STATIONARY DIFFUSION PROCESSES
WITH DISCONTINUOUS DRIFT COEFFICIENTS

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ABSTRACT. The paper is devoted to the stationary Fokker–Planck equation \( \Delta u - \text{div}(uf) = 0 \) with a locally bounded measurable vector field \( f \) defined on the entire \( \mathbb{R}^n \). The existence of a positive (not necessarily integrable) solution is proved. Various conditions on the vector field \( f \) are deduced that suffice for the existence of a solution that is a probability density. Under these conditions, the corresponding diffusion has an invariant probability measure with density \( u \). In particular, the assumptions of the theorems proved here are fulfilled for certain vector fields \( f \) with trajectories tending to infinity. In this situation, the approach in question turns out to be more efficient than the earlier methods based on the construction of a Lyapunov function.

§1. INTRODUCTION

The present paper continues the work started in [1, 2, 3, 4] on the solvability of the stationary Fokker–Planck equation

\begin{equation}
\Delta u - \text{div}(uf) = 0,
\end{equation}

or

\[
\sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u(x)f_i(x)) = 0;
\]

the solution \( u \) and the vector field \( f \) are usually considered on the entire space \( \mathbb{R}^n \). Principal attention is paid to the existence of a solution that is a probability density. The presence of such a solution guarantees the stability of the system \( dx/dt = f(x) \) under noise and occurs not for all \( f \). For instance, equation (1.1) has no probability solutions when \( f \equiv 0 \) and for some other \( f \) (see the remark to Theorem 3 in [4]). Our main purpose in the present paper is in obtaining some conditions on \( f \) that suffice for the existence of a positive solution of class \( L^1(\mathbb{R}^n) \). It is essential that the phase space here is the entire \( \mathbb{R}^n \) rather than a compact manifold because, by [5], in the latter case equation (1.1) has a positive \( L^1 \)-solution for an arbitrary (sufficiently smooth) field \( f \).

The problem posed above is closely related to the question of existence of an invariant probability measure for the dynamical system \( dx/dt = f(x) \) perturbed by an appropriate noise. For this reason, until a recent time (before the release of the paper [4]), the problem had been studied mainly by probability methods. The first result in this direction is the Khas’minskii theorem on the existence of an invariant probability measure in the case where there is a Lyapunov function \( \psi \) with the following properties: \( \psi(x) \to +\infty \),

\[
\Delta \psi + \sum_{i=1}^{n} f_i \frac{\partial \psi}{\partial x_i} \to -\infty \quad \text{as} \quad |x| \to +\infty
\]
of the integral identity (4.1), provided by Theorem 4.1 may or may not belong to existence of a positive solution for equation (1.1) with a fairly arbitrary vector field stages. At the first stage, by using earlier results, we prove Theorem 4.1 about the than the random process. The method suggested dates back to [4] and consists of two
and invokes new methods of study.

The presence of such trajectories is an essential obstruction to building a Lyapunov function
including certain vector fields with trajectories going to infinity (see Theorem 6.3). The

should be noted that, at this stage, we do not impose any restrictions on the direction

is positive. We introduce the vector field

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locally nonbounded) vector fields (see [6]). Next, in [7] that theorem was extended to singular (discontinuous and even locally nonbounded) vector fields \( \mathbf{f} \). In the present paper, we supplement the Lyapunov–Khas’minskii method by a new approach that covers a fairly wide class of vector fields, including certain vector fields with trajectories going to infinity (see Theorem 6.3). The

In the present paper, a pure mathematical investigation concerns equation (1.1) rather
than the random process. The method suggested dates back to [4] and consists of two
stages. At the first stage, by using earlier results, we prove Theorem 4.1 about the

dependence of the vector field. Therefore, for a locally bounded measurable \( \mathbf{f} \), a positive solution of (1.1) provided by Theorem 4.1 may or may not belong to \( L^1(\mathbb{R}^n) \), depending on the direction of \( \mathbf{f} \).

At the second stage, by using identity (2.3), its integral form (2.2), and the averaging
method, we study the asymptotics of the positive solution at infinity. So, in Theorems
5.1, 6.1, and 6.2 we obtain various conditions on the vector field \( \mathbf{f} \). In the final part of the paper, we prove an upper estimate for solutions in terms of a decaying exponential. In particular, this estimate
holds true for some systems \( \frac{dx}{dt} = \mathbf{f}(x) \) with trajectories going to infinity. The result obtained is interpreted as a stabilization of a nonstable dynamical system by an external noise added to its right-hand side.

\section{Auxiliary lemmas}

\begin{lemma}
Let a vector field \( \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \) be locally bounded and measurable. Suppose
that a positive continuous function \( u(x) \) is a weak solution of equation (1.1) in the sense
of the integral identity (4.1), and that a bounded measurable function \( \varphi(x) \) vanishes outside some ball. Assume that the function

\[
V(x) = \int_{\mathbb{R}^n} u(x + y) \varphi(y) \, dy_1 \ldots dy_n
\]

is positive. We introduce the vector field

\[
\mathbf{F}(x) = \frac{1}{V(x)} \int_{\mathbb{R}^n} u(x + y) \mathbf{f}(x + y) \varphi(y) \, dy_1 \ldots dy_n.
\]

Then \( \mathbf{F}(x) \) is measurable and locally bounded, and the function \( V(x) \) is continuous and
is a weak solution of the equation \( \Delta V = \text{div}(V \mathbf{F}) = 0 \) in the following sense:

\[
\int_{\mathbb{R}^n} V(x) \left( \Delta \psi(x) + \sum_{i=1}^n F_i(x) \partial \psi(x)/\partial x_i \right) \, dx_1 \ldots dx_n = 0 \quad \text{for all} \quad \psi \in C_0^\infty(\mathbb{R}^n).
\]

\end{lemma}

\begin{proof}
For \( \psi \in C_0^\infty(\mathbb{R}^n) \), we have

\[
\int_{\mathbb{R}^n} V(x) \left( \Delta \psi(x) + \sum_{i=1}^n F_i(x) \partial \psi(x)/\partial x_i \right) \, dx = \int_{\mathbb{R}^n} \Delta \psi(x) \int_{\mathbb{R}^n} u(x + y) \varphi(y) \, dy \, dx
\]

\[
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x + y) \varphi(y) \left( \sum_{i=1}^n f_i(x + y) \partial \psi(x)/\partial x_i \right) \, dy \, dx
\]

\[
= \int_{\mathbb{R}^n} \varphi(y) \int_{\mathbb{R}^n} u(x + y) \left( \Delta \psi(x) + \sum_{i=1}^n f_i(x + y) \partial \psi(x)/\partial x_i \right) \, dx \, dy.
\]
\end{proof}
For every fixed \( y \), the inner integral is equal to zero because by a change of variables it is reduced to

\[
\int_{\mathbb{R}^n} u(x) \left( \Delta \psi(x - y) + \sum_{i=1}^{n} f_i(x) \partial \psi(x - y) / \partial x_i \right) \, dx.
\]

The last integral is zero by (4.1).

