Алгебра и анализ
Том 24 (2012), № 5

St. Petersburg Math. J.
Vol. 24 (2013), No. 5, Pages 759–781
S 1061-0022(2013)01268-4
Article electronically published on July 24, 2013

NEWTONIAN AND AFFINE CLASSIFICATIONS OF IRREDUCIBLE CUBICS

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Abstract. The equivalence classes of Newton's classification of irreducible cubics are described in terms of families of standard equations. An affine classification of irreducible cubics is given.

§0. Introduction

In the unpublished draft *Geometriæ libri duo* [1], Isaac Newton gave a complete classification of irreducible cubics, consisting of 59 equivalence classes. He began with five standard cubics (we give them modern names): $y^2 = x^3$, a cubic with cusp; $y^2 = x^2(x+1)$, a cubic with node; $y^2 = x^2(x-1)$, a cubic with an isolated point; $y^2 = x[(x-a)^2+1]$ with $a \in \mathbb{R}$, a simple cubic; and $y^2 = x(x+1)(x+a)$ with $a > 1$, a cubic with oval. Newton constructed sections of five cubic cones whose directors are these standard cubics in the following way. For each of them, he considered the unions of the cubic and certain straight lines which Newton called *the horizontals*. Next, for each horizontal he considered a cone over the standard cubic and introduced an auxiliary plane passing through the horizontal and the apex of the cone. Finally, he considered a plane that is parallel to the auxiliary plane and cut the cone along the desired cubic section. Under projection from the apex, the horizontal is mapped to the line at infinity in the plane of the cubic section. This means that the choice of a horizontal is equivalent to the choice of the line at infinity.

Felix Klein made the following remark about Newton: "...he had a very clear conception of projective geometry; for he said that all curves of the third order can be derived by central projection from five fundamental types" [2].

The cubics with cusp and those with node have one real flex. The cubics with an isolated point, the simple cubics, and the cubics with oval have three real flexes lying on the odd branch. The flexes lie on a straight line. If a line intersects the odd branch at three distinct regular points (and therefore divides it into three arcs), then one of the arcs has no flexes, the other arc has one flex, and the third arc has two flexes. Newton chose the horizontals so that the locations of the flexes in the unions of the standard cubic and the horizontals be different.

For the five standard cubics enumerated above, Newton got respectively 9, 14, 12, 9, and 15 equivalence classes of cubic sections, which are called the Newtonian classes in accordance with the following definition.

Let $\mathbb{R}^2 = \mathbb{R}P^2 \setminus L$ be an affine chart of the projective plane, let L be the line at infinity, and let $C_0, C_1 \subset \mathbb{R}^2$ be affine cubics.

Affine cubics C_0 and C_1 realize the same *Newtonian class* if there exists an isotopy $F_t : (\mathbb{R}P^2, C_0 \cup L, L) \to (\mathbb{R}P^2, C_1 \cup L, L)$ such that

2010 *Mathematics Subject Classification.* Primary 14P25, 14R05, 14H50. Secondary 51N10, 01A45, 32Q57.

Key words and phrases. Isotopy and affine classification of cubics.

1) for every $t \in [0,1]$, the image of the restriction $F_t\big|_{C_0 \cup L}(C_0 \cup L)$ is the union of an irreducible cubic C_t and the line L;

2) if $x \in C_0$ is either a regular point of the cubic C_0, or its flex, or its singular point, then every point of the path $F_t\big|_x(x)$ is a point of the same sort of the cubic C_t.

In this paper, the Newtonian classes are presented as parametric families of standard equations. The number of parameters in the equation is called the *modality* of the family. Newton's enumeration is preserved: in item $i.j$, the jth Newtonian class of the ith standard cubic is considered with respect to the list above. Relative positions of the horizontals and standard cubics described in [1] are depicted in Figure 0.

Theorem. *There exist only* 59 *Newtonian classes of affine irreducible cubics*; *these classes are presented on the list of parametric families of standard equations. There only exist the following affine classes of affine irreducible cubics*: 15 *classes of modality* 0, 23 *families* (*of classes*) *of modality* 1, 16 *families of modality* 2, 5 *families of modality* 3, *and these families are presented on the list of standard equations below.*

In each item of the list, the dimension dim of the respective stratum of the 9-dimensional space of affine cubics is given. Representatives of the classes are depicted in Figures 1–5, where the real asymptotes of cubics different from the coordinate axes are shown as dashed lines.

1.1. $y^2 = x^3$, $\dim = 5$.
1.2. $y = x^3$, $\dim = 5$.
1.3. $x^2 y = 1$, $\dim = 6$.
1.4. $x^2 y = 1 - x$, $\dim = 6$.
1.5. $(x-1)y^2 + x^3 = 0$, $\dim = 6$.
1.6. $(x+1)y^2 = x^3$, $\dim = 6$.
1.7. $(x-1)y^2 - ax^2 y + x^3 = 0$, where $0 < a < 2$; $\dim = 7$.
1.8. $y^2 - x^2 y - x^3 = 0$, $\dim = 6$.
1.9. $(x+1)y^2 - ax^2 y + x^3 = 0$, where $a > 0$; $\dim = 7$.
2.1. $y^2 = x^2(x+1)$, $\dim = 6$.
2.2. $\left(1 - \frac{x}{a}\right)y^2 = x^2(x+1)$, where $a > 0$; $\dim = 7$.
2.3. $(1-x)y^2 = x^2$, $\dim = 6$.
2.4. $\left(1 - \frac{x}{a}\right)y^2 = x^2(1-x)$, where $0 < a < 1$; $\dim = 6$.
2.5. $(x+1)(x-1)y + 1 = 0$, $\dim = 6$.
2.6. $\left(\frac{x}{a} + 1\right)y^2 = x^2(x+1)$, where $a > 1$; $\dim = 7$.
2.7. $(x-1)y^2 = bx(x+a)(y-x)$, where $a > 0$ and $0 \le b < 4$; $\dim = 8$.
2.8. $(x-1)y^2 = ax(y-x)$, where $a > 0$; $\dim = 7$.
2.9. $xy = (x-1)^3$, $\dim = 6$.
2.10. $(x-1)y^2 = ax(y-x)$, where $a < -4$; $\dim = 7$.
2.11. $(1-x)y^2 = bx(x-a)(y-x)$, where $a > 1$ and $b > 0$; $\dim = 8$.
2.12. $(x+1)(x-a)y + x = 0$, where $a > 0$; $\dim = 7$.
2.13. $(x+1)(x-a)y + x^2 = 0$, where $a > 0$ and $a \neq 1$; $\dim = 7$.
2.14. $\left(\frac{x}{a} + 1\right)y^2 - bx^2 y = x^2(x+1)$, where $a > 0$ and $b > 0$; $\dim = 8$.
3.1. $y^2 = x^2(x-1)$, $\dim = 6$.
3.2. $\left(1 - \frac{x}{a}\right)y^2 = x^2(x-1)$, where $a > 1$; $\dim = 7$.
3.3. $(x^2+1)y = x^2$, $\dim = 6$.
3.4. $\left(\frac{x}{a} + 1\right)y^2 = -x^2(x+1)$, where $0 < a < 1$; $\dim = 7$.
3.5. $(x+1)y^2 = -x^2$, $\dim = 6$.
3.6. $\left(\frac{x}{a} + 1\right)y^2 = x^2(x-1)$, where $0 < a < \frac{1}{3}$; $\dim = 7$.
3.7. $(3x+1)y^2 = x^2(x-1)$, $\dim = 6$.
3.8. $\left(\frac{x}{a} + 1\right)y^2 = x^2(x-1)$, where $a > \frac{1}{3}$; $\dim = 7$.

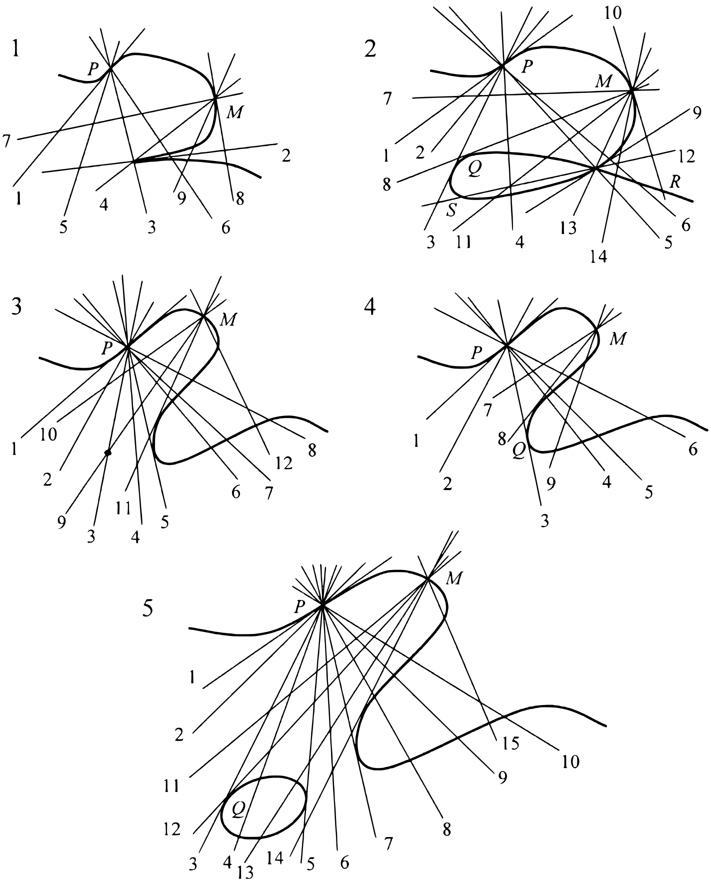

FIGURE 0. Standard cubics and horizontals.

3.9. $y(x^2 + ax + 1) = x$, where $0 \le a < 2$; $\dim = 7$.

3.10. $(1 - x)y^2 + bx(x - a)(y - x) = 0$, where $0 < a < 1$ and $0 < b < 4$; $\dim = 8$.

3.11. $(x + 1)y^2 + ax(y + x) = 0$, where $0 < a < 4$; $\dim = 7$.

3.12. $(x + 1)y^2 - bx(x - a)(y + x) = 0$, where $a > 0$, $b > 0$, and $a < x_2(a, b)$ (see the notation in item 3.12 of §3); $\dim = 8$.

4.1. $y^2 = x[(x - a)^2 + 1]$, where $a \in \mathbb{R}$; $\dim = 7$.

4.2. $xy^2 = -(x + a)[(x - b)^2 + 1]$, where $a > 0$ and $b \in \mathbb{R}$; $\dim = 8$.

4.3. $xy^2 = -[(x - a)^2 + 1]$, where $a \in \mathbb{R}$; $\dim = 7$.

4.4. $xy^2 = -(x - a)[(x - b)^2 + 1]$, where $a > 0$ and $b < \frac{a^2-4}{4a}$; $\dim = 8$.

4.5. $xy^2 = -(x - a)\big[\big(x - \frac{a^2-4}{4a}\big)^2 + 1\big]$, where $a > 0$; $\dim = 7$.

4.6. $xy^2 = -(x - a)[(x - b)^2 + 1]$, where $a > 0$ and $b > \frac{a^2-4}{4a}$; $\dim = 8$.

4.7. $xy^2 = c[(x - a)^2 + 1](y + x - b)$, where $a \in \mathbb{R}$, $b \ge 0$, and $-4 < c < 0$; $\dim = 9$.

4.8. $xy^2 = -4[(x - a)^2 + 1](y + x - b)$, where $a \in \mathbb{R}$, $b \ge 0$, and $b \ne 2a$; $\dim = 8$.

4.9. $xy^2 = c[(x - a)^2 + 1](y + x - b)$, where $a \in \mathbb{R}$, $b \ge 0$, $\sqrt{c(c + 4)(a^2 + 1)} < ac - 2b$, $0 < c < 2\big[a^2 - ab + 1 + \sqrt{(a^2 - ab + 1)^2 + b^2}\big]$, and $x_1^0(a, b, c) < x_2(a, b, c)$ (see the notation in item 4.9 of §4); $\dim = 9$.

