem of existence of an averaged wave operator for a pair of unitary operators whose spectral measures are singular with respect to the Lebesgue measure can be rewritten in terms of Cauchy-type integrals. The approach to the problem presented in the paper is based upon truncated Toeplitz operators, convergence is analyzed in terms of their symbols, and the results obtained are applied to the boundary behavior of functions belonging to \*$\text{-invariant subspaces of the Hardy class }H^2$.

§1. Introduction

For a finite singular Borel measure \(\mu\) on the unit circle and for \(f \in L^2(\mu)\), we study the behavior of the functions \(H_r f \in L^2(\mu)\) defined by

\[
(H_r f)(z) = \int \frac{f(z) - f(\xi)}{1 - rz} d\mu(\xi),
\]

where \(r \in (0, 1)\). The limit of the functions \(H_r f\) as \(r \nearrow 1\), whenever it exists, can be regarded as the Hilbert transform of \(f\). Denote by \(B(\mu)\) the subclass of \(L^2(\mu)\) that consists of all functions \(f\) such that the norms of the operators

\[
h \mapsto \int \frac{f(z) - f(\xi)}{1 - rz} h(\xi) d\mu(\xi), \quad h \in L^2(\mu),
\]

are bounded uniformly for \(r \in (0, 1)\). In particular, if \(f \in B(\mu)\), then the norms of the functions \(H_r f\) are bounded uniformly on \((0, 1)\). In accordance with Proposition 4.1 in \[1\], \(f \in B(\mu)\) if and only if there exists an operator \(X\) on \(L^2(\mu)\) for which

\[
(XU - UX = (\cdot, \bar{f})1 - (\cdot, 1)f,
\]

where \(U\) is the unitary operator of multiplication by the independent variable on \(L^2(\mu)\).

Let \(C(\mu)\) be the subset of \(B(\mu)\) that consists of all functions \(f \in B(\mu)\) such that the functions \(H_r f\) have a weak limit as \(r \nearrow 1\). If \(f \in B(\mu)\) and \(X\) is an operator satisfying \(2\), then \(f \in C(\mu)\) if and only if the limit of the Abel means of the sequence \((U^n X U^{-n})_{n \geq 0}\) exists in the weak operator topology (see \[1\], Proposition 4.2). By \[1\], Theorem 4.3, for any singular measure \(\mu\) without atoms, there exist functions \(f \in B(\mu)\) such that \(f \notin C(\mu)\). This leads us naturally to the question about describing the functions of class \(C(\mu)\), which will be discussed in \[4\].

We will need truncated Toeplitz operators on the space \(K_\theta = H^2 \ominus \theta H^2\), where \(H^2\) is the Hardy space, and \(\theta\) is an inner function in the unit disk, that is, \(\theta \in H^2\) and \(|\theta| = 1\) almost everywhere on the unit circle. The spaces \(K_\theta\) are invariant under the backward shift operator \(h \mapsto \frac{h - h(0)}{z}\), which is the operator adjoint to the shift \(h \mapsto zh\)

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on $H^2$. These spaces play an important role in operator theory, they are used substantially in the functional model theory for contractions on Hilbert spaces. The theory of the model operator on $K_\theta$ was presented in the book [2]. Like the Toeplitz operators on $H^2$, truncated Toeplitz operators on $K_\theta$ act as the superposition of multiplication by a function called the symbol, and the orthogonal projection onto $K_\theta$. In Theorem 3.2 it will be shown that $f \in \mathcal{B}(\mu)$ if and only if $f$ is associated with some truncated Toeplitz operator. Thus, the class $\mathcal{B}(\mu)$ turns out to be rather broad, which looks somewhat surprising, and the demonstration of this fact is one of our main purposes in this paper; see the discussion in the last paragraph of §5.

If a function $f$ is sufficiently smooth, then the limit in $L^2(\mu)$ of the functions $H_rf$ as $r \nearrow 1$ exists and has the form $\int \frac{f(\xi) - f(\xi z)}{1 - \xi z} d\mu(\xi)$. In Lemma 4.1 it will be shown that the functions $f$ associated with truncated Toeplitz operators whose symbols are continuous belong to the class $\mathcal{C}(\mu)$. A wider sufficient condition will be established in Theorem 4.3.

On the other hand, all bounded measurable functions on the unit circle are symbols of bounded truncated Toeplitz operators, and for some truncated Toeplitz operators with bounded symbols the convergence in question fails (see Theorem 4.4). Therefore, it would be natural to study this convergence in terms of continuity properties of symbols of truncated Toeplitz operators.

In §4 we touch on the unitary equivalence of operators realized by a wave operator, provided it exists. The results of the preceding sections show that in some situations the weak averaged wave operator may fail to exist, but nevertheless, some parts of the unitary operators under consideration turn out to be unitarily equivalent. From this viewpoint, the situation where the wave operator fails to exist is “better” than the case where it exists and equals zero.

Finally, §5 is devoted to some consequences of our results related to the boundary behavior of functions in $K_\theta$. For an inner function $\theta$ and complex numbers $\alpha$, $|\alpha| = 1$, consider the family of singular Clark measures $\sigma_\alpha$ on the unit circle, determined by the relation

$$1 - |\theta(z)|^2 |1 - \alpha \theta(z)|^2 = \int \frac{1 - |z|^2}{|\xi - z|^2} d\sigma_\alpha(\xi).$$

(The expression on the left is a positive harmonic function in the unit disk, so that it can be represented as the Poisson transform of a nonnegative Borel measure $\sigma_\alpha$ on the unit circle. Since $\theta$ is an inner function, the boundary values of the left-hand side vanish almost everywhere with respect to the Lebesgue measure; therefore, $\sigma_\alpha$ is a singular measure.) Since $\theta$ has nontangential limits equal to $\alpha$ at $\sigma_\alpha$-almost all points of the unit circle, we can consider the operator that takes functions belonging to a dense subset of $K_\theta$ to their boundary functions in $L^2(\sigma_\alpha)$. Clark [3] showed that this operator can be extended by continuity to a unitary operator. Poltoratski [4] proved that angular boundary values exist $\sigma_\alpha$-almost everywhere for all functions in $K_\theta$. More precisely, if $\varphi \in K_\theta$ and $\varphi_r(z) = \varphi(rz)$, then the functions $\varphi_r$ have a limit in $L^2(\sigma_\alpha)$ as $r \nearrow 1$, and moreover, the angular limits of $\varphi$ exist at $\sigma_\alpha$-almost all points. This result allows us to consider the values of $\varphi$ at the points of the unit circle where the limit exists. A natural problem arises: to estimate the difference $\varphi - \varphi_r$. We introduce the functions $g_r$,

$$g_r(z) = \frac{\varphi(z) - \varphi(rz)}{\theta(z) - \theta(rz)},$$

take $\alpha = 1$, and denote by $\mu$ the Clark measure $\sigma_1$ of the inner function $\theta$. Let $f \in L^2(\mu)$ be the function such that $\varphi = f \mu$-almost everywhere. It turns out that, viewed as elements of the space $L^2(\mu)$, the functions $g_r$ coincide with $H_rf$ (see §6), and this allows
us to apply our results about the uniform boundedness of norms and the convergence of the functions \( J_r f \) to the question about the boundary behavior of functions of class \( K_\theta \).

§2. Functions of unitary operators on \( K_\theta \)

Before working with the class of all truncated Toeplitz operators, consider the important subclass that consists of functions of certain natural unitary operators \( U_\alpha \) on \( K_\theta \).

Let \( \theta \) be an inner function in the unit disk, \( \theta(0) = 0 \), and let \( \alpha \) be a unimodular complex number. Then relation (8) is equivalent to the following:

\[
\frac{1 + \bar{\alpha} \theta(z)}{1 - \bar{\alpha} \theta(z)} = \int \frac{1 + \bar{\xi} z}{1 - \bar{\xi} z} d\sigma_\alpha(\xi).
\]

Indeed, formula (8) can be obtained by taking the real parts in (5). Therefore, if (3) is fulfilled, the expressions in (5) may differ only by a pure imaginary constant, but taking \( z = 0 \) shows that this constant is zero. The substitution of \( z = 0 \) in (5) also shows that \( \sigma_\alpha \) is a probability measure.