**Theorem 2.1.** Suppose that a vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) is locally bounded and measurable, and a positive continuous function \( u(x) \) satisfies equation (1.1) in the weak sense, i.e., (4.1) is fulfilled. If \( R > 0 \), let \( x_0 \in \mathbb{R}^n \), \( \sup_{|x-x_0|<R} |f(x)| \leq M \) and \( 0 < a < b \), then \( u \) satisfies the following relations:

\[
u(x_0) e^{-MR} \int_{|x|<R} u(x) dx_1 \ldots dx_n \leq \int_{|x-x_0|<R} u(x) dx_1 \ldots dx_n
\]

(2.1)

\[
\int_{|x|=1} u(bx) \, dS_x - \int_{|x|=1} u(ax) \, dS_x = \int_{a \leq |x| \leq b} |x|^{-n} u(x)(f(x), x) \, dx_1 \ldots dx_n.
\]

(2.2)

**Proof.** We apply the operation of averaging. Consider a nonnegative function \( \varphi_1(x) \) of class \( C_0^\infty(\mathbb{R}^n) \) supported on the unit ball with center at zero such that \( \int_{\mathbb{R}^n} \varphi_1(x) \, dx = 1 \). Put

\[
\varphi_k(x) = k^n \varphi_1(kx), \quad u_k(x) = \int_{\mathbb{R}^n} u(x+y) \varphi_k(y) \, dy_1 \ldots dy_n,
\]

\[
f_k(x) = \frac{1}{u_k(x)} \int_{\mathbb{R}^n} u(x+y) f(x+y) \varphi_k(y) \, dy_1 \ldots dy_n.
\]

Then \( u_k(x) \) and \( f_k(x) \) are infinitely differentiable and, by Lemma 2.1, are related by the formula \( \Delta u_k - \text{div}(u_k f_k) = 0 \).

Since \( \int_{\mathbb{R}^n} \varphi_k(y) \, dy = 1 \), we see that the quantity

\[
u_k(x) = \int_{\mathbb{R}^n} u(x+y) \varphi_k(y) \, dy_1 \ldots dy_n = \int_{|y|<1/k} u(x+y) \varphi_k(y) \, dy_1 \ldots dy_n
\]

is between the minimum and the maximum of \( u(y) \) on the closed ball of radius \( 1/k \) and centered at \( x \). Since the continuous function \( u \) is uniformly continuous on each compact set, we see that the \( u_k \) converge to \( u \) uniformly on compact sets.

For smooth \( u \) and \( f \), formula (2.2) is obtained by integration of the formula

\[
d \int_{|x|=1} u(tx) \, dS_x / dt = \int_{|x|=1} u(tx)(f(tx), x) \, dS_x
\]

(2.3)

taken from Lemma 3 in [3]:

\[
\int_{|x|=1} u(bx) \, dS_x - \int_{|x|=1} u(ax) \, dS_x = \int_{a}^{b} \int_{|x|=1} u(tx)(f(tx), x) \, dS_x \, dt
\]

\[
= \int_{a \leq |x| \leq b} |x|^{-n} u(x)(f(x), x) \, dx_1 \ldots dx_n.
\]

Since (2.2) has already been verified for each \( u_k \), we deduce it for \( u \) by a limit passage: the uniform convergence of the \( u_k \) to \( u \) on the compact sets implies the convergence of

\[
\int_{|x|=1} u_k(bx) \, dS_x - \int_{|x|=1} u_k(ax) \, dS_x \quad \text{to} \quad \int_{|x|=1} u(bx) \, dS_x - \int_{|x|=1} u(ax) \, dS_x.
\]
The convergence of
\[ \int_{a \leq |x| \leq b} |x|^{-n} u_k(x)(f_k(x), x) \, dx_1 \ldots dx_n \to \int_{a \leq |x| \leq b} |x|^{-n} u(x)(f(x), x) \, dx_1 \ldots dx_n \]
is a consequence of the convergence of the $u_k f_k$ to $u f$ in $L^1(\{x : a < |x| < b\})$, which, in its turn, occurs because
\[ u_k(x)f_k(x) = \int_{\mathbb{R}^n} u(x + y)f(x + y)\varphi_k(y) \, dy_1 \ldots dy_n \]
and by the properties of the averaging operation (see [2] p. 54).

For smooth $u$ and $f$, inequality (2.1) was obtained in Lemma 5 of [2]. In the general case, first we prove it with $R > 0$ replaced by $R - \delta$, $\delta > 0$. Fix $\delta > 0$. Then $|f_k(x)| \leq M$ for $|x| < R - \delta$ and all sufficiently large $k$. Indeed,
\begin{equation}
|f_k(x)| = \frac{1}{u_k(x)} \left| \int_{\mathbb{R}^n} u(x + y)f(x + y)\varphi_k(y) \, dy_1 \ldots dy_n \right| \\
\leq \frac{1}{u_k(x)} \int_{\mathbb{R}^n} u(x + y)\varphi_k(y) \, dy_1 \ldots dy_n \sup_{|z - x| < 1/k} |f(z)| = \sup_{|z - x| < 1/k} |f(z)| \leq M,
\end{equation}
because the support of $\varphi_k(y)$ is included in the ball of radius $1/k$ and centered at zero, whereas for large $k$ and $|x| < R - \delta$, the supremum is taken over a set included in a ball of radius $R$ and centered at zero. Thus, for large $k$ (it depends on $\delta$ how large they should be), (2.1) holds true for $u_k$ and $f_k$ under the condition that $R > 0$ is replaced by $R - \delta$, $\delta > 0$. Specifically, in this situation it is obtained by a uniform limit passage, and another limit passage as $\delta \to 0$ implies the initial claim. The theorem is proved. \hfill \Box

§3. Estimates of positive solutions

**Theorem 3.1.** Let $f$ be locally bounded and measurable and let a function $u$ be continuous and positive. Suppose that on $\mathbb{R}^n$ equation (1.1) is fulfilled in the sense of the integral identity (4.1). Then for every points $x_1$ and $x_2$ in $\mathbb{R}^n$ we have
\begin{equation}
u(x_1)(1 + \sqrt{\delta})^n \exp(-M(\delta + 2\sqrt{\delta})) \leq u(x_2) \end{equation}
\begin{equation}
\leq u(x_1)((1 + \sqrt{\delta})^n \exp(M(\delta + 2\sqrt{\delta}))) - 1),
\end{equation}
where $\delta = \delta(x_1, x_2) = |x_1 - x_2|$, and $M = M(x_1, x_2) = \sup_{|y| < |x_1| + |x_2| + \delta + \sqrt{\delta}} |f(y)|$.