5.1. $y^2 = x(x + 1)(x + a)$, where $a > 1$; $\dim = 7$.

5.2. $xy^2 = -(x + 1)(x + a)(x + b)$, where $1 < a < b$; $\dim = 8$.

5.3. $xy^2 = -(x + 1)(x + a)$, where $a > 1$; $\dim = 7$.

5.4. $xy^2 = -(x+a)(x+1)(x-b)$, where $a > 1$ and $b > 0$; dim = 8.
5.5. $xy^2 = (x+1)(x-a)$, where $a > 0$; dim = 7.
5.6. $xy^2 = -(x+1)(x-a)(x-b)$, where $0 < a < b$; dim = 8.
5.7. $xy^2 = -(x-1)(x-a)$, where $a > 1$; dim = 7.
5.8. $xy^2 = (x-a)(x-1)(x-b)$, where $0 < a < 1 < b$ and $b > (\sqrt{a}+1)^2$; dim = 8.
5.9. $xy^2 = (x-a)(x-1)[x-(\sqrt{a}+1)^2]$, where $0 < a < 1$; dim = 7.
5.10. $xy^2 = (x-a)(x-1)(x-b)$, where $0 < a < 1 < b$ and $b < (\sqrt{a}+1)^2$; dim = 8.
5.11. $xy^2 = c(x+1)(x+a)(x+y-b)$, where $a > 1$, $b > -1$, and $-4 < c < 0$; dim = 9.
5.12. $xy^2 = b(x+1)(x+y-a)$, where $a > -1$ and $b \in \mathbb{R}$; dim = 8.
5.13. $xy^2 = c(x+1)(x-a)(x+y-b)$, where $a > 0$, $-1 < b < a$, and $c > 0$; dim = 9.
5.14. $xy^2 = -4(x+1)(x+a)(x+y-b)$, where $a > 1$ and $b > -1$; dim = 8.
5.15. $xy^2 = c(x-a)(x-1)(x+y-b)$, where $0 < a < 1 < b$, $c > 0$, and $x_2(a,b,c) > x_1^0(a,b,c)$ (see item 5.15 in §5); dim = 9.

The proof is presented in §§1–5. For every item $i.j$ in those sections, Newton's choice of the horizontal is shown, and we choose an affine (x,y)-coordinate system so that, in the projective coordinates $(x:y:z)$, the line at infinity has equation $z=0$ and the cubic passes through the point $(0:1:0)$. Up to the end of the paper, we preserve the same notation for the coefficients of monomials used in the equation

$$Ax^3 + Bx^2 + Cx + D + Ey + Fxy + Gx^2y + Hy^2 + Ixy^2 = 0.$$

Once an affine (x,y)-coordinate system is selected, we use Newton's polygon to write the equation of the cubic. The monomials corresponding to the vertices of Newton's polygon must have nonzero coefficients, and one of them is presupposed to be equal to 1 or -1.

Since the cubic passes through the point $(0:1:0)$, the equation of the cubic with respect to y is either quadratic (if $H \neq 0$ or $I \neq 0$) or linear (if $H = I = 0$). If the equation is quadratic and its discriminant $\mathcal{D}_2(x)$ vanishes for $x = x_0$, then the line $x = x_0$ either intersects the cubic at the singular point, or has a point of contact with it. If the line at infinity either intersects the cubic at three distinct points, or is tangent to the cubic at a point other than $(0:1:0)$, or has two complex conjugate points of intersection with the cubic, then $G^2 - 4AI > 0$, or $G^2 - 4AI = 0$, or $G^2 - 4AI < 0$, respectively.

Finally, in each item $i.j$, to get a standard equation of the cubic, some suitable generic scaling $(x,y) \mapsto (k_1x,\, k_2y)$ is applied to the affine plane.

Many equations of the cubics occurring in our presentation are widely known. One can find them in the voluminous literature given in [3]. In the survey [4], a correspondence is presented between the 59 Newtonian classes and the 78 species of Newton's classification of the cubics, as given in *Enumeratio linearum tertij ordinis* [5].

§1. Cubics with cusp

A cubic with cusp is depicted in Figure 0.1, where P is the flex and M is an arbitrary regular point. Newton chose nine lines for the role of horizontals, they have numbers 1–9 in Figure 0.1:

line 1 is the flexional tangent to the cubic at P,
line 2 is the cuspidal tangent,
line 3 passes through the cusp and the flex P,
line 4 passes through the cusp and the point M,
line 5 passes through the flex P without contact and does not intersect the cubic at other real points,
line 6 passes through the flex P without contact and intersects the cubic at two other real points,

line 7 passes through the point M without contact and does not intersect the cubic at other real points,

line 8 is the tangent at the point M,

line 9 passes through the point M and intersects the cubic at two other real regular points.

These horizontals provide nine distinct Newtonian classes of cubics. It can be checked that these classes exhaust the Newtonian classes that result from cubics with cusp.

In items 1.1–1.9, the affine coordinate system is chosen in the following way:

in items 1.1, 1.5, 1.6, the x-axis is line 2 and the y-axis is line 3,

in item 1.2, the x-axis is line 1 and the y-axis is line 3,

in item 1.3, the x-axis is line 1 and the y-axis is line 2,

in item 1.4, the x-axis is line 8 and the y-axis is line 2, and

in items 1.7–1.9, the x-axis is line 2 and the y-axis is line 4.

The Newtonian and affine classes provided by the cubic with cusp are shown in Figure 1.

1.1. In the affine coordinate system chosen, the equation of the cubic is[1] $y^2+Ax^3 = 0$. The transformation $(x,y)\mapsto\big(-\sqrt[3]{A}x,\ y\big)$ turns it into the standard equation 1.1.

1.2. $y+Ax^3=0$. The transformation $(x,y)\mapsto\big(-\sqrt[3]{A}x,\ y\big)$ turns it into the standard equation 1.2.

1.3. $x^2y+D=0$. The transformation $(x,y)\mapsto\big(x,\ -\frac{1}{D}\,y\big)$ turns it into the standard equation 1.3.

1.4. $x^2y+Cx+D=0$. Transformation $(x,y)\mapsto\big(\frac{D}{C}\,x,\ \frac{C^2}{D}\,y\big)$ turns it into standard equation 1.4.

1.5. $y^2+Ax^3+Gx^2y+Ixy^2=0$, where $G^2-4AI<0$.

Lemma 1. *A cubic with such an equation has a flex at the point* $(0:1:0)$ *if and only if* $G=0$.

Proof. The equation of the cubic in the affine chart $\mathbb{R}P^2\setminus\{y=0\}$ is $z+Ax^3+Gx^2+Ix = 0$. The condition $z''(0)=0$ is equivalent to $G=0$. □

By Lemma 1, the equation under study becomes $y^2+Ax^3+Ixy^2=0$. The transformation $(x,y)\mapsto\big(-I\,x,\ -I\sqrt{\frac{I}{A}}\,y\big)$ turns it into the standard equation 1.5.

1.6. $y^2+Ax^3+Gx^2y+Ixy^2=0$, where $G^2-4AI>0$.

By Lemma 1, $G=0$ and the equation becomes $y^2+Ax^3+Ixy^2=0$. The transformation $(x,y)\mapsto\big(I\,x,\ I\sqrt{-\frac{I}{A}}\,y\big)$ turns it into the standard equation 1.6. The oblique asymptotes are $y=\pm(x-\frac{1}{2})$.

1.7. $y^2+Ax^3+Gx^2y+Ixy^2=0$, where $G^2-4AI<0$. By Lemma 1, we have $G\neq 0$. The transformation $(x,y)\mapsto\big(-I\,x,\ -I\sqrt{\frac{I}{A}}\,y\big)$ turns it into the standard equation 1.7, where $a=-\frac{G}{\sqrt{AI}}$. The conditions $G^2-4A<0$ and $G\neq 0$ turn into $a^2-4<0$ and $a\neq 0$, respectively. The symmetry $(x,y)\mapsto(x,-y)$ preserves the affine and Newtonian classes and changes the sign only of the monomial ax^2y, whence $0<a<2$.

1.8. $y^2+Ax^3+Gxy^2=0$. The transformation $(x,y)\mapsto\big(-\frac{G^2}{A}\,x,\ -\frac{G^3}{A^2}\,y\big)$ turns it into the standard equation 1.8. The oblique asymptote is $y=-x+1$.

1.9. $y^2+Ax^3+Gx^2y+Ixy^2=0$. The asymptote $x=-\frac{1}{I}$ and the cuspidal arc are located on opposite sides of the y-axis; thus, $AI<0$ and the condition $G^2-4AI>0$ is always fulfilled. By Lemma 1, we have $G\neq 0$. The transformation $(x,y)\mapsto\big(I\,x,\ I\sqrt{-\frac{I}{A}}\,y\big)$

[1] For brevity, in all items below the words *In the affine coordinate system chosen, the equation of the cubic is* are omitted.

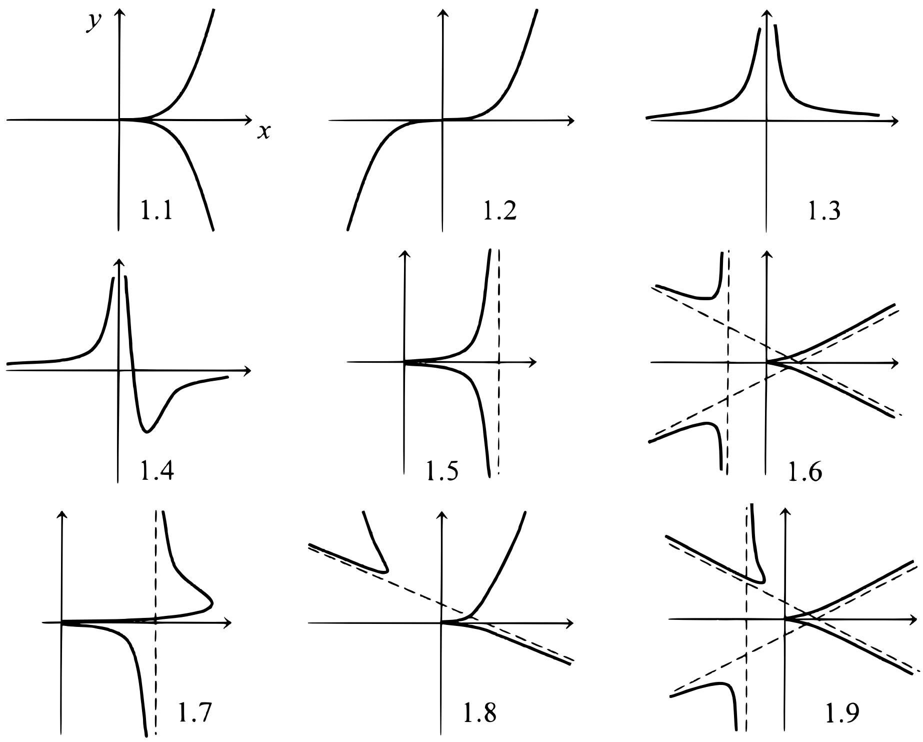

FIGURE 1. Classes generated by cubics with cusp.

yields the standard equation 1.9, where $a = -\frac{G}{\sqrt{-\frac{I}{A}}}$. The symmetry $(x, y) \mapsto (x, -y)$ preserves the affine and Newtonian classes and changes the sign only of the monomial ax^2y, whence $a > 0$. The equations of the oblique asymptotes are $y = \alpha_{1,2}x + \beta_{1,2}$, where

$$\alpha_{1,2} = \frac{a \pm \sqrt{a^2+4}}{2}$$

and

$$\beta_{1,2} = \mp\frac{a^2+2}{\sqrt{a^2+4}} - \frac{a}{2},$$

respectively.

