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STEIN–TOMAS THEOREM FOR A TORUS AND THE PERIODIC SCHRÖDINGER OPERATOR WITH SINGULAR POTENTIAL

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Abstract. A discrete version of the Stein–Tomas theorem for a torus is proved, except for the endpoint case. The result makes it possible to establish the absolute continuity of the spectrum of the periodic Schrödinger operator with a δ-like potential concentrated on a hypersurface of nonzero curvature.

§1. Introduction

Let Σ be a compact C^2-hypersurface in $\mathbb{R}^d$, $d \geq 3$, with or without boundary. There exist orthogonal local coordinates in which Σ is given by the equation

$$x_d = w(x_1, \dots, x_{d-1}). \tag{1}$$

The $(d-1)\times(d-1)$-Hessian $\det\left\{\frac{\partial^2 u}{\partial x_i \partial x_j}\right\}$ is called the *Gaussian curvature* of Σ and does not depend on the choice of local orthogonal coordinates. We assume this curvature to be nonzero at any point. In what follows, by "curvature" we mean the Gaussian curvature.

Let dS be the $(d-1)$-dimensional Lebesgue measure on Σ, and let $\widehat{f}$ and $\check{f}$ be the Fourier transform and its inverse,

$$\widehat{f}(\xi) = (2\pi)^{-d/2}\int f(x)e^{-i\xi x}\,dx, \quad \check{f}(x) = (2\pi)^{-d/2}\int f(\xi)e^{i\xi x}\,d\xi.$$

The following result is due to Stein and Tomas, see, e.g., [16] and [14, Chapter VIII–IX].

Theorem 1. *Let Σ be a compact C^∞-hypersurface in $\mathbb{R}^d$, $d \geq 3$, whose curvature is nonzero everywhere. Then*

$$\|\check{f}\|_{L_2(\Sigma)} \leq C\|f\|_{L_{p'}(\mathbb{R}^d)}, \quad 1 \leq p' \leq \frac{2d+2}{d+3}, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}$ is the Schwartz class. The constant C may depend on Σ but not on f.

If $1 \leq p' < \frac{2d+2}{d+3}$, then the proof is much more elementary; see, e.g., [18]. In that paper it was also shown that the exponent $p' = \frac{2d+2}{d+3}$ cannot be improved.

The main result of the present paper is Theorem 8, which can be viewed as an analog of Theorem 1 for a hypersurface Σ in a d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\Gamma$, where

$$\Gamma = \{l_1 b_1 + \dots + l_d b_d,\ l_1, \dots, l_d \in \mathbb{Z}\} \tag{2}$$

is a lattice. Here $b_1, \dots, b_d$ is a fixed (not necessarily orthonormal) basis in $\mathbb{R}^d$, $d \geq 2$. The space $L_{p'}(\mathbb{R}^d)$ is replaced with $l_{p'}(\Gamma')$ for the dual lattice Γ'. The general idea of

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the proof is the same as in Theorem 1. Therefore, the result may happen to be known to experts, although we have not managed to find it in literature.

In §3 we use Theorem 8 to study the spectrum of the periodic Schrödinger operator. Let $d \geq 3$, let Σ be a Γ-periodic system of Lipschitz hypersurfaces in $\mathbb{R}^d$ (this means that Σ is invariant with respect to shifts by the elements of Γ). For such Σ, the notion of a Γ-periodic function on Σ is well defined. Let σ be a Γ-periodic function such that $\sigma \in L_{d-1,\mathrm{loc}}(\Sigma)$. In $L_2(\mathbb{R}^d)$, consider the quadratic form

$$h_\sigma[u,u] = \int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx + \int_\Sigma \sigma(x)|u(x)|^2\,dS(x)$$

defined on $H^1(\mathbb{R}^d)$. It is well known (see, e.g., [13]) that the form h_σ is closed and semibounded from below. Therefore, it corresponds to a selfadjoint operator

$$H_\sigma = -\Delta + \sigma(x)\delta_\Sigma(x), \quad x \in \mathbb{R}^d,$$

which is called the Schrödinger operator with a δ-like potential σ concentrated on a periodic hypersurface Σ. We prove the following result.