**Proof.** Fixing $x_1$ and $x_2$, we consider the ball of radius $R = \sqrt{\delta}$ and centered at $x_2$. Then Theorem 2.1 yields the inequality
\[ u(x_2) e^{-MR} \int_{|x| < R} dx_1 \ldots dx_n \leq \int_{|x - x_2| < R} u(x) \, dx_1 \ldots dx_n. \]
Applying Theorem 2.1 to the ball of radius $R + \delta$ and centered at $x_1$, we obtain
\[ \int_{|x - x_1| < R + \delta} u(x) \, dx_1 \ldots dx_n \leq u(x_1) e^{M(R + \delta)} \int_{|x| < R + \delta} dx_1 \ldots dx_n. \]
Since $u$ is positive, and the $(R + \delta)$-ball centered at $x_1$ includes the $R$-ball centered at $x_2$, we see that
\[ \int_{|x - x_2| < R} u(x) \, dx_1 \ldots dx_n \leq \int_{|x - x_1| < R + \delta} u(x) \, dx_1 \ldots dx_n. \]
Theorem 4.1. For every locally bounded measurable vector field \( \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \) there exists a positive continuous function \( u \) that solves equation (1.1) in a weak sense:

\[
\int_{\mathbb{R}^n} u(\mathbf{x}) \left( \Delta \psi(\mathbf{x}) + \sum_{i=1}^n f_i(\mathbf{x}) \frac{\partial \psi(\mathbf{x})}{\partial x_i} \right) \, dx_1 \ldots dx_n = 0 \quad \text{for all} \quad \psi \in C_0^\infty(\mathbb{R}^n),
\]

and satisfies (2.2), (2.1), (3.1), and (3.2).
Proof. Under the additional assumption that $f \in C^\infty(\mathbb{R}^n)$, this readily follows from Theorem 2 in \cite{4}. In order to lift the restriction $f \in C^\infty(\mathbb{R}^n)$, we construct a sequence $\{f_k\}_{k=1}^\infty$ of vector fields of class $C^\infty(\mathbb{R}^n)$ that converges to $f$ in the $L^1$-metric on every ball in $\mathbb{R}^n$. To this end, we apply averaging. Consider a nonnegative function $\varphi_1(x)$ of class $C^\infty_0(\mathbb{R}^n)$ supported on the unit ball with center at zero and satisfying $\int_{\mathbb{R}^n} \varphi_1(x) \, dx = 1$. Put
\[
\varphi_k(x) = k^n \varphi_1(kx), \quad f_k(x) = \int_{\mathbb{R}^n} f(x + y) \varphi_k(y) \, dy_1 \ldots dy_n.
\]
Local convergence in the $L^1$-metric was proved in \cite{5} p. 90]. Note also that the numerical function $\sup_{k \leq N} |f_k(x)|$ of the variable $x$ is bounded on every ball in $\mathbb{R}^n$; this can be verified by simple calculations similar to (2.4) but with $u$ dropped.

Consider the sequence $\{u_k(x)\}_{k=1}^\infty$ of infinitely differentiable positive solutions of the equations $\Delta u_k - \text{div}(u_k f_k) = 0$, $u_k(0) = 1$ (these solutions exist by Theorem 2 in \cite{4}). By Theorem 3.1 and the local boundedness of $\sup_{k \leq N} |f_k(x)|$, for every ball there is a constant $M$ such that inequalities (3.1) and (3.2) hold true on this ball for $u = u_k$ with every $k$. They imply that the functions $\{u_k(x)\}_{k=1}^\infty$ are uniformly bounded and equicontinuous on each ball. We take a nesting sequence of balls that cover $\mathbb{R}^n$ and apply the Arzelà–Ascoli theorem on each ball. Then, by the diagonal procedure, we obtain a subsequence of the sequence $\{u_k(x)\}_{k=1}^\infty$ that converges uniformly on compact sets to some continuous function $u(x)$, which is positive by (3.1). For simplicity, we still denote this subsequence by the same symbol $\{u_k(x)\}_{k=1}^\infty$.

We want to know whether $u(x)$ is a weak solution of equation (1.1). The formula $\Delta u_k - \text{div}(u_k f_k) = 0$ implies the integral identity
\[
(4.2) \quad \int_{\mathbb{R}^n} u_k(\Delta \psi + (f_k, \nabla \psi)) \, dx_1 \ldots dx_n = 0 \quad \text{for all} \quad \psi \in C^\infty_0(\mathbb{R}^n).
\]
Fix some function $\psi \in C^\infty_0(\mathbb{R}^n)$. Since the sequence $\{u_k(x)\}_{k=1}^\infty$ converges uniformly to $u(x)$ on the support of $\psi(x)$ and the sequence $\{f_k\}_{k=1}^\infty$ converges to $f$ in the $L^1$-metric on the support of $\psi(x)$, we can pass to the limit in (4.2). As a result, we arrive at (4.1). This proves the theorem.