§2. CUBICS WITH NODE

The real part of a projective cubic with node can be subdivided into three non-intersecting pieces: the node, the loop, and the odd component. It has one real flex on the odd component. A cubic with node is depicted in Figure 0.2, where the point P is the flex and M is an arbitrary regular point on the odd component. Newton chose the following fourteen positions of the horizontal (see Figure 0.2.):

line 1 is the flexional tangent to the cubic at P,

line 2 passes through the flex P without contact and does not intersect the cubic at other real points,

line 3 passes through the flex P and is tangent to the loop,

line 4 passes through the flex P and intersects the loop at two distinct points,

line 5 passes through the flex P and the node,

line 6 passes through the flex P and intersects the odd component at two other points, line 7 passes through the point M without contact and does not intersect the cubic at other real points,

line 8 passes through the point M and touches the loop at the point Q,

line 9 is the tangent to one of two branches passing through the node,

line 10 is the tangent to the odd component at the point M and intersects it at the point R,

line 11 passes through the point M and intersects the loop at two points,

line 12 passes through the node and intersects the loop at the point S,

line 13 passes through the point M and the node, and

line 14 passes through the point M and intersects the odd component at two regular points, say at M_1 and M_2.

These fourteen horizontals provide fourteen distinct Newtonian classes of cubics. It can be checked that these classes exhaust the Newtonian classes provided by cubics with nodes.

For the choice of affine coordinate systems we need the following lines:

line 15 passes through the node and the point of contact between the loop and line 3,

line 16 passes through the node and the point Q,

line 17 is the tangent to another branch passing through the node with respect to line 9,

line 18 passes through the node and the point R,

line 19 is the tangent to the loop at the point S,

line 20 passes through the node and the point of intersection of line 19 with the odd component.

In items 2.1–2.14, the affine coordinate system is chosen in the following way: in items 2.1–2.4, 2.6, 2.7 the x-axis is line 15 and the y-axis is line 5, in item 2.5 the x-axis is line 1 and the y-axis is line 15, in items 2.8, 2.11, 2.14 the x-axis is line 16 and the y-axis is line 13, in item 2.9 the x-axis is line 1 and the y-axis is line 17, in item 2.10 the x-axis is line 13 and the y-axis is line 18, in item 2.12 the x-axis is line 19 and the y-axis is line 20, in item 2.13 the x-axis is line 8 and the y-axis is line 16.

If the horizontal does not pass through the node (items 2.1–2.8, 2.10, 2.11, 2.13, and 2.14), the origin of the (x, y)-plane is chosen the at the node, so that $E = C = D = 0$ and $F^2 - 4B > 0$ (we choose $H = 1$).

The Newtonian and affine classes provided by the cubics with node are shown in Figure 2.

2.1. $y^2 + Ax^3 + Bx^2 + Fxy = 0$.

Lemma 2. *For a cubic with such an equation, the line $x = -\frac{B}{A}$ is its tangent at the point $\left(-\frac{B}{A}, 0\right)$ if and only if $F = 0$.*

Proof. The line $x = -\frac{B}{A}$ is the tangent if and only if the discriminant

$$\mathcal{D}_2(x) = F^2x^2 - 4x^2(Ax + B)$$

vanishes when $x = -\frac{B}{A}$, i.e., if and only if $F = 0$. □

By Lemma 2, the equation becomes $y^2 + Ax^3 + Bx^2 = 0$, where $B < 0$. The transformation $(x, y) \mapsto \left(\frac{A}{B}x, \frac{A}{B\sqrt{-B}}y\right)$ turns it into the standard equation 2.1.

2.2. The equation of the cubic is

$$y^2 + Ax^3 + Bx^2 + Fxy + Gx^2y + Ixy^2 = 0, \tag{1}$$

where $F^2 - 4B > 0$ and $G^2 - 4AI < 0$.

Lemma 3. *An irreducible cubic with equation* (1) *has a flex at the point* $(0 : 1 : 0)$ *if and only if $G = FI$.*

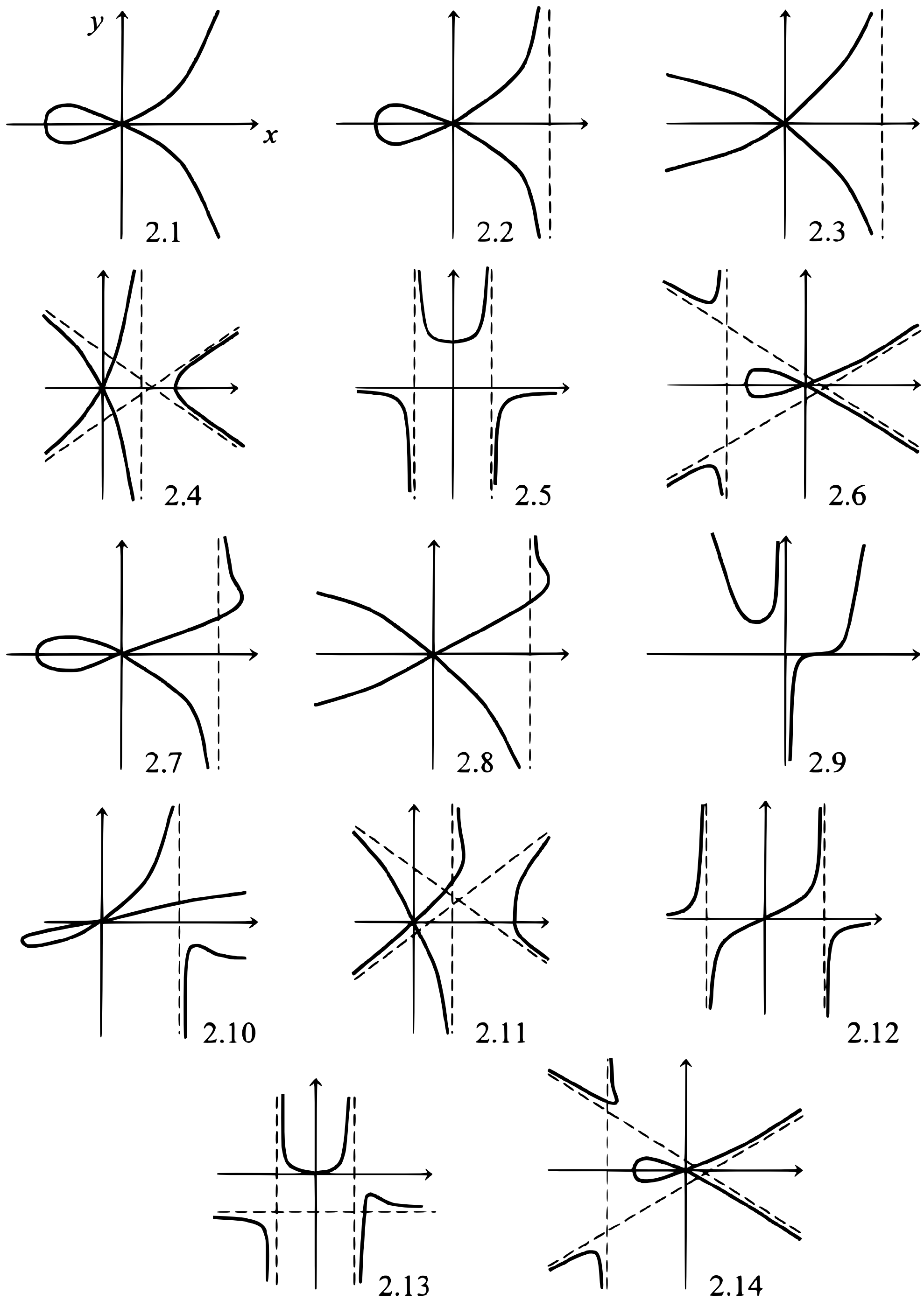

FIGURE 2. Classes generated by cubics with node.

Proof. In the affine chart $\mathbb{R}P^2 \setminus \{y=0\}$, the cubic has equation

$$z + Ax^3 + Bx^2z + Fxz + Gx^2 + Ix = 0.$$

Condition $z''(0) = 0$ is equivalent to $G = FI$. □

Lemma 4. *For an irreducible cubic with equation* (1) *where* $F \neq 0$, *the line* $x = -\frac{B}{A}$ *is the tangent to the cubic at the point* $\left(-\frac{B}{A}, 0\right)$ *if and only if* $AF = BG$.

Proof. The equation is quadratic in y, and its discriminant is

$$\mathcal{D}_2(x) = x^2[(Gx + F)^2 - 4(Ix + 1)(Ax + B)].$$

The identity $\mathcal{D}_2\left(-\frac{B}{A}\right) = 0$ is equivalent to $AF = BG$. □

Lemma 5. *If an irreducible cubic has equation* (1), $G = FI$, *and* $AF = BG$, *then* $F = G = 0$.

Proof. Suppose the contrary: the cubic is irreducible and $F \neq 0$. Then $A = BI$. Substitution of $G = FH$ and $A = BI$ into equation (1) turns it into the equation $(Ix + 1)(y^2 + Fxy + Bx^2) = 0$ of a reducible cubic. □

By Lemmas 3–5, $F = G = 0$ and equation (1) becomes $y^2 + Ax^3 + Bx^2 + Ixy^2 = 0$, where $B < 0$ and $AI > 0$.

The transformation $(x, y) \mapsto \left(\frac{A}{B}x, \frac{A}{B\sqrt{-B}}y\right)$ turns it into the standard equation 2.2, where $a = -\frac{A}{BI} > 0$.

2.3. $y^2 + Bx^2 + Fxy + Ixy^2 = 0$, where $B < 0$.

Lemma 6. *A cubic with such an equation has a flex at the point* $(0 : 1 : 0)$ *if and only if* $F = 0$.

The proof is similar to that of Lemma 1.

By Lemma 6, the equation becomes $y^2 + Bx^2 + Ixy^2 = 0$. The transformation $(x, y) \mapsto \left(-Ix, \frac{I}{\sqrt{-B}}\, y\right)$ turns it into the standard equation 2.3.

2.4. The cubic has equation (1), where $F^2 - 4B > 0$ and $G^2 - 4AI > 0$. The asymptote $x = -\frac{1}{I}$ passes between the origin and the tangent $x = -\frac{B}{A}$ to the cubic, whence $0 < \frac{A}{BI} < 1$.

By Lemmas 3–5, we have $F = G = 0$, and equation (1) becomes $y^2 + Ax^3 + Bx^2 + Ixy^2 = 0$, where $B < 0$ and $AI < 0$. The transformation $(x, y) \mapsto \left(-\frac{A}{B}\, x, -\frac{A}{B\sqrt{-B}}\, y\right)$ converts the equation under study into the standard equation 2.4, where $a = \frac{A}{BI}$ and $0 < a < 1$. The oblique asymptotes are $y = \pm\left(x - \frac{1-a}{2}\right)$.

2.5. $x^2y + Fxy + Ey + D = 0$. Since $y'(0) = DF = 0$, we have $F = 0$ and the equation becomes $x^2y + Ey + D = 0$. The asymptotes parallel to the y-axis are located on opposite sides of the y-axis, whence $E < 0$. The transformation $(x, y) \mapsto \left(\frac{1}{\sqrt{-E}}\, x, -\frac{E}{D}\, y\right)$ converts the equation of the cubic into the standard equation 2.5.

2.6. The cubic has equation (1), where $F^2 - 4B > 0$ and $G^2 - 4AI > 0$. The tangent $x = -\frac{B}{A}$ to the loop is located between the origin and the asymptote $x = -\frac{1}{I}$, whence $\frac{A}{BI} > 1$.

By Lemmas 3–5, we have $F = G = 0$, and equation (1) becomes

$$y^2 + Ax^3 + Bx^2 + Ixy^2 = 0,$$

where $B < 0$ and $AI < 0$. The transformation $(x, y) \mapsto \left(\frac{A}{B}\, x, \frac{A}{B\sqrt{-B}}\, y\right)$ converts it into the standard equation 2.6, where $a = \frac{A}{BI}$ and $a > 1$. The oblique asymptotes are $y = \pm\sqrt{a}\left(x - \frac{a-1}{2}\right)$.