We define unitary operators \( U_\alpha \) on \( K_\theta \) by

\[
U_\alpha h = P_\theta zh + \alpha(h, \bar{z}\theta)1,
\]

where \( P_\theta \) stands for the orthogonal projection onto \( K_\theta \). The fact that all functions of \( U_\alpha \) are truncated Toeplitz operators, was established in §12 of [5].

The operators \( U_\alpha \) are unitarily equivalent to the operators of multiplication by \( z \) on \( L^2(\sigma_\alpha) \) (see [3]); the unitary equivalence is realized by the embedding

\[
K_\theta \hookrightarrow L^2(\sigma_\alpha)
\]

defined via the angular boundary values of functions in \( K_\theta \) (see [4]). Here and in what follows the term embedding means a mapping that takes functions belonging to the initial space \( K_\theta \) to their values regarded as elements of another \( L^2 \)-space.

The values of functions of class \( K_\theta \) inside the unit disk can be recovered by their values \( \sigma_\alpha \)-almost everywhere with the help of the formula

\[
\varphi(z) = (1 - \bar{\alpha} \theta(z)) \int \frac{\varphi(\xi) d\sigma_\alpha(\xi)}{1 - \bar{\xi} z}.
\]

If \( q \in L^\infty(\sigma_\alpha) \), the operator \( q(U_\alpha) \) multiplies the boundary values of functions in \( K_\theta \) on the set where \( \theta = \alpha \) by \( q \); more precisely, for \( h \in K_\theta \), \( q(U_\alpha)h \) is the function in \( K_\theta \) whose boundary values are equal \( \sigma_\alpha \)-almost everywhere to those of \( h \) multiplied by \( q \).

Let \( \varphi \in K_\theta \) be the function determined by the condition that \( \varphi \) equals \( q \cdot \sigma_\alpha \)-almost everywhere. Comparing the values of functions \( \sigma_\alpha \)-almost everywhere, we obtain \( \varphi = q(U_\alpha)1 \); similarly, the relation \( q(U_\alpha)^* \bar{z}\theta = \bar{z}\theta \bar{\varphi} \) is a consequence of the fact that \( \bar{z}\theta \bar{\varphi} \) is the function in \( K_\theta \) whose values are equal \( \sigma_\alpha \)-almost everywhere to the values of \( q \cdot \bar{z}\theta \).

Since the operator \( q(U_\alpha) \) commutes with \( U_\alpha \) and \( U_1 = U_\alpha + (1 - \alpha)(\cdot, \bar{z}\theta)1 \), we obtain

\[
q(U_\alpha)U_1 - U_1 q(U_\alpha) = (1 - \alpha)((\cdot, \bar{z}\theta)q(U_\alpha)1 - (\cdot, q(U_\alpha)^* \bar{z}\theta)1)
= (1 - \alpha)((\cdot, \bar{z}\theta)\varphi - (\cdot, \bar{z}\theta \bar{\varphi})1).
\]

Now we take a singular probability measure \( \mu \) on the unit circle and construct an inner function \( \theta \) by formula (3) with \( \alpha = 1 \) and \( \mu = \sigma_1 \):

\[
\frac{1 + \theta(z)}{1 - \theta(z)} = \int \frac{1 + \bar{\xi} z}{1 - \bar{\xi} z} d\mu(\xi).
\]

Let \( V \) be the unitary embedding

\[
V : K_\theta \hookrightarrow L^2(\mu) = L^2(\sigma_1).
\]
Denote by $U$ the operator of multiplication by the independent variable on $L^2(\mu)$; we have

$$VU_1V^* = U.$$  

For $q \in L^\infty(\sigma_\alpha)$, we introduce the operator $X : L^2(\mu) \to L^2(\mu)$ by the formula

$$X = \frac{1}{1 - \alpha} Vq(U_\alpha)V^{-1}. \tag{10}$$

From (5), we can easily obtain the following formula for the commutator:

$$(11) \quad K = XU - UX = (\cdot, \bar{z})f - (\cdot, \bar{z}\bar{f})1,$$

where $f = V\varphi \in L^2(\mu)$, i.e., $\varphi = f \mu$-almost everywhere. For nonnegative integers $n$ we have

$$X - U^{n+1}XU^{-(n+1)} = \sum_{m=0}^{n} U^m KU^{-(m+1)}. \tag{12}$$

Let $B_r$ denote the Abel means of the sequence (12):

$$B_r = X - (1 - r) \sum_{n=0}^{\infty} r^n U^{n+1}XU^{-(n+1)} = \sum_{m=0}^{\infty} r^m U^m KU^{-(m+1)}. \tag{13}$$

Formula (11) can be rewritten as $(Kh)(z) = \int (f(z) - f(\xi))\xi h(\xi) \, d\mu(\xi)$, whence by (13) we obtain

$$(14) \quad (B_r h)(z) = \sum_{m=0}^{\infty} r^m z^m \int (f(z) - f(\xi))\xi^{-m} h(\xi) \, d\mu(\xi) = \int \frac{f(z) - f(\xi)}{1 - rz} h(\xi) \, d\mu(\xi).$$

Thus, the $B_r$ coincide with the operators (11). Since the set of all vectors $h \in L^2(\mu)$ for which $B_r h$ converge as $r \nearrow 1$ is a subspace reducing $U$, convergence for $h \equiv 1$ yields convergence for all $h \in L^2(\mu)$. Therefore, for the operator $X$ defined by (10), the Abel means of the sequence of the operators $U^nXU^{-n}$ have a limit if and only if the functions $\mathcal{H}_r f$ converge.

§3. AVERAGED WAVE OPERATORS
AND TRUNCATED TOEPLITZ OPERATORS

Taking an inner function $\theta$ with $\theta(0) = 0$, we consider $K_\theta = H^2 \ominus \theta H^2$. For a function $\psi \in L^2$, the truncated Toeplitz operator $A_\psi$ on $K_\theta$ is defined by

$$A_\psi h = P_\theta \psi h, \quad h \in K_\theta.$$  

This operator is defined initially on the dense subset of all bounded functions in $K_\theta$, and the symbol $\psi$ is assumed to be such that $A_\psi$ is a bounded operator, thus defined on the entire space. The basic properties of truncated Toeplitz operators were studied in [5].

Let $\mu$ be the Clark measure $\sigma_\theta$ of $\theta$, as defined by (11). Since $\theta(0) = 0$, $\mu$ is a probability measure. Take an operator $X$ on $L^2(\mu)$ and assume that formula (11) is fulfilled: $XU - UX = (\cdot, \bar{z})f - (\cdot, \bar{z}\bar{f})1$ for some $f \in L^2(\mu)$. Construct $\varphi \in K_\theta$ so that $\varphi = f \mu$-almost everywhere; set

$$A = V^* XV,$$

where $V$ is the unitary embedding $K_\theta \hookrightarrow L^2(\mu)$. Define $U_1$ by formula (6) with $\alpha = 1$. Since $VU_1 = UV$, for the commutator we have the formula

$$AU_1 - U_1 A = V^* XVU_1 - U_1 V^* XV = V^* (XU - UX)V$$

$$= (\cdot, V^* \bar{z})V^* f - (\cdot, V^*(\bar{z}\bar{f}))V^*.\tag{14}$$
The boundary values of the functions $\bar{z}\theta, \varphi, \bar{z}\theta\bar{\varphi}, 1 \in K_\theta$ coincide $\mu$-almost everywhere with the functions $\bar{z}, f, \bar{zf}, 1$, respectively. Therefore,

$$AU_1 - U_1 A = (\cdot, \bar{z}\theta)\varphi - (\cdot, \bar{z}\theta\bar{\varphi})1.$$  

**Theorem 3.1.** An operator $A$ on $K_\theta$ is a truncated Toeplitz operator if and only if relation (15) is fulfilled for some function $\varphi \in K_\theta$.