As we saw in the example (3.5), under the assumptions of Theorem 4.1 a positive solution may or may not belong to $L^1(\mathbb{R}^n)$, depending on the direction of the vector field $f$. Conditions on $f$ sufficient for the integrability of a positive solution will be obtained in subsequent sections, and also can be found in Theorem 3 of \cite{4}. A remark to that theorem also presented in \cite{4} yields a condition for the absence of a positive solution of class $L^1(\mathbb{R}^n)$. A nonintegrable positive solution has no probabilistic sense, but is of general interest, as is Theorem 4.1 itself. In connection with Theorem 4.1, it should be noted that the author is not aware of any its analog for an arbitrary second order linear elliptic equation. It is also not clear whether such an equation has a nontrivial (not necessarily nonnegative) solution defined on the entire space $\mathbb{R}^n$. But if we consider a compact manifold rather than $\mathbb{R}^n$, it is possible to construct an equation different from (1.1) that has only a trivial solution. As is well known, the equation $\Delta u(x) + V(x)u(x) = \lambda u(x)$ has a nontrivial solution if and only if the parameter $\lambda$ belongs to a certain countable set. In other words, for a typical function $V$, the equation $\Delta u(x) + V(x)u(x) = 0$ on the torus has no nontrivial solutions. On the hand, by the paper \cite{5}, equation (1.1) on the torus has a positive solution for every (sufficiently smooth) vector field $f$, and the methods of \cite{6} make it possible to prove the nontrivial solvability of (1.1) with a complex vector field on the torus. Thus, equation (1.1) possesses properties quite different from those of other linear elliptic equations.
§5. Existence of a solution that is a probability density

**Theorem 5.1.** Suppose that a locally bounded measurable vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) has the property that the integral
\[
\int_0^{+\infty} r^{n-1} \exp \left( \int_0^r q(t) \, dt \right) \, dr, \quad \text{where} \quad q(t) = \lim_{\varepsilon \to 0} \sup_{t-\varepsilon < |x| < t+\varepsilon} \frac{(f(x), x)}{|x|},
\]
converges. Then there exists a positive continuous function \( u \in L^1(\mathbb{R}^n) \) such that
\[
\int_{\mathbb{R}^n} u(x) \left( \Delta \psi(x) + \sum_{i=1}^n f_i(x) \partial \psi(x) / \partial x_i \right) \, dx_1 \ldots dx_n = 0 \quad \text{for all} \quad \psi \in C_0^\infty(\mathbb{R}^n)
\]
and
\[
\int_{\mathbb{R}^n} u(x) \, dx_1 \ldots dx_n = 1.
\]
Moreover, an arbitrary positive and continuous weak solution \( u \) in the sense of (4.1) belongs to \( L^1(\mathbb{R}^n) \) and enjoys the inequality
\[
(5.1) \quad \int_{|x|=1} u(rx) \, dS_x \leq u(0) \left( \int_{|x|=1} dS_x \right) \exp \left( \int_0^r q(t) \, dt \right).
\]

**Proof.** For every positive continuous weak solution \( u \) of equation (1.1) (at least one exists by Theorem 4.1), we must prove that \( u \in L^1(\mathbb{R}^n) \).

As in the proof of Theorem 3 in [4], we introduce the spherical means
\[
v(t) = \int_{|x|=1} u(tx) \, dS_x.
\]
Let \( 0 < a < b \). Consider the difference analog of the logarithmic derivative of \( v(t) \) and apply the integral formula (2.2), which is valid in the case in question by Theorem 2.1:
\[
\left( \ln \left( \int_{|x|=1} u(bx) \, dS_x \right) - \ln \left( \int_{|x|=1} u(ax) \, dS_x \right) \right) / (b - a)
\]
\[
= \frac{1}{b - a} \ln \left( \int_{|x|=1} u(bx) \, dS_x / \int_{|x|=1} u(ax) \, dS_x \right)
\]
\[
= \frac{1}{b - a} \ln \left( 1 + \left( \int_{|x|=1} u(bx) \, dS_x - \int_{|x|=1} u(ax) \, dS_x \right) / \int_{|x|=1} u(ax) \, dS_x \right)
\]
\[
\leq \left( \int_{|x|=1} u(bx) \, dS_x - \int_{|x|=1} u(ax) \, dS_x \right) / (b - a) \int_{|x|=1} u(ax) \, dS_x
\]
\[
= \int_{a \leq |x| \leq b} |x|^{-n+1} u(x) (f(x), x/|x|) \, dx_1 \ldots dx_n / \left( b - a \right) \int_{|x|=1} u(ax) \, dS_x
\]
\[
\leq \sup_{a < |y| < b} (f(y), y/|y|) \int_{a \leq |x| \leq b} |x|^{-n+1} u(x) \, dx_1 \ldots dx_n / \left( b - a \right) \int_{|x|=1} u(ax) \, dS_x.
\]
(We have used the inequality \( \ln(1 + t) \leq t \) for \( t > -1 \).)

We estimate the ratio of the integrals from above. We have
\[
\int_{a \leq |x| \leq b} |x|^{-n+1} u(x) \, dx_1 \ldots dx_n = \int_a^b \int_{|x|=1} u(tx) \, dS_x \, dt = \int_{|x|=1} \int_a^b u(tx) \, dt \, dS_x.
\]
For every fixed \( x \) with \( |x| = 1 \), we have
\[
\int_a^b u(tx) \, dt \leq (b - a)(u(ax) + \varepsilon), \quad \varepsilon = \sup_{|y-z| \leq b-a, \ |y| \leq b, \ |z| \leq b} |u(y) - u(z)|,
\]
because the argument of \( u \) in the integrand runs through an interval of length \( b - a \). Therefore,

\[
\int_{a \leq |x| \leq b} |x|^{-n+1} u(x) \, dx_1 \ldots dx_n \leq (b - a) \left( \int_{|x| = 1} u(ax) \, dS_x + \varepsilon \int_{|x| = 1} dS_x \right).
\]

Thus,

\[
\int_{a \leq |x| \leq b} |x|^{-n+1} u(x) \, dx_1 \ldots dx_n / \left( (b - a) \int_{|x| = 1} u(ax) \, dS_x \right) \\
\leq 1 + \varepsilon \int_{|x| = 1} dS_x / \left( \int_{|x| = 1} u(ax) \, dS_x \right) \leq 1 + \varepsilon \inf_{|y| \leq b} u(y).
\]

Combining this with the preceding inequalities, we obtain

\[
\ln \left( \int_{|x| = 1} u(bx) \, dS_x \right) - \ln \left( \int_{|x| = 1} u(ax) \, dS_x \right) \leq (b - a) \left( 1 + \varepsilon / \inf_{|y| \leq b} u(y) \right) \sup_{a < |x| < b} \frac{(f(x), x)}{|x|}.
\]