2.7. The cubic has equation (1), where $F^2 - 4B > 0$ and $G^2 - 4AI < 0$. The origin is located between the tangent $x = -\frac{B}{A}$ to the loop and asymptote $x = -\frac{1}{I}$, whence $\frac{BI}{A} < 0$.

By Lemma 4, we have $AF = BG$, and equation (1) can be written in the form $(Ix + 1)y^2 + x(x + \frac{B}{A})(Gy + Ax) = 0$. The transformation $(x, y) \mapsto \left(-Ix, \frac{GI}{A}\, y\right)$ converts this equation into the standard equation 2.7, where $a = -\frac{BI}{A} > 0$ and $b = \frac{G^2}{AI}$. The condition $G^2 - 4AI < 0$ turns into $0 < b < 4$. The condition $F^2 - 4B > 0$ is fulfilled whenever $a > 0$ and $0 < b < 4$.

2.8. $y^2 + Bx^2 + Fxy + Ixy^2 = 0$, where $F^2 - 4B > 0$. The transformation $(x, y) \mapsto \big(-Ix, -\frac{FI}{B}y\big)$ converts it into the standard equation 2.8, where $a = -\frac{F^2}{B}$. The condition $F^2 - 4B > 0$ turns into $a^2 + 4a > 0$. For $a < -4$, the cubic belongs to the Newtonian class 2.10, whence $a > 0$.

2.9. $xy + A(x - x_0)^3 = 0$, where $A \neq 0$ and $x_0 \neq 0$. The transformation $(x, y) \mapsto \big(\frac{1}{x_0}x, -\frac{1}{Ax_0^2}y\big)$ converts this equation into the standard equation 2.9.

2.10. $y^2 + Bx^2 + Fxy + Ixy^2 = 0$, where $F^2 - 4B > 0$. The transformation $(x, y) \mapsto \big(-Ix, -\frac{FI}{B}y\big)$ converts this equation into the standard equation 2.8, where $a = -\frac{F^2}{B}$. The condition $F^2 - 4B > 0$ turns into $a^2 + 4a > 0$. Since for $a > 0$ this cubic belongs to the Newtonian class 2.8, we have $a < -4$.

2.11. The cubic has equation (1), where $F^2 - 4B > 0$ and $G^2 - 4AI > 0$. The asymptote $x = -\frac{1}{I}$ passes between the origin and the tangent $x = -\frac{B}{A}$ to the cubic, whence $\frac{BI}{A} > 1$.

By Lemma 4, we have $AF = BG$, and equation (1) can be written in the form $(Ix + 1)y^2 + x(x + \frac{B}{A})(Gy + Ax) = 0$. The transformation $(x, y) \mapsto \big(-Ix, \frac{GI}{A}y\big)$ converts it into the standard equation 2.11, where $a = \frac{BI}{A}$ and $b = -\frac{G^2}{AI}$. The condition $G^2 - 4AI > 0$ turns into $b^2 + 4b > 0$. For $b < -4$, this cubic belongs to the Newtonian class 2.11 but does not satisfy the choice of an affine coordinate system, therefore $b > 0$. The oblique asymptotes are

$$y = \frac{-b \pm \sqrt{b^2 + 4b}}{2}x + \frac{b(a-1)}{2}\Big(1 \mp \frac{b+2}{\sqrt{b^2+4b}}\Big).$$

2.12. $x^2y + Cx + Ey + Fxy = 0$. The asymptotes $x = \frac{1}{2}\big(-F \pm \sqrt{F^2 - 4E}\big)$ are located on opposite sides of the y-axis, so that $E < 0$. The equation of the cubic can be written in the form $y(x - x_1)(x - x_2) + Cx = 0$, where $x_1 < 0 < x_2$. The transformation $(x, y) \mapsto \big(-\frac{1}{x_1}x, -\frac{x_1}{C}y\big)$ converts it into the standard equation 2.12, where $a = -\frac{x_2}{x_1} > 0$.

2.13. $x^2y + Bx^2 + Ey + Fxy = 0$. The asymptotes $x = \frac{1}{2}\big(-F \pm \sqrt{F^2 - 4E}\big)$ are located on opposite sides of the y-axis, whence $E < 0$.

Lemma 7. *A cubic satisfying such an equation has a flex at the point* $(0 : 1 : 0)$ *if and only if* $F = 0$.

The proof is similar to that of Lemma 1.

By Lemma 7, we have $F \neq 0$, and the equation of the cubic can be written in the form $y(x-x_1)(x-x_2)+Bx^2 = 0$, where $x_2x_1 < 0$. The transformation $(x, y) \mapsto \big(-\frac{1}{x_1}x, \frac{1}{B}y\big)$ converts it into the standard equation 2.13, where $a = -\frac{x_2}{x_1} > 0$. The condition $F \neq 0$ turns into $a \neq 1$.

2.14. The cubic has equation (1), where $F^2 - 4B > 0$ and $G^2 - 4AI > 0$. The tangent $x = -\frac{B}{A}$ to the loop is located between the asymptote $x = -\frac{1}{I}$ and the origin, whence $0 < \frac{BI}{A} < 1$.

By Lemma 4, we have $AF = BG$, and equation (1) can be written in the form

$$(Ix + 1)y^2 + x\Big(x + \frac{B}{A}\Big)(Gy + Ax) = 0.$$

The transformation $(x, y) \mapsto \big(Ix, \frac{GI}{A}y\big)$ converts it into standard equation 2.14, where $a = \frac{BI}{A}$, $b = -\frac{G^2}{I}$, and $0 < a < 1$. The condition $G^2 - 4AI > 0$ turns into $b^2 + 4b > 0$.

If $b < -4$, the condition $F^2 - 4B > 0$ turns into $b < -\frac{4}{a}$. If $b > 0$, the condition $F^2 - 4B > 0$ turns into $b > 0$.

If $b < -\frac{4}{a}$, then the cubic belongs to the Newtonian class 2.14 but it does not fit the choice of an affine coordinate system relative to the position of the flex points on arcs of the odd branch. The oblique asymptotes are

$$y = \frac{b \pm \sqrt{b^2 + 4b}}{2}\, x + \frac{b(a-1)}{2}\Big(1 \pm \frac{b+2}{\sqrt{b^2+4b}}\Big).$$

§3. Cubics with an isolated point

The real part of a projective cubic with an isolated point consists of the isolated point and the odd branch with three flexes. It is depicted in Figure 0.3, where the point P is one of the flexes and M is an arbitrary regular point.

Newton chose twelve lines for the role of horizontals, they have the numbers 1–12 in Figure 0.3. There are four special lines in the pencil of real lines passing through the flex P:

line 1 is the flexional tangent to the cubic at P,

line 3 passes through the isolated point of the cubic,

line 5 is the tangent to the odd branch at a point different from P,

line 7 passes through three real flexes.

These lines divide this pencil into four open nonintersecting segments $(1, 3)$, $(3, 5)$, $(5, 7)$, and $(7, 1)$. The horizontals having numbers 2, 4, 6, and 8 are arbitrary lines belonging to these segments, respectively.

Newton chose four more lines for the role of horizontals. These lines pass through the point M:

line 9 passes through the isolated point,

line 10 does not intersect the cubic at other real points,

line 11 is the tangent to the odd branch at a point different from M,

line 12 passes through M and intersects the cubic at two other real regular points.

These horizontals provide twelve distinct Newtonian classes of cubics. It can be checked that these classes exhaust the Newtonian classes provided by cubics with an isolated point.

For the choice of affine coordinate systems we need the following lines:

line 13 passes through the isolated point and the point of contact between line 5 and the odd branch,

line 14 is the tangent at the point M,

line 15 passes through the isolated point and the point of transversal intersection of line 14 and the odd branch,

line 16 passes through the isolated point and the point of contact between the odd branch and line 11.

In items 3.1–3.12 the affine coordinate system is chosen in the following way: in items 3.1, 3.2, 3.4–3.8 the x-axis is line 13 and the y-axis is line 3, in item 3.3 the x-axis is line 5 and the y-axis is line 13, in item 3.9 the x-axis is line 14 and the y-axis is line 15, in items 3.10–3.12 the x-axis is line 16 and the y-axis is line 9.

If the horizontal passes through the isolated point of the cubic (items 3.3 and 3.9), then the coordinate system is chosen so that the isolated point coincides with the point $(0 : 1 : 0)$. If the horizontal does not pass through the isolated point (the remaining items), the origin of the (x, y)-plane is chosen at the isolated point and then $F^2 - 4BH < 0$. The Newtonian and affine classes provided by cubics with an isolated point are shown in Figure 3.

3.1. $y^2 + Ax^3 + Bx^2 + Fxy = 0$, where $F^2 - 4B < 0$.

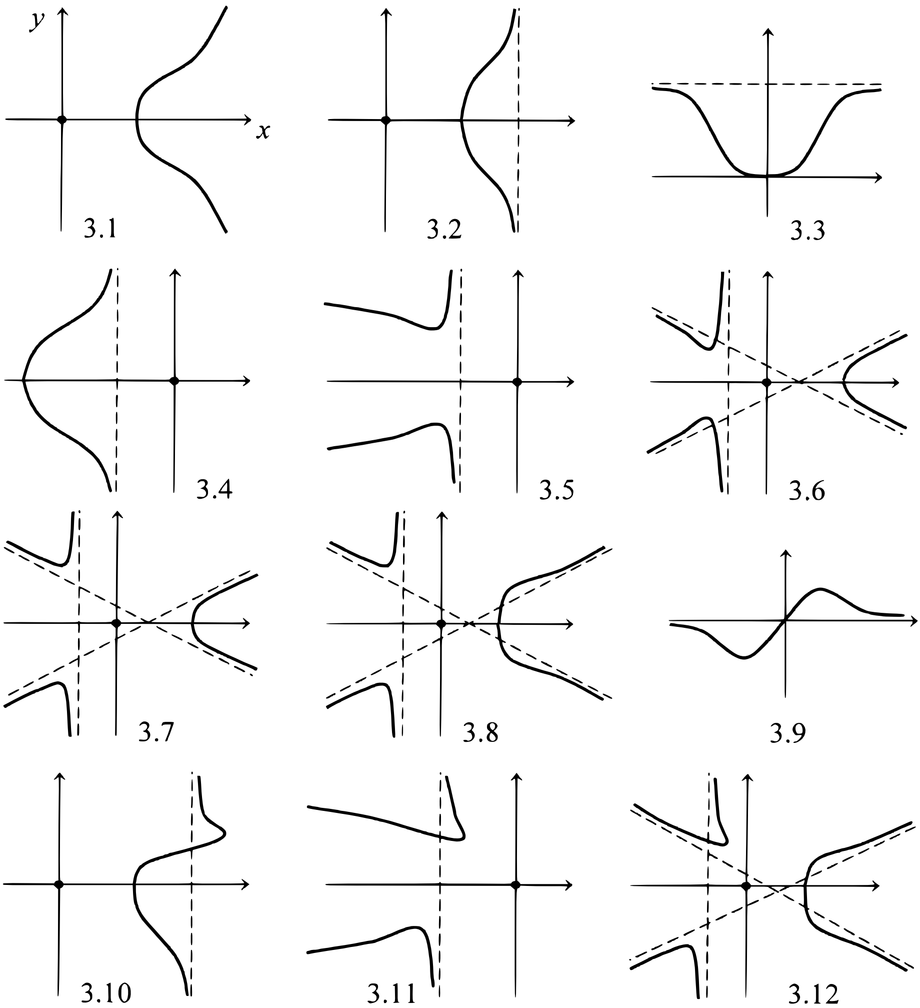

FIGURE 3. Classes generated by cubics with an isolated point.

By Lemma 2, we have $F = 0$, and the equation becomes $y^2 + Ax^3 + Bx^2 = 0$, where $B > 0$. The transformation $(x, y) \mapsto \left(-\frac{A}{B}x,\ \frac{A}{B\sqrt{B}}y\right)$ converts it into the standard equation 3.1.