Theorem 2. *Let $d = 4$, let $\Sigma \subset \mathbb{R}^d$ be a Γ-periodic system of C^4-hypersurfaces such that their curvatures vanish nowhere (including the boundaries). Assume that $\sigma \in L_{p,\mathrm{loc}}(\Sigma)$, $p > 6$, and that σ is Γ-periodic. Then the spectrum of the corresponding operator H_σ is absolutely continuous.*

For periodic Schrödinger operators, the absolute continuity of the spectrum is a natural conjecture based on certain considerations from solid-state physics. The case of a usual electric potential has been studied in large generality (see, e.g., [2, 12]). The two-dimensional case with δ-like potential was considered in [3, 17], where it was assumed that $\sigma \in L_{p,\mathrm{loc}}(\Sigma)$, $p > 1$. The case of $d = 3$ was treated in [13] for $\sigma \in L_{2,\mathrm{loc}}(\Sigma)$; see also [5]. Both results are optimal in the L_p-scale. The paper [13] also dealt with higher dimensions, but an additional geometric condition was imposed on Σ: it was assumed that there exists a direction transversal to Σ at all points.

In the present paper we address the case where $d = 4$ under a different geometrical assumption on Σ: we assume that its curvature does not vanish. Note that the condition imposed in [13] and our condition do not cover each other and, unfortunately, together do not cover all possible cases: a polygon (repeated periodically) only satisfies the first condition, a sphere only satisfies the second, and the surface of a cylinder satisfies none of them.

Remark 3. Theorem 2 only addresses the case of a purely singular potential. It is possible to include an additional electric potential in the same way as in [2] or [9], or a magnetic potential, or a singular electric potential as in [13].

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§2. Stein–Tomas theorem for a torus

The classical proof of the Stein–Tomas theorem for $1 \leq p' < \frac{2d+2}{d+3}$ is based on the following decay property of the Fourier transform of surface measures; this property was first established in [6] (see also [14, §VIII.3]). For a hypersurface Σ, by $C_0^l(\Sigma)$ we denote the subspace in $C^l(\Sigma)$ consisting of functions compactly supported in $\Sigma \setminus \partial\Sigma$.

Proposition 4. *Let $\psi \in C_0^\infty(\Sigma)$, where $\Sigma \subset \mathbb{R}^d$ is a compact C^∞-hypersurface whose curvature does not vanish on the support of ψ. Then*

$$\left| \int_\Sigma e^{-i\langle \xi, x\rangle} \psi(x)\, dS(x) \right| \le C(1+|\xi|)^{-(d-1)/2}, \quad \xi \in \mathbb{R}^d. \tag{3}$$

Remark 5. In the references mentioned above, Σ was assumed to be infinitely smooth. It turns out that the proof given in [14] can be done for even d if we assume that $\Sigma \in C^l$, $\psi \in C_0^l(\Sigma)$, $l = \frac{d+4}{2}$. In the case of odd d, it can be done for $l = \frac{d+5}{2}$, or for $l = \frac{d+3}{2}$, but with the right-hand side replaced by

$$C(1+|\xi|)^{-(d-1)/2} \ln(1+|\xi|).$$

In the Appendix we give an outline of the proof with the corresponding modifications. Therefore, in Theorem 1 we may assume (for $p' < \frac{2d+2}{d+3}$) that $\Sigma \in C^l$. The question of the minimal smoothness sufficient to prove the result is open. An example in [7] shows that only the existence and positivity of the curvature without any other smoothness assumptions do not suffice. Nor do we know if the estimate for odd d holds true without the factor $\ln(1+|\xi|)$. This factor does not affect the results of §§2 and 3.

Consider the torus $\mathbb{T}^d = \mathbb{R}^d/\Gamma$, where Γ is the lattice (2). The functions on $\mathbb{T}^d$ are in one-to-one correspondence with the Γ-periodic functions on $\mathbb{R}^d$ or functions on the elementary cell

$$\Omega = \{y_1 b_1 + \cdots + y_d b_d,\ 0 \le y_1, \dots, y_d < 1\} \tag{4}$$

of Γ. During computations in coordinates, it is convenient to identify the set $\mathbb{T}^d$ with Ω. The elements

$$\varphi_n(y) = |\Omega|^{-1/2} e^{i\langle n, y\rangle}, \quad n \in \Gamma', \tag{5}$$

where Γ' is the dual lattice

$$\Gamma' = \left\{ n = \sum_{j=1}^d n_j b_j',\ n_j \in \mathbb{Z} \right\}, \quad \langle b_k, b_j' \rangle = 2\pi \delta_{kj}, \tag{6}$$

form an orthonormal basis in $L_2(\mathbb{T}^d)$. The Fourier transform

$$\widehat{f}(n) = |\Omega|^{-1/2} \int_\Omega f(x) e^{-i\langle n, x\rangle}\, dx, \quad n \in \Gamma', \tag{7}$$

is a bijection between $C^\infty(\mathbb{T}^d)$ and the Schwartz class $\mathcal{S}(\Gamma')$ (consisting of functions decaying faster than any power) and extends up to an isometric isomorphism of $L_2(\mathbb{T}^d)$ and $l_2(\Gamma')$. The inverse Fourier transform is

$$\check{f}(x) = |\Omega|^{-1/2} \sum_{n \in \Gamma'} f(n) e^{i\langle n, x\rangle}, \quad x \in \Omega.$$