Consider a partition of the interval \([a, b]\) into smaller intervals by points \( a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b \) and apply the result obtained to the elements of the partition:

\[
\ln \left( \int_{|x| = 1} u(bx) \, dS_x \right) - \ln \left( \int_{|x| = 1} u(ax) \, dS_x \right) = \sum_{i=1}^{k} \left( \ln \left( \int_{|x| = 1} u(t_i x) \, dS_x \right) - \ln \left( \int_{|x| = 1} u(t_{i-1} x) \, dS_x \right) \right)
\]

\[
\leq \left( 1 + \varepsilon / \inf_{|y| \leq b} u(y) \right) \sum_{i=1}^{k} (t_i - t_{i-1}) \sup_{t_{i-1} < |x| < t_i} \frac{(f(x), x)}{|x|},
\]

where \( \varepsilon = \sup_{|y-z| \leq \delta} |u(y) - u(z)|, \delta = t_i - t_{i-1}, i = 1, 2, \ldots, k. \)

We introduce the following sequence of partitions: to each natural number \( l \), we assign the partition of \([a, b]\) into \( 2^l \) parts of equal length. We pass to the limit as \( l \to \infty \) in the last inequality. The first factor tends to 1 because \( \varepsilon \to 0 \) by the uniform continuity of the function \( u(x) \) on the ball of radius \( b \). The sum is the integral of the piecewise constant function taking the value \( \sup_{t_{i-1} < |x| < t_i} \frac{(f(x), x)}{|x|} \) on the interval \((t_{i-1}, t_i)\), \( i = 1, 2, \ldots, 2^l \), and equal to zero at the ends of these intervals; such a function arises for each \( l \). As \( l \to \infty \), this sequence of functions tends monotonically to the function \( q(t) = \lim_{\varepsilon \to 0} \sup_{t-\varepsilon < |x| < t+\varepsilon} \frac{(f(x), x)}{|x|} \) on the interval \([a, b]\) everywhere except, probably, the countable set of points that determine the partitions. Therefore, the function \( q(t) \) is measurable. The vector field \( f \) being locally bounded, the Lebesgue dominated convergence theorem shows that, as \( l \to \infty \), the sums tend to \( \int_a^b q(t) \, dt \). Thus, for \( 0 < a < b \) we have proved the inequality

\[
\ln \left( \int_{|x| = 1} u(bx) \, dS_x \right) - \ln \left( \int_{|x| = 1} u(ax) \, dS_x \right) \leq \int_a^b q(t) \, dt.
\]

Letting \( a \to 0 \), we arrive at (5.1). So, the convergence of the integral assumed in the theorem implies the convergence of the integral

\[
\int_0^{+\infty} r^{n-1} \int_{|x| = 1} u(rx) \, dS_x \, dr = \int_{\mathbb{R}^n} u(x) \, dx_1 \ldots dx_n,
\]

i.e., the relation \( u \in L^1(\mathbb{R}^n) \). This proves the claim. \( \Box \)
Theorem 5.1 extends Theorem 3 in [4] to the case of discontinuous vector fields $f$. This extension leads to complications both in the statement and in the proof: in place of identity (2.3), its integral form (2.2) is employed, and the function $\sup_{|x|=1}(f(tx), x)$ in the claim of Theorem 3 in [4] is replaced by the function
\[ q(t) = \lim_{\varepsilon \to 0} \sup_{0 < |x| < t + \varepsilon} (f(x), x)/|x|. \]

However, for continuous bounded vector fields $f$, the functions indicated above (and their integrals) coincide. These functions may differ substantially only for vector fields $f$ that have “too many” discontinuity points.

A not quite immediate example of a discontinuous vector field $f$ satisfying the assumptions of Theorem 5.1 is given by the formula
\[ f(x) = -(1 + 2 \text{sgn}(|x|))x/|x| + g(x), \]
in which $g$ is a locally bounded measurable vector field such that for every $x \in \mathbb{R}^n$ we have $(g(x), x) \leq 0$. Clearly, it suffices to verify the assumptions of Theorem 5.1 for the zero vector field $g$, when the function $q(t)$ is piecewise-linear and can easily be integrated. It should be noted that the study of equation (1.1) with the vector field (5.2) by the Lyapunov–Has’minskii method (see [6] and [7]) encounters serious difficulties. These difficulties are related to the fact that, even for $g$ not very big in the absolute value, the radial component (5.2) takes some negative and some positive values in an arbitrary neighborhood of infinity. Smooth oscillating vector fields similar to (5.2) and a bounded measurable vector field $f$ have “too many” discontinuities.

§6. Averaging method

**Theorem 6.1.** Suppose that $\varepsilon > 0$, $\delta > 0$, $Q_\delta(x) = \{y : |y_i - x_i| \leq \delta/2, i = 1, 2, \ldots, n\}$, and a bounded measurable vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ has the property that, for $h(x) = \delta^{-n}\int_{Q_\delta(x)} f(y) dy_1 \ldots dy_n$, the following integral converges:
\[ \int_0^{+\infty} r^{n-1} \exp \left( \varepsilon r + \int_0^r \sup_{|y|=1} (h(ty), y) dt \right) dr. \]

Let also $\sup_{x \in \mathbb{R}^n} |f(x)| \leq M$, and let the numbers $M$, $n$, $\delta$, and $\varepsilon$ obey the inequality
\[ M \left( (1 + \sqrt{n}^{1/4}) \exp(M(\delta \sqrt{n} + 2\sqrt{n}^{1/4})) - 1 \right) \leq \varepsilon. \]

Then there exists a positive continuous function $u \in L^1(\mathbb{R}^n)$ such that it is a weak solution of equation (1.1) in the sense of the integral identity (4.1) and $\int_{\mathbb{R}^n} u(x) dx_1 \ldots dx_n = 1$.

**Remark.** The vector field $h$ is continuous: this is an immediate consequence of the absolute continuity of the integral. Therefore, the inner integral in the statement of the theorem is well defined and we could use the maximum in place of the supremum.

**Remark.** It is convenient to perceive Theorem 6.1 in the following way. The numbers $\delta$ and $\varepsilon$ should be thought of as small, and the convergence of the integral (6.1) should be treated as a condition “slightly stronger” than the assumption of Theorem 5.1. Moreover, inequality (6.2) should be interpreted as follows: the number $\delta$ is small, and the extent to which it must by small depends on $M$, $n$, and $\varepsilon$. Indeed, for fixed $M$ and $n$, the left-hand side of (6.2) tends to zero as $\delta \to 0$. In what follows, examples will be given in support of the fact that Theorem 6.1 is substantial.