3.2. The cubic has equation (1), where $F^2 - 4B < 0$ and $G^2 - 4AI < 0$. By Lemmas 3–5, we have $F = G = 0$, and equation (1) becomes $y^2 + Ax^3 + Bx^2 + Ixy^2 = 0$, where $B > 0$ and $AI > 0$. The tangent $x = -\frac{B}{A}$ is located between the origin and the asymptote $x = -\frac{1}{I}$, whence $\frac{A}{BI} > 1$. The transformation $(x, y) \mapsto \left(-\frac{A}{B}x,\ -\frac{A}{B\sqrt{B}}y\right)$ converts the equation into the standard equation 3.2, where $a = \frac{A}{BI} > 1$.

3.3. $x^2 + Ey + Fxy + Gx^2y = 0$, where $F^2 - 4EG < 0$.

Lemma 8. *A cubic with such an equation has a flex at the point* $(1 : 0 : 0)$ *if and only if* $F = 0$.

The proof is similar to that of Lemma 1.

By Lemma 8, the equation becomes $x^2 + Ey + Gx^2y = 0$, where $EG > 0$. The transformation $(x, y) \mapsto \left(\sqrt{\frac{G}{E}}x,\ -G\,y\right)$ converts it into the standard equation 3.3.

3.4. The cubic has equation (1), where $F^2-4B<0$ and $G^2-4AI<0$. The asymptote $x=-\frac{1}{I}$ is located between the origin and the tangent $x=-\frac{B}{A}$, whence $0<\frac{A}{BI}<1$.

By Lemmas 3–5, we have $F=G=0$, and equation (1) becomes

$$y^2+Bx^2+Ax^3+Ixy^2=0,$$

where $B>0$ and $AI>0$. The transformation $(x,y)\mapsto\left(\frac{A}{B}\,x,\ \frac{A}{B\sqrt{B}}\,y\right)$ converts it into the standard equation 3.4, where $a=\frac{A}{BI}$.

3.5. $y^2+Bx^2+Fxy+Ixy^2=0$, where $F^2-4B<0$. By Lemma 6, we have $F=0$, and the equation becomes $y^2+Bx^2+Ixy^2=0$, where $B>0$. The transformation $(x,y)\mapsto\left(Ix,\ \frac{I}{\sqrt{B}}\,y\right)$ converts it into the standard equation 3.5.

3.6. The cubic has equation (1), where $F^2-4B<0$ and $G^2-4AI>0$. The horizontal intersects the odd branch at the flex P and two other real regular points, say, at points M_1 and M_2. The points P, M_1, and M_2 divide the odd branch into three arcs so that the arcs PM_1 and PM_2 have one flex each and M_1M_2 has no flexes.

The origin is located between the asymptote $x=-\frac{1}{I}$ and the tangent $x=-\frac{B}{A}$, whence $\frac{A}{BI}<0$.

By Lemmas 3–5, we have $F=G=0$, and equation (1) becomes

$$y^2+Ax^3+Bx^2+Ixy^2=0,$$

where $B>0$ and $AI<0$. The transformation $(x,y)\mapsto\left(-\frac{A}{B}\,x,\ -\frac{A}{B\sqrt{B}}\,y\right)$ converts it into the standard equation 3.6, where $a=-\frac{A}{BI}>0$. The oblique asymptotes are $y=\pm\sqrt{a}\left(x-\frac{a+1}{2}\right)$. If $a\neq\frac{1}{3}$, the two points of intersection of the asymptotes and the cubic have abscissa $x=\frac{a(a+1)}{3a-1}$. The flexes are located on the arcs PM_1 and PM_2 if and only if the oblique asymptotes intersect the arcs PM_1 and PM_2, which happens if and only if $0<a<\frac{1}{3}$.

3.7. The cubic has equation (1), where $F^2-4B<0$ and $G^2-4AI>0$. The situation is similar to that in item 3.6 and provides the standard equation 3.7 with $a=\frac{1}{3}$.

3.8. The cubic has equation (1), where $F^2-4B<0$ and $G^2-4AI>0$. The horizontal passes through the flex P, and intersects the odd branch at two distinct points, say at M_1 and M_2, so that the two other flexes lie on the arc M_1M_2. The further analysis in this case is similar to that in item 3.6 and provides the standard equation 3.8 with $a>\frac{1}{3}$.

3.9. $x+Ey+Fxy+Gxy^2=0$, where $F^2-4EG<0$. The transformation $(x,y)\mapsto\left(-\sqrt{\frac{G}{E}}\,x,\ \sqrt{EG}\,y\right)$ converts the equation into the standard equation 3.9, where $a=-\frac{F}{EG}$. The condition $F^2-4EG<0$ turns into $-2<a<2$. The symmetry $(x,y)\mapsto(-x,-y)$ preserves the affine and Newtonian classes and changes the sign only of the monomial axy, so that the last condition reduces to $0\le a<2$.

3.10. The cubic has equation (1), where $F^2-4B<0$ and $G^2-4AI<0$. The tangent $x=-\frac{B}{A}$ passes between the origin and the asymptote $x=-\frac{1}{I}$, whence $0<\frac{BI}{A}<1$.

By Lemma 4, we have $AF=BG$, and equation (1) can be written in the form $(1+Ix)y^2+x\left(x+\frac{B}{A}\right)(Gx+Ax)=0$. The transformation $(x,y)\mapsto\left(-Ix,\ \frac{GI}{A}y\right)$ converts it into the standard equation 3.10, where $a=\frac{BI}{A}$ and $b=\frac{G^2}{AI}$. The condition $G^2-4AI<0$ turns into $0<b<4$. The condition $F^2-4B<0$ turns into $b<\frac{4}{a}$.

3.11. $y^2+Bx^2+Fxy+Ixy^2=0$, where $F^2-4B<0$. The transformation $(x,y)\mapsto\left(Ix,\ \frac{FI}{\sqrt{B}}\,y\right)$ converts it into the standard equation 3.11, where $a=\frac{F^2}{B}$. The condition $F^2-4B<0$ turns into $0<a<4$.

3.12. $y^2+Ax^3+Bx^2+Fxy+Gx^2y+Ixy^2=0$, where $F^2-4B<0$ and $G^2-4AI>0$. The horizontal passes through the point M and intersects the odd branch at two other regular points, say, M_1 and M_2, and divides the odd branch into three arcs. For definiteness, let points M_1 and M_2 be denoted so that, in the chosen affine coordinate system, the arc MM_2 has one flex and the arc M_1M_2 has two flexes.

The origin is located between the tangent $x = -\frac{B}{A}$ and the asymptote $x = -\frac{1}{I}$, whence $\frac{BI}{A} < 0$.

By Lemma 4, we have $AF = BG$, and the equation can be written in the form $(1 + Ix)y^2 + x\big(x + \frac{B}{A}\big)(Gy + Ax) = 0$. The transformation $(x, y) \mapsto \big(Ix,\ \frac{GI}{A}y\big)$ converts it into the standard equation 3.12, where $a = -\frac{BI}{A} > 0$ and $b = -\frac{G^2}{AI}$. The condition $G^2 - 4AI > 0$ turns into $b^2 + 4b > 0$. If $b > 0$, the condition $F^2 - 4B < 0$ turns into $b < \frac{4}{a}$. If $b < -4$, the condition $F^2 - 4B < 0$ turns into $b > \frac{4}{a}$, which is impossible.

The oblique asymptotes are $y = \alpha_{1,2}x + \beta_{1,2}$, where

$$\alpha_{1,2} = \frac{b \pm \sqrt{b^2 + 4b}}{2} \quad \text{and} \quad \beta_{1,2} = \frac{b(a+1)\,(\alpha_{1,2} + 1)}{b - 2\alpha_{1,2}}.$$

The equation for finding the abscissas of the intersection points of the oblique asymptotes and the cubic is

$$(\beta_{1,2} + 2\alpha_{1,2} + ab)\,x + \beta_{1,2} = 0.$$

If $\beta_{1,2} + 2\alpha_{1,2} + ab = 0$ and $\beta_{1,2} \neq 0$, then the cubic has flexes lying on the line at infinity at the points $\big(1 : \frac{b \pm \sqrt{b^2+4b}}{2} : 0\big)$, respectively. If $\beta_{1,2} + 2\alpha_{1,2} + ab \neq 0$, then the abscissae are

$$x_{1,2}(a, b) = \frac{(a+1)(b + 2 \pm \sqrt{b^2 + 4b})}{(a+1)\left(-b - 2 \pm \sqrt{b^2 + 4b}\right) + 2(b+4)},$$

respectively.

The flexes are located on the arc M_1M_2 if and only if the two oblique asymptotes intersect the arc M_1M_2. This means that $a < x_{1,2}(a, b)$. On the other hand, one real flex is located in the second quadrant on the arc MM_2, the flex with the abscissa $x_1(a, b)$ is in the first quadrant on the arc M_1M_2, and the flex with the abscissa $x_2(a, b)$ is in the fourth quadrant on the arc $M_1\ M_2$; therefore, $a < x_1(a, b) < x_2(a, b)$ and the last condition can be simplified to $a\ <\ x_2(a, b)$.

§4. Simple cubics

A simple cubic is nonsingular, its real part consists of the odd branch having three flexes. It is depicted in Figure 0.4, where the point P is one of the flexes and M is an arbitrary regular point on the odd branch.

Newton chose nine lines for the role of horizontals, they have numbers 1–9 in Figure 0.4. There are three special lines in the pencil of real lines passing through the flex P:

line 1 is the flexional tangent to the cubic at P,

line 3 is the tangent to the odd branch at a point different from P,

line 5 passes through three flexes.

These lines divide this pencil into three open nonintersecting segments $(1, 3)$, $(3, 5)$, and $(5, 1)$. The horizontals with numbers 2, 4, and 6 are arbitrary lines belonging to these segments, respectively.

Newton chose three more lines for the role of horizontals that pass through the point M:

line 7 intersects the odd branch at M transversally and does not intersect the cubic at other real points,

line 8 is the tangent to the odd branch at M, and

line 9 passes through M and intersects the cubic at two other real regular points.

These horizontals provide nine distinct Newtonian classes of cubics. It can be checked that these classes exhaust the Newtonian classes provided by simple cubics.

For the choice of affine coordinate systems we need the following lines:

line 10 passes through two complex conjugate points of contact of two complex conjugate tangents to the cubic that pass through the flex P,

line 11 passes through two complex conjugate points of contact of two complex conjugate tangents to the cubic that pass through the point M,

line 12 is the tangent at the point M.

In items 4.1–4.9 the affine coordinate system is chosen in the following way: in item 4.1 the x-axis is line 10 and the y-axis is line 3, in items 4.2–4.6 the x-axis is line 10 and the y-axis is line 1, in items 4.7–4.9 the x-axis is line 11 and the y-axis is line 12.

The Newtonian and affine classes provided by simple cubics are shown in Figure 4.