**Proof.** Theorem 8.1 in [5] says that $A$ is a truncated Toeplitz operator if and only if for any pairs of vectors $f, g \in K_\theta$ with $zf, zg \in K_\theta$ we have $(AZf, zg) = (Af, g)$. For $f \in K_\theta$, the property $zf \in K_\theta$ means that $(f, \bar{z}\theta) = 0$. The fact that $h \in K_\theta$ has the form $h = zg$ for $g \in K_\theta$ is equivalent to the condition $(h, 1) = 0$. Therefore, $A$ is a truncated Toeplitz operator if and only if the relations $(f, \bar{z}\theta) = 0$, $(h, 1) = 0$ imply $(Azf, h) = (Af, \bar{zh})$, or equivalently, $((AU_1 - U_1 A)f, h) = 0$. Obviously, this is the case if (15) is fulfilled.

Conversely, if $A$ is a truncated Toeplitz operator, then, as was shown above, the relations $(f, \bar{z}\theta) = 0$, $(h, 1) = 0$ imply $((AU_1 - U_1 A)f, h) = 0$. Hence, the operator $AU_1 - U_1 A$ can be written in the form

$$AU_1 - U_1 A = (\cdot, \bar{z}\theta)\varphi + (\cdot, \gamma)1$$

for some $\varphi, \gamma \in K_\theta$. For the operator $X = VAV^*$ on $L^2(\mu)$, the commutator $XU - UX$ is a rank-two operator on $L^2(\mu)$, the formula for which can be obtained from the right-hand side of (16) if we replace the functions belonging to $K_\theta$ by their boundary values $\mu$-almost everywhere. Theorem 6.1 in [6] says that if $\mu$ is a singular measure on the unit circle and $X$ is an operator on $L^2(\mu)$ such that the commutator $XU - UX$ is a finite sum $\sum (\cdot, \bar{\nu}_k)v_k$, then $\sum u_kv_k = 0$ $\mu$-almost everywhere. Therefore, $\bar{z}\theta\varphi + \gamma = 0$ at $\mu$-almost all points. Hence, $\gamma$ is the function of class $K_\theta$ whose values $\mu$-almost everywhere are equal to the values of the function $-\bar{z}\theta\bar{\varphi}$. Since $\bar{z}\theta\bar{\varphi} \in K_\theta$, we obtain $\gamma = -\bar{z}\theta\bar{\varphi}$, and relation (15) follows. \qed

For a truncated Toeplitz operator $A$, define $\varphi \in K_\theta$ by formula (15), and let $f \in L^2(\mu)$ be the function for which $\varphi = f$ $\mu$-almost everywhere. We say that $f$ is associated with $A$. Notice that the functions $\varphi$ and $f$ are determined up to an additive constant; indeed, for a constant function $\varphi$ we obtain zero on the right-hand side of (15). The standard choice of $\varphi$ can be determined by the condition $\varphi(0) = 0$.

**Theorem 3.2.** Let $\mu$ be a singular Borel probability measure on the unit circle. A function $f \in L^2(\mu)$ belongs to the class $B(\mu)$ if and only if $f$ is associated with some bounded truncated Toeplitz operator on $K_\theta$, where $\theta$ is defined by formula (9).

**Proof.** By Proposition 4.1 of the paper [11], the norms of the operators (11) are bounded uniformly in $r$ if and only if there exists an operator $X$ on $L^2(\mu)$ for which $XU - UX = (\cdot, f)1 - (\cdot, f)f$, or, if $X$ is replaced by $-XU$, $XU - UX = (\cdot, \bar{z})f - (\cdot, \bar{zf})1$. For $A = V^*XV$, this is equivalent to (15), which means that $f$ is associated with $A$. \qed

Now we discuss how the function $\varphi$ occurring in the expression for the commutator (15) can be found from the symbol $\psi$ of a truncated Toeplitz operator. Obviously, the symbols in $\theta H^2 + \overline{\theta H^2}$ yield the zero operator, so one can consider only symbols $\psi$ that belong to $K_\theta + \overline{K_\theta}$. For $\psi \in K_\theta + \overline{K_\theta}$, write

$$\psi_+ = P_+ \psi = A1 \in K_\theta, \quad \psi_- = P_- \psi \in \overline{K_\theta}, \quad \psi_0 = (\psi, 1) = (A1, 1) = (A(\bar{z}\theta), \bar{z}\theta).$$

Also we have

$$(\psi_, 1) = 0, \quad A(\bar{z}\theta) = (\psi_+ + \psi_0)\bar{z}\theta.$$ 

For the symbol $\psi$ we obtain

$$\psi = \psi_+ + \psi_- = A1 + (z\theta \cdot A(\bar{z}\theta) - \psi_0) = A1 + z\theta \cdot A(\bar{z}\theta) - (A1, 1).$$
Proposition 3.3. 1. If $A$ is a truncated Toeplitz operator on $K_\theta$ with $\psi_0 = (A_1, 1) = 0$, then

$$\varphi = A_1 - zA(\bar{z}\theta)$$

up to an additive constant.

2. If $A = A_\psi$ with $\psi \in K_\theta + \overline{K_\theta}$, then

$$\varphi = \psi_+ - \theta\psi_-$$

(also up to an additive constant); the operator $A_\psi$ commutes with $U_1$ if and only if $\varphi = \psi_+ - \theta\psi_- \equiv \text{const.}$

From relation (18) it follows that if $\psi \in K_\theta + \overline{K_\theta}$, then $\varphi$ is the function in $K_\theta$ that coincides with $\psi$ $\sigma_\alpha$-almost everywhere.

Proof. 1. Take a function $\varphi$ with $\varphi(0) = 0$ and apply (15) to the vector $\bar{z}\theta$:

$$(AU_1 - U_1 A)(\bar{z}\theta) = (\bar{z}\theta, \bar{z}\theta)\varphi - (\bar{z}\theta, \bar{z}\theta)(\bar{z}\theta) = \varphi - \varphi(0) = \varphi.$$ 

On the other hand, since $U_1(\bar{z}\theta) = 1$, we have

$$(AU_1 - U_1 A)(\bar{z}\theta) = A_1 - U_1 A(\bar{z}\theta) = A_1 - zA(\bar{z}\theta),$$

where the relation $U_1 A(\bar{z}\theta) = zA(\bar{z}\theta)$ is a consequence of the assumption $(A(\bar{z}\theta), (\bar{z}\theta)) = \psi_0 = 0$. Comparing the right-hand sides, we obtain formula (18).

2. The case where $\psi$ is a constant function is trivial; without loss of generality we may assume that $\psi_0 = 0$. By property (15) for $A = A_\psi$ with $\psi \in K_\theta + \overline{K_\theta}$, we have

$$\varphi = A_\psi 1 - zA_\psi (\bar{z}\theta) = \psi_+ - z(\psi_0 + \psi_0)\bar{z}\theta = \psi_+ - \theta\psi_-,$$

and formula (19) is proved. \qed

Take $q \in L^\infty(\sigma_\alpha)$, $|\alpha| = 1$. The fact that $q(U_\alpha)$ is a truncated Toeplitz operator was proved in [5, §12]; this also follows directly from Theorem 3.1 and formula (8).