Theorem 6. *Let $\Sigma \subset \mathbb{T}^d$ be a compact $(d-1)$-dimensional C^l-submanifold with or without boundary, and let $l = \left[\frac{d+4}{2}\right]$. Assume that the curvature of Σ does not vanish anywhere (including the boundary). Let $\psi \in C_0^l(\Sigma)$ be nonnegative. Then*

$$\|\check{f}\|_{L_2(\Sigma, \psi dS)} \le C_{p'} \|f\|_{l_{p'}(\Gamma')}, \quad 1 \le p' < \frac{2d+2}{d+3}, \quad f \in \mathcal{S}(\Gamma'). \tag{8}$$

To prove Theorem 6, we need the following simple lemma.

Lemma 7. *Let $a \ge 2$. Then*

$$\int_0^{1/2} \left| \frac{\sin(\pi a x)}{\sin(\pi x)} \right| dx \le C \ln a.$$

Proof. The inequalities

$$\sin(\pi x) \geq 2x \quad \text{for} \quad 0 \leq x \leq \frac{1}{2}, \qquad \frac{\sin(\pi a x)}{\sin(\pi x)} \leq a \quad \text{for} \quad 0 \leq x \leq \frac{1}{a}$$

yield

$$\int_0^{1/2} \Big|\frac{\sin(\pi a x)}{\sin(\pi x)}\Big|\, dx = \int_0^{1/a} \frac{\sin(\pi a x)}{\sin(\pi x)}\, dx + \int_{1/a}^{1/2} \Big|\frac{\sin(\pi a x)}{\sin(\pi x)}\Big|\, dx$$
$$\leq 1 + \int_{1/a}^{1/2} \frac{dx}{2x} = 1 - \frac{\ln 2}{2} + \frac{\ln a}{2}. \qquad \square$$

Proof of Theorem 6. We define the convolution of two functions on Γ' by

$$(h_1 * h_2)(n) = |\Omega|^{-1/2} \sum_{l \in \Gamma'} h_1(l) h_2(n-l).$$

It is well known that $\widehat{(fg)} = \widehat{f} * \widehat{g}$. Inequality (8) is a consequence of the estimate

$$\|\widehat{S_0} * f\|_{l_p(\Gamma')} \leq C \|f\|_{l_{p'}(\Gamma')}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{9}$$

where

$$\widehat{S_0}(n) = |\Omega|^{-1/2} \int_\Sigma e^{-i\langle n, x\rangle} \psi(x)\, dS(x), \quad n \in \Gamma',$$

is the Fourier transform of the measure $\psi\, dS$. Indeed, assume that (9) is true. The square of the left-hand side of (8) equals

$$\Big| \int_\Sigma \check{f} \bar{\check{f}} \psi\, dS \Big| = \Big| \sum_{n \in \Gamma'} \big(\widehat{S_0} * f\big)(n) \overline{f(n)} \Big| \leq \|\widehat{S_0} * f\|_{l_p(\Gamma')} \|f\|_{l_{p'}(\Gamma')} \leq C \|f\|^2_{l_{p'}(\Gamma')}.$$

Now we prove (9). Let

$$R_j = \{n \in \Gamma' \colon\ 2^j < \max\{|n_1|, \dots, |n_d|\} \leq 2^{j+1}\}, \quad j \in \mathbb{N};$$
$$K_j(n) = \begin{cases} \widehat{S}_0(n) & \text{if } \ n \in R_j; \\ 0 & \text{otherwise.} \end{cases}$$

Also, put $K_0(n) = \widehat{S}_0(n) - \sum_j K_j(n)$. Note that $|K_0(n)| \leq C$ and $K_0(n) \neq 0$ only for a finite set of $n \in \Gamma'$.