**Proof of Theorem 6.1.** For a positive continuous weak solution $u(x)$ of the equation $\Delta u - \text{div}(af) = 0$ (such a solution exists by Theorem 4.1), we must prove the inclusion $u \in L^1(\mathbb{R}^n)$. 


Put

\[ V(x) = \int_{Q_\delta(x)} u(y) \, dy_1 \ldots dy_n, \quad Q_\delta(x) = \{ y : |y_i - x_i| \leq \delta/2, \quad i = 1, 2, \ldots, n \}. \]

We introduce the vector field

\[ F(x) = \frac{1}{V(x)} \int_{Q_\delta(x)} u(y) f(y) \, dy_1 \ldots dy_n. \]

By Lemma 2.1, the function \( V(x) \) is continuous and is a weak solution of the equation \( \Delta V - \text{div}(VF) = 0 \) in the sense of the integral identity in Lemma 2.1. To apply Lemma 2.1, we should make a linear change of variables in the integrals and then introduce a function \( \varphi(y) \) equal to one for \( y \in Q_\delta(0) \) and to zero otherwise.

Fix \( x \in \mathbb{R}^n \) and consider the difference

\[ F(x) - \delta^{-n} \int_{Q_\delta(x)} f(y) \, dy_1 \ldots dy_n \]

\[ = \frac{1}{V(x)} \left( \int_{Q_\delta(x)} u(y) f(y) \, dy_1 \ldots dy_n - \delta^{-n} \int_{Q_\delta(x)} f(y) \, dy_1 \ldots dy_n \right) \]

\[ = \frac{1}{V(x)} \int_{Q_\delta(x)} f(y)(u(y) - \delta^{-n}V(x)) \, dy_1 \ldots dy_n. \]

Combined with the inequality

\[ \delta^{-n} \int_{Q_\delta(x)} u(y) \, dy_1 \ldots dy_n \geq \inf_{y \in Q_\delta(x)} u(y), \]

this identity yields the estimate

\[ \left| F(x) - \delta^{-n} \int_{Q_\delta(x)} f(y) \, dy_1 \ldots dy_n \right| \leq \delta^n M \left( \sup_{y \in Q_\delta(x)} u(y) - \inf_{y \in Q_\delta(x)} u(y) \right) / V(x) \]

\[ \leq M \left( \sup_{y \in Q_\delta(x)} u(y) - \inf_{y \in Q_\delta(x)} u(y) \right) / \inf_{y \in Q_\delta(x)} u(y) = M |u(x_1) - u(x_2)| / u(x_1) \]

\[ \leq M \left( (1 + \sqrt{n})^n \exp(M(\delta\sqrt{n} + 2\sqrt{n} 1/4)) - 1 \right) \leq \varepsilon, \]

where \( x_1, x_2 \in Q_\delta(x) \) depend on \( x \) and are such that \( u(x_1) = \inf_{y \in Q_\delta(x)} u(y), \ u(x_2) = \sup_{y \in Q_\delta(x)} u(y) \). The last inequality is condition (6.2); the next to the last follows from (3.2) and Theorem 3.1 together with the condition \( |x_1 - x_2| \leq \delta \sqrt{n} \). So, \( |F(x) - h(x)| \leq \varepsilon \) for every \( x \in \mathbb{R}^n \); therefore

\[ (6.3) \quad \sup_{|y|=1} (F(ty), y) \leq \varepsilon + \sup_{|y|=1} (h(ty), y) \quad \text{for} \quad t \in \mathbb{R}. \]

Combined with the convergence of the integral in the assumption of the theorem, this implies the convergence of the integral

\[ \int_0^\infty r^{n-1} \exp \left( \int_0^r \sup_{|y|=1} (F(ty), y) \, dt \right) \, dr, \]

where the inner integral is well defined and

\[ \sup_{|y|=1} (F(ty), y) = \lim_{\varepsilon \to 0} \sup_{t-\varepsilon < |x| < t+\varepsilon} \frac{(F(x), x)}{|x|}, \]

because the vector field \( F \) is continuous. Therefore, the positive continuous function \( V(x) = \int_{Q_\delta(x)} u(y) \, dy_1 \ldots dy_n \), which is a weak solution of the equation \( \Delta V - \text{div}(VF) = 0 \) in the sense of the integral identity in Lemma 2.1, is a weak solution of the equation \( \Delta V - \text{div}(VF) = 0 \) in the sense of the integral identity in Lemma 2.1.
For this, it suffices to observe that
\[
\lim_{k \to \infty} \left( (1 + \sqrt{\delta_k} n^{1/4})^n \exp(M_k(\delta_k \sqrt{n} + 2\sqrt{\delta_k} n^{1/4})) - 1 \right) M_k = 0,
\]
where \(M_k = k + C\), \(\delta_k = 2\pi k^{-5}\). Thus, for \(\varepsilon = 1/4\), the quantities \(M_k\) and \(\delta_k\) indicated above, and the vector fields \(f_k(x)\) defined by (6.4), the assumptions of Theorem 6.1 are
fulfilled for all sufficiently large $k$. Therefore, for all sufficiently large $k$ the equation \( \Delta u - \text{div}(u_k) = 0 \) has a positive weak solution that is a probability density.

We observe that the vector field $b$ defined by (6.4) and constructed already in [4] satisfies the assumptions of Theorem 5.1, and if $g \in C^{\infty}(\mathbb{R}^n)$, then it also satisfies the assumptions of Theorem 3 in [4]. It is remarkable how the study complicates if we add the vector field $ak \prod_{i=1}^n \text{sgn}(k^5 x_i)$ to $b$. Though Theorem 5.1 and Theorem 3 in [4] are yet capable to “isolate” a sinusoid (of arbitrary amplitude) only depending on $|x|$, they turn out to be inapplicable to vector fields $f_k(x)$ because of their irregularity: for $f_k(x)$, the function $\sup_{x \in \mathbb{R}^n} (f'(x), x)$ is positive. Probably, the case of such (and more complicated) vector fields might be covered by a certain version of Theorem 5.1 in which the supremum would be taken over surfaces adjusted in a way to the vector field, rather than over spheres. The Lyapunov–Has’minskii method (see [6] and [7]) is more flexible than Theorem 5.1 in this respect: it applies not merely to spherical Lyapunov functions. For the time being, such refinements of Theorem 5.1 and Theorem 3 in [4] are a subject of research and development, but the averaging method based on Theorem 6.1 gives a positive result when applied to (6.4). Yet another example related to averaging is provided by the following theorem.