The symbol of the truncated Toeplitz operator $q(U_\alpha)$ is the function $(1 + \alpha\theta_\alpha)\varphi$, where $\varphi \in K_\theta$, $\varphi = q$ $\sigma_\alpha$-almost everywhere. Indeed, this formula is obvious if $q$ is a constant. Hence we may assume that $0 = \int q\,d\sigma_\alpha = \varphi(0)$, and, thus, $\bar{z}\varphi \in K_\theta$. The operator $q(U_\alpha)$ multiplies the boundary values at $\sigma_\alpha$-almost all points by the function $q$. Therefore, $q(U_\alpha)1$ is the function equal to $q$ at $\sigma_\alpha$-almost all points, that is, $q(U_\alpha)1 = \varphi$. Similarly, the values of the function $\varphi$ coincide with $\alpha\bar{z}$ $\sigma_\alpha$-almost everywhere; hence the values of the function $q(U_\alpha)(\bar{z}\theta)$ are equal $\sigma_\alpha$-almost everywhere to $q \cdot \alpha \bar{z}$, and we obtain $q(U_\alpha)(\bar{z}\theta) = \alpha \cdot \bar{z}\varphi$. By formula (17) with $A = q(U_\alpha)$, we have

$$\psi = q(U_\alpha)1 + z\theta \cdot q(U_\alpha)(\bar{z}\theta) - (q(U_\alpha)1, 1) = \varphi + z\theta \cdot \alpha \bar{z}\varphi - \int q\,d\sigma_\alpha = (1 + \alpha\theta)\varphi,$$

as required.

To obtain the function $f \in L^2(\mu)$ associated with the truncated Toeplitz operator $A = \frac{1}{1 - \alpha} q(U_\alpha)$, we should take the $\mu$-almost all boundary values for the function $\varphi \in K_\theta$ whose values coincide with $q$ $\sigma_\alpha$-almost everywhere. Since the operators that take functions from $K_\theta$ to their boundary functions in $L^2(\sigma_\alpha)$ and in $L^2(\mu) = L^2(\sigma_1)$ are unitary, the operator sending $q \in L^\infty(\sigma_\alpha)$ to $f \in L^2(\mu)$ extends by continuity to a unitary operator $L^2(\sigma_\alpha) \rightarrow L^2(\mu)$. Therefore, by Theorem 3.2 the class $B(\mu)$ contains the image of $L^\infty(\sigma_\alpha)$; thus, the class $B(\mu)$ contains the “unitary copies” of the spaces $L^\infty(\sigma_\alpha)$ for any unimodular $\alpha \neq 1$; in this sense this class is very wide. However, the class $L^\infty(\mu) = L^\infty(\sigma_1)$ itself is not a subset of $B(\mu)$; namely, there exist continuous (and hence bounded) functions on the unit circle that, being viewed as elements of the space $L^\infty(\mu)$, do not belong to $B(\mu)$ (see [7]). One can compare this result with the following
consequence of Lemma 4.1 below: if $q$ is a continuous function viewed as an element of the space $L^\infty(\sigma_\alpha)$ with $\alpha \neq 1$, then the corresponding function $f$ belongs to $C(\mu)$.

§4. Convergence conditions

As usual, $\mu$ is a probability singular measure on the unit circle, $\theta$ is the inner function determined by relation (9). Now we study sequences $(U^nXU^{-n})$ of operators on $L^2(\mu)$, where $X$ is an operator for which the commutator $XU - UX$ has the form (11) for some function $f \in L^2(\mu)$. By Theorems 3.1 and 3.2 this is equivalent to the relation $X = VAV^*$, where $A$ is a truncated Toeplitz operator on $K_\theta$, and then $f \in B(\mu)$ is associated with $A$; the weak convergence of the Abel means of the sequence $(U^nXU^{-n})$ is equivalent to that $f \in C(\mu)$.

Sufficient conditions for the averaged convergence of the operators $U^nXU^{-n}$ can be stated in terms of the continuity of the symbols of truncated Toeplitz operators and of functions applied to the unitary operators $U_\alpha$. Recall that $V = K_\theta \hookrightarrow L^2(\mu)$ is the unitary embedding.

Lemma 4.1. Let $q$ be a continuous function on the unit circle. Assume that either $X = VT_qV^*$, where $T_q$ is a truncated Toeplitz operator with symbol $q$, or $X = Vq(U_\alpha)V^*$, where $U_\alpha$ is the unitary operator defined by formula (6). Then the Cesàro means of the sequence of operators $(U^nXU^{-n})_{n \geq 0}$ have a limit in the strong operator topology. The limit operator commutes with $U$ and coincides with $q(U)$ on the orthogonal complement to all eigenvectors of $U$.

Proof. The operator $T_q - q(U_\alpha)$ is compact if $|\alpha| = 1$ and $q$ is a continuous function on the unit circle. Indeed, for $q(z) = z^n$, $T_q - q(U_\alpha)$ is a finite-rank operator: if $n \geq 0$, then $T_z^n = T^u_z$, and since $T_z - U_\alpha$ is a rank-one operator, we see that $\text{rank}(T_z^n - U_\alpha^n) \leq n$; similarly, $\text{rank}(T_z - U_\alpha^*) = 1$, and for $n < 0$ we have $T_z^{-n} = T_{\bar{z}}^{-n}$ and $\text{rank}(T_z^{-n} - U_\alpha^{-n}) \leq -n$. Therefore, if $q$ is a trigonometric polynomial, then $T_q - q(U_\alpha)$ is a finite-rank operator. Any continuous function can be uniformly approximated by polynomials, and the limit in norm of finite-rank operators is a compact operator.

Thus, in both cases $V^*XV - q(U_1)$ is a compact operator, and since $Vq(U_1)V^* = q(U)$, the operator $X - q(U)$ is also compact.

First, we consider eigenvectors of the operator $U$. Suppose that $Uh = \omega h$, $|\omega| = 1$. For $L = (\cdot,a)b$ we have $U^nLU^{-n}h = \bar{\omega}^n(h,a)U^n b$; the convergence in question follows from the fact that the Cesàro means of the sequence $\bar{\omega}^nU^n b$ tend to the projection of the vector $b$ to the eigensubspace of $U$ corresponding to the eigenvalue $\omega$. If the operator $L$ is compact, it belongs to the closure in norm of all linear combinations of rank-one operators, and we get convergence on the subspace generated by all eigenvectors of $U$.

Now consider the Cesàro means of the restrictions of $(U^nLU^{-n})_{n \geq 0}$ to the subspace orthogonal to all eigenvectors of $U$, where $L$ is a compact operator. It is well known that these Cesàro means converge to zero in the strong operator topology; the author was not able to find a reference to the original source; the proof can be found, e.g., in §2 of [1].

Set $L = X - q(U)$, then $U^nXU^{-n} = q(U) + U^nLU^{-n}$, and the claim follows. \( \square \)

Corollary 4.2. Let $\theta$ be an inner function with $\theta(0) = 0$, and assume that the measure $\sigma_1$ has no atoms. Take a truncated Toeplitz operator $A$ on $K_\theta$, and let the function $\varphi \in K_\theta$ be defined by formula (15). If $\varphi$ coincides with a continuous function $q_\alpha$ almost everywhere for some unimodular $\alpha \neq 1$, then the strong limit of the sequence of the Cesàro means of the operators $U_1^nAU_1^{-n}$ exists and is equal to $A - \frac{1}{1-\alpha}(q(U_\alpha) - q(U_1))$. 
Proof. From formula (8) it follows that the operator $A - \frac{1}{1 - \alpha} q(U\alpha)$ commutes with $U_1$. Hence,

$$U_1^n A U_1^{-n} = U_1^n \left( A - \frac{1}{1 - \alpha} q(U\alpha) \right) U_1^{-n} + U_1^n \cdot \frac{1}{1 - \alpha} q(U\alpha) U_1^{-n}$$

$$= \left( A - \frac{1}{1 - \alpha} q(U\alpha) \right) + \frac{1}{1 - \alpha} V^* U^n (V q(U\alpha) V^*) U^{-n} V.$$ 

By Lemma 4.1, the Cesàro means of the second summand in the last expression tend to $\frac{1}{1 - \alpha} V^* q(U)V = \frac{1}{1 - \alpha} q(U_1)$, and the proof is complete. \qed

Now we show that weak convergence survives if we allow some discontinuities. For simplicity, it is convenient to assume that the measure $\mu$ has no atoms. This assumption is not too restrictive because on the subspace of eigenvectors we always have strong convergence of the Cesàro means, and also because this condition is imposed when studying convergence of the functions $H_r$.