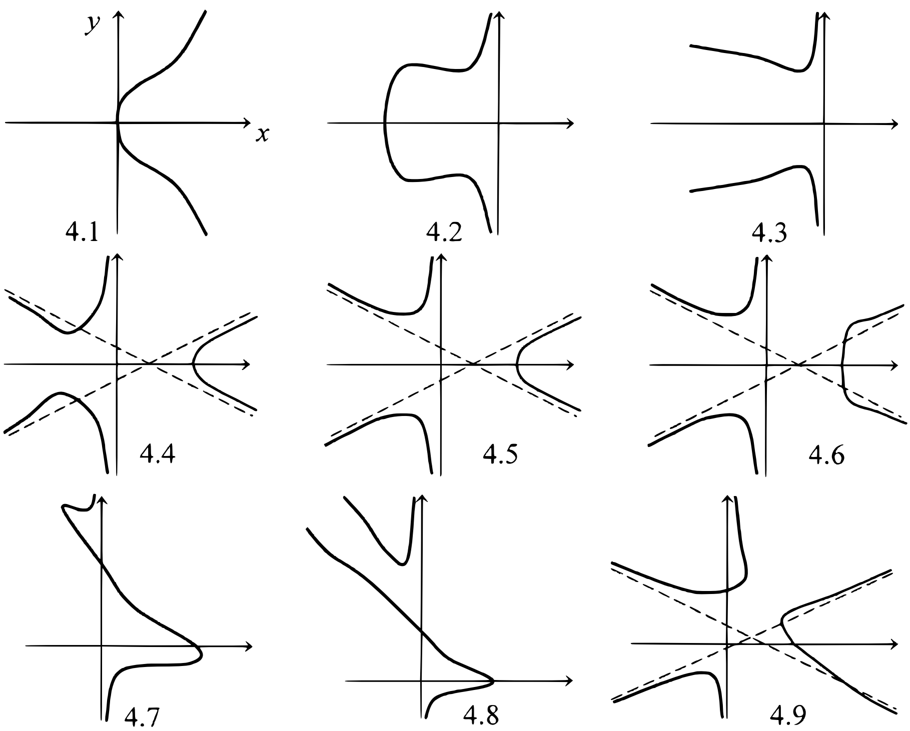

FIGURE 4. Classes generated by simple cubics.

4.1. $-y^2 + Ey + Fxy + Ax^3 + Bx^2 + Cx + D = 0$.

Lemma 9. *If a cubic has such an equation, if the polynomial $Ax^3 + Bx^2 + Cx + D$ has three distinct roots x_1, x_2, x_3, and if two of the three lines $x = x_1$, $x = x_2$, $x = x_3$ are tangent to the cubic, then $E = F = 0$, the third of these lines is also tangent to the cubic, and $D = 0$.*

Proof. Without loss of generality, suppose that $x = x_1$ and $x = x_2$ are tangent to the cubic. The discriminant $\mathcal{D}_2(x) = (Fx + E)^2 + 4(Ax^3 + Bx^2 + Cx + D)$ vanishes for $x = x_1$ and $x = x_2$, so that $E = F = 0$. Thus, $\mathcal{D}_2(x_3) = 0$, and $x = x_3$ is tangent to the cubic. By the choice of an affine coordinate system, one of these lines is $x = 0$, whence $D = 0$. □

By Lemma 9, the equation of the cubic becomes $-y^2 + Ax^3 + Bx^2 + Cx$ or $y^2 = Ax\bigl[(x-p)^2 + q^2\bigr]$, where $p \in \mathbb{R}$, $q > 0$, and without loss of generality we may assume that $A > 0$. The lines $x = p \pm qi$ are complex conjugate tangents. The transformation $(x, y) \mapsto \bigl(\frac{1}{q}\,x,\ \frac{1}{q\sqrt{Aq}}\,y\bigr)$ converts the equation of the cubic into the standard equation 4.1, where $a = \frac{p}{q}$.

4.2. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A < 0$.

Lemma 10. *If a cubic has such an equation, if the polynomial $Ax^3 + Bx^2 + Cx + D$ has three distinct roots x_1, x_2, x_3, and if two of the three lines $x = x_1$, $x = x_2$, $x = x_3$ are tangent to the cubic, then $F = G = 0$, and the third of these lines is also tangent to the cubic.*

The proof is similar to that of Lemma 9.

By Lemma 10, the equation of the cubic becomes $-xy^2 + Ax^3 + Bx^2 + Cx + D = 0$, and since the polynomial $Ax^3 + Bx^2 + Cx + D$ has only one real root, it can be written in the form $xy^2 = A(x + x_1)\bigl[(x-p)^2 + q^2\bigr]$, where $A < 0$, $x_1 > 0$, $p \in \mathbb{R}$, and $q > 0$. The transformation $(x, y) \mapsto \bigl(\frac{1}{q}\,x,\ \frac{1}{q\sqrt{-Aq}}\,y\bigr)$ converts it into the standard equation 4.2, where $a = \frac{x_1}{q}$ and $b = \frac{p}{q}$.

4.3. $-xy^2 + Bx^2 + Cx + D + Fxy \;=\; 0.$

Lemma 11. *If a cubic has such an equation, if the polynomial $Bx^2 + Cx + D$ has two distinct roots x_1, x_2, and if one of the lines $x = x_1$, $x = x_2$ is tangent to the cubic, then $F = 0$, and the second line is tangent to the cubic.*

The proof is similar to that of Lemma 9.

By Lemma 11, the equation can be written in the form $-xy^2 + Bx^2 + Cx + D = 0$ or $xy^2 = B\bigl[(x-p)^2 + q^2\bigr]$, where $p \in \mathbb{R}$, $q > 0$, and without loss of generality we may assume that $B < 0$. The transformation $(x, y) \mapsto \bigl(\frac{1}{q}\,x,\ \frac{1}{\sqrt{-Bq}}\,y\bigr)$ converts it into the standard equation 4.3, where $a = \frac{p}{q}$.

4.4. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + A > 0$. The horizontal intersects the odd branch at the flex P and at two other real regular points, say, at points M_1 and M_2. The points P, M_1, and M_2 divide the odd branch into three arcs so that the arcs PM_1 and PM_2 have one flex each.

Lemma 12. *If a cubic has an equation as above, if the polynomial $Ax^3 + Bx^2 + Cx + D$ has one real root x_0 and two complex conjugate roots $x_{2,3} = p \pm qi$, and if the lines $x = p \pm qi$ are tangent to the cubic at the points $(p \pm qi, 0)$, then $F = G = 0$, and the line $x = x_0$ is tangent to the cubic.*

The proof is similar to that of Lemma 9.

By Lemma 12, the equation of the cubic can be written in the form

$$xy^2 = A(x - x_0)\left[(x-p)^2 + q^2\right],$$

where $p \in \mathbb{R}$, $q > 0$, and $A > 0$. The transformation $(x, y) \mapsto \bigl(\frac{1}{q}\,x,\ \frac{1}{q\sqrt{A}}\,y\bigr)$ converts it into the standard equation 4.4, where $a = \frac{x_0}{q}$, $b = \frac{p}{q}$.

The oblique asymptotes are $y = \pm\bigl[x - \frac{1}{2}(a + 2b)\bigr]$. If $4ab - a^2 + 4 \neq 0$, they intersect the cubic in the affine plane when $x = \frac{4a(b^2+1)}{4ab - a^2 + 4} < 0$. The arcs PM_1 and PM_2 have one flex each, implying that $b < \frac{a^2-4}{4a}$.

4.5. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + A > 0$. The analysis in this case is similar to that in item 4.4 and provides the standard equation 4.5 with $b = \frac{a^2-4}{4a}$.

4.6. $-xy^2+Fxy+Gx^2y+Ax^3+Bx^2+Cx+D=0$, where $G^2+A>0$. The horizontal intersects the odd branch at the flex P and at two other real regular points, say, at points M_1 and M_2. The points P, M_1, and M_2 divide the odd branch into three arcs such that the other two flexes are located on the arc M_1M_2. The further analysis in this case is similar to that in item 4.4 and provides the standard equation 4.6 with $b>\frac{a^2-4}{4a}$.

4.7. $-xy^2+(Gx^2+Fx+E)y+Ax^3+Bx^2+Cx+D=0$, where $G^2+4A<0$. By the choice of an affine coordinate system, the polynomial Ax^3+Bx^2+Cx+D has one real root, say x_0, and two complex conjugate ones, say $p\pm qi$, and the lines $x=p\pm qi$ are the tangents to the cubic at the points $(p\pm qi,0)$, respectively.

Lemma 13. *If a nonsingular cubic has an equation as above, and if the lines $x=p\pm qi$ are the tangents to the cubic at the points $(p\pm qi,0)$, then*:

1) *if $G\neq 0$, then $p\pm qi$ are the roots of polynomial Gx^2+Fx+E,*

2) *if $G=0$, then $E=F=0$, and the cubic does not belong to the Newtonian class* 4.7.

Proof. 1) $G\neq 0$. If $x=p\pm q$ are the tangents at the points $(p\pm qi,0)$, then the discriminant $\mathcal{D}_2(x)=(Gx^2+Fx+E)^2+4x(Ax^3+Bx^2+Cx+D)$ vanishes for $x=p\pm qi$. Thus, $Gx^2+Fx+E=G\big[(x-p)^2+q^2\big]$.

2) If $G=0$, then $E=F=0$, and the cubic has a flex at $(0:1:0)$ with the flexional tangent $x=0$. □

By Lemma 13, the equation of the cubic becomes $xy^2=[(x-p)^2+q^2][Gy+A(x-x_0)]$. The transformation $(x,y)\mapsto\big(\frac{1}{q}\,x,\ \frac{G}{qA}\,y\big)$ converts it into the standard equation 4.7, where $a=\frac{p}{q}$, $b=\frac{x_0}{q}$, and $c=\frac{G^2}{A}$. The condition $G^2+4A<0$ turns into $-4<c<0$. Since the transformation $(x,y)\mapsto(-x,-y)$ does not change the Newtonian and affine classes and acts on the equation as $(a,b)\mapsto(-a,-b)$, we can choose any open half of the plane of parameters (a,b), in particular, $a\in\mathbb{R}$ and $b\geq 0$.

4.8. $-xy^2+(Gx^2+Fx+E)y+Ax^3+Bx^2+Cx+D=0$, where $G^2+4A=0$. By the choice of an affine coordinate system, the polynomial Ax^3+Bx^2+Cx+D has one real root, say x_0, and two complex conjugate ones, say $p\pm qi$, and the lines $x=p\pm qi$ are the tangents to the cubic at the points $(p\pm qi,0)$, respectively.

By Lemma 13, the equation of the cubic can be written in the form

$$xy^2=[(x-p)^2+q^2][Gy+A(x-x_0)].$$

The transformation $(x,y)\mapsto\big(\frac{1}{q}\,x,\ \frac{G}{qA}\,y\big)$ converts it into the standard equation 4.8, where $a=\frac{p}{q}$, $b=\frac{x_0}{q}$, $c=\frac{G^2}{A}$, $a\in\mathbb{R}$, and $b\geq 0$ (see item 4.7). The condition $G^2+4A=0$ turns into $c=-4$. If $b\neq 2a$, the line at infinity is the tangent to the cubic at the point $N=((-1):2:0)$. If $b<2a$ (respectively, $b>2a$), the cubic touches the line at infinity in the second (respectively, the fourth) quadrant. If $b=2a$, the cubic deforms into a cubic with a node at the point N.

4.9. $-xy^2+(Gx^2+Fx+E)y+Ax^3+Bx^2+Cx+D=0$, where $G^2+4A>0$. The horizontal intersects the odd branch at the point M and at two other real regular points, say, M_1 and M_2, and divides the odd branch into three arcs. For definiteness, let the points M_1 and M_2 be denoted in such a way that, in the chosen affine coordinate system, the arc MM_1 has no flexes, MM_2 has one flex, and M_1M_2 has two flexes.

By the choice of an affine coordinate system, the polynomial Ax^3+Bx^2+Cx+D has one real root, say x_0, and two complex conjugate ones, say $p\pm qi$, and the lines $x=p\pm qi$ are the tangents to the cubic at the points $(p\pm qi,0)$.

By Lemma 13, the equation becomes $xy^2=[(x-p)^2+q^2][Gy+A(x-x_0)]$. The transformation $(x,y)\mapsto\big(\frac{1}{q}\,x,\ \frac{G}{qA}\,y\big)$ turns it into the standard equation 4.9, where $a=\frac{p}{q}$, $b=\frac{x_0}{q}$, $c=\frac{G^2}{A}$, $a\in\mathbb{R}$, and $b>0$ (see item 4.7). For $b=0$, the cubic has an oval and does not belong to the Newtonian class 4.9. The condition $G^2 + 4A > 0$ converts into $c^2 + 4c > 0$. For $c < -4$, the arc M_2M_3 does not have two flexes; therefore, $c > 0$.