**Theorem 6.2.** Suppose that $n \geq 2$, a vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is measurable and $\sup_{x \in \mathbb{R}^n} |g(x)| \leq 1$. Then there exists a number $\alpha_0 > 0$ depending on $n$ such that, for every positive $\alpha < \alpha_0$, for the vector field \begin{equation}
(6.5) \quad f(x) = \begin{cases} -x/|x| & \text{if } |x| \geq \alpha/2, \\ g(x) & \text{if } |x| < \alpha/2, \end{cases}
\end{equation}
there exists a continuous positive weak solution $u \in L^1(\mathbb{R}^n)$ of equation (1.1) in the sense of the integral identity (4.1), and moreover, $\int_{\mathbb{R}^n} u(x) \, dx_1 \ldots dx_n = 1$.

**Proof.** We apply Theorem 6.1. Put $M = 1$, $\varepsilon = 1/2$, and choose $\delta > 0$ so as to ensure (6.2). This is possible because, for $M$ and $n$ fixed, the right-hand side of (6.2) tends to zero as $\delta \to 0$.

We start the construction of $\alpha \leq \delta$ such that the vector field $f$ and the corresponding vector field $h$ make the integral (6.1) convergent. If $\alpha \leq \delta$, we have
\[
\mathbf{h}(x) = \delta^{-n} \int_{Q_\delta(x)} f(y) \, dy_1 \ldots dy_n = \delta^{-n} \int_{\Pi_1(x)} f(y) \, dy_1 \ldots dy_n + \delta^{-n} \int_{\Pi_2(x)} f(y) \, dy_1 \ldots dy_n,
\]
where $\Pi_2(x) = \{ y : |y_1| < \alpha/2 \} \cap Q_\delta(x)$, $\Pi_1(x) = Q_\delta(x) \setminus \Pi_2(x)$.

The Lebesgue measure $\mu(\Pi_2(x))$ of the set $\Pi_2(x)$ does not exceed $\alpha \delta^{n-1}$; consequently, $|\delta^{-n} \int_{\Pi_2(x)} f(y) \, dy_1 \ldots dy_n| \leq \alpha/\delta$. Now we compare $h(x)$ and the “focusing” vector field $-\delta^{-n} \mu(\Pi_1(x)) x/|x|$. We have
\[
\left| \int_{\Pi_1(x)} f(y) \, dy_1 \ldots dy_n + \mu(\Pi_1(x)) \frac{x}{|x|} \right| \leq \mu(\Pi_1(x)) \sup_{y \in \Pi_1(x)} \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leq \mu(\Pi_1(x)) \sup_{y \in Q_\delta(x)} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|,
\]
which implies the required estimate
\[
\left| h(x) + \delta^{-n} \mu(\Pi_1(x)) \frac{x}{|x|} \right| \leq \frac{\alpha}{\delta} + \delta^{-n} \mu(\Pi_1(x)) \sup_{y \in Q_\delta(x)} \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leq \frac{\alpha}{\delta} + \sup_{y \in Q_\delta(x)} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|.
\]

Put $\alpha \leq \delta/8$, then $\delta^{-n} \mu(\Pi_1(x)) \geq 7/8$; therefore, for every $x \in \mathbb{R}^n$ we have
\[
\left( h(x), \frac{x}{|x|} \right) \leq \left( -\delta^{-n} \mu(\Pi_1(x)) \frac{x}{|x|}, \frac{x}{|x|} \right) + \frac{\alpha}{\delta} + \sup_{y \in Q_\delta(x)} \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leq \frac{7}{8} + \frac{1}{8} \sup_{y \in Q_\delta(x)} \left| \frac{x}{|x|} - \frac{y}{|y|} \right|.
\]

Since the first derivatives of the vector field $x/|x|$ are infinitesimal for $|x|\to\infty$, there exists a positive number $r_0$ such that the supremum is smaller than $1/8$ for $|x| \geq r_0$

(6.6) \[(h(x), x/|x|) \leq -5/8,
\]
implies the convergence of the integral (6.1) for $\varepsilon = 1/2$. Next, we apply Theorem 6.1. The proof is complete.  

The above arguments show that Theorem 6.2 remains true if in its statement we replace the vector field $x/|x|$ by $g(x)$ on a countable set of narrow strips of the form \[\{x : |x_1 - \beta_k| < \alpha\}\] that lie sufficiently far from one another (so that the cube $Q_\delta(x)$ in the proof intersect at most one strip).

We present an immediate consequence of Theorem 6.2.

**Theorem 6.3.** Let $n \geq 2$. Then there exists a number $\alpha_0 > 0$ depending only on $n$ such that for every positive $\alpha < \alpha_0$ and the vector field

(6.7) \[f(x) = \begin{cases}
-x/|x| & \text{if } |x_1| \geq \alpha, \\
(0;1;0;\ldots;0) & \text{if } |x_1| < \alpha,
\end{cases}
\]
there exists a continuous positive weak solution $u \in L^1(\mathbb{R}^n)$ of equation (1.1) in the sense of the integral identity (4.1); moreover, $\int_{\mathbb{R}^n} u(x) \, dx_1 \ldots dx_n = 1$.

Under the assumptions of Theorem 6.3, the trajectories of the system $dx/dt = f(x)$ in the domain $\{y : |y_1| < \alpha\}$ tend to infinity. The presence of such trajectories hampers stabilization and strongly impedes application of the Lyapunov method to diffusion processes, as was elaborated by R. Z. Khas’minski˘ı. The problem is resolved radically by Theorem 6.3. We can suggest the following probabilistic interpretation of the result obtained.

A nonstable system $dx/dt = f(x)$ is stabilized by an appropriate noise added to its right-hand part. Perturbed by the noise, this system possesses an invariant probability measure whose density obeys (1.1) and, as will be shown below, decays exponentially at infinity.

It should be noted that not merely the discontinuous vector field (6.7), but also its smoother analogs possess the above properties. Theorem 6.2 implies that Theorem 6.3 remains true if we change arbitrarily the vector field (6.7) on the set $\{y : \alpha/2 < |y_1| < \alpha\}$, retaining the condition $\sup_{x \in \mathbb{R}^n} |f(x)| \leq 1$. Thus, by smoothing, we can construct a smooth vector field (6.5) all trajectories of which tend to infinity. By 6.2, stabilization by a noise will also occur in this case.