**Theorem 4.3.** Let $\mu$ be a singular probability measure on the unit circle. Suppose that $\mu$ has no point masses and construct the inner function $\theta$ for which $\mu$ is the Clark measure $\sigma_1$. Let $q$ be a bounded function continuous on an open set the complement of which satisfies $\mu e = 0$, and assume that either $X = VTq V^*$, where $T_q$ is a truncated Toeplitz operator with symbol $q$, or $X = q(U\alpha)$, where $\alpha \neq 1$ and $U\alpha$ is the unitary operator defined by formula (6). Then the Cesàro means of the sequence $(U^n X U^{-n})_{n \geq 0}$ tend to the operator $q(U)$ in the weak operator topology.

If we allow $\mu$ to have atoms, the result remains valid with the conclusion as in Lemma 4.1.

Proof. Take an arbitrary function $h \in K\theta$; it suffices to show that the Cesàro means of the sequence whose elements have the form

$$(U^n X U^{-n} h, h) = (V^* X V V^* U^{-n} h, V^* U^{-n} h) = \int q \cdot |V^* U^{-n} h|^2 d\nu$$

converge to $(q(U) h, h) = \int q |h|^2 d\mu$, where $\nu$ is either Lebesgue measure in the case of truncated Toeplitz operators, or $\nu = \sigma_\alpha$ in the case where $X = q(U\alpha)$. The Cesàro means of $(U^n X U^{-n} h, h)$ can be written as $\int q w_n d\nu$, where the $w_n$ are the Cesàro means of the functions $|V^* U^{-n} h|^2$. If $u$ is a continuous function on the entire unit circle, then by Lemma 4.1 we have $\int u w_n d\nu \to \int u d\tilde{\mu}$, where $d\tilde{\mu} = |h|^2 d\mu$.

From the assumption it follows that for any $\epsilon > 0$ there exists a continuous function $u$ on the unit circle with values on the segment $[0, 1]$ and such that $u = 1$ on $e$, $\tilde{\mu}({u \neq 0}) < \epsilon$. We may assume that $|q| \leq 1$ everywhere. Write $q = q_1 + q_2$, where $q_1 = (1 - u)q$ is a continuous function, so that $\int q_1 w_n d\nu$ tends to $\int q_1 d\tilde{\mu}$; the function $q_2 = uq$ satisfies $|q_2| \leq u$, and moreover,

$$\int u d\tilde{\mu} \leq \tilde{\mu}({u \neq 0}) < \epsilon.$$ 

We have $|\int q_2 w_n d\nu| \leq \int u w_n d\nu$; the right-hand side tends to $\int u d\tilde{\mu} < \epsilon$. Therefore, $\int q_2 w_n d\nu$ becomes small for large $n$, whence the required convergence follows easily. \qed

Now we show that sometimes the convergence in question may fail.

**Theorem 4.4.** Let $\theta$ be an inner function with $\theta(0) = 0$, assume that $\mu = \sigma_1$ has no atoms and let $U$ denote the operator of multiplication by $z$ on $L^2(\mu)$.

1) There exists a truncated Toeplitz operator $T_q$ on $K\theta$ with bounded symbol $q$ such that the Abel means of the sequence of operators $(U^n X U^{-n})$, where $X = VT_q V^*$, have no limit in the weak operator topology.
2) For any unimodular complex number \( \alpha \neq 1 \), there exist functions \( q \in L^\infty(\sigma_\alpha) \) such that the Abel means of the sequence of operators \( (U^nXU^{-n}) \) with \( X = Vq(U_\alpha)V^* \) have no limit in the weak operator topology.

The second part of this theorem is equivalent to Lemma 3.1 in [1].

In both cases the commutator \( XU - UX \) has the form \( \sum U^s \) for some function \( f \in L^2(\mu) \). If convergence fails, we obtain \( f \in B(\mu) \setminus C(\mu) \).

**Proof.** Suppose that, conversely, the Abel means always converge, and in particular, the Abel means of the numbers

\[
(U^nXU^{-n}1, 1) = (V^* XV V^*U^{-n}1, V^*U^{-n}1) = \int q \cdot |V^*U^{-n}1|^2 \, d\nu
\]

have a limit for any function \( u \in L^\infty(\nu) \), where \( \nu \) is the normalized Lebesgue measure on the unit circle in the first case, and \( \nu = \sigma_\alpha \) in the second case. Then, since the space \( L^1(\nu) \) is weakly sequentially complete, the Abel means of the functions \( |V^*U^{-n}1|^2 \) must have a limit \( s \in L^1(\nu) \). On the other hand, from Lemma 4.1 it follows that for any continuous function \( q \), the Cesàro means of the numbers \( (U^nXU^{-n}1, 1) \) tend to \( (q(U)1, 1) = \int q \, d\mu \). Hence, \( \mu = s \cdot \nu \), but this is impossible because the measures \( \mu \) and \( \nu \) are mutually singular. The contradiction obtained proves the theorem.

Now we discuss the construction of functions \( f \in B(\mu) \setminus C(\mu) \). Such a function \( f \) will be constructed as a function associated with a truncated Toeplitz operator either in terms of its symbol, or in terms of \( q \in L^\infty(\sigma_\alpha) \) for the operator \( q(U_\alpha) \); moreover, our construction starts with objects (e.g., weights \( w_n \)) whose formulas cannot be explicitly written in a simple form. Thus, it would be of interest to find more clear examples. Nevertheless, our construction seems to shed some light on the phenomenon where the limit in question fails to exist.

Moreover, the same arguments work not only for the Abel summation method, but also for any summation method described in §5 of [1]; namely, the existence of a stronger matrix summation method is required, which is also stronger than the Cesàro arithmetical means method.

For any sequence, the convergence of its Cesàro means implies that of the Abel means. Since the sequence of the operators \( (U^nXU^{-n}) \) is bounded, the converse is also true for it (see [8]).

In the proofs of Theorems 4.3 and 4.4 we construct a sequence of nonnegative functions \( w_n \in L^1(\nu) \) that are averages of the functions \( |U_1^{-n}V^*1|^2 \), where either \( \nu \) is the Lebesgue measure, or \( \nu = \sigma_\alpha \) for some \( \alpha \neq 1 \). Then \( \int w_n \, d\nu = 1 \) for any \( n \), and \( w_n, \nu \rightarrow \mu \) \( * \)-weakly: \( \int qw_n \, d\nu \rightarrow \int q \, d\mu \) for any continuous function \( q \). Since the measures \( \mu \) and \( \nu \) are mutually singular, there exist functions \( q \in L^\infty(\nu) \) for which convergence fails. Take the truncated Toeplitz operator with symbol \( q \) in the case where \( \nu \) is the Lebesgue measure, or the operator \( q(U_\alpha) \) if \( \nu = \sigma_\alpha \). Then the associated function \( f \in L^2(\mu) \) satisfies \( f \in B(\mu) \setminus C(\mu) \).