The standard equation 4.9 is quadratic in y and has the discriminant

$$\mathcal{D}_2(x) = c\big[(x-a)^2+1\big]\big[(c+4)x^2 - 2(ac+2b)x + c(a^2+1)\big].$$

There are four tangents to the cubic that are parallel to the y-axis: complex conjugate ones $x = a \pm i$ and if $\mathcal{D}_1(c) = (ac-2b)^2 - c(c+4)(a^2+1) > 0$ and $ac - 2b > 0$, i.e., if $\sqrt{c(c+4)(a^2+1)} < ac - 2b$, then there are two real tangents $x = x^0_{1,2}(a,b,c) \equiv \frac{ac+2b\pm\sqrt{\mathcal{D}_1(c)}}{c+4}$ to the arcs M_2M_3 and M_1M_3, respectively. If $\mathcal{D}_1(c) = 0$, these arcs merge at the node, so that

$$0 < c < 2\big[a^2 - ab + 1 + \sqrt{(a^2-ab+1)^2 + b^2}\big].$$

The oblique asymptotes are $y = \alpha_{1,2}x + \beta_{1,2}$, where

$$\alpha_{1,2} = \frac{c \pm \sqrt{c^2+4c}}{2} \quad \text{and} \quad \beta_{1,2} = -c\left[a \pm \frac{a(c+2)+b}{\sqrt{c^2+4c}}\right],$$

respectively.

Substitution of $y = \alpha_{1,2}x + \beta_{1,2}$ into the standard equation 4.9 gives the abscissae of the points of intersection of these asymptotes and the cubic:

$$x_{1,2}(a,b,c) = \frac{(c+4)(a^2+1)(\beta_{1,2}-b)}{(2a-b)^2 - (c+4)(a^2+1)(\alpha_{1,2}+1)},$$

respectively. The flexes are located on the arc M_1M_2 if and only if asymptotes intersect the arc M_1M_2 (cf. item 3.12). This means that $x^0_1(a,b,c) < x_{1,2}(a,b,c)$. On the other hand, two real flexes are located in the first quadrant on the arcs MM_2 and M_1M_2, and the third one in the fourth quadrant on the arc M_1M_2; therefore, $x^0_1(a,b,c) < x_1(a,b,c) < x_2(a,b,c)$, and the condition from the preceding sentence can be simplified to $x^0_1(a,b,c) < x_2(a,b,c)$.

§5. Cubics with an oval

The real part of this cubic consists of an oval and an odd branch with three flexes. It is shown in Figure 0.5, where the point P is a flex and M is an arbitrary regular point on the odd branch.

Newton chose fifteen lines for the role of horizontals, they have numbers 1–15 in Figure 0.5.

There are five special lines in the pencil of real lines passing through the flex P:

line 1 is the flexional tangent to the cubic at P,

lines 3 and 5 are the tangents to the oval,

line 7 is the tangent to the odd branch at a point different from P,

line 9 passes through three real flexes.

They divide this pencil into five open nonintersecting segments $(1,3)$, $(3,5)$, $(5,7)$, $(7,9)$, and $(9,1)$.

The horizontals with numbers 2, 4, 6, 8, and 10 are arbitrary lines belonging to these segments, respectively.

Newton chose five more positions of the horizontals that pass through the point M:

line 11 does not intersect the cubic at other real points,

line 12 is one of the tangents to the oval at the point Q (see Figure 0.5),

line 13 intersects the oval at two points,

line 14 is the tangent to the odd branch at another point,

line 15 intersects the cubic at two other regular points M_1 and M_2 lying on the odd branch.

These horizontals provide fifteen distinct Newtonian classes of cubics. It can be checked that these classes exhaust the Newtonian classes provided by cubics with an oval.

For the choice of affine coordinate systems we need the following lines:

line 16 passes through the points of contact between the oval and lines 3 and 5,

line 17 passes through the point of contact between the oval and line 3 and the point of contact between the odd branch and line 5,

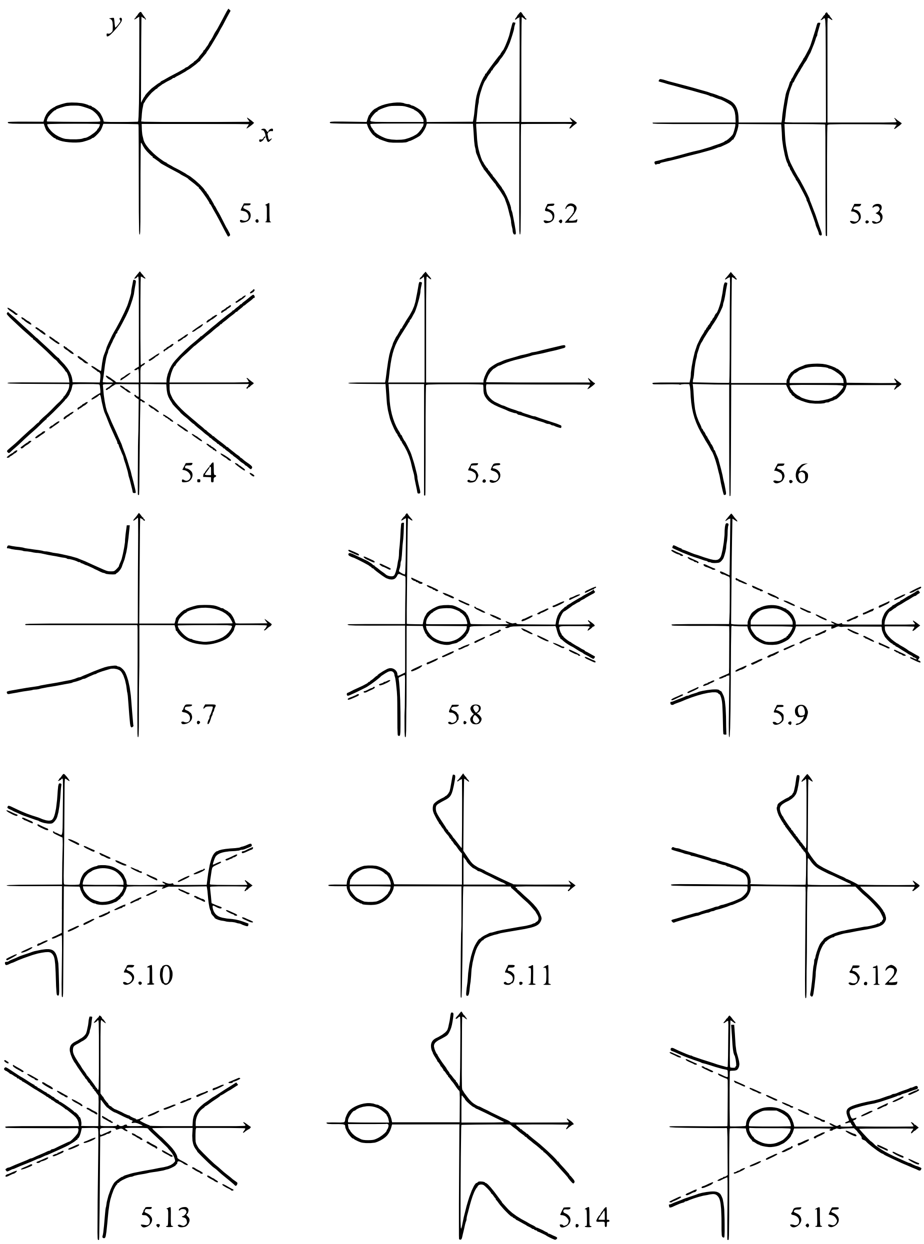

FIGURE 5. Classes generated by cubics with oval.

line 18 passes through the points of contact of the tangents to the oval passing through the point M,

line 19 is the tangent to the odd branch at M.

In items 5.1–5.15 the affine coordinate system is chosen in the following way: in item 5.1 the x-axis is line 16 and the y-axis is line 7, in items 5.2–5.10 the x-axis is line 16 and the y-axis is line 1, in items 5.11–5.15 the x-axis is line 18 and the y-axis is line 19.

The Newtonian and affine classes provided by the cubic with oval are shown in Figure 5.

5.1. $-y^2+Ey+Fxy+Ax^3+Bx^2+Cx+D = 0$. By Lemma 9, we have $E = F = D = 0$, and the equation can be written in the form $y^2 = Ax(x-x_1)(x-x_2)$, where the quantities x_1, x_2, and $-A$ have one and the same sign. For definiteness, let $|x_1| > |x_2|$. The transformation $(x,y) \mapsto \big(-\frac{1}{x_2}\,x,\, -\frac{1}{x_2\sqrt{-Ax_2}}\,y\big)$ converts it into the standard equation 5.1, where $a = \frac{x_1}{x_2} > 1$.

5.2. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A < 0$. By Lemma 10, we have $F = G = 0$, and the equation can be written in the form

$$xy^2 = A(x - x_1)(x - x_2)(x - x_3),$$

where $A < 0$ and the roots x_1, x_2, and x_3 have one and the same sign. For definiteness, let $|x_3| < |x_2| < |x_1|$. The transformation $(x,y) \mapsto \big(-\frac{1}{x_3}\,x,\, -\frac{1}{x_3\sqrt{-A}}\,y\big)$ converts the equation into the standard equation 5.2, where $a = \frac{x_2}{x_3}$ and $b = \frac{x_1}{x_3}$.

5.3. $-xy^2 + Fxy + Bx^2 + Cx + D = 0$. By Lemma 11, we have $F = 0$, and the equation can be written in the form $xy^2 = B(x - x_1)(x - x_2)$, where the numbers B, x_1, and x_2 have one and the same sign. For definiteness, let $|x_2| < |x_1|$. The transformation $(x,y) \mapsto \big(-\frac{1}{x_2}\,x,\, \frac{1}{\sqrt{Bx_2}}\,y\big)$ converts the equation into the standard equation 5.3, where $a = \frac{x_1}{x_2} > 1$.

5.4. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A > 0$. By Lemma 10, we have $F = G = 0$, and the equation can be written in the form $xy^2 = A(x-x_1)(x-x_2)(x-x_3)$, where $A > 0$, two of the roots x_1, x_2, and x_3 have one and the same sign, and the third root has the opposite sign. Up to the symmetry $x \mapsto -x$, we can choose $x_1 < x_3 < 0 < x_2$. The transformation $(x,y) \mapsto \big(-\frac{1}{x_3}\,x,\, -\frac{1}{x_3\sqrt{A}}\,y\big)$ converts the equation in question into the standard equation 5.4, where $a = \frac{x_1}{x_3} > 1$ and $b = -\frac{x_2}{x_3} > 0$. The oblique asymptotes are $y = \pm\big(x - \frac{b-a-1}{2}\big)$.

5.5. $-xy^2+Fxy+Bx^2+Cx+D = 0$. By Lemma 11, we have $F = 0$, and the equation can be written in the form $xy^2 = B(x - x_1)(x - x_2)$, where x_1 and x_1 have opposite signs. For definiteness, let the odd branch pass through the point $(x_1, 0)$ and the oval through the point $(x_2, 0)$. This choice implies $Bx_1 < 0$. The transformation $(x,y) \mapsto \big(-\frac{1}{x_1}\,x,\, \frac{1}{\sqrt{-Bx_1}}\,y\big)$ converts the equation in question into the standard equation 5.5, where $a = -\frac{x_2}{x_1} > 0$.

5.6. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A < 0$. By Lemma 10, we have $F = G = 0$, and the equation can be written in the form $xy^2 = A(x - x_1)(x - x_2)(x - x_3)$, where $A < 0$ and two of the roots x_1, x_2, x_3, say x_2 and x_3, have one and the same sign and the points $(x_2, 0)$ and $(x_3, 0)$ are situated on the oval, while x_1 has the opposite sign and the point $(x_3, 0)$ is situated on the odd branch. For definiteness, let $|x_2| < |x_3|$. The transformation $(x,y) \mapsto \big(-\frac{1}{x_1}\,x,\, -\frac{1}{x_1\sqrt{-A}}\,y\big)$ yields the standard equation 5.6, where $a = -\frac{x_2}{x_1} > 0$ and $b = -\frac{x_3}{x_1} > a$.