Finishing the section, we indicate the paper \[\text{[11]}\], in which uniqueness theorems for equation (1.1) were obtained. The authors of \[\text{[11]}\] cleared whether the solvability of equation (1.1) implies the existence of an invariant probability measure for the corresponding diffusion.

§7. UPPER ESTIMATE FOR SOLUTION. A REMARK ON POSITIVE HARMONIC FUNCTIONS

**Theorem 7.1.** For the solutions obtained in Theorems 6.2 and 6.3, we have

(7.1) \[u(x) < C \exp(-|x|/10)
\]
with a constant $C$ independent of $x$. 

Proof. We continue the arguments started in the proofs of Theorems 6.1 and 6.2. Recall that $Q_{\delta}(x) = \{y : |y_i - x_i| \leq \delta/2, i = 1, 2, \ldots, n\}$,

$$\begin{align*}
V(x) &= \int_{Q_{\delta}(x)} u(y) \, dy_1 \ldots dy_n, \quad h(x) = \delta^{-n} \int_{Q_{\delta}(x)} f(y) \, dy_1 \ldots dy_n, \\
F(x) &= \frac{1}{V(x)} \int_{Q_{\delta}(x)} u(y)f(y) \, dy_1 \ldots dy_n,
\end{align*}$$

$M = 1$, $\varepsilon = 1/2$, and $\delta$ satisfies (6.2). We observe that $\sup_{x \in \mathbb{R}^n} |F(x)| \leq 1$ because $\sup_{x \in \mathbb{R}^n} |f(x)| \leq 1$ and inequalities similar to (2.4) hold true. From the proof of Theorem 6.1, we deduce (6.3) and also the formula $\Delta V - \text{div}(V F) = 0$, which is understood in the sense of the integral identity in Lemma 2.1. Inequality (6.3), namely,

$$\sup_{|y|=1} (F(ty), y) \leq 1/2 + \sup_{|y|=1} (h(ty), y), \quad t \in \mathbb{R},$$

combined with (6.6) yields $(F(x), x/|x|) \leq -1/8$ for $|x| \geq r_0$. In combination with (5.1) and Theorem 5.1 applied to the equation $\Delta V - \text{div}(V F) = 0$, the last inequality leads to the estimate

$$\int_{|x|=1} V(rx) \, dS_x \leq C_1 \exp(-r/8),$$

valid for all $r > 0$ with some constant $C_1 > 0$ independent of $r$.

We obtain a pointwise estimate for $V$. For this, we fix an arbitrary $y$ with $|y| > 1/2$ and consider the spherical shell $\{x : |y| - 1/2 \leq |x| \leq |y| + 1/2\}$, which contains the ball of radius 1/2 centered at $y$. Then (7.2), the relation $\sup_{x \in \mathbb{R}^n} |F(x)| \leq 1$, and inequality (2.1) in Theorem 2.1 applied to the equation $\Delta V - \text{div}(V F) = 0$ show that

$$\begin{align*}
V(y) \exp(-1/2) \int_{|x| \leq 1/2} dx_1 \ldots dx_n &\geq \int_{|x-y| \leq 1/2} V(x) \, dx_1 \ldots dx_n \\
&\geq \int_{|y|-1/2 \leq |x| \leq |y|+1/2} V(x) \, dx_1 \ldots dx_n = \int_{|y|-1/2} |y|+1/2 n^{n-1} \int_{|x|=1} V(rx) \, dS_x \, dr \\
&\leq C_1 \int_{|y|-1/2} |y|+1/2 n^{n-1} \exp(-r/8) \, dr \leq C_1 (|y| + 1/2)^{n-1} \exp((1/2 - |y|)/8).
\end{align*}$$

This readily implies the inequality

$$V(y) \leq C_2 \exp(-|y|/10)$$

for all $y$, with some constant $C_2 > 0$ independent of $y$. On the other hand, by inequality (2.1) in Theorem 2.1 applied to the equation $\Delta u - \text{div}(uF) = 0$, we have

$$V(y) = \int_{Q_{\delta}(x)} u(x) \, dx_1 \ldots dx_n \geq \int_{|x-y| \leq \delta/2} u(x) \, dx_1 \ldots dx_n \geq u(y) e^{-\delta/2} \int_{|x-y| \leq \delta/2} dx_1 \ldots dx_n.$$

Combined with (7.3), these estimates yield (7.1).

It should be noted that the lower estimate follows from (3.1) if we put $x_1 = 0$, $x_2 = x$, $\delta = |x|$, and $M = 1$.

**Theorem 7.2.** If $n \geq 2$, then there exists a positive harmonic function $u(x_1, x_2, \ldots, x_n)$ defined on the set

$$\{x : |x_1| < \alpha\}$$

with some $\alpha > 0$ and satisfying the inequality

$$u(x) \leq C \exp(-|x|),$$

where $C$ does not depend on $x$. 

Proof. Put $g(x) \equiv 0$ in Theorem 6.2 and apply Theorem 7.1. To deduce (7.4) from (7.1), we make a change of variables.

Note that, in the case of dimension 2, the function $u$ can be obtained explicitly by the methods of complex analysis. For example, the function

$$u(x, y) = \Re \exp(-\sqrt{z^2 + 1}), \quad z = x + iy, \quad u(x, 0) = \exp(-\sqrt{x^2 + 1}),$$

when considered in the strip $\{(x, y) : |y| < 1\}$, is harmonic and satisfies (7.4); it is positive in the strip $\{(x, y) : |y| < \alpha\}$ for all sufficiently small $\alpha > 0$. But in higher dimensions, it is hardly possible to write out such a function explicitly; however, it exists by Theorem 7.2.

In conclusion, it should be noted that we are aware of no positive harmonic function $u$ in the strip $\{x : |x_1| < \alpha\}$ such that $u(x) \leq C \exp(-|x|^\beta)$ for some $\beta > 1$. The following example supports the conjecture about the absence of such a function. The function

$$u(x, y) = \Re \exp(- (x + iy)^2) = \exp(y^2 - x^2) \cos(2xy)$$

is harmonic on $\mathbb{R}^2$ and is of variable sign on the strip $\{(x, y) : |y| < \alpha\}$ for any $\alpha > 0$; at the same time, $u(x, 0) = \exp(-x^2)$.

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