To understand the behavior of the sequence \( (U^nXU^{-n}1, 1) \) in the case where convergence fails, consider the construction of the corresponding function \( q \in L^\infty(\nu) \). The properties of the measures \( \mu, \nu \) and the weights \( w_n \) described above imply that there exists an increasing sequence of indices \( (n_k) \) and a sequence \( (e_k) \) of mutually disjoint subsets of the unit circle for which

\[
\int_{e_k} w_{n_k} \, d\nu \geq \frac{3}{4}.
\]

This can be proved by arguments used in the proof of Theorem 6 in [9] (see also the proof of Lemma 5 in [10]), to which S. V. Kisliakov kindly drew the author’s attention.
Assuming that the union of the sets \((e_k)\) coincides with the entire circle, we define a function \(q\) by the formula
\[
q = \sum_k (-1)^k \chi_{e_k},
\]
where \(\chi_{e_k}\) stands for the indicator of the set \(e_k\). We have
\[
\left| (-1)^k \int_T qw_{n_k} d\nu - \int_{e_k} w_{n_k} d\nu \right| \leq \int_{T \setminus e_k} qw_{n_k} d\nu \leq \int_{T \setminus e_k} w_{n_k} d\nu \leq \frac{1}{4}.
\]
Therefore,
\[
(-1)^k \int_T qw_{n_k} d\nu \geq \frac{1}{2},
\]
whence we see that the subsequence \((\int qw_{n_k})\) has no limit.

\section{Wave operators and unitary equivalence}

The construction of the wave operator involves a pair of unitary (or selfadjoint) operators acting, in general, in two different spaces, and the “identification operator” acting from the first space to the second. If \(U_1 : H_1 \to H_1\), \(U_2 : H_2 \to H_2\) are unitary operators and \(X : H_1 \to H_2\) is a bounded operator, then the wave operator is a limit of the sequence \(U_2^n X U_1^{-n}\); one can also consider averaged wave operators corresponding to various summation methods. In the classic pattern of scattering theory, \(X\) is an isometric operator; then the wave operator \(Y\), which is a strong limit of isometric operators, is also isometric and intertwines the operators \(U_1\) and \(U_2\): \(YU_1 = U_2Y\). Hence, the restriction of \(U_2\) to the image of \(Y\) turns out to be unitarily equivalent to \(U_1\).

In many more general situations the intertwining relation \(YU_1 = U_2Y\) remains valid, although \(Y\) may fail to be an isometry; moreover, \(Y\) may have nonzero kernel. Consider the polar decomposition \(Y = V_Y \cdot |Y|\), where \(|Y| = (Y^*Y)^{1/2}\) and \(V_Y\) is a partial isometry, i.e., an operator acting as an isometry on the orthogonal complement to its kernel. Then \(V_Y U_1 = U_2 V_Y\), and the restriction of \(U_1\) to the orthogonal complement of the kernel of \(Y\) turns out to be unitarily equivalent to the restriction of \(U_2\) to the closed range of \(Y\).

It would be of interest to look from this point of view at the situation where the averaged wave operator does not exist. Our central idea here is that the absence of the limit that defines the wave operator can nevertheless give us nontrivial intertwining relations. This case is, for instance, more informative than the case where the limit exists but is equal to the zero operator.

Spectral measures of two unitary operators are said to be \textit{mutually singular} if there exist two disjoint Borel sets on which they are supported; and we say that they are \textit{mutually absolutely continuous} if the collections of Borel sets for which the corresponding spectral projections vanish coincide for both operators. The following lemma contains arguments similar to those used in \cite{[1]} and generalizes Theorem 5.1 of \cite{[6]}, where the commutator \(XU_1 - U_2X\) was assumed to have finite rank.

\textbf{Lemma 5.1.} \textit{Assume that the spectral measures of unitary operators \(U_1, U_2\) are mutually singular. Then for any bounded operator \(X\), the Cesàro means of the operators \(U_2^n X U_1^{-n}\) tend to the zero operator in the weak operator topology.}

\textbf{Proof.} The Cesàro means of the sequence \(U_2(U_2^n X U_1^{-n}) - (U_2^n X U_1^{-n})U_1\) are of the form \(\frac{1}{n}(U_2^{n+1} X U_1^{-n} - U_2X)\); hence, they tend to the zero operator in norm. Therefore, convergence of any subsequence for the Cesàro means of the sequence \((U_2^n X U_1^{-n} h)\) to some vector \(k\) is equivalent to the fact that the subsequence with the same indices of the Cesàro means of the vectors \((U_2^n X U_1^{-n})(U_1 h)\) tends to \(U_2 k\).
Theorem 5.2. Suppose that for some vector $h_0 \neq 0$ the Cesàro means of the sequence $(U_2^n X U_1^{-n} h_0)$ do not tend weakly to zero. Since the sequence is bounded, we can find its subsequence that tends to a nonzero vector. The set of vectors $h$ for which the limit of the Cesàro means of the vectors $(U_2^n X U_1^{-n} h)$ exists along this subsequence, is a nontrivial closed subspace reducing $U_1$. The corresponding limits give rise to a nonzero operator on this reducing subspace, and we get a nonzero operator intertwining a reducing part of $U_1$ with $U_2$. Hence, a nontrivial reducing part of $U_1$ is unitarily equivalent to a reducing part of $U_2$. But this contradicts the assumption that the spectral measures of the operators $U_1, U_2$ are mutually singular. \hfill $\Box$

As a straightforward consequence of this lemma, we obtain the following result.

**Theorem 5.2.** Let $U_1 : H_1 \to H_1, U_2 : H_2 \to H_2$ be unitary operators, and $X : H_1 \to H_2$ a bounded operator. Denote by $C_n$ the Cesàro means of the sequence $(U_2^n X U_1^{-n})$. Consider the subspaces

$$\{ h \in H_1 : (C_n h, k) \to 0 \quad \forall k \in H_2 \} \quad \text{and} \quad \{ k \in H_2 : (C_n h, k) \to 0 \quad \forall h \in H_1 \};$$

they reduce the operators $U_1, U_2$, respectively. Then the spectral measures of the restrictions of $U_1, U_2$ to the orthogonal complements of these subspaces are mutually absolutely continuous.

**Proof.** By Lemma 5.1, all parts of the operator $U_1$ whose spectral measures are singular with respect to the spectral measure of $U_2$, are contained in the subspace on which the operators $C_n$ tend weakly to zero. To get a similar fact for the parts of $U_2$ with spectral measures singular with respect to the spectral measure of $U_1$, we can apply the same argument to the adjoint operators. \hfill $\Box$

Observe that the situation described by the theorem covers the cases where the averaged wave operator fails to exist. An interesting open question remains about conditions that guarantee unitary equivalence, which means that the spectral measures are not only mutually absolutely continuous, but also the functions that describe the local spectral multiplicity coincide. Note that, in the paper [1], pairs of unitary operators with rank-two difference for which the averaged wave operator does not exist were constructed so that the operators is the pairs were unitarily equivalent to each other.

§6. Boundary behavior of functions in $K_\theta$

As above, $\theta$ is an inner function, but we also consider the case where $\theta(0) \neq 0$; $K_\theta = H^2 \ominus \theta H^2$. The Clark measures $\sigma_\alpha$ are defined by formula (5). For a function $\varphi \in K_\theta$, we define $g_\varphi$ by formula (11): $g_\varphi(z) = \frac{\varphi(z) - \varphi(rz)}{\theta(z) - \theta(rz)}$. If $\varphi = f$ $\mu$-almost everywhere, by formula (7) for $\mu$-almost all $z$ we have

$$g_\varphi(z) = \frac{\varphi(z) - \varphi(rz)}{1 - \theta(rz)} = \varphi(z) \cdot \frac{1}{1 - \theta(rz)} - \frac{\varphi(rz)}{1 - \theta(rz)} = f(z) \cdot \int \frac{d\mu(\xi)}{1 - \xi \cdot rz} - \int \frac{f(\xi) d\mu(\xi)}{1 - \xi \cdot rz} = (H_\varphi f)(z),$$

which by (14) is equal to $B_\varphi h$ with $h \equiv 1$:

$$g_\varphi = B_\varphi 1.$$  

In this section we study consequences of the fact that the functions $g_\varphi$ and $H_\varphi f$ coincide.

**Theorem 6.1.** Let $\varphi \in K_\theta$, and let $\alpha_1, \alpha_2$ be distinct unimodular complex numbers. Assume that $\varphi$ coincides with a bounded function $q$ $\sigma_{\alpha_1}$-almost everywhere. Then the norms of $g_\varphi$ in $L^2(\sigma_{\alpha_2})$ are bounded uniformly in $r$. 