5.7. $-xy^2 + Fxy + Bx^2 + Cx + D = 0$. By Lemma 11, we have $F = 0$, and the equation can be written in the form $xy^2 = B(x - x_1)(x - x_2)$, where x_1 and x_2 have one and the same sign and $Bx_1 < 0$. For definiteness, let $|x_1| < |x_2|$. The transformation $(x,y) \mapsto \big(\frac{1}{x_1}\,x,\, \frac{1}{\sqrt{-Bx_1}}\,y\big)$ leads to the standard equation 5.7, where $a = \frac{x_2}{x_1} > 1$.

5.8. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A > 0$. The horizontal intersects the odd branch at the flex P and at two other regular points, say, M_1 and M_2. The points P, M_1, and M_2 divide the odd branch into three arcs so that the arcs PM_1 and PM_2 have one flex each and the arc M_1M_2 has no flexes.

By Lemma 10, we have $F = G = 0$, and the equation can be written in the form $xy^2 = A(x - x_1)(x - x_2)(x - x_3)$, where $A > 0$ and the roots x_1, x_2, x_3 have one and the same sign. For definiteness, let $|x_1| < |x_2| < |x_3|$. The transformation $(x, y) \mapsto \big(\frac{1}{x_2}\, x,\ \frac{1}{x_2\sqrt{A}}\, y\big)$ converts the equation under study into the standard equation 5.8, where $a = \frac{x_1}{x_2}$, $b = \frac{x_3}{x_2}$, and $0 < a < 1 < b$.

The equations of the oblique asymptotes are $y = \pm\big(x - \frac{a+b+1}{2}\big)$.

Substitution of this expression for y in the standard equation gives the abscissa $x = \frac{4ab}{4ab-(a+b-1)^2}$ of the points of intersection of the oblique asymptotes and the cubic.

The flexes are located on the arcs PM_1 and PM_2 each if and only if the points of intersection of the oblique asymptotes and the cubic are located on the arcs PM_1 and PM_2, and if and only if $b > (\sqrt{a} + 1)^2$.

5.9. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A > 0$. The analysis in this case is similar to that in item 5.8 and provides the standard equation 5.9 with the condition $b = (\sqrt{a} + 1)^2$.

5.10. $-xy^2 + Fxy + Gx^2y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A > 0$. The horizontal intersects the odd branch at the flex P and at two other regular points, say, M_1 and M_2. The points P, M_1, and M_2 divide the odd branch into three arcs so that the arc M_1M_2 has two flexes. The analysis in this case is similar to that in item 5.8 and provides the standard equation 5.10 with the condition $b < \big(\sqrt{a} + 1\big)^2$.

5.11. $-xy^2 + (Gx^2 + Fx + E)y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A < 0$. By the choice of an affine coordinate system, the polynomial $Ax^3 + Bx^2 + Cx + D$ has three real roots, say x_0, x_1, and x_2. For definiteness, suppose that the lines $x = x_1$ and $x = x_2$ are tangent to the oval at the points $(x_1, 0)$ and $(x_2, 0)$, and that $(x_0, 0)$ is the point of intersection of the odd branch and the x-axis. The roots x_1 and x_2 have one and the same sign. For definiteness, let $|x_1| < |x_2|$. If x_1 and x_2 are negative, then $x_2 < x_1 < x_0$. If x_1 and x_2 are positive, then $x_0 < x_1 < x_2$.

Lemma 14. *If a nonsingular cubic has an equation as above, and if the lines $x = x_1$ and $x = x_2$ are the tangents to the oval at the points $(x_1, 0)$ and $(x_2, 0)$, then*:

1) *if $G \neq 0$, then x_1 and x_2 are the roots of the polynomial $Gx^2 + Fx + E$;*
2) *if $G = 0$, then the cubic is not of the Newtonian class* 5.11.

The proof is similar to that of Lemma 13.

By Lemma 14, the equation of the cubic can be written in the form

$$xy^2 = (x - x_1)(x - x_2)[Gy + A(x - x_0)].$$

The transformation $(x, y) \mapsto \big(-\frac{1}{x_1}\, x,\ -\frac{G}{Ax_1}\, y\big)$ converts it into the standard equation 5.11, where $a = \frac{x_2}{x_1} > 1$, $b = -\frac{x_0}{x_1} > -1$, and $c = \frac{G^2}{A}$. The condition $G^2 + 4A < 0$ turns into $-4 < c < 0$.

For $b = -1$, the cubic has a node, and if $b = -a$, then it has an isolated point. For $-a < b < -1$ or $b < -a$, the cubic belongs to the Newtonian class 5.11, but its location contradicts the choice of the affine coordinate system with respect to the location of flexes on arcs of the odd branch.

5.12. $-xy^2 + Fxy + Ey + Bx^2 + Cx + D = 0$. The polynomial $Bx^2 + Cx + D$ has two real roots, say x_0 and x_1. Let the line $x = x_1$ be tangent to the oval.

Lemma 15. *If a nonsingular cubic has an equation as above, and if the line $x = x_1$ is the tangent to the cubic at the point $(x_1, 0)$, then*:

1) *if* $F \neq 0$, *then* x_1 *is the root of the polynomial* $Fx + E$,
2) *if* $F = 0$, *the cubic is not of the Newtonian class* 5.12.

The proof is similar to that of Lemma 13.

By Lemma 15, the equation of the cubic can be written in the form

$$xy^2 = (x - x_1)[Fy + B(x - x_0)].$$

The transformation $(x, y) \mapsto \big(-\frac{1}{x_1}\,x,\ -\frac{F}{Bx_1}\,y\big)$ turns it into the standard equation 5.12, where $a = -\frac{x_0}{x_1} > 0$ and $b = -\frac{F^2}{Bx_1}$.

For $a = -1$, the cubic becomes nodal. For $a < -1$, the cubic belongs to the Newtonian class 5.12, but its location contradicts the choice of an affine coordinate system with respect to the location of the flexes on arcs of the odd branch.

5.13. $-xy^2 + (Gx^2 + Fx + E)y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A > 0$.

The points of intersection of the cubic and the x-axis are the roots of the polynomial $Ax^3 + Bx^2 + Cx + D$, and two of them, say x_1 and x_2, are located on the oval, and the third, say x_0, is on the odd branch. The lines $x = x_1$ and $x = x_2$ are the tangents to the oval at the points $(x_1, 0)$ and $(x_2, 0)$. The roots x_1 and x_2 have opposite signs, and the root x_0 is located between them. For definiteness, let $x_1 < x_0 < x_2$.

By Lemma 14, the equation of the cubic can be written in the form

$$xy^2 = (x - x_1)(x - x_2)[Gy + A(x - x_0)].$$

The transformation $(x, y) \mapsto \big(-\frac{1}{x_1}\,x,\ -\frac{G}{Ax_1}\,y\big)$ turns it into the standard equation 5.13, where $a = -\frac{x_2}{x_1}$, $b = -\frac{x_0}{x_1}$, $c = \frac{G^2}{A}$, $a > 0$, $-1 < b < a$.

The condition $G^2 + 4A > 0$ turns into $c^2 + 4c > 0$. For $c < -4$, either the cubic is singular, or its location contradicts the choice of a coordinate system with respect to the location of the flexes on arcs of the odd branch. Thus, $c > 0$. For $c > 0$ and $b < -1$ or for $c > 0$ and $b > -a$, the location of the cubic contradicts the choice of an affine coordinate system with respect to the location of the flexes on arcs of the odd branch.

5.14. $-xy^2 + (Gx^2 + Fx + E)y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A = 0$. The analysis this case is similar to that in item 5.11 with minor modification: the condition $G^2 + 4A = 0$ turns into $c = -4$ and leads to the standard equation 5.14 with $a > 1$, $b > -1$. The point of contact between the cubic and the line at infinity is $(1 : (-2) : 0)$.

5.15. $-xy^2 + (Gx^2 + Fx + E)y + Ax^3 + Bx^2 + Cx + D = 0$, where $G^2 + 4A > 0$. The horizontal intersects the odd branch at the point M and at two other regular points, say, M_1 and M_2, and divides the cubic into three arcs. For definiteness, let the points M_1 and M_2 be denoted so that, in the chosen affine coordinate system, the arc MM_1 has no flexes, the arc MM_2 has one flex, and M_1M_2 has two flexes.

The abscissas of the points of intersection of the x-axis and the cubic are roots of the polynomial $Ax^3 + Bx^2 + Cx + D$, and two of them, say x_1 and x_2, are located on the oval, and the third, say x_0, is on the arc M_1M_2. The lines $x = x_1$ and $x = x_2$ are the tangents to the oval at the points $(x_1, 0)$ and $(x_2, 0)$. The roots x_0, x_1, and x_2 have one and the same sign. For definiteness, let $|x_1| < |x_2| < |x_0|$.

By Lemma 14, the equation of the cubic can be written in the form

$$xy^2 = (x - x_1)(x - x_2)[Gy + A(x - x_0)].$$

The transformation $(x, y) \mapsto \big(\frac{1}{x_2}\,x,\ \frac{G}{Ax_2}\,y\big)$ converts it into the standard equation 5.15, where $a = \frac{x_1}{x_2}$, $b = \frac{x_0}{x_2}$, $c = \frac{G^2}{A}$, and $0 < a < 1 < b$. The condition $G^2 + 4A > 0$ turns into $c^2 + 4c > 0$.

For $c < -4$ the cubic contradicts the choice of a coordinate system with respect to the location of the flexes on arcs of the odd branch; therefore, $c > 0$.

The standard equation 5.15 is quadratic in y and has the discriminant

$$\mathcal{D}_2(x) = c(x-a)(x-1)\bigl\{(c+4)x^2 - \bigl[c(a+1)+4b)\bigr]x + ac\bigr\}.$$

There are four tangents to the cubic parallel to the y-axis: two tangents $x = a$ and $x - 1$ to the oval and two tangents to the arcs M_1M_2 and MM_2,

$$x = x^0_{1,2}(a,b,c) = \frac{c(a+1) + 4b \pm \sqrt{\mathcal{D}_1(c)}}{2(c+4)},$$

respectively, where

$$\mathcal{D}_1(c) = \bigl[c(a+1)+4b)\bigr]^2 - 4ac(c+4) = [c(a-1)+4b]^2 + 16c(b-a) > 0.$$

The equations of the oblique asymptotes are $y = \alpha_{1,2}x + \beta_{1,2}$, where

$$\alpha_{1,2} = \frac{c \pm \sqrt{c^2+4c}}{2}, \qquad \beta_{1,2} = \frac{c\bigl[b + (a+1)\bigl(\alpha_{1,2}+1\bigr)\bigr]}{c - 2\alpha_{1,2}}.$$

Substitution of $y = \alpha_{1,2}x + \beta_{1,2}$ in the standard equation gives the abscissae

$$x_{1,2}(a,b,c) = \frac{ac(\beta_{1,2} - b)}{\beta^2_{1,2} - c\bigl[a\bigl(\alpha_{1,2}+1\bigr) - (a+1)(\beta_{1,2}-b)\bigr]}$$

of the points of intersection of the cubic and the oblique asymptotes, respectively. The arc M_1M_2 has two flexes if and only if the two oblique asymptotes intersect the arc M_1M_2 (cf. item 3.12). This means that $x^0_1(a,b,c) < x_{1,2}(a,b,c)$. On the other hand, two flexes are located in the first quadrant on the arcs MM_2 and M_1M_2, one on each arc, and the third flex is in the fourth quadrant on the arc M_1M_2; therefore, $x^0_1(a,b,c) < x_1(a,b,c) < x_2(a,b,c)$, and the condition in the preceding sentence can be simplified to $x_2(a,b,c) > x^0_1(a,b,c)$.

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Received 5/OCT/2011

Translated by THE AUTHOR