We are going to estimate the convolutions $K_j * f$ for each $j > 0$ separately. The estimate in $l_\infty(\Gamma')$ is straightforward,

$$\|K_j * f\|_{l_\infty(\Gamma')} \leq |\Omega|^{-1/2} \|K_j\|_{l_\infty(\Gamma')} \|f\|_{l_1(\Gamma')} \leq C j 2^{-j(d-1)/2} \|f\|_{l_1(\Gamma')}, \tag{10}$$

where the last inequality follows from Proposition 4, Remark 5, and the definition of K_j. For odd d, the estimate is also valid without the factor j on the right-hand side in (10). So, we have

$$\|K_j * f\|_{l_2(\Gamma')} = \|\check{K}_j \cdot \check{f}\|_{L_2(\Omega)} \leq \|\check{K}_j\|_{L_\infty(\Omega)} \|\check{f}\|_{L_2(\Omega)} = \|\check{K}_j\|_{L_\infty(\Omega)} \|f\|_{l_2(\Gamma')}. \tag{11}$$

The main technical part of the proof is the estimate

$$\|\check{K}_j\|_{L_\infty(\Omega)} \leq C 2^j j^{d-1}. \tag{12}$$

Assuming that (12) is proved, we can apply the Riesz–Torin interpolation theorem (see, e.g., [1]) to (10), (11). For $2 \leq p \leq \infty$ we have

$$\|K_j * f\|_{l_p(\Gamma')} \leq C 2^{2j/p} j^{2(d-1)/p} 2^{-j(d-1)(1-2/p)/2} j^{1-2/p} \|f\|_{l_{p'}(\Gamma')}. \tag{13}$$

Note that a similar estimate is also fulfilled for K_0,

$$\|K_0 * f\|_{l_p(\Gamma')} \le C\|f\|_{l_{p'}(\Gamma')}.$$

If $1 \le p' < \frac{2d+2}{d+3}$, then the exponent in (13) is negative, and (9) can be obtained by summing over j. □

Proof of estimate (12). By construction, we have

$$\check{K}_j(x) = |\Omega|^{-1/2} \sum_{n\in R_j} \widehat{S}_0(n) e^{i\langle n,x\rangle} = |\Omega|^{-1} \int_\Sigma \sum_{n\in R_j} e^{i\langle n,x-y\rangle}\psi(y)\,dS(y). \tag{14}$$

From (4) and (6) it follows that the modulus of the integrand equals

$$\begin{aligned} \psi(y)\Big|\sum_{n\in R_j}\prod_{l=1}^d e^{2\pi i n_l(x_l-y_l)}\Big| &= \psi(y)\Big|\prod_{l=1}^d \sum_{|n_l|\le 2^{j+1}} e^{i2\pi n_l(x_l-y_l)} - \prod_{l=1}^d \sum_{|n_l|\le 2^j} e^{i2\pi n_l(x_l-y_l)}\Big| \\ &\le \psi(y)\Big(\prod_{l=1}^d \Big|\frac{\sin\big(\pi(2^{j+2}+1)(x_l-y_l)\big)}{\sin(\pi(x_l-y_l))}\Big| + \prod_{l=1}^d \Big|\frac{\sin\big(\pi(2^{j+1}+1)(x_l-y_l)\big)}{\sin(\pi(x_l-y_l))}\Big|\Big). \end{aligned} \tag{15}$$

Integration in (14) is over $\Sigma \subset \Omega$. The cell Ω corresponds to the cube $[0;1)^d$ in the coordinates y_l, see (4). There exists a finite open covering $\Sigma \subset \cup_{k=1}^N B_k$ together with the corresponding partition of unity $\{\psi_k\}$ such that the projection $\Sigma_k = \Sigma \cap B_k$ onto the coordinate hyperplane $y_{m_k} = 0$ is a diffeomorphism onto its image U_k. It suffices to estimate the integrals over Σ_k for each k separately. Without loss of generality, we may assume that $m_k = d$, $U_k \subset (0;1)^{d-1}$. Both terms on the right-hand side of (15) are estimated in the same way, so we shall estimate only the second. The last factor in this term attains its maximal value at $y_l = x_l$, and therefore, does not exceed $2^{j+1}+1$. Thus, the integral (14) can be estimated by

$$\begin{aligned} &C\,(2^{j+1}+1)\int_{U_k}\prod_{l=1}^{d-1}\Big|\frac{\sin\big(\pi(2^{j+1}+1)(x_l-y_l)\big)}{\sin(\pi(x_l-y_l))}\Big| g_k(y')\,dy_1\dots dy_{d-1} \\ &\qquad \le C'\,2^j\int_{[0;1]^{d-1}}\prod_{l=1}^{d-1}\Big|\frac{\sin\big(\pi(2^{j+1}+1)(x_l-y_l)\big)}{\sin(\pi(x_l-y_l))}\Big|\,dy_1\dots dy_{d-1}, \end{aligned} \tag{16}$$

where g_k is the Jacobian of the projection, U_k is the image of Σ_k, and $y' = (y_1,\dots,y_{d-1})$. The last integral does not depend on x, so we can assume that $x = 0$. By Lemma 7,

$$\Big(\int_0^1 \Big|\frac{\sin(\pi(2^{j+1}+1)t)}{\sin(\pi t)}\Big|\,dt\Big)^{d-1} = 2^{d-1}\Big(\int_0^{1/2}\Big|\frac{\sin(\pi(2^{j+1}+1)t)}{\sin(\pi t)}\Big|\,dt\Big)^{d-1} \le Cj^{d-1}. \tag{17}$$