We have the following sufficient condition for $g_r$ to have a limit at a point of the unit circle.

**Theorem 6.2.** Under the conditions of Theorem 6.1, if $\sigma_{\alpha_2}$ has an atom at a point $\omega$, then $g_r(\omega)$ has a limit as $r \to 1$,

$$\lim g_r(\omega) = -\sigma_{\alpha_2}(\{\omega\})(\bar{\alpha}_2 - \bar{\alpha}_1) \int \frac{q(z) d\sigma_{\alpha_1}(z)}{|1 - \bar{\omega}z|^2} - \bar{\alpha}_1 \int \frac{q(z) d\sigma_{\alpha_1}(z)}{1 - \bar{\omega}z}. \quad (21)$$

The convergence of the integrals in (21) is guaranteed by the property $\int \frac{d\sigma_{\alpha_2}(z)}{|1 - \bar{\omega}z|^2} < \infty$ (see Proposition 4 in [11]); for completeness of the presentation it will be checked below when proving the theorem.

Since the norms of $g_r$ in $L^2(\sigma_{\alpha_2})$ are bounded by Theorem 6.1, the pointwise convergence established in Theorem 6.2 implies $L^2$-convergence on the part of the space $L^2(\sigma_{\alpha_2})$ corresponding to point masses of the measure $\sigma_{\alpha_2}$. To get convergence on the entire space $L^2(\sigma_{\alpha_2})$, we replace the boundedness assumption in Theorem 6.2 by continuity.

**Theorem 6.3.** Under conditions of Theorem 6.1, if $\varphi$ coincides $\sigma_{\alpha_1}$-almost everywhere with a continuous function $q$, then the limit of the functions $g_r$ as $r \to 1$ exists in $L^2(\sigma_{\alpha_2})$ and coincides with the function $\frac{\varphi - q}{\alpha_2 - \alpha_1}$ almost everywhere with respect to the continuous part of $\sigma_{\alpha_2}$.

This result can be compared with formula (2.1) in [12], where stronger assumptions were imposed on a function on the real line. Both formulas connect extension by continuity, the Hilbert transform, and the mapping that takes the boundary values of a function in $K_\theta$ corresponding to the measure $\sigma_{\alpha_1}$ to the boundary values of the same function $\sigma_{\alpha_2}$-almost everywhere.

The continuity property of the function $q$ can be relaxed in the light of Theorem 4.3.

If $\alpha_1 = \alpha_2$, the assertion of Theorem 6.3 is no longer true; moreover, even the conclusion of Theorem 6.1 fails. Namely, as an example in [7] shows, $\varphi$ can coincide with a continuous function $\sigma_1$-almost everywhere, but the norms of $g_r$ in $L^2(\sigma_1)$ can be unbounded.

Apparently, if a point $\omega$ with $\sigma_{\alpha_2}(\omega) > 0$ lies in the support of the measure $\sigma_{\alpha_1}$, the limit of $g_r(\omega)$ exists by Theorem 6.2, but its value may fail to coincide with the value computed in accordance with Theorem 6.3.

Without loss of generality we may assume that $\alpha_2 = 1$. Indeed, otherwise we can consider the inner function $\tilde{\alpha}_2 \theta$ in place of $\theta$ and the unimodular number $\alpha = \alpha_1 \tilde{\alpha}_2$ in place of $\alpha_1$. Thus, fix $\alpha$, $\alpha \neq 1$; we shall use the notation $\mu = \sigma_1$; for $\mu$-almost all $z$ we have $\theta(z) = 1$. The angular boundary function of $\varphi$ on the subset of the circle where $\theta = 1$ is denoted by $f : f \in L^2(\mu)$, $\varphi = f$ $\mu$-almost everywhere.

**Proof of Theorem 6.1.** The case where $\theta(0) = 0$. Then $\mu = \sigma_1$ is a probability measure. Let $\varphi \in K_\theta$ coincide $\sigma_\alpha$-almost everywhere with a function $q \in L^\infty(\sigma_\alpha)$. By (20), $g_r = B_r 1$. By construction we have $\|B_r\| \leq 2 \|X\|$, so that the norms of $g_r$ in $L^2(\mu)$ are bounded uniformly in $r$.

The case where $\theta(0) \neq 0$ will be reduced to the preceding case. Set $\lambda = \theta(0)$, define $\theta_\lambda = \frac{1 - \lambda}{1 + \lambda}, \frac{\sigma - \lambda}{1 - \lambda}$ Then $\theta_\lambda(0) = 0$, and it can be checked that

$$\frac{1 + \theta}{1 - \theta} = \frac{1 - |\lambda|^2}{|1 - \lambda|^2} \cdot \frac{1 + \theta_\lambda}{1 - \theta_\lambda} + 2i \frac{\text{Im} \lambda}{|1 - \lambda|^2}.$$ 

Therefore, taking the real parts, we see that the measure $\sigma_1$ of the function $\theta$ and the corresponding measure of the function $\theta_\lambda$ differ only by a constant factor of $\frac{1 - |\lambda|^2}{|1 - \lambda|^2}$. 

Take φ ∈ Kθ and define φλ = \frac{\varphi - \bar{\varphi}}{1-\bar{\varphi}} ∈ K_{\lambda}. The values of φλ differ σα-almost everywhere from the values of φ by the factor 1 - \bar{\lambda}α, the assumption that φ coincides σα-almost everywhere with a bounded function is equivalent to the same property of the function φλ. Then the already proved part of the theorem shows that the norms of the functions \( \varphi_{\lambda} - (\varphi_{\lambda})_r \) in \( L^2(\mu) \) are bounded, \( \mu = \sigma_1 \). At \( \mu \)-almost all points we have \( \theta = 1 \), and

\[
\frac{\varphi_{\lambda} - (\varphi_{\lambda})_r}{1 - ((\theta_{\lambda})_r)} = \frac{\varphi_{\lambda} - \varphi_{\lambda}}{1 - \bar{\lambda} \varphi_{\lambda}} = \frac{1 - \lambda}{1 - |\lambda|^2} \cdot \frac{\varphi - \varphi_r}{1 - \theta_r} + \frac{1 - \lambda}{1 - |\lambda|^2} \cdot \varphi.
\]

By this relation, the norms of the functions \( g_r = \frac{\varphi - \varphi_r}{1 - \theta_r} \) in \( L^2(\mu) \) are also bounded. □

**Proof of Theorem 6.2.** Set \( \alpha = \alpha_1 \alpha_2 \) and consider the pair \( \alpha, 1 \) in place of \( \alpha_1, \alpha_2 \). Assume that \( \sigma_1(\{\omega\}) > 0 \) and that, for \( \varphi ∈ K_\theta \), we have \( \varphi \equiv q \) σα-almost everywhere, with \( q ∈ L^\infty(\sigma_\alpha), \alpha ≠ 1 \). We must prove that \( g_r(\omega) \) tends to the limit (21) as \( r ↑ 1 \).

If \( \lambda \) is a point of the open unit disk, then the reproducing kernel \( k_\lambda \) for the space \( K_\theta \) has the form

\[
k_\lambda(z) = \frac{1 - \bar{\theta}(\lambda)\theta(z)}{1 - \lambda z};
\]

for \( \varphi ∈ K_\theta \) we have \( \varphi(\lambda) = (\varphi, k_\lambda) \).

We use the results of [13] rewritten in terms of reproducing kernels. The fact that \( \sigma_1(\{\omega\}) > 0 \) means that the functional \( \varphi \mapsto \varphi(\omega) \) is bounded on \( K_\theta \). Then the reproducing kernel \( k_\lambda \) is also well defined for \( \lambda = \omega \) by the same formula, and \( k_\omega ∈ K_\theta \). Hence, the boundary function of \( k_\omega \) belongs to \( L^2(\sigma_\alpha) \), namely, \( \int |k_\omega(z)|^2 d\sigma_\alpha < \infty \).

Since \( \theta(z) = \alpha ≠ 1 \) σα-almost everywhere and \( \theta(\omega) = 1 \), for \( \sigma_\alpha \)-almost all \( z \) we have \( k_\omega(z) = \frac{\varphi - \varphi(z)}{1 - \bar{\omega}z}, \) whence

\[
\int \frac{d\sigma_\alpha(z)}{|1 - \bar{\omega}z|^2} < \infty.
\]

Now it is easily seen that the integrals in (21) converge.

Moreover, it is well known that then the radial derivative

\[
\theta'(\omega) = \lim_{r \to 1} \frac{\theta(\omega) - \theta(r\omega)}{\omega - r\omega}
\]

exists, and

\[
\theta'(\omega) = \frac{1}{\omega\sigma_1(\{\omega\})} ≠ 0.
\]

Thus, the existence of the limit of \( g_r(\omega) \) at \( \omega \) is equivalent to the existence of the radial derivative of \( \varphi \) at \( \omega \), and

\[
\lim_{r ↑ 1} g_r(\omega) = \frac{\varphi'(\omega)}{\theta'(\omega)} = \omega\sigma_1(\{\omega\})\varphi'(\omega).
\]

(From [13] it follows that the fact that \( \varphi'(\omega) \) exists for all \( \varphi ∈ K_\theta \) is stronger than the existence of values at \( \omega \). Therefore, in the general case, if \( \sigma_1(\{\omega\}) > 0 \), then there may exist functions \( \varphi ∈ K_\theta \) for which the derivative \( \varphi'(\omega) \) does not exist.)

We have

\[
\omega\varphi'(\omega) = \omega \cdot \lim_{r ↑ 1} \frac{\varphi(\omega) - \varphi(r\omega)}{\omega - r\omega} = \lim_{r ↑ 1} \frac{(\varphi, k_\omega) - (\varphi, k_{r\omega})}{1 - r}.
\]

Since the embedding \( K_\theta ↪ L^2(\sigma_\alpha) \) is a unitary operator, the scalar product of functions of class \( K_\theta \) can be rewritten as the scalar product in \( L^2(\sigma_\alpha) \):

\[
\omega\varphi'(\omega) = \lim_{r ↑ 1} \int \frac{\varphi k_\omega d\sigma_\alpha - \int \varphi k_{r\omega} d\sigma_\alpha}{1 - r} = \lim_{r ↑ 1} \int q \cdot \frac{k_\omega - k_{r\omega}}{1 - r} d\sigma_\alpha.
\]
For $\sigma_\alpha$-almost all $z$ we have $\theta(z) = \alpha$; thus, by the formulas for the reproducing kernels at such points we obtain $k_\omega(z) = \frac{1-\alpha}{1-z^z}$, $k_{r\omega}(z) = \frac{1-\theta(r\omega)\alpha}{1-r\omega}$. Therefore,

$$
\frac{k_\omega(z) - k_{r\omega}(z)}{1 - r} = \frac{1}{1 - r} \left( \frac{1 - \bar{\alpha}}{1 - \bar{z}z} - \frac{1 - \theta(r\omega)\bar{\alpha}}{1 - r\omega} \right)
$$

$$
= -\frac{1 - \bar{\alpha}}{(1 - \bar{z}z)(1 - r\omega)} - \frac{1 - \theta(r\omega)\bar{\alpha}}{1 - r\omega} \cdot \frac{\bar{\alpha}}{1 - r\omega}.
$$

Since the function $\theta$ has radial derivative at the point $\omega$, we see that the function $\frac{1 - \theta(r\omega)}{1 - r\omega}$ is bounded, and the absolute value of the right-hand side of (24) can be estimated by $\frac{\text{const}}{|1 - \omega^z|}$ uniformly in $r$. By (22), to show that the integral in (23) converges, it suffices to establish the convergence of $\frac{k_\omega - k_{r\omega}}{1 - r}$ $\sigma_\alpha$-almost everywhere. Thus,

$$
\lim_{r \nearrow 1} g_r(\omega) = \sigma_1(\{\omega\}) \cdot \omega \varphi'(\omega) = \sigma_1(\{\omega\}) \cdot \int q \cdot \lim_{r \nearrow 1} \frac{k_\omega - k_{r\omega}}{1 - r} d\sigma_\alpha.
$$

From (24) we obtain

$$
\lim_{r \nearrow 1} \frac{k_\omega(z) - k_{r\omega}(z)}{1 - r} = -\frac{1 - \bar{\alpha}}{(1 - \bar{z}z)(1 - \bar{\omega}z)} - \omega \varphi'(\omega) \cdot \frac{\bar{\alpha}}{1 - \bar{\omega}z}
$$

$$
= -\frac{1 - \bar{\alpha}}{|1 - \bar{z}z|^2} - \frac{\bar{\alpha}}{\sigma_1(\{\omega\}) \cdot (1 - \bar{\omega}z)}
$$

for $\sigma_\alpha$-almost all $z$. If we plug this in (25), we obtain (21) with $\alpha_1 = \alpha$, $\alpha_2 = 1$. \qed

Proof of Theorem 6.3. As above, we assume that $\alpha_2 = 1$ (the general case can be reduced to this by the consideration of $\alpha = \alpha_1\bar{\alpha}_2$ in place of $\alpha_1$). First, we consider the case where $\theta(0) = 0$.

Suppose $\varphi \in K_\theta$, $\mu = \sigma_1$, and let $f \in L^2(\mu)$ be the boundary function of $\varphi : \varphi = f$ $\mu$-almost everywhere. The functions $g_r$ are defined by formula (4), and $g_r$ coincides with $H_r f$ as an element of the space $L^2(\mu)$. By assumption, the function $\varphi \in K_\theta$ coincides $\sigma_\alpha$-almost everywhere with a continuous function $q$ for some $\alpha \neq 1$. We must prove that $g_r$ have a limit in $L^2(\mu)$ and that

$$
g_r \to \frac{f - q}{1 - \alpha} \quad \text{in } L^2(\mu_c) \quad \text{as } r \nearrow 1,
$$

where $\mu_c$ is the continuous part of $\mu$. Convergence on the atomic part of $\mu$ is a consequence of Theorem 6.2 if we recall Theorem 6.1 by which the norms of the $g_r$ in $L^2(\mu)$ are bounded.

By Theorem 4.1 the Cesàro means, and hence also the Abel means, of the operators $U_1^n q(U_n) U_1^{-n}$ have a strong limit, and on the orthogonal complement $K$ to all eigenvectors of $U_1$ this limit coincides with $q(U_n) U_1^{-n}$. Thus, the limit of the operators $B_r$ defined in (22) exists and coincides on $K$ with $X - \frac{q(U_1)}{1 - \alpha}$. Applying this convergence to the vector $h \equiv 1 \in L^2(\mu)$ and using the fact that $B_r 1 = g_r$, we conclude that the functions $g_r$ have a limit that coincides $\mu_c$-almost everywhere with $(X - \frac{q(U_1)}{1 - \alpha}) 1$. In its turn, this coincides with the boundary values $\mu_c$-almost everywhere (recall that $\mu = \sigma_1$) of the function $(\frac{q(U_1)}{1 - \alpha} - \frac{q(U_1)}{1 - \alpha}) 1 \in K_\theta$. Comparing the boundary values $\sigma_\alpha$-almost everywhere, we obtain $q(U_1) 1 = f$. Since $\varphi = f$ and $q(U_1) 1 = q \mu$-almost everywhere, we see that $\lim_{r \nearrow 1} g_r = \frac{f - q}{1 - \alpha}$ $\mu_c$-almost everywhere, as required.

The reduction of the case where $\theta(0) \neq 0$ to the case analyzed above is similar to that in the proof of Theorem 6.1. \qed
References


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