Now, estimate (12) is a consequence of (14), (15), (16), and (17). □

The following standard construction allows us to extend Theorem 6 to the case where $\psi = 1$.

Theorem 8. *Let $\Sigma \subset \mathbb{T}^d$ be a compact $(d-1)$-dimensional C^l-submanifold with or without boundary, and let $l = \big[\frac{d+4}{2}\big]$. Assume that the curvature of Σ does not vanish anywhere, including $\partial\Sigma$. Then*

$$\|\check{f}\|_{L_2(\Sigma)} \le C_{p'}\|f\|_{l_{p'}(\Gamma')}, \quad 1 \le p' < \frac{2d+2}{d+3}, \quad f \in \mathcal{S}(\Gamma'). \tag{18}$$

Proof. By definition, the hypersurface Σ is a union of hypersurfaces Σ_k of the following type: for each k a domain $D_k \subset \mathbb{R}^{d-1}$ is given with a C^l-smooth boundary and such that Σ_k is a graph of a C^l-smooth function of $d-1$ variables,

$$\Sigma_k = \{x : \ x_{m_k} = w_k(x'), x' \in \bar{D}_k\}, \text{ where } x' = (x_1, \dots, x_{m_k-1}, x_{m_k+1}, \dots, x_d).$$

The curvature of Σ being nonzero means that the Hesse matrix $\frac{\partial^2 w_k}{\partial x_i \partial x_j}$ is nonsingular.

It suffices to prove the theorem for each Σ_k separately. Let $\widetilde{D}_k$ be a neighborhood of $\bar{D}_k$. The function w_k can be extended into $\widetilde{D}_k$ with the same smoothness, which gives an extension of Σ_k. Since the curvature is a continuous function, there exists a neighborhood D'_k such that $\bar{D}_k \subset D'_k \subset \widetilde{D}_k$ and the curvature in D'_k is still nonzero. Now, consider a smooth function ψ_k equal to 1 on $\bar{D}_k$ and supported in D'_k. Clearly, we have constructed a hypersurface $\Sigma'_k \supset \Sigma_k$ and a function $\psi_k \in C_0^l(\Sigma'_k)$ satisfying the assumptions of Theorem 6. Therefore,

$$\|\check{f}\|_{L_2(\Sigma_k)} \le \|\check{f}\|_{L_2(\Sigma'_k, \psi_k dS)} \le C_{p'}^{(k)} \|f\|_{l_{p'}(\Gamma')}. \qquad \square$$

Corollary 9. *Under the assumptions of Theorem* 8*, we have*

$$\|\check{f}\|_{L_q(\Sigma)} \le C_{q,r} \|f\|_{l_r(\Gamma')}, \ \ 1 \le r < \frac{2d+2}{d+3}, \ \ 2 \le q < \Big(\frac{d-1}{d+1}\Big) r', \ \ \frac{1}{r} + \frac{1}{r'} = 1. \tag{19}$$

Proof. The Fourier transform satisfies

$$\|\check{f}\|_{L_\infty(\Sigma)} \le |\Omega|^{-1/2} \|f\|_{l_1(\Gamma')}.$$

Using the Riesz–Torin theorem, we can interpolate this inequality and (8) (assuming that p' is sufficiently close to $\frac{2d+2}{d+3}$) to get (19). $\square$

§3. Proof of Theorem 2

The periodic hypersurface Σ can be regarded as a hypersurface in the torus $\mathbb{T}^d = \mathbb{R}^d/\Gamma$ satisfying the assumptions of Theorem 8, or a hypersurface in the cell Ω given by (4), which is also identified with the torus. For periodic functions, the class $L_{p,\mathrm{loc}}(\Sigma)$ coincides with $L_p(\Sigma \cap \Omega)$. The well-known Thomas approach (see, e.g., [15, 2, 11]) gives a sufficient condition for the spectrum of H_σ to be absolutely continuous. Namely, introduce a family of operators $H_\sigma(\xi)$, where $\xi \in \mathbb{C}^d$ is called the quasimomentum. Consider the following family of quadratic forms

$$h_\sigma(\xi)[v,v] = \int_\Omega \big\langle (\nabla + i\xi)v(x), (\nabla + i\bar{\xi})v(x) \big\rangle \, dx + \int_{\Sigma \cap \Omega} \sigma(x) |v(x)|^2 \, dS(x)$$

defined on the periodic Sobolev space $\widetilde{H}^1(\Omega)$. These forms are sectorial (and also real for real ξ) in the sense of Kato [8]. Therefore, they determine a family of m-sectorial operators $H_\sigma(\xi)$. Assume that $|b_1| = 1$ (this is not really a restriction because the statements of the results are dilation-invariant). Then the Thomas criterion states that the spectrum of H_σ will be absolutely continuous if the following theorem holds.

Theorem 10. *Assume that the conditions of Theorem* 2 *are satisfied. Then for any* $\lambda \in \mathbb{C}$ *and* $\xi \in \mathbb{R}^m$, $\xi \perp b_1$, *there exists* τ_0 *such that for any* $|\tau| > \tau_0$ *the operator* $\big(H_\sigma((\pi + i\tau)b_1 + \xi) - \lambda I\big)$ *is invertible and*

$$\Big\| \big(H_\sigma((\pi + i\tau)b_1 + \xi) - \lambda I\big)^{-1} \Big\| \le C|\tau|^{-1}. \tag{20}$$

For simplicity, we denote $H_\sigma((\pi + i\tau)b_1 + \xi)$ by $H(\tau)$. Let $H_0(\tau)$ denote the free operator (i.e., $H(\tau)$ with $\sigma = 0$). In the Fourier representation (7), the operator $H_0(\tau)$ acts as an operator of multiplication by the symbol

$$h_n(\tau) = |n + \pi b_1 + \xi|^2 - \tau^2 + 2i\tau \langle n + \pi b_1, b_1 \rangle, \quad n \in \Gamma',$$

in $L_2(\Gamma')$. We have

$$|h_n(\tau)| \geq |\operatorname{Im} h_n(\tau)| = |2\tau\left(\langle n, b_1\rangle + \pi\right)| \geq 2\pi|\tau| \tag{21}$$

because $\langle n, b_1\rangle \in 2\pi\mathbb{Z}$. Therefore, for $|\tau| > 0$, the operator $H_0(\tau)$ is invertible and

$$\|H_0(\tau)^{-1}\| \leq (2\pi|\tau|)^{-1}, \quad \tau \neq 0.$$

The following proposition is a particular case of [2, Theorem 3.1].

Proposition 11. *Suppose $d \geq 4$, $\gamma > 0$. Then*

$$\left\|\{h_n(\tau)^{-1}\}_{n\in\Gamma'}\right\|_{l_{d-2+\gamma}(\Gamma')} \leq C(\gamma, \varkappa)|\tau|^{-\varkappa} \tag{22}$$

for any $\varkappa$ with

$$0 < \varkappa < \varkappa_0(\gamma) = \frac{\gamma}{d-2+\gamma}.$$

The constant C does not depend on τ.

For $\tau > 0$, we introduce the operator $|H_0(\tau)|^{-1/2}$ with the symbol $|h_n(\tau)|^{-1/2}$. The following lemma is the key part of the proof; it employs the results of §2.

Lemma 12. *Suppose that the hypersurface Σ satisfies the conditions of Theorem 2. Let $u \in L_2(\Omega)$. Then the trace $\left(|H_0(\tau)|^{-1/2}u\right)\big|_\Sigma$ is well defined, and for $p > 6$ we have*

$$\left\||H_0(\tau)|^{-1/2}u\right\|_{L_{2p'}(\Sigma\cap\Omega)} \leq C(\tau)\|u\|_{L_2(\Omega)}, \quad C(\tau) \to 0, \quad |\tau| \to +\infty. \tag{23}$$

Proof. For any fixed $\tau \neq 0$ we have $|h_n(\tau)| \geq C(\tau)|n|^2$. This means that for $u \in L_2(\Omega)$ we have $|H_0(\tau)|^{-1/2}u \in \widetilde{H}^1(\Omega)$, and the elements of the last Sobolev space admit well-defined traces on Σ.

Note that $p > 6$, whence $2 \leq 2p' < \frac{12}{5}$. Using Corollary 9, Hölder inequality, and Proposition 11, we get

$$\begin{aligned}\left\||H_0(\tau)|^{-1/2}u\right\|_{L_{2p'}(\Sigma)} &\leq C\left\|\{|h_n(\tau)|^{-1/2}\widehat{u}_n\}\right\|_{l_r(\Gamma')}\\ &\leq C\left\|\{|h_n(\tau)|^{-1}\}\right\|^{1/2}_{l_{2+\gamma}(\Gamma')}\|\widehat{u}\|_{l_2(\Gamma')} \leq C(\gamma,\varkappa)|\tau|^{-\varkappa/2}\|u\|_{L_2(\Omega)}.\end{aligned}$$

Here r is defined by

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{4+2\gamma},$$

and $\gamma > 0$ is sufficiently small to satisfy

$$r < \frac{2d+2}{d+3} = \frac{10}{7}, \quad 2p' < \left(\frac{d-1}{d+1}\right)r' = \frac{3r'}{5}.$$

This choice of γ is possible because the two inequalities are valid for $\gamma = 0$ (they become $r = \frac{4}{3} < \frac{10}{7}$ and $p > 6$). □

Proof of Theorem 10. The remaining part of the proof is standard (see, e.g., [2] or [9]). Consider the polar decomposition

$$H_0(\tau) = \Phi_0(\tau)|H_0(\tau)|.$$

Let $v_1 \in \operatorname{Dom}(H(\tau))$, $\|v_1\| = 1$. Also, put

$$v_2 = \Phi_0(\tau)v_1. \tag{24}$$

We have

$$(H_0(\tau)v_1, v_2) = \left\||H_0(\tau)|^{1/2}v_1\right\|^2_{L_2(\Omega)} = \left\||H_0(\tau)|^{1/2}v_2\right\|^2_{L_2(\Omega)}.$$

From (21) it follows that

$$(H_0(\tau)v_1, v_2) = (|H_0(\tau)|v_1, v_1) \geq 2\pi|\tau|.$$

To prove Theorem 10, it suffices to show that for any $v_1 \in \mathrm{Dom}(H(\tau))$, $\|v_1\| = 1$, there exists v_2 (which in fact will be given by (24)), $\|v_2\| = 1$, such that

$$|(H(\tau)v_1, v_2) - \lambda(v_1, v_2)| \geq C|\tau|.$$

The Hölder inequality and Lemma 12 imply

$$\begin{aligned}\left|\int_{\Sigma\cap\Omega} \sigma(x)v_1(x)\overline{v_2(x)}\,dS(x)\right| &\leq \|\sigma\|_{L_p(\Sigma\cap\Omega)}\|v_1\|_{L_{2p'}(\Sigma\cap\Omega)}\|v_2\|_{L_{2p'}(\Sigma\cap\Omega)}\\ &\leq \widetilde{C}(\tau)\big\||H_0(\tau)|^{1/2}v_1\big\|_{L_2(\Omega)}\big\||H_0(\tau)|^{1/2}v_2\big\|_{L_2(\Omega)},\end{aligned}$$

where $\widetilde{C}(\tau) = C(\tau)^2\|\sigma\|_{L_p(\Sigma\cap\Omega)}$, and $C(\tau)$ is the constant occurring in (23). Therefore,

$$\begin{aligned}|(H(\tau)v_1, v_2) - \lambda(v_1, v_2)| &\geq (H_0(\tau)v_1, v_2) - |\lambda| - \left|\int_{\Sigma\cap\Omega} \sigma(x)v_1(x)\overline{v_2(x)}\,dS(x)\right|\\ &\geq (H_0(\tau)v_1, v_2) - |\lambda| - \widetilde{C}(\tau)\|\,|H_0(\tau)|^{1/2}v_1\|_{L_2(\Omega)}\|\,|H_0(\tau)|^{1/2}v_2\|_{L_2(\Omega)}\\ &= (H_0(\tau)v_1, v_2) - |\lambda| - \widetilde{C}(\tau)(H_0(\tau)v_1, v_2) \geq 2\pi|\tau|(1 - \widetilde{C}(\tau)) - |\lambda|,\end{aligned}$$

and this gives the required result because $\widetilde{C}(\tau) \to 0$, and λ is fixed. □

§4. Appendix: Sketch of the proof of Proposition 4

We are going to briefly explain the appearance of the additional factor $\log(1 + |\xi|)$ in (3) if we do not assume Σ to be infinitely smooth. For this, we give the main steps of the proof of Proposition 4; the full proof (for the C^∞-case) can be found in [14].

Assume that $\Sigma \in C^l$. Using a partition of unity and the representation (1), we can rewrite the integral (3) as a sum of integrals of the form

$$\int_{\mathbb{R}^{d-1}} e^{-i\lambda\Phi(x,\eta)}\psi_1(x)\,dx, \quad \Phi(x,\eta) = x_1\eta_1 + \cdots + x_{d-1}\eta_{d-1} + w(x)\eta_d, \tag{25}$$

where $x = (x_1, \dots, x_{d-1})$, $\psi_1 \in C_0^{l-1}(\mathbb{R}^{d-1})$, $\xi = \lambda\eta$, $|\eta| = 1$, and $\lambda > 0$. The original problem reduces to studying the asymptotics of the integrals (25) as $\lambda \to +\infty$. This can be done by using the multidimensional stationary phase method. It is possible to choose a partition of unity in such a way that, for each η, the function Φ has at most one critical point (i.e., a point where $\nabla_x\Phi(x,\eta) = 0$). This critical point will be nondegenerate because of the nonzero curvature assumption. Using the Morse lemma (see, e.g., [10] for the case of finite smoothness; in this case the Jacobi matrix of the coordinate transform is C^{l-2}-smooth), we can reduce the study of the asymptotics in the case of one critical point to the following lemma with $n = d - 1$, $r = l - 2$.

Lemma 13. *Suppose $\psi \in C_0^r(\mathbb{R}^n)$, $n \geq 2$, $r = \left[\frac{n+1}{2}\right]$. Let*

$$Q(x) = x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2.$$

Then

$$\left|\int_{\mathbb{R}^n} e^{i\lambda Q(x)}\psi(x)\,dx\right| \leq C\lambda^{-n/2}\ln\lambda.$$

For odd n, this estimate holds true with $C\lambda^{-n/2}$ on the right-hand side.

Proof. Let $h \in C_0^\infty(\mathbb{R}^n)$, $h(x) = 1$ for $|x| \leq 1$, $h(x) = 0$ for $|x| \geq 2$. Consider also the nonnegative homogeneous functions $h_j(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $h_j(\lambda x) = \lambda h_j(x)$, $\lambda > 0$, such that $h_j(x) = 0$ for $|x_j|^2 \leq \frac{|x|^2}{2n}$, $\sum_j \eta_j(x) = 1$ for $x \neq 0$. Because of homogeneity, we have

$$\left|\left(\frac{\partial}{\partial x_j}\right)^m h_j(x)\right| \leq C_m|x|^{-m}. \tag{26}$$

Let $\varepsilon = \lambda^{-1/2}$. It suffices to estimate the integrals

$$\int_{\mathbb{R}^n} e^{i\lambda Q(x)}\psi(x)h_j(x)\,dx$$
$$= \int_{\mathbb{R}^n} e^{i\lambda Q(x)}\psi(x)h_j(x)h(x/\varepsilon)\,dx + \int_{\mathbb{R}^n} e^{i\lambda Q(x)}\psi(x)h_j(x)(1-h(x/\varepsilon))\,dx.$$

The first term does not exceed $C\varepsilon^n = C\lambda^{-n/2}$. To estimate the second term, we integrate by parts, obtaining

$$\int_{|x|\geq\varepsilon} e^{i\lambda Q(x)}\Big(\frac{\partial}{\partial x_j}\frac{1}{\pm 2i\lambda x_j}\Big)^N \{\psi(x)h_j(x)(1-h(x/\varepsilon))\}\,dx,$$

where the signs $\pm$ are the same as the signs at the x_j in $Q(x)$. The integrand can be estimated by $\lambda^{-N}|x|^{-2N}$. Indeed, the support of the integrand is a subset of the set on which $h_j(x) \neq 0$, and the last inequality can only be true if $|x| \leq \sqrt{2n}|x_j|$. Note that if $m > 0$, then (26) is also true for the function $h(x/\varepsilon)$, because on the support of the left hand side we have $\varepsilon < |x| < 2\varepsilon$. Therefore, each integration by parts adds at most one factor of the order of $|x|^{-1}$. Hence, the absolute value of the integral can be estimated by

$$C\lambda^{-N}\int_{\varepsilon}^{R} t^{n-1-2N}\,dt,$$

where $R = \operatorname{diam}\operatorname{supp}\psi$. The choice $N = \left[\frac{n+1}{2}\right]$ completes the proof. If n is odd, then we have a logarithmic factor, which will disappear if we increase N by 1 (but this will require ψ to be more smooth). □

If the domain of integration does not contain critical points, then the integral (25) can be estimated in a standard way by using integration by parts. This does not impose any additional smoothness assumptions.

Added in proof. After the paper was sent into print, the author found out that the result on absolute continuity can be improved by using Theorem 3 from [4]. This will be published in a separate paper